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An allocation rule for graph machine scheduling problems

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Abstract

This paper studies a new type of interactive Operations Research problem, called a graph machine scheduling problem (GMS-problem). A GMS-problem combines aspects from minimum cost spanning tree problems and sequencing problems. Given a graph, we aim to first establish a connection order on the players such that the total cost of connecting them to a source is minimal and second to find a cost allocation of such an optimal order among the players involved. We restrict attention to GMS-problems on trees and propose a recursive method to solve these tree GMS-problems integrated with an allocation approach. This latter mechanism consistently and recursively uses myopic reference orders to determine potential cost savings, which will then be appropriately allocated. Interestingly, the transition process from a myopic reference order to an optimal one will be smooth using the switching of blocks of agents based on the basic notion of merge segments.

Keywords: Scheduling; Connection problems; Sequencing problems; Graph machine scheduling problems; Cost allocation

1. Introduction

As argued in Bergantiños et al. (2014), there are many real-life problems that require the construction of infrastructures to connect a set of agents to a source, either directly or indirectly. One of them is the urban supply of water from a general reservoir to certain points of interest (agents), which involves building pipelines throughout a city. Installing pipes between two points takes a certain amount of time. The first problem that arises in this kind of situation is the question of where the pipelines should be. The objective will thus be to connect all the agents to the network in such a way that the total time involved is minimized. The construction time can be interpreted as a cost to be minimized. To address this problem, the standard minimum cost spanning tree (MCST) setting has been widely applied (see, for example, Curiel, 1997 or Bergantiños and Lorenzo,

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2004). In this way, the focus is on determining so-called MCSTs. A tree is a set of edges such that there is a single path from the source to each of the agents, and the cost of a tree is the sum of the costs of all the edges belonging to it. Other real-world situations that can be modeled as an MCST problem can be found in Claus and Kleitman (1973). MCSTs can be computed in a polynomial time and the most common methods are Kruskal's algorithm (Kruskal, 1956) and Prim's algorithm (Prim, 1957). A further issue of relevance in an interactive optimization setting is the allocation of costs among the different agents involved. An adequate allocation will serve to establish and maintain cooperation between the agents. Out of the extensive literature on MCST problems that approach the cost allocation issue, we mention Claus and Kleitman (1973), Bird (1976), Granot and Huberman (1981, 1984), Feltkamp et al. (1994), Curiel (1997), Kar (2002), Dutta and Kar (2004), Fernández et al. (2004), Norde et al. (2004), Moretti et al. (2004), Tijs et al. (2006), Estévez-Fernández and Reijnierse (2014), and Gómez-Rúa and Vidal-Puga (2017). Bergantiños and Vidal-Puga (2021) is a recent review.

Looking back at the urban supply of water, it is often essential for the agents to be provided with water at all times (e.g., a hospital), so they have to contract an external service company for as long as the water supply does not reach them. Thus, each agent has an associated coefficient that indicates the cost per unit of time in the system, i.e., per unit of time for which the pipes that connect it to the source are not yet constructed. The total construction time will depend on *when* this agent is connected to the source: for example, if agent 2 is connected to the source via agent 1, then the pipeline connecting the source to agent 1 must be constructed first, followed by the pipeline connecting agent 1 to agent 2. Hence, the total time required to connect agent 2 to the source would be the sum of the construction times of the pipelines connecting the source to agent 1, and agent 1 to agent 2. Thus, the objective is to minimize the total aggregate costs instead of just the total construction time for the project as a whole.

Situations such as those described above result in a new type of problem, which we have called the graph machine scheduling problem (GMS-problem). One issue we would like to highlight is the proximity of our problem to a sequencing problem, of which we will give a brief description below.

In deterministic one-machine sequencing problems, a set of jobs needs to be processed on a machine. Each of these jobs is identified with one agent and has associated with it: a processing time, i.e., the time needed by the machine to process that specific job, and a cost function, which indicates how costly it is for that agent to spend a unit of time in the system. The main objective in sequencing problems consists in finding an optimal order, i.e., an order on the jobs that minimizes the total aggregated cost of all agents. The cost of an agent will naturally depend on its completion time, and this dependence is linear in the classical model (Smith, 1956). There are, however, numerous variants of sequencing problems that allow for an adaptation to real situations. One of them is the sequencing problem with precedence constraints, where some jobs need to be processed before others, as analyzed in Baker (1971), Sidney (1975), and Hamers et al. (2005). Other variants can be found in Baker and Su (1974), Cheng and Gupta (1989), Serafini (1996), and Schmidt (2000). In most interactive sequencing problems, an initial order is assumed as a starting point and the focus is on the allocation of cost savings with respect to this initial order. For the classical model, Curiel et al. (1989)

introduces and axiomatically characterizes an allocation rule, the *equal gain splitting rule* (EGS-rule), based on neighbor switches to derive an optimal order from the initial one, which was later generalized in Hamers et al. (1996). Sequencing problems have been extensively dealt with from a game-theoretic perspective, see Borm et al. (2002), Calleja et al. (2002, 2006), Slikker (2005), Estévez-Fernández et al. (2008), Çiftçi et al. (2013), Curiel (2015), Musegaas et al. (2018), Saavedra-Nieves et al. (2020), and Schouten et al. (2021).

A GMS-problem is closely related to an MCST problem. However, in a GMS-problem the costs are computed in a different way. In particular, the order in which the edges are activated has a substantial effect on the cost in our setup, whereas this order is irrelevant in calculating the costs in an MCST problem. Besides, the GMS-problem is deeply linked to sequencing problems with precedence constraints. Although interactive sequencing problems with precedence constraints have been treated in the literature before (see Hamers et al., 2005), the approach under which we will study them here has, to the best of our knowledge, never been adopted. In our setting, we start from a graph $(N \cup \{0\}, E)$ where N is the set of nodes corresponding to the agents, 0 is the source node that must serve all agents, and E is a set of edges connecting the nodes. Each edge has a specific activation time, and each agent has a cost depending on the time it gets connected. We will aim, on the one hand, to find an optimal connection order on the agents such that the corresponding total aggregate connection costs over all players are minimized. Note that a connection order on the agents induces an activation order on the edges. On the other hand, we aim to find a fair allocation of these costs.

In this paper, we start by motivating the GMS-problem through a particular example and by formally describing the general problem in detail. The difficulty in obtaining an optimal connection order for the GMS-problem results in the restriction to GMS-problems on trees. We first focus and discuss a procedure to find an optimal order for network structures consisting of 2 lines arising from the source, integrated with an allocation approach. The proposed solution algorithm for these $2-$lines GMS-problems is a reformulation of the work of Sidney (1975), but including a more elaborate procedure and additional ingredients like merge segments that will be essential for our cost allocation procedure. The $2-$lines algorithm and allocation procedure serve as the basis to recursively solve $n-$lines and general tree GMS-problems. For the allocation procedure, we use a myopic reference order that will depend on the problem at hand and show that it is possible to go from the reference order to an optimal order by non-negative savings by switching blocks of agents appropriately selected on the basis of merge segments.

The remainder of this paper is organized as follows. Section 2 focuses on the general GMS-problem. Section 3 provides a solution algorithm and an allocation rule for $2-$lines GMS-problems. Sections 4 and 5 generalize the previous procedures to $n-$lines and tree GMS-problems, respectively. Finally, Section 6 summarizes the main conclusions of this work.

2. Problem description and motivation

The problem we address in this paper is the following. There is a node 0 that provides a certain service, a finite set of nodes N that must be (directly or indirectly) connected to 0 in order to receive the service, and a set of edges $E \subseteq \{\{i, j\} \mid i, j \in N \cup \{0\}, i \neq j\}$ such that $(N \cup \{0\}, E)$ is a connected graph. The edges are initially "inactive" and each has an activation time. In addition, each node has an associated parameter indicating the unit cost of not being connected to node 0. The objective pursued is twofold:

1. First, we aim to choose a *connection order* on the nodes of N so that the sum of the costs over all nodes, the *total connection cost*, is as low as possible. This order will lead to a set of edges in E to be activated that connects all nodes to 0. Once we have found an optimal solution, we know the costs required to connect each node of N to 0 according to that solution: the *minimal connection cost*.
2. The second objective is to propose a fair allocation of the minimal connection cost among the nodes of N.

Let us now treat a simple example that motivates the need to address these two issues, and also to better appreciate the complexity of the problem at hand.

Example 2.1. Consider a situation with three cities (1, 2, and 3) that need to be connected to a service node 0 as soon as possible. The graph in Figure 1 represents the possible connections between the nodes and the times required to activate each of these connections. The unit cost parameters associated with each city is 1. As mentioned above, the first objective is to connect the three cities to the service node (the 0 node) while minimizing the total connection cost. Table 1 displays the possible connection orders on the nodes, along with the corresponding edges to be sequentially activated, the individual cost vector (whose coordinates represent the cost of cities 1, 2, and 3, respectively), and the total aggregated costs.

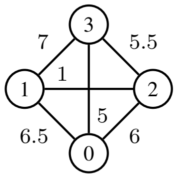


Fig. 1: Example of a GMS-problem.

Order	Edges to be activated	Individual cost vector	Total cost
$(1,2,3)$	$\{0,1\},\{1,2\},\{0,3\}$	$(6.5, 7.5, 12.5)$	26.5
$(1,3,2)$	$\{0,1\},\{0,3\},\{1,2\}$	$(6.5, 12.5, 11.5)$	30.5
$(2,1,3)$	$\{0,2\},\{2,1\},\{0,3\}$	$(7, 6, 12)$	25
$(2,3,1)$	$\{0,2\},\{0,3\},\{2,1\}$	$(12, 6, 11)$	29
$(3,1,2)$	$\{0,3\},\{0,1\},\{1,2\}$	$(11.5, 12.5, 5)$	29
$(3,2,1)$	$\{0,3\},\{3,2\},\{2,1\}$	$(11.5, 10.5, 5)$	27

Table 1: Possible orders for the GMS-problem from Figure 1.

By considering the 6 possible orders, one sees that the optimal connection order is $(2,1,3)$. Correspondingly, the edges are activated in the order $\{0,2\},\{2,1\}$, and $\{0,3\}$. The optimal connection cost is 25, which is obtained by adding the costs of the nodes, 6 for player 2, 7 for player 1, and 12 for player 3. Note that the greedy myopic order (i.e., the order in which at each step the edge incident on the component containing 0 with the lowest time is constructed) is not optimal. This myopic order is $(3,2,1)$, which has an associated cost of 27. Nevertheless, this myopic order could be used as a reference order for cost allocation by subtracting an appropriate allocation of the savings of 2 from the myopic reference individual costs $11.5, 10.5$, and 5. △

Below we will formally present the problem under consideration, as well as some essential definitions to address it. From now on we will use the terms machine and players instead of source and nodes, respectively.

A *graph machine scheduling problem*, GMS-problem, can be summarized by a tuple $\mathcal{G} = (N, 0, E, \gamma, \alpha)$, where N is a finite set of jobs or players, 0 represents the machine, E is a set of available (precedence) edges between players and machine, i.e., $E \subseteq \{\{i,j\} \mid i,j \in N \cup \{0\}, i \neq j\}$, such that $(N \cup \{0\}, E)$ is a connected graph, $\gamma\colon E \to \mathbb{R}_+$ with $\gamma_{ij} = \gamma(\{i,j\})$ representing the activation time of the edge $\{i,j\} \in E$, and, finally, $\alpha\colon N \to \mathbb{R}_+$, with $\alpha(i)$ representing the linear cost coefficient to spend one time unit in the system for player $i \in N$.

The main assumption is that a player $i \in N$ can only be processed by the machine if all players on a path in E from i to the machine have been processed before. A processing or connection order is described by a bijection $\sigma\colon N \to \{1,\dots,|N|\}$, and $\Pi(N)$ denotes the set of all processing orders.[1] A processing order $\sigma \in \Pi(N)$ is called feasible if the aforementioned condition is met for all players. Given $\sigma \in \Pi(N)$ and $i \in N$, let $P_\sigma(i) = \{j \in N \mid \sigma(j) < \sigma(i)\}$ denote the set of predecessors of i in σ. Also, let $P^0_\sigma(i) = P_\sigma(i) \cup \{0\}$. Formally, $\sigma \in \Pi(N)$ is feasible if there exists $j \in P^0_\sigma(i)$ such that $\{i,j\} \in E$ for all $i \in N$. Let $\mathcal{F}(N)$ denote the set of all feasible orders.

Definition 2.1. Let $(N, 0, E, \gamma, \alpha)$ be a GMS-problem, and let $\sigma \in \mathcal{F}(N)$. Given $i \in N$, we define the *completion time of player i with respect to σ*, $C_i(\sigma)$, as follows:

$$C_i(\sigma) = \sum_{k \in P_\sigma(i)} C_k(\sigma) + \min\left\{\gamma_{ij} \mid j \in P^0_\sigma(i) \text{ and } \{i,j\} \in E\right\}.$$

Given $i \in N$, we define the *cost of player i with respect to σ*, $c_i(\sigma)$, as follows:

$$c_i(\sigma) = \alpha(i) \cdot C_i(\sigma).$$

Let $c(\sigma) = (c_i(\sigma))_{i \in N}$ denote the individual cost vector with respect to σ. We define the *total cost of σ*, $TC(\sigma)$, as follows:

$$TC(\sigma) = \sum_{i \in N} c_i(\sigma). \tag{1}$$

Among other things, this paper aims to determine an optimal order $\hat{\sigma} \in \mathcal{F}(N)$ that minimizes the total costs among all feasible processing orders. It is important to note that the problem of finding an optimal connection order for general graphs $(N \cup \{0\}, E)$ is hard. The optimal solution in Example 2.1 already shows some unexpected peculiarities, highlighting the potential complexity of the GMS-problem. In this paper, we will focus on GMS-problems such that $(N \cup \{0\}, E)$ is a tree.

It should be stressed that by restricting the problem to trees, there will be only one path between any two nodes. With the purpose of simplifying the notation, the GMS-problems treated from now on will be denoted by a tuple $(N, 0, E, \gamma, \alpha)$, where γ now is a function on N. In particular, for $\gamma\colon N \to \mathbb{R}_+$ we have that $\gamma(i) = \gamma_{ip^0_E(i)}$, where $p^0_E(i)$ is the first player on the unique path between the machine and i. In this way, it is easily seen that equation (1) can be reformulated as

$$TC(\sigma) = \sum_{k=1}^{|N|} \gamma(\sigma^{-1}(k)) \cdot \left(\sum_{\{j \in N \mid \sigma(j) \geq k\}} \alpha(j) \right).$$

[1]We will use the terms processing order and connection order interchangeably.

3. $2-$lines GMS-problems

In this section we describe and analyze 2-lines GMS-problems. We will first tackle the question of how to find an optimal connection order for these problems, and second how to allocate the cost of such a minimal connection order among the players involved.

3.1. Optimal orders

A GMS-problem $(N, 0, E, \gamma, \alpha)$ is called a $2-$*lines GMS-problem* if there exists a partition $\langle A, B\rangle$ of N with $A = \{a_1, \ldots, a_{\tilde{s}}\}$ and $B = \{b_1, \ldots, b_{\tilde{t}}\}$ with $\tilde{s} + \tilde{t} = |N|$, such that

$$E = \big\{\{0, a_1\}, \{a_1, a_2\}, \ldots, \{a_{\tilde{s}-1}, a_{\tilde{s}}\}\big\} \cup \big\{\{0, b_1\}, \{b_1, b_2\}, \ldots, \{b_{\tilde{t}-1}, b_{\tilde{t}}\}\big\}.$$

The sets A and B are called branches. For this particular case, a feasible order is described by a bijection $\sigma\colon A \cup B \to \{1, 2, \ldots, \tilde{s} + \tilde{t}\}$ such that

$$\sigma(a_k) < \sigma(a_l) \Rightarrow k < l;$$
$$\sigma(b_k) < \sigma(b_l) \Rightarrow k < l.$$

Let $\mathcal{F}(A \cup B)$ denote the set of all such feasible orders. A graphical representation of a $2-$lines GMS-problem can be seen in Figure 2.

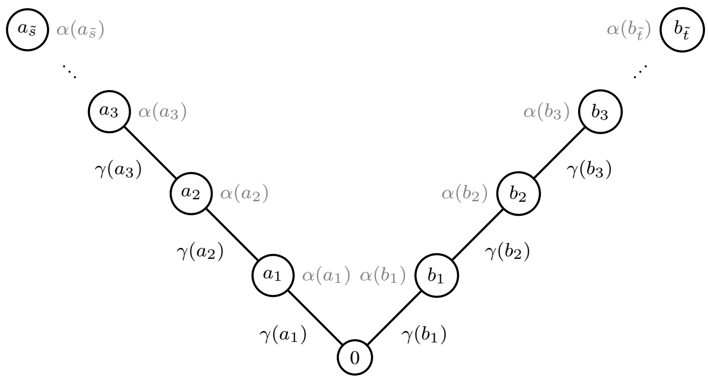


Fig. 2: Graphical representation of a $2-$lines GMS-problem.

Definition 3.1. Let $(N, 0, E, \gamma, \alpha)$ be a $2-$lines GMS-problem, and let $\sigma \in \mathcal{F}(A \cup B)$. We define a *segment of A from h to l* as the following subset of A:

$$Q_{hl} = \{a_h, a_{h+1}, \ldots, a_l\},$$

where $1 \leq h \leq l \leq \tilde{s}$. Analogously, we define a *segment of B from h to l* as the following subset of B:

$$R_{hl} = \{b_h, b_{h+1}, \ldots, b_l\},$$

where $1 \leq h \leq l \leq \tilde{t}$.

When we do not specify which branch a certain segment belongs to, we will use the notation X or Y instead of Q and R. A segment from the beginning of a branch is called a *head*.

Definition 3.2. Given a segment X, we define the *cost weighted average time per edge of* X, $CAT(X)$, as

$$CAT(X) = \frac{\sum_{i\in X}\gamma(i)}{\sum_{i\in X}\alpha(i)}.$$

Moreover, given a segment X, we define its *urgency*, $U(X)$, as

$$U(X) = \frac{1}{CAT(X)}.$$

Given a 2−lines GMS-problem, $(N, 0, E, \gamma, \alpha)$, our objective is to solve the following problem:

$$\begin{aligned} \min \quad & TC(\sigma) \\ \text{s.t.} \quad & \sigma \in \mathcal{F}(A \cup B). \end{aligned}$$

In Algorithm 1 we formally present the algorithm to solve 2−lines GMS-problems. Steps 1–3 constitute an iteration of the algorithm. The output of Algorithm 1 is a *merge order*:

$$\hat{\sigma} = (M_1, M_2, \ldots, M_{m-1}, M_m),$$

with $m \geq 2$, where $M_1, M_2, \ldots, M_m$ are called *merge segments*. It could be possible that both M_k and M_{k+1} belong to the same branch (because they might be merged at different steps). Of course, a merge order $\hat{\sigma}$ corresponds to one order on all players. If we do not want to highlight the merge segments, we will use the notation $\hat{\tau}$ for this order.

Example 3.1 shows the main ideas of this algorithm.

Example 3.1. Consider the 2−lines GMS-problem presented in Figure 3a. Figure 3b shows the steps to be followed by Algorithm 1 in the first iteration. First, $CAT(Q_{11})$ and $CAT(R_{11})$ are compared, and Q_{11} is selected as the first pivot. Since $CAT(Q_{11}) > CAT(R_{12})$, the pivot is changed to R_{12} (red arrows represent pivot transitions). After making the appropriate comparisons (see blue arrows), Q_{14} is chosen as the next pivot and, since there are no other segment in branch B with lower CAT than Q_{14}, it becomes the first merge segment, that is, $M_1 = Q_{14}$. We merge Q_{14} to the machine and renumber the nodes. This leads to the second iteration, presented in Figure 4. Since $CAT(R_{11}) < CAT(Q_{11})$, R_{11} is selected as the first pivot of this iteration and becomes the second merge segment, as there is no other possible comparison. The same happens in the third iteration, as can be seen in Figure 5. After adequately renumbering the nodes, Q_{11} and R_{11} are compared in the fourth iteration, as a result of which Q_{11} is selected as the pivot. This still needs to be compared with R_{12}. Since $CAT(Q_{11}) \leq CAT(R_{12})$, Q_{11} becomes a merge segment. Finally, we are left with one branch, so each individual is considered a separate merge segment.

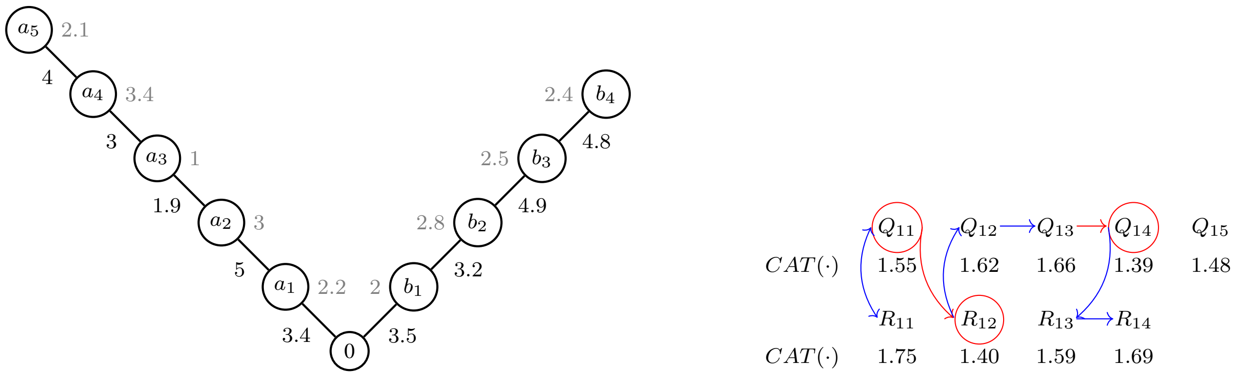
Algorithm 1 Algorithm to solve a 2−lines GMS-problem

0. Initialize $k = 1, i = 1$.
1. Consider the 2−lines GMS-problem $(N, 0, E, \gamma, \alpha)$. Initialize $l = 1, l' = 1$.
   - If $\dfrac{\gamma(a_1)}{\alpha(a_1)} \leq \dfrac{\gamma(b_1)}{\alpha(b_1)}$, select pivot $P_i = \{a_1\}$.
   - If $\dfrac{\gamma(a_1)}{\alpha(a_1)} > \dfrac{\gamma(b_1)}{\alpha(b_1)}$, select pivot $P_i = \{b_1\}$.
2. (a) If $P_i \subseteq A$, take $l = l + 1$.
   - If $l \leq \tilde{t}$, compare $CAT(P_i)$ to $CAT(R_{1l})$.
     — If $CAT(P_i) \leq CAT(R_{1l})$, go back to step 2.
     — If $CAT(P_i) > CAT(R_{1l})$, then $i = i + 1$, $P_i = R_{1l}$. Go back to step 2.
   - If $l > \tilde{t}$, go to step 3.

   (b) If $P_i \subseteq B$, take $l' = l' + 1$.
   - If $l' \leq \tilde{s}$, compare $CAT(P_i)$ to $CAT(Q_{1l'})$.
     — If $CAT(P_i) \leq CAT(Q_{1l'})$, go back to step 2.
     — If $CAT(P_i) > CAT(Q_{1l'})$, then $i = i + 1$, $P_i = Q_{1l'}$. Go back to step 2.
   - If $l' > \tilde{s}$, go to step 3.
3. Set $M_k = P_i$.

   (a) If $M_k \subseteq A$.
   - If $A \setminus M_k = \emptyset$, then $M_{k+j} = \{b_j\}$ for all $j \in \{1, \ldots, \tilde{t}\}$. The algorithm is finished.
   - Otherwise, let $A = r(A \backslash M_k)$, where $r\colon A \backslash M_k \to A$ is a renumbering function such that $r(a_h) = a_{h-l'}$, for all $h \in \{l'+1, \ldots, \tilde{s}\}$. Let $\gamma(a_h) = \gamma(a_{h+l'})$ and $\alpha(a_h) = \alpha(a_{h+l'})$ for all $h \in \{1, \ldots, \tilde{s} - l'\}$. Set $N = A \cup B$, $k = k + 1$ and $i = i + 1$. Go back to step 1.

   (b) If $M_k \subseteq B$.
   - If $B \setminus M_k = \emptyset$, then $M_{k+j} = \{a_j\}$ for all $j \in \{1, \ldots, \tilde{s}\}$. The algorithm is finished.
   - Otherwise, let $B = \tilde{r}(B \backslash M_k)$, where $\tilde{r}\colon B \backslash M_k \to B$ is a renumbering function such that $r(b_h) = b_{h-l}$, for all $h \in \{l+1, \ldots, \tilde{t}\}$. Let $\gamma(b_h) = \gamma(b_{h+l})$ and $\alpha(b_h) = \alpha(b_{h+l})$ for all $h \in \{1, \ldots, \tilde{t} - l\}$. Set $N = A \cup B$, $k = k + 1$ and $i = i + 1$. Go back to step 1.

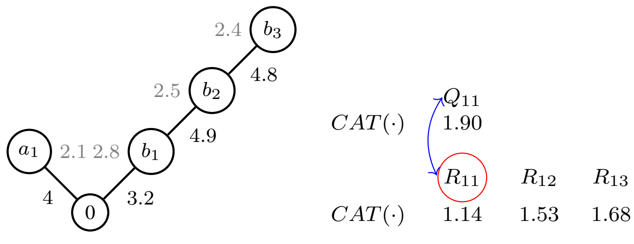


(a) Graphical representation. (b) Steps of Algorithm 1.

Fig. 3: First iteration.

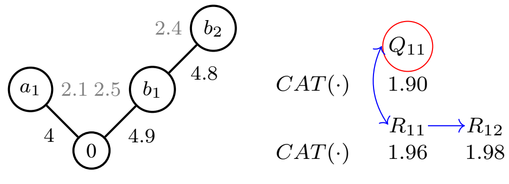
(a) Graphical representation. (b) Steps of Algorithm 1.

Fig. 4: Second iteration.



(a) Graphical representation. (b) Steps of Algorithm 1.

Fig. 5: Third iteration.



(a) Graphical representation. (b) Steps of Algorithm 1.

Fig. 6: Fourth iteration.

The algorithm outputs the following merge order[2]:

$$\hat{\sigma} = (M_1, M_2, M_3, M_4, M_5, M_6),$$

where $M_1 = \{a_1a_2a_3a_4\}$[3], $M_2 = \{b_1\}$, $M_3 = \{b_2\}$, $M_4 = \{a_5\}$, $M_5 = \{b_3\}$, and $M_6 = \{b_4\}$, which has an associated cost of $TC(\hat{\sigma}) = 381.33$. △

Next, we will show a series of results to prove that Algorithm 1 leads to an optimal order.

Remark 3.1. Given a set $Z \subseteq N$, we will use the notation $\gamma[Z] = \sum_{i \in Z} \gamma(i)$ and $\alpha[Z] = \sum_{i \in Z} \alpha(i)$. Hence, if X is a segment we can write $CAT(X) = \frac{\gamma[X]}{\alpha[X]}$.

The following proposition shows that an optimal order cannot have two consecutive segments of the same branch separated by nodes of the other branch when the CAT of the first segment is greater than the CAT of the next segment.

Proposition 3.1. *Let $(N, 0, E, \gamma, \alpha)$ be a $2-$lines GMS-problem. Let X and Y be two segments that belong to the same branch, and let Z be a segment from the other branch. If $CAT(X) > CAT(Y)$, then*

$$\tau = (\sim, X, Z, Y, \sim)$$

is not optimal.

Proof. Consider:

$$\tau_1 = (\sim, X, Y, Z, \sim);$$
$$\tau_2 = (\sim, Z, X, Y, \sim).$$

Note that

$$\begin{aligned} TC(\tau) - TC(\tau_1) &= \alpha[Y] \cdot \gamma[Z] - \alpha[Z] \cdot \gamma[Y] = \alpha[Y] \cdot \alpha[Z] \cdot \frac{\gamma[Z]}{\alpha[Z]} - \alpha[Z] \cdot \alpha[Y] \cdot \frac{\gamma[Y]}{\alpha[Y]} \\ &= \alpha[Y] \cdot \alpha[Z] \cdot (CAT(Z) - CAT(Y)), \end{aligned}$$

[2]Note that Sidney-components (see Hamers et al., 2005 for details) are not the same as merge segments. It can be checked that the Sidney-components would be $S_1 = \{a_1a_2a_3a_4\}$, $S_2 = \{b_1b_2\}$, $S_3 = \{a_5\}$, and $S_4 = \{b_3b_4\}$.
[3]We use this notation to emphasize the order of the players in the set M_1.

and

$$TC(\tau) - TC(\tau_2) = \alpha[Z] \cdot \gamma[X] - \alpha[X] \cdot \gamma[Z] = \alpha[Z] \cdot \alpha[X] \cdot \frac{\gamma[X]}{\alpha[X]} - \alpha[X] \cdot \alpha[Z] \cdot \frac{\gamma[Z]}{\alpha[Z]}$$
$$= \alpha[Z] \cdot \alpha[X] \cdot (CAT(X) - CAT(Z)).$$

Suppose for the sake of contradiction that τ is optimal. Then,

$$TC(\tau) - TC(\tau_1) \leq 0 \quad \text{and} \quad TC(\tau) - TC(\tau_2) \leq 0,$$

and hence, using the equalities above,

$$CAT(X) \leq CAT(Z) \leq CAT(Y),$$

which is a contradiction. Thus, τ cannot be optimal. □

Definition 3.3. Let $\hat{\sigma} = (M_1, M_2, \ldots, M_m)$ be the output of Algorithm 1 for a 2−lines GMS-problem and let τ be a feasible order. Let $k \in \{1, 2, \ldots, m\}$. A *component of* M_k *in* τ is defined as a maximal connected subset of M_k with respect to τ. Denote by M_k/τ the *set of components of* M_k *in* τ, and let $M_k/\tau = \{G_1, G_2, G_3, \ldots, G_{m_k}\}$ be the different components in the order they appear in τ, that is:

$$\tau = (\sim, G_1, \ldots, G_2, \ldots, G_3, \ldots, G_{m_k}, \sim).$$

Obviously, all merge segments and their components are segments. The following lemma presents different properties of merge segments and their components. Specifically, we will see that the last component is decisive for a certain segment to become a merge segment.

Lemma 3.1. *Let $\hat{\sigma} = (M_1, M_2, \ldots, M_m)$ be the output of Algorithm 1 for a 2−lines GMS-problem. Take $k \in \{1, 2, \ldots, m\}$ such that $|M_k| > 1$. Let τ be a feasible order such that $\left|M_k/\tau\right| > 1$. Then, the following holds:*

i) $CAT(G_{m_k}) < CAT(G_1 \cup G_2 \cup \cdots \cup G_{m_k - 1})$*;*
ii) $CAT(G_{m_k}) < CAT(M_k)$*;*
iii) $CAT(G_{m_k}) < CAT(G_1)$.

Proof. Since $|M_k| > 1$, it holds that M_k is not the first pivot in Algorithm 1. In order for M_k to become the new pivot and hence, the merge segment, it must have lower CAT than the previous pivot. Furthermore, the CAT of the previous pivot is less than or equal to the CAT of the combination of the first $m_k - 1$ components of M_k. This might be a direct comparison, but it could also be an indirect comparison via several other pivots. That is,

$$CAT(M_k) < CAT(G_1 \cup G_2 \cup \cdots \cup G_{m_k - 1}). \tag{2}$$

Then,

$$\frac{\gamma[G_1]+\gamma[G_2]+\cdots+\gamma[G_{m_k-1}]+\gamma[G_{m_k}]}{\alpha[G_1]+\alpha[G_2]+\cdots+\alpha[G_{m_k-1}]+\alpha[G_{m_k}]} \underset{(2)}{<} \frac{\gamma[G_1]+\gamma[G_2]+\cdots+\gamma[G_{m_k-1}]}{\alpha[G_1]+\alpha[G_2]+\cdots+\alpha[G_{m_k-1}]},$$

and, consequently,

$$\gamma[G_{m_k}]\cdot(\alpha[G_1]+\alpha[G_2]+\cdots+\alpha[G_{m_k-1}]) < (\gamma[G_1]+\gamma[G_2]+\cdots+\gamma[G_{m_k-1}])\cdot\alpha[G_{m_k}]. \tag{3}$$

Hence,

$$\frac{\gamma[G_{m_k}]}{\alpha[G_{m_k}]} < \frac{\gamma[G_1]+\gamma[G_2]+\cdots+\gamma[G_{m_k-1}]}{\alpha[G_1]+\alpha[G_2]+\cdots+\alpha[G_{m_k-1}]}.$$

Thus, $CAT(G_{m_k}) < CAT(G_1 \cup G_2 \cup \cdots \cup G_{m_k-1})$, proving i).

To prove ii), we add $\gamma[G_{m_k}]\cdot\alpha[G_{m_k}]$ on both sides of equation (3):

$$\begin{aligned}&\gamma[G_{m_k}]\cdot(\alpha[G_1]+\alpha[G_2]+\cdots+\alpha[G_{m_k-1}])+\gamma[G_{m_k}]\cdot\alpha[G_{m_k}]\\ &< (\gamma[G_1]+\gamma[G_2]+\cdots+\gamma[G_{m_k-1}])\cdot\alpha[G_{m_k}]+\gamma[G_{m_k}]\cdot\alpha[G_{m_k}],\end{aligned}$$

which results in

$$\begin{aligned}&\gamma[G_{m_k}]\cdot(\alpha[G_1]+\alpha[G_2]+\cdots+\alpha[G_{m_k-1}]+\alpha[G_{m_k}])\\ &< (\gamma[G_1]+\gamma[G_2]+\cdots+\gamma[G_{m_k-1}]+\gamma[G_{m_k}])\cdot\alpha[G_{m_k}].\end{aligned}$$

Consequently,

$$\frac{\gamma[G_{m_k}]}{\alpha[G_{m_k}]} < \frac{\gamma[G_1]+\gamma[G_2]+\cdots+\gamma[G_{m_k-1}]+\gamma[G_{m_k}]}{\alpha[G_1]+\alpha[G_2]+\cdots+\alpha[G_{m_k-1}]+\alpha[G_{m_k}]},$$

and thus $CAT(G_{m_k}) < CAT(G_1 \cup G_2 \cup \cdots \cup G_{m_k}) = CAT(M_k)$, proving ii).

To prove iii), note that $CAT(G_{m_k}) < CAT(M_k) < CAT(G_1)$, where the first inequality follows from ii) and the second inequality from Algorithm 1, see also equation (2). □

Given a non-connected merge segment, the following lemma guarantees that there will be at least one pair of consecutive components in a feasible order such that the CAT of the first component is strictly greater than the CAT of the next component.

Lemma 3.2. *Let $\hat{\sigma} = (M_1, M_2, \ldots, M_m)$ be the output of Algorithm 1 for a $2-$lines GMS-problem. Take $k \in \{1,2,\ldots,m\}$ such that $|M_k| > 1$. And let τ be a feasible order such that $\left|M_k/\tau\right| > 1$. Then, there exists $k \in \{2,3,\ldots,m_k\}$ such that $CAT(G_{k-1}) > CAT(G_k)$.*

Proof. Suppose for the sake of contradiction that $CAT(G_{k-1}) \leq CAT(G_k)$ for all $k \in \{2,3,\ldots,m_k\}$. That is,

$$CAT(G_1) \leq CAT(G_2) \leq \cdots \leq CAT(G_{m_k-1}) \leq CAT(G_{m_k}).$$

This implies that $CAT(G_1) \leq CAT(G_{m_k})$, contradicting iii) of Lemma 3.1. Hence, there exists $k \in \{2, 3, \ldots, m_k\}$ such that $CAT(G_{k-1}) > CAT(G_k)$. □

The following lemma tells us that there can be no optimal order in which merge segments have more than one component.

Lemma 3.3. *Let $\hat{\sigma} = (M_1, M_2, \ldots, M_m)$ be the output of Algorithm 1 for a 2−lines GMS-problem. It holds that the elements of $M_k, k \in \{1, \ldots, m\}$, are consecutive in any optimal order.*

Proof. Let τ be an optimal order. Suppose that there exists $k \in \{1, \ldots, m\}$ such that players from M_k are separated by other players in τ. Also consider the set of components of M_k in τ, i.e., $M_k/\tau = \{G_1, G_2, \ldots, G_{m_k}\}$, so we would have:

$$\tau = (\sim, G_1, \ldots, G_2, \ldots, G_{m_k}, \sim).$$

From Lemma 3.2, we know that there exists $\tilde{k} \in \{2, \ldots, m_k\}$ such that $CAT(G_{\tilde{k}-1}) > CAT(G_{\tilde{k}})$. Rewrite τ as follows:

$$\tau = (\sim, G_{\tilde{k}-1}, Z, G_{\tilde{k}}, \sim),$$

where Z is a segment from the other branch. From Proposition 3.1, τ is not optimal, which is a contradiction. Hence, $|M_k/\tau| = 1$ for all $k \in \{1, \ldots, m\}$. □

The proposition below states that at least one optimal order has to start with the first merge segment obtained by applying Algorithm 1.

Proposition 3.2. *Let $(N, 0, E, \gamma, \alpha)$ be a 2−lines GMS-problem. Let $\hat{\sigma} = (M_1, M_2, \ldots, M_m)$ be the output of Algorithm 1 for such problem. There always exists an optimal order that starts with M_1.*

Proof. Consider $\tau \in \mathcal{F}(A \cup B)$ such that τ does not start with M_1. We will first prove that there always exists τ^* starting with M_1 such that $TC(\tau^*) \leq TC(\tau)$.

Assume w.l.o.g. that $M_1 \subseteq A$. During Algorithm 1, we compared the CAT of M_1 with the $CATs$ of all possible segments $R_{1l}, l \in \{1, \ldots, \tilde{t}\}$. It holds that:

$$\begin{aligned} &CAT(M_1) \leq CAT(R_{11}); \\ &CAT(M_1) \leq CAT(R_{12}); \\ &\qquad\vdots \\ &CAT(M_1) \leq CAT(R_{1\tilde{t}}). \end{aligned} \tag{4}$$

In case $|M_1/\tau| = 1$, M_1 is a connected component in τ. If τ does not start with M_1, then it must start with some players from branch B, followed by M_1:

$$\tau = (R_{1\ell}, M_1, \sim),$$

where $1 \leq \ell \leq \tilde{t}$. Now, consider the following order:

$$\tau_1 = (M_1, R_{1\ell}, \sim),$$

in which we have swapped the positions of M_1 and $R_{1\ell}$. Then,

$$\begin{aligned} TC(\tau) - TC(\tau_1) &= \alpha[M_1] \cdot \alpha[R_{1\ell}] \cdot CAT(R_{1\ell}) - \alpha[R_{1\ell}] \cdot \alpha[M_1] \cdot CAT(M_1) \\ &= \alpha[M_1] \cdot \alpha[R_{1\ell}] \cdot (CAT(R_{1\ell}) - CAT(M_1)) \underset{(4)}{\geq} 0, \end{aligned} \tag{5}$$

and hence, τ is not better than τ_1.

In case $|M_1/\tau| > 1$, then we can write τ as

$$\tau = (\sim, G_1, \ldots, G_2, \ldots, G_3, \ldots, G_{m_1}, \sim),$$

with $m_1 > 1$, where $M_1/\tau = \{G_1, G_2, G_3, \ldots, G_{m_1}\}$ denotes the set of components of M_1 in τ.

From Lemma 3.3, τ cannot be optimal since the elements of M_1 are not consecutive. In particular, we know there exists a bijection

$$\rho \colon \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m\}$$

such that $\tau' = (M_{\rho(1)}, \ldots, M_{\rho(m)})$ is an optimal order. Take $\hat{i} \in \{1, \ldots, m\}$ such that $\rho(\hat{i}) = 1$. Next consider

$$\tau^* = (M_{\rho(\hat{i})}, M_{\rho(1)}, M_{\rho(2)}, \ldots, M_{\rho(\hat{i}-1)}, M_{\rho(\hat{i}+1)}, \ldots, M_{\rho(m)}).$$

Then,

$$TC(\tau^*) \leq TC(\tau') < TC(\tau),$$

where the first inequality follows from a similar reasoning that leads to (5) and the second inequality from Lemma 3.3. □

The following proposition shows that the specific structure of an optimal order leads to an optimal order of a subproblem.

Proposition 3.3. *Let $(N, 0, E, \gamma, \alpha)$ be a $2-$lines GMS-problem, and let $\tilde{\sigma}^N = (M_1, \ldots, M_m)$ be the output of Algorithm 1. Let τ^N be an optimal order. If τ^N starts with M_1, it holds that $\tau^N|_{N \setminus M_1}$ is an optimal order* *for the subproblem on $N \setminus M_1$.*

Proof. W.l.o.g., we assume that $M_1 \subseteq A$. If $M_1 = A$, then it is clear that $\tau^N = (M_1, b_1, b_2, \ldots, b_{\tilde{t}})$, where $\tau^N|_{N\setminus M_1} = (b_1, b_2, \ldots, b_{\tilde{t}})$ is the optimal order of the $1-$line GMS-problem with set of players $N \setminus M_1 = \{b_1, b_2, \ldots, b_{\tilde{t}}\}$. Let us suppose now that $M_1 \subsetneq A$. For the sake of contradiction, assume that $\tau^N|_{N\setminus M_1}$ is not an optimal order for the aforementioned subproblem. Then, there would exist $\hat{\tau}^{N\setminus M_1}$ such that

$$TC(\hat{\tau}^{N\setminus M_1}) - TC(\tau^N|_{N\setminus M_1}) < 0. \tag{6}$$

Consider the order $\hat{\tau}^N = (M_1, \hat{\tau}^{N\setminus M_1})$. Note that

$$TC(\hat{\tau}^N) - TC(\tau^N) = TC(\hat{\tau}^{N\setminus M_1}) - TC(\tau^N|_{N\setminus M_1}) \underset{(6)}{<} 0,$$

which is a contradiction because τ^N is optimal. Thus, $\tau^N|_{N\setminus M_1}$ is an optimal order for the problem with set of players $N \setminus M_1$. □

The next result describes a reverse version of Proposition 3.3: if we have a specific optimal order for a subproblem, we can derive an optimal order for the general problem.

Lemma 3.4. *Let $(N, 0, E, \gamma, \alpha)$ be a $2-$lines GMS-problem, and let $\tilde{\sigma}^N = (M_1, \ldots, M_m)$ be the output of Algorithm 1. Let $\tau^{N\setminus M_1}$ be an optimal order for the problem on $N \setminus M_1$. It holds that $\tau^N = (M_1, \tau^{N\setminus M_1})$ is an optimal order for $(N, 0, E, \gamma, \alpha)$.*

Proof. From Proposition 3.2, we know there exists an optimal order $\hat{\tau}^N$ for $(N, 0, E, \gamma, \alpha)$ that starts with M_1, so $\hat{\tau}^N = (M_1, \hat{\tau}^N|_{N\setminus M_1})$. From Proposition 3.3, $\hat{\tau}^N|_{N\setminus M_1}$ is an optimal order for the subproblem with set of players $N \setminus M_1$. For the sake of contradiction, suppose that τ^N is not optimal. Then,

$$TC(\hat{\tau}^N) - TC(\tau^N) = TC(\hat{\tau}^N|_{N\setminus M_1}) - TC(\tau^{N\setminus M_1}) < 0,$$

which contradicts $\tau^{N\setminus M_1}$ being optimal for the problem with set of players $N \setminus M_1$. □

We present below the main result of this subsection, which indicates that Algorithm 1 always leads to an optimal order.

Theorem 3.1. *Let $(N, 0, E, \gamma, \alpha)$ be a $2-$lines GMS-problem, and let $\hat{\tau}$ be the order provided by Algorithm 1. Then, $TC(\hat{\tau}) \leq TC(\tau)$ for all $\tau \in \mathcal{F}(A \cup B)$.*

Proof. The proof uses induction to the number of players, $|N|$.

Consider $|N| = 2$. We present this situation in Figure 7. In such a case, there are two possible orders, $\tau_1 = (a_1, b_1)$ and $\tau_2 = (b_1, a_1)$. Note that:

$$TC(\tau_1) = (\alpha(a_1) + \alpha(b_1)) \cdot \gamma(a_1) + \alpha(b_1) \cdot \gamma(b_1);$$
$$TC(\tau_2) = (\alpha(b_1) + \alpha(a_1)) \cdot \gamma(b_1) + \alpha(a_1) \cdot \gamma(a_1),$$

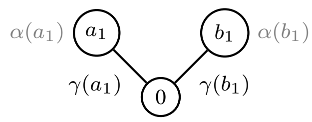


Fig. 7: A 2−lines GMS-problem with 2 players.

and thus

$$TC(\tau_1) - TC(\tau_2) = \alpha(b_1) \cdot \gamma(a_1) - \alpha(a_1) \cdot \gamma(b_1) = \alpha(b_1) \cdot \alpha(a_1) \cdot \left(\frac{\gamma(a_1)}{\alpha(a_1)} - \frac{\gamma(b_1)}{\alpha(b_1)} \right). \tag{7}$$

Algorithm 1 compares $\frac{\gamma(a_1)}{\alpha(a_1)}$ to $\frac{\gamma(b_1)}{\alpha(b_1)}$ in order to choose the first merge segment, which in this case will consist of a single node. From (7), we can see that the optimal order will be determined by the exact same comparison, thus Algorithm 1 leads to an optimal order.

Now assume that Algorithm 1 leads to an optimal order if the number of players is $k < |N|$.

Now, take $k = |N|$. Let $\hat{\sigma} = (M_1, M_2, \ldots, M_m)$ be the output of Algorithm 1 corresponding to $\hat{\tau}$. Naturally, $\hat{\sigma}|_{N \setminus M_1} = (M_2, \ldots, M_m)$ will be an output of our procedure for the problem with set of players $N \setminus M_1$. Using our induction hypothesis, $\hat{\sigma}|_{N \setminus M_1}$ is optimal for such subproblem. From Lemma 3.4, the order $(M_1, \hat{\sigma}|_{N \setminus M_1})$ is optimal. Clearly, $\hat{\sigma} = (M_1, \hat{\sigma}|_{N \setminus M_1})$, finishing the proof. □

3.2. Allocating the minimal cost

This subsection introduces the κ rule as a cost allocation rule for 2−lines GMS-problems. The κ rule takes as a reference point a myopic connection order and its corresponding cost vector and will subtract a specific allocation vector of the cost savings as given by the block splitting rule (BSR), which will be described later. The underlying allocation procedure is closely tied to the theoretical results presented in Subsection 3.1. In particular, the merge segments will be the foundation of the allocation procedure that we will discuss. Below, we explain the ideas behind the κ rule in more detail.

Let $\mathcal{G} = (N, 0, E, \gamma, \alpha)$ be a 2−lines GMS-problem. The order that we will use as a reference point is an endogenous and myopic order τ_0. It will depend on the particular problem we are considering, in the following way: at each step, the machine selects the player that has a higher urgency, always taking into account the existing precedence relations. For expositional simplicity, we will assume that $\frac{\alpha(i)}{\gamma(i)} \neq \frac{\alpha(j)}{\gamma(j)}$ for all $i, j \in N, i \neq j$, i.e., all players' urgencies are different. Thus, τ_0 is unique[4]. Moreover, given an optimal order $\hat{\tau}$, the total amount that will be saved is $g^N = TC(\tau_0) - TC(\hat{\tau})$. The allocation approach starts from the reference order, τ_0, and "repairs" it until the optimal order found by Algorithm 1, $\hat{\tau}$, is reached. In order to guarantee that these repairs lead to non-negative cost savings and there is a local incentive to perform each step, we exchange blocks of players related to the merge segments.[5]

Below we provide a 2−lines GMS-problem that illustrates the ideas behind our allocation procedure.

[4]In case of ties, any possible reference order is considered with a certain probability. We will elaborate on this issue in Section 6.
[5]This was one of our primary motivations to modify the algorithm of Sidney (1975).

Example 3.2. Consider the 2−lines GMS-problem from Example 3.1. In order to compute the reference order, we start by comparing $\frac{\gamma(a_1)}{\alpha(a_1)} = \frac{3.4}{2.2} = 1.55$ to $\frac{\gamma(b_1)}{\alpha(b_1)} = \frac{3.5}{2} = 1.75$. Since the urgency of player a_1 is higher, we select a_1 as the first player of τ_0. The second step consists in comparing $\frac{\gamma(a_2)}{\alpha(a_2)} = 1.67$ to $\frac{\gamma(b_1)}{\alpha(b_1)} = 1.75$, leading to a_2 being the second player in τ_0. We repeat these comparisons until all players have been included in τ_0. The reference order is $\tau_0 = (a_1, a_2, b_1, b_2, a_3, a_4, a_5, b_3, b_4)$, with $c(\tau_0) = (7.48, 25.2, 17, 68, 50.4, 23.8, 42.28, 72.25, 80.88)$ and $TC(\tau_0) = 387.29$. We have seen in Example 3.1 that $\hat{\tau} = (a_1, a_2, a_3, a_4, b_1, b_2, a_5, b_3, b_4)$ is an optimal order, with $TC(\hat{\tau}) = 381.33$. Thus, τ_0 is not optimal, and there is a saving of $g^N = 387.29 - 381.33 = 5.96$ from τ_0 to $\hat{\tau}$.

How to allocate the savings of going from τ_0 to $\hat{\tau}$? Which players should be compensated for such savings? We know that we cannot switch b_2 with a_3 (the first point at which we have two consecutive players from different branches that are misplaced with respect to the optimal order) since this leads to a negative switch: in the reference order b_2 goes before a_3, this means that b_2 has a higher urgency than a_3. But we can switch specific consecutive blocks of players simultaneously. In this particular case, we could switch $X = \{b_1 b_2\}$ with $Y = \{a_3 a_4\}$, thus obtaining the optimal order. The gain resulted from switching them is denoted by g_{XY} and equals 5.96. Regarding a savings allocation rule, we propose to allocate $\frac{1}{2} g_{XY}$ to all players in X equally and $\frac{1}{2} g_{XY}$ to all players in Y equally. Here, our proposal would be to allocate a saving of $\frac{1}{2} \cdot \left(\frac{5.96}{2}\right) = 1.49$ to each of the players a_3, a_4, b_1, and b_2. Subsequently, these savings should be subtracted from $c(\tau_0)$, i.e., the κ cost allocation rule would lead to the following cost allocation vector: $(7.48, 25.2, 15.51, 66.51, 50.4, 22.31, 40.79, 72.25, 80.88)$. △

The above example raises the following question: how do we choose these blocks in general in a unique way such that non-negative switching gains are guaranteed in each step? The determination of these blocks cannot be carried out simply by observing the orders τ_0 and $\hat{\tau}$, but will be done iteratively. To do that, we will present in detail the *block splitting rule* (BSR). The key point of this approach is to determine, at each step, which blocks are to be swapped. Note that given two consecutive blocks, X and Y with X before Y, the gain resulted from switching them is:

$$g_{XY} = \alpha[Y] \cdot \gamma[X] - \alpha[X] \cdot \gamma[Y] = \alpha[Y] \cdot \alpha[X] \cdot \frac{\gamma[X]}{\alpha[X]} - \alpha[X] \cdot \alpha[Y] \cdot \frac{\gamma[Y]}{\alpha[Y]}$$
$$= \alpha[Y] \cdot \alpha[X] \cdot (CAT(X) - CAT(Y)) .$$

The merge segments play a fundamental role in defining the BSR, since by conveniently using their properties along with Algorithm 1 we will be able to guarantee non-negative savings at each iteration. Thus, this procedure consists of two main stages: firstly, we will repair those merge segments whose players are not consecutive, and secondly reorder them as in $\hat{\tau}$. To this end, we will need to consider $\hat{\sigma}$, the corresponding merge order to $\hat{\tau}$. Algorithm 2 shows the scheme of this procedure.

Let X^{it} and Y^{it} be the misplaced blocks switched at iteration "it" of Algorithm 2. Then, we define:

$$\text{BSR}^{\text{it}}(\tau_0, \hat{\sigma}) = \frac{1}{2} g_{X^{\text{it}} Y^{\text{it}}} \left(\frac{1}{|X^{\text{it}}|} e^{X^{\text{it}}} + \frac{1}{|Y^{\text{it}}|} e^{Y^{\text{it}}} \right),$$

Algorithm 2 Algorithm to allocate the gains of a 2−lines GMS-problem

0. Obtain τ_0 and apply Algorithm 1 to get $\hat{\sigma} = (M_1, M_2, \dots, M_m)$. Initialize $k = 1$, $r = 1$, it $= 1$, and $\tau' = \tau_0$.
1. (a) If $|M_k/\tau'| = 1$, take $k = k + 1$.
   - If $k \leq m - 1$, go back to step 1.
   - If $k = m$, go to step 2.

   (b) If $|M_k/\tau'| > 1$, take $\tilde{k} \in \{2, \dots, m_k\}$ such that $CAT(G_{\tilde{k}-1}) > CAT(G_{\tilde{k}})$ (we know there exists such a pair of components from Lemma 3.2). It is clear that between $G_{\tilde{k}-1}$ and $G_{\tilde{k}}$ there are only players from the other branch, i.e.,

   $$\tau' = (\sim, G_{\tilde{k}-1}, Z, G_{\tilde{k}}, \sim),$$

   where Z is a segment from the opposite branch of M_k. From Proposition 3.1, we know that either the order $\tau_1 = (\sim, G_{\tilde{k}-1}, G_{\tilde{k}}, Z, \sim)$ or the order $\tau_2 = (\sim, Z, G_{\tilde{k}-1}, G_{\tilde{k}}, \sim)$ has a lower total cost than τ'. Take $\tau'' = \arg\min_{\tau}\{TC(\tau) \colon \tau \in \{\tau_1, \tau_2\}\}$.
   - If $\tau'' = \tau_1$, then the blocks that have been switched are Z and $G_{\tilde{k}}$. Players from Z should receive $\frac{1}{2|Z|} g_{ZG_{\tilde{k}}}$, while players from $G_{\tilde{k}}$ should receive $\frac{1}{2|G_{\tilde{k}}|} g_{ZG_{\tilde{k}}}$.
   - If $\tau'' = \tau_2$, then the blocks that have been switched are $G_{\tilde{k}-1}$ and Z. Players from $G_{\tilde{k}-1}$ should receive $\frac{1}{2|G_{\tilde{k}-1}|} g_{G_{\tilde{k}-1}Z}$, while players from Z should receive $\frac{1}{2|Z|} g_{G_{\tilde{k}-1}Z}$.

   Set $\tau' = \tau''$, and take it $=$ it $+ 1$. Go back to step 1.
2. (a) If $r \leq m$, consider the bijection

   $$\rho \colon \{1, 2, \dots, m\} \to \{1, 2, \dots, m\}$$
   $$i \mapsto \rho(i) = j,$$

   such that $\tau' = (M_{\rho(1)}, \dots, M_{\rho(m)})$. We need to go from τ' to $\hat{\tau}$.
   i. If $\rho(r) = r$, take $r = r + 1$. Go back to step 2.
   ii. If $\rho(r) \neq r$, take $\tilde{r} \in \{r + 1, \dots, m\}$ such that $\rho(\tilde{r}) = r$ (this means that M_r is on position $\tilde{r}$, i.e., $M_r = M_{\rho(\tilde{r})}$). By Algorithm 1, it holds that

   $$CAT(M_r) \leq CAT\left(M_r^{\cup}\right),$$

   where $M_r^{\cup} = \bigcup_{l=r}^{\tilde{r}-1} M_{\rho(l)}$. Hence, the order

   $$\tau'' = (\sim, M_{\rho(\tilde{r})}, M_{\rho(r)}, \dots, M_{\rho(\tilde{r}-1)}, M_{\rho(\tilde{r}+1)}, \sim)$$

   that consists in moving $M_{\rho(\tilde{r})}$ to the front of $M_{\rho(r)}$ (so that $M_{\rho(\tilde{r})} \equiv M_r$ is now on position r) has lower total cost than τ'. The blocks that have been switched are $M_r^{\cup}$ and M_r. Allocate $\frac{1}{2|M_r^{\cup}|} g_{M_r^{\cup} M_r}$ to the players in $M_r^{\cup}$ and $\frac{1}{2|M_r|} g_{M_r^{\cup} M_r}$ to the players in M_r. Set $\tau' = \tau''$, and take $r = r + 1$ and it $=$ it $+ 1$. Go back to step 2.

   (b) If $r > m$, then $\tau' = \hat{\tau}$ and $I =$ it $- 1$. The algorithm is finished. The outputs of the algorithm are the allocation and the misplaced blocks at each iteration it $\in \{1, \dots, I\}$.

where, for $S \subseteq N$, e^S is the vector in $\mathbb{R}^N$ satisfying $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ otherwise. $\text{BSR}^{\text{it}}(\tau_0, \hat{\sigma})$ represents the allocation obtained at iteration it, hence $\text{BSR}(\tau_0, \hat{\sigma}) = \sum_{\text{it}=1}^{I} \text{BSR}^{\text{it}}(\tau_0, \hat{\sigma})$, where I represents the total number of iterations needed. Subsequently, we define the cost allocation rule, κ, by setting

$$\kappa(\mathcal{G}) = c(\tau_0) - \text{BSR}(\tau_0, \hat{\sigma}),$$

for a 2−lines GMS-problem $\mathcal{G}$.

The following example shows how to obtain the κ rule for a 2−lines GMS-problem by applying the above procedure.

Example 3.3. Consider the 2−lines GMS-problem $\mathcal{G}$ from Figure 8. Assuming that $\alpha(i) = 1$ for all $i \in A \cup B$, these numbers are not incorporated in the figure for clarity. The reference order would be $\tau_0 = (b_1, b_2, b_3, b_4, a_1, a_2, b_5, a_3, a_4, a_5)$, with an associated individual cost vector of $c(\tau_0) = (89.5, 108, 151.5, 161.5, 180.5, 15, 34.5, 52.5, 69.5, 129.5)$ and total cost of $TC(\tau_0) = 992$. The optimal solution provided by Algorithm 1 would be $\hat{\tau} = (b_1, a_1, a_2, a_3, a_4, b_2, b_3, b_4, a_5, b_5)$, with an associated cost

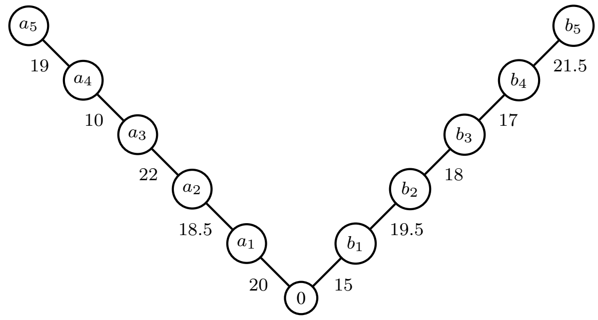


Fig. 8: Example of a 2−lines GMS-problem.

of $TC(\hat{\tau}) = 972$. The corresponding merge order is $\hat{\sigma} = (M_1, M_2, M_3, M_4, M_5, M_6)$, where $M_1 = \{b_1\}$, $M_2 = \{a_1a_2a_3a_4\}$, $M_3 = \{b_2b_3\}$, $M_4 = \{b_4\}$, $M_5 = \{a_5\}$, and $M_6 = \{b_5\}$.

Table 2 summarizes the steps followed by Algorithm 2 to go from τ_0 to $\hat{\sigma}$. Note that, in the first iteration, M_2 is non-connected in τ_0. In fact, $M_2/\tau_0 = \{G_1, G_2\}$, where $G_1 = \{a_1a_2\}$ and $G_2 = \{a_3a_4\}$. These components have $Z = \{b_5\}$ in between. By Lemma 3.1, we know that $CAT(G_1) > CAT(G_2)$. By Proposition 3.1, we know that either moving G_2 to the front of Z or moving G_1 to the back of Z would lead to positive savings. Thus, we need to compare the following two orders:

$$\tau_1 = (b_1, b_2, b_3, b_4, a_1, a_2, \underbrace{a_3, a_4}_{G_2}, \overbrace{b_5}^{Z}\, a_5) \quad \text{and} \quad \tau_2 = (b_1, b_2, b_3, b_4, \overbrace{b_5}^{Z}, \underbrace{a_1, a_2}_{G_1}, a_3, a_4, a_5).$$

Since $TC(\tau_1) = 981$ and $TC(\tau_2) = 996.5$, we choose $\tau' = \tau_1$. Hence, blocks Z and G_2 have been switched, as shown in Table 2. The resulting saving of 11 needs to be split equally between the two blocks that have been exchanged, and subsequently equally to the number of players in each block. Thus, players a_1, a_2, and b_5 have savings of 2.75, 2.75 and 5.5, respectively. After this iteration, all the merge segments are connected and all that remains is to reorder them sequentially, as is done in iterations 2 and 3. The savings allocation process is the same: first between the blocks that need to be switched and then between players of each block.

it	Case	Switched blocks	Resulting order	Gain	$\text{BSR}^{\text{it}}(\tau_0, \hat{\sigma})$
1	Non-connected merge segment $M_2/\tau_0 = \{\{a_1a_2\}, \{a_3a_4\}\}$	$\{a_1a_2\}$ $\{b_5\}$	$(M_1, M_3, M_4, M_2, M_6, M_5)$	11	$(0, 0, 2.75, 2.75, 0, 0, 0, 0, 0, 5.5)$
2	Non-ordered merge segments	M_2 $M_3 \cup M_4$	$(M_1, M_2, M_3, M_4, M_6, M_5)$	6.5	$(0.813, 0.813, 0.813, 0.813, 0, 0, 1.083, 1.083, 1.083, 0)$
3	Non-ordered merge segments	M_5 M_6	$\hat{\sigma}$	2.5	$(0, 0, 0, 0, 1.25, 0, 0, 0, 1.25)$

Table 2: Steps of Algorithm 2 to go from τ_0 to $\hat{\sigma}$.

Therefore, the final savings allocation rule provided by Algorithm 2 is the following:

$$\text{BSR}(\tau_0, \hat{\sigma}) = \sum_{\text{it}=1}^{3} \text{BSR}^{\text{it}}(\tau_0, \hat{\sigma}) = (0.813, 0.813, 3.563, 3.563, 1.25, 0, 1.083, 1.083, 1.083, 6.75).$$

Hence,

$$\kappa(\mathcal{G}) = (88.687, 107.187, 147.937, 157.937, 179.25, 15, 33.417, 51.417, 68.417, 122.75).$$

The vector above is a proposal for allocating the total cost of an optimal order. One could also view the outcome of $\kappa(\mathcal{G})$ in the following way. Consider the individual cost vector corresponding to the optimal order, $c(\hat{\tau}) = (35, 53.5, 75.5, 85.5, 159, 15, 105, 123, 140, 180.5)$. To reach $\kappa(\mathcal{G})$, we can take a vector of compensations, $(53.687, 53.687, 72.437, 72.437, 20.25, 0, -71.583, -71.583, -71.583, -57.75)$. Players a_1, a_2, a_3, a_4, and a_5 have positive numbers, which indicate that they have to pay compensations for being handled earlier than other players in the optimal order, with respect to what was supposed to happen according to the reference order. On the contrary, players b_2, b_3, b_4, and b_5 get compensated for delaying their processing in $\hat{\tau}$ compared to τ_0. $\triangle$

4. $n-$lines GMS-problems

This section generalizes the optimization and allocation results for the $2-$lines GMS-problems to $n-$lines GMS-problems.

4.1. Optimal orders

A GMS-problem $(N, 0, E, \gamma, \alpha)$ is called an $n-$*lines GMS-problem* if there exists a partition $\langle A^1, \ldots, A^n \rangle$ of N with $A^k = \{a_1^k, \ldots, a_{\tilde{s}_k}^k\}$ for all $k \in \{1, \ldots, n\}$ with $\sum_{k=1}^{n} \tilde{s}_k = |N|$, such that

$$E = \bigcup_{k=1}^{n} \{\{0, a_1^k\}, \{a_1^k, a_2^k\}, \ldots, \{a_{\tilde{s}_k-1}^k, a_{\tilde{s}_k}^k\}\}.$$

As for the $2-$lines GMS-problems, the sets $A^k, k \in \{1, \ldots, n\}$, are called branches. A feasible order is described by a bijection $\sigma \colon N \to \{1, 2, \ldots, |N|\}$ such that $\sigma(a_h^k) < \sigma(a_l^k) \Rightarrow h < l$, for all $k \in \{1, \ldots, n\}$. Let $\mathcal{F}(N)$ denote the set of all such feasible orders. The definitions regarding the segments and $CATs$ can be directly extended to this generalization. A graphical representation of an $n-$lines GMS-problem is provided in Figure 9.

To solve an $n-$lines GMS-problem, we propose an algorithm that combines the concept of recursion with Algorithm 1. This procedure is based on the following idea: given an $n-$lines GMS-problem $(N, 0, E, \gamma, \alpha)$, we consider a $2-$lines GMS-problem, $(A^h \cup A^l, 0, E|_{A^h \cup A^l}, \gamma, \alpha)$, where $h, l \in \{1, \ldots, n\}$. Note that $E|_{A^h \cup A^l}$

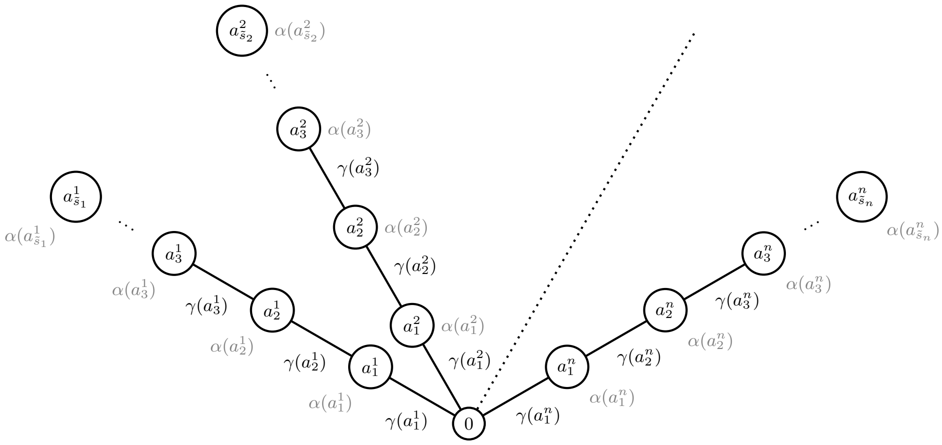


Fig. 9: Graphical representation of an $n-$lines GMS-problem.

is the restriction of E to the players of $A^h \cup A^l$. $\hat{\sigma}_{hl}$ will denote the output of Algorithm 1 for such a subproblem in which the merge segments are specified. $\hat{\tau}_{hl}$ will refer to the corresponding order on all players involved. Furthermore, $A^{hl}_{\hat{\tau}}$ will denote the branch formed by the nodes from A^h and A^l following the order specified by $\hat{\tau}_{hl}$. We informally present the algorithm below.

Algorithm 3 Algorithm to solve an $n-$lines GMS-problem

1. Consider an $n-$lines GMS-problem $(N, 0, E, \gamma, \alpha)$. Select branches A^1 and A^2.
2. Apply Algorithm 1 to solve the corresponding $2-$lines GMS-problem, $(A^1 \cup A^2, 0, E|_{A^1 \cup A^2}, \gamma, \alpha)$. This leads to an optimal order, $\hat{\tau}_{12}$. Replace A^1 and A^2 with the branch $A^{12}_{\hat{\tau}}$. We get a new problem with one branch less. Renumber the branches adequately.
3. (a) If there are still more than two branches left, go back to step 1.
   (b) If there are two branches left, apply Algorithm 1. The order obtained is the solution.

The following example illustrates how to apply Algorithm 3 to solve an $n-$lines GMS-problem.

Example 4.1. Consider the $3-$lines GMS-problem presented in Figure 10. We will avoid renumbering the nodes for a better understanding of the algorithmic process. Also, we will denote the branches by A, B, and C for clarity, instead of A^1, A^2, and A^3, respectively.

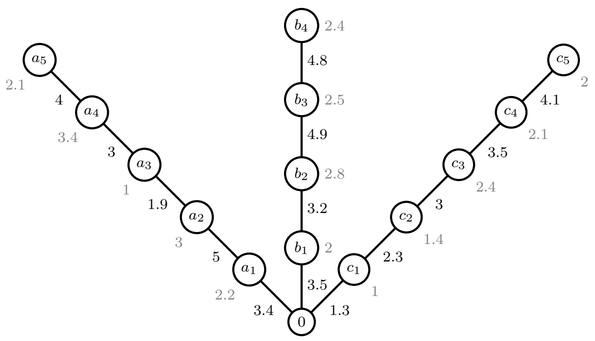


Fig. 10: Example of a $3-$lines GMS-problem.

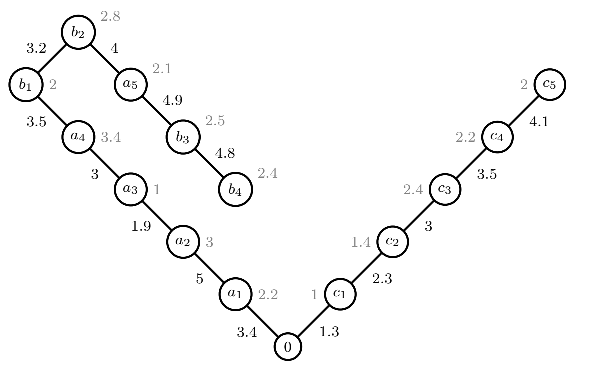


Fig. 11: Resulting $2-$lines GMS-problem after solving $(A \cup B, 0, E|_{A \cup B}, \gamma, \alpha)$.

As indicated in Algorithm 3, we first select branches A and B. The $2-$lines GMS-problem

$(A \cup B, 0, E|_{A\cup B}, \gamma, \alpha)$ has been solved in Example 3.1, leading to the optimal order $\hat{\tau}_{12} = (a_1, a_2, a_3, a_4, b_1, b_2, a_5, b_3, b_4)$. We replace branches A and B with one branch respecting the order of $\hat{\tau}_{12}$, as illustrated in Figure 11. Next we solve this $2-$lines GMS-problem by applying Algorithm 1. The output would be:

$$\hat{\tau} = (c_1, a_1, a_2, a_3, a_4, c_2, c_3, b_1, b_2, c_4, a_5, b_3, b_4, c_5),$$

finishing Algorithm 3. △

Along similar lines as in Lemma 3.3 one can show the following result.

Lemma 4.1. *Let $(N, 0, E, \gamma, \alpha)$ be an $n-$lines GMS-problem, and $\hat{\sigma}_{12} = (M_1, M_2, \ldots, M_m)$ be the output of Algorithm 1 for the $2-$lines GMS-problem $(A^1 \cup A^2, 0, E|_{A^1\cup A^2}, \gamma, \alpha)$. It holds that the elements of $M_k, k \in \{1, \ldots, m\}$, are consecutive in any optimal order for $(N, 0, E, \gamma, \alpha)$.*

Moreover, the result below guarantees that obtaining an optimal order for an $n-$lines GMS-problem, the first two branches can be replaced by one branch that reflects the optimal order found by Algorithm 1 for the corresponding $2-$lines subproblem.

Proposition 4.1. *Let $(N, 0, E, \gamma, \alpha)$ be an $n-$lines GMS-problem, and let τ^*_{12} be the output of Algorithm 1 for the $2-$lines GMS-problem $(A^1 \cup A^2, 0, E|_{A^1\cup A^2}, \gamma, \alpha)$. There exists an optimal order $\hat{\tau}$ for $(N, 0, E, \gamma, \alpha)$ such that $\hat{\tau}(i) < \hat{\tau}(j)$ for all $i, j \in A^1 \cup A^2$ for which $\tau^*_{12}(i) < \tau^*_{12}(j)$.*

Proof. See Appendix A. □

The following result states that Algorithm 3 leads to an optimal order.

Theorem 4.1. *Let $(N, 0, E, \gamma, \alpha)$ be an $n-$lines GMS-problem, and let $\hat{\tau}$ be an order provided by Algorithm 3. Then, $TC(\hat{\tau}) \leq TC(\tau)$ for all $\tau \in \mathcal{F}(N)$.*

Proof. We will use induction in the number of branches, n.

If $n = 2$, Algorithm 3 coincides with Algorithm 1.

Suppose that Algorithm 3 leads to an optimal solution for all $k < n$.

Now, take $k = n$. We can select branches A^1 and A^2 and apply Algorithm 1 to obtain a relative order, τ_{12}. Using Proposition 4.1, there exists an optimal order $\hat{\tau}$ for $(N, 0, E, \gamma, \alpha)$ that maintains the order induced from branches A^1 and A^2. We can convert these two branches into one, A^{12}_{τ}, reducing the dimension of our problem by 1. Now we have an $(n-1)-$lines GMS-problem, for which is clear that $\hat{\tau}$ is also an optimal order. By the induction hypothesis, Algorithm 3 leads to an optimal solution, τ^*, for the $(n-1)-$lines GMS-problem. It is straightforward to prove that τ^* is also an optimal solution for the $n-$lines GMS-problem. For, if not, there would exist an optimal order $\tilde{\tau}$ such that $TC(\tilde{\tau}) < TC(\tau^*)$. Thus, $TC(\hat{\tau}) = TC(\tilde{\tau}) < TC(\tau^*)$, contradicting the induction hypothesis. □

4.2. Allocating the minimal cost

We will illustrate how to extend the ideas of the allocation procedure as described by κ for $2-$lines GMS-problems into a rule on $n-$lines GMS-problems. Consider an $n-$lines GMS-problem $\mathcal{G} = (N, 0, E, \gamma, \alpha)$. Again, take as starting point of the allocation mechanism a reference order on all players, τ_0^N, in the same way as before. Let $\hat{\tau}^N$ be the optimal order provided by Algorithm 3. The total savings of $g^N = TC(\tau_0^N) - TC(\hat{\tau}^N)$ need to be adequately subtracted from $c(\tau_0^N)$ to obtain the final cost allocation. To determine the proportion of g^N for which each player is responsible, we apply a procedure similar to that in Algorithm 3: we will recursively allocate the local savings obtained at the $2-$lines GMS-problems that comprise our $n-$lines. The sum of these local savings however is not necessarily equal to g^N. Instead, we use these numbers to determine the *relative* importance of the different subproblems (and, mainly, of the players involved in these problems) in the final savings obtained.

Let $\tau_0^k, k \in \{2, \ldots, n\}$ be the reference order for the $2-$lines GMS-problem induced by branches $A^{1 \ldots k-1}$ and A^k, and let $\hat{\tau}^k$ and $\hat{\sigma}^k$ be the optimal order and its corresponding merge order provided by Algorithm 1 for such $2-$lines GMS-problem. The local cost savings are defined by $g^k = TC(\tau_0^k) - TC(\hat{\tau}^k)$ and they will be allocated in the same way as before for each subproblem. That is, given $k \in \{2, \ldots, n\}$, going from τ_0^k to $\hat{\sigma}^k$ leads to a saving of g^k that is allocated among the players involved using Algorithm 2, thus obtaining $\text{BSR}(\tau_0^k, \hat{\sigma}^k)$. Each of these vectors are now complemented with 0's on those coordinates that refer to non-involved players. Figure 12 summarizes this procedure.

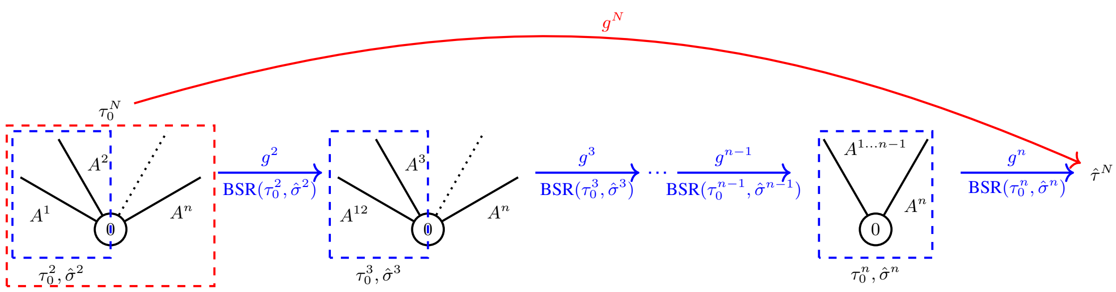


Fig. 12: Steps of the allocation procedure in an $n-$lines GMS-problem.

We define the cost allocation rule, κ, by setting:

$$\kappa(\mathcal{G}) = c(\tau_0^N) - \frac{g^N}{\sum_{k=2}^{n} g^k} \cdot \sum_{k=2}^{n} \text{BSR}(\tau_0^k, \hat{\sigma}^k),$$

for an $n-$lines GMS-problem $\mathcal{G}$.

The following example illustrates how to obtain the cost allocation rule for a $3-$lines GMS-problem.

Example 4.2. Consider the $3-$lines GMS-problem presented in Figure 13. The general reference order would be $\tau_0^N = (b_1, c_1, b_2, b_3, b_4, a_1, a_2, c_2, c_3, c_4, b_5, a_3, a_4, a_5)$, with an associated individual cost vector of $c(\tau_0^N) = (107, 125.5, 226, 236, 255, 15, 52, 70, 87, 204, 32.5, 146, 161.5, 182.5)$ and total cost of $TC(\tau_0^N) = 1900$.

First, select branches A and B. From Example 3.3, we know that $g^2 = 20$ and $\text{BSR}(\tau_0^2, \hat{\sigma}^2) = (0.813, 0.813, 3.563, 3.563, 1.25, 0, 1.083, 1.083, 1.083, 6.75, 0, 0, 0, 0)$.

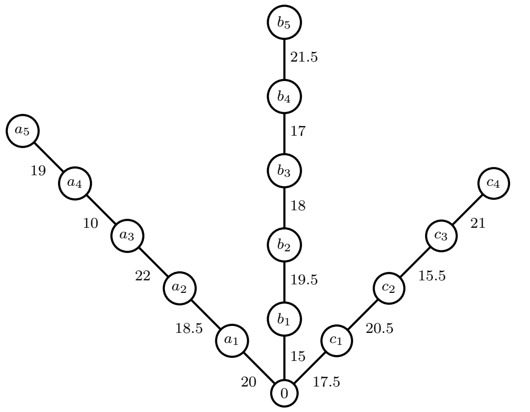


Fig. 13: Example of a 3−lines GMS-problem.

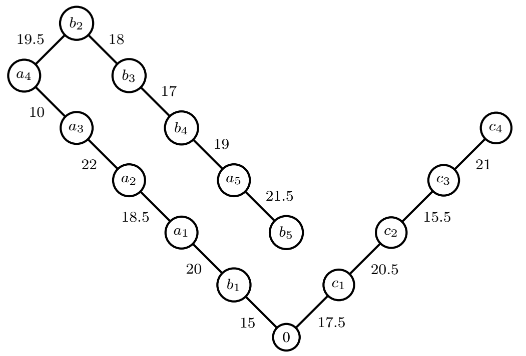


Fig. 14: Resulting 2−lines GMS-problem.

Now, select branches $A \cup B$ and C. The local reference order for the resulting 2−lines GMS-problem of Figure 14 is given by $\tau_0^3 = (b_1, c_1, a_1, a_2, c_2, c_3, c_4, a_3, a_4, b_2, b_3, b_4, a_5, c_5)$, with $TC(\tau_0^3) = 1887.5$, and $\hat{\tau}^3 = (b_1, c_1, a_1, a_2, a_3, a_4, c_2, c_3, b_2, b_3, b_4, a_5, c_4, b_5)$, with $TC(\hat{\tau}^3) = 1859$. Thus, $g^3 = 28.5$. Also, $\hat{\tau}^N = \hat{\tau}^3$, so there is a total saving of $g^N = TC(\tau_0^N) - TC(\hat{\tau}^N) = 1900 - 1859 = 41$ that needs to be adequately allocated among the players. Note that $g^N \neq g^2 + g^3$. The corresponding merge order to $\hat{\tau}^3$ is given by $\hat{\sigma}^3 = (M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8, M_9, M_{10})$, with $M_1 = \{b_1\}, M_2 = \{c_1\}, M_3 = \{a_1a_2a_3a_4\}, M_4 = \{c_2c_3\}, M_5 = \{b_2\}, M_6 = \{b_3\}, M_7 = \{b_4\}, M_8 = \{a_5\}, M_9 = \{c_4\}$, and $M_{10} = \{b_5\}$. Table 3 summarizes the steps of Algorithm 2 to go from τ_0^3 to $\hat{\sigma}^3$.

it	Case	Switched blocks	Resulting order	Gain	BSR$^{\text{it}}(\tau_0^3, \hat{\sigma}^3)$
1	Non-connected merge segment $M_3/\tau_0^3 = \{\{a_1a_2\}, \{a_3a_4\}\}$	$\{c_2c_3c_4\}$ $\{a_3a_4\}$	$(M_1, M_2, M_3, M_4, M_9, M_5, M_6, M_7, M_8, M_{10})$	18	$(0, 0, 4.5, 4.5, 0, 0, 0, 0, 0, 0, 0, 3, 3, 3)$
2	Non-ordered merge segments	M_9 M_5	$(M_1, M_2, M_3, M_4, M_5, M_9, M_6, M_7, M_8, M_{10})$	1.5	$(0, 0, 0, 0, 0, 0, 0.75, 0, 0, 0, 0, 0, 0, 0.75)$
3	Non-ordered merge segments	M_9 M_6	$(M_1, M_2, M_3, M_4, M_5, M_6, M_9, M_7, M_8, M_{10})$	3	$(0, 0, 0, 0, 0, 0, 0, 1.5, 0, 0, 0, 0, 0, 1.5)$
4	Non-ordered merge segments	M_9 M_7	$(M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_9, M_8, M_{10})$	4	$(0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 0, 0, 0, 2)$
5	Non-ordered merge segments	M_9 M_8	$\hat{\sigma}^3$	2	$(0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1)$

Table 3: Steps of Algorithm 2 to go from τ_0^3 to $\hat{\sigma}^3$.

Therefore,

$$\text{BSR}(\tau_0^3, \hat{\sigma}^3) = (0, 0, 4.5, 4.5, 1, 0, 0.75, 1.5, 2, 0, 0, 3, 3, 8.25).$$

Hence,

$$\begin{aligned}\kappa(\mathcal{G}) &= (107, 125.5, 226, 236, 255, 15, 52, 70, 87, 204, 32.5, 146, 161.5, 182.5)\\ &\quad - \frac{41}{20+28.5}\cdot\big[(0.813, 0.813, 3.563, 3.563, 1.25, 0, 1.083, 1.083, 1.083, 6.75, 0, 0, 0, 0)\\ &\qquad\qquad + (0, 0, 4.5, 4.5, 1, 0, 0.75, 1.5, 2, 0, 0, 3, 3, 8.25)\big]\\ &= (106.313, 124.813, 219.184, 229.184, 253.098, 15, 50.45, 67.816, 84.394, 198.294,\\ &\quad 32.5, 143.464, 158.964, 175.526). \qquad\triangle\end{aligned}$$

5. Tree GMS-problems

We will now extend the results we have seen for n-lines GMS-problems to the case of tree GMS-problems, both for optimization and cost allocation.

5.1. Optimal orders

A GMS-problem $(N, 0, E, \gamma, \alpha)$ is called a *tree GMS-problem* if $(N \cup \{0\}, E)$ is a tree.

Definition 5.1. Let $(N, 0, E, \gamma, \alpha)$ be a tree GMS-problem. We define the *degree of a node* $a \in N$, $\deg(a)$, as the number of edges incident on that node.

Definition 5.2. Let $(N, 0, E, \gamma, \alpha)$ be a tree GMS-problem. A *sub-source* will be either the machine, 0, or a node with degree at least 3. Let $\mathcal{S}$ be the set of sub-sources.

Given the sub-sources of a tree, we are interested in knowing their *level*.

Definition 5.3. Let $(N, 0, E, \gamma, \alpha)$ be a tree GMS-problem. The *level* $\ell(s)$ *of a sub-source* $s \in \mathcal{S}$ is the number of sub-sources in the path between 0 and s, including 0. Thus, the machine 0 is the only sub-source with level 1.

We assume an ordering on the sub-sources, from level 1 to the highest level, v. Given a level $l \in \{2, \dots, v\}$, there are m_l sub-sources. Thus, we can write

$$\mathcal{S} = \{0, s_1^2, \dots, s_{m_2}^2, s_1^3, \dots, s_{m_3}^3, \dots, s_1^v, \dots, s_{m_v}^v\},$$

where s_k^l denotes the $k-$th sub-source from level l. Figure 15 provides an illustration of the sub-sources of a tree and their levels.

The theoretical results for $n-$lines GMS-problems can be extended to the general case of trees. In particular, given a tree GMS-problem, the elements of the merge segments obtained when solving a $2-$lines GMS-problem of the highest level remain consecutive in any optimal order. Furthermore, there always exists an optimal order that maintains the order induced by the aforementioned subproblem. With all these ingredients, it is immediate to prove that Algorithm 4 below leads to an optimal order. Here, a recursive methodology is adopted, starting with the $n-$lines GMS-problems at the highest level. For each of them, the $2-$lines GMS-

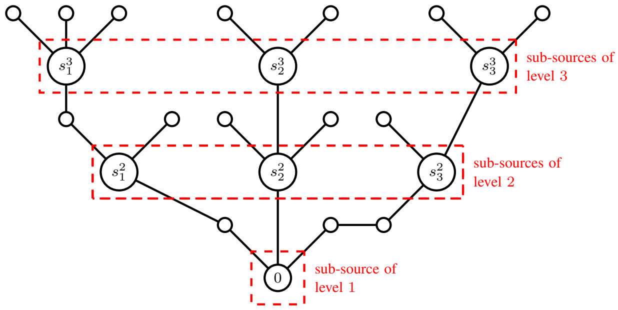


Fig. 15: Sketch of the sub-sources of a tree and their levels.

problems that comprise it are solved recursively until these $n-$lines are converted into a single line, thus reducing the dimension.

Algorithm 4 Algorithm to solve a tree GMS-problem

1. Consider the tree GMS-problem $(N, 0, E, \gamma, \alpha)$.
2. Let $k \in \{1, \ldots, m_v\}$. Consider the n_k^v-lines GMS-problem arising from s_k^v. Apply Algorithm 3 to obtain $\tau_{s_k^v}$. Replace the n_k^v branches arising from s_k^v with one branch using the order of $\tau_{s_k^v}$. We have a new tree GMS-problem with one level less since all the subproblems of the highest level have been converted into lines. Renumber the nodes adequately. The highest level has now been reduced by 1.
3. (a) If $\mathcal{S} \neq \{0\}$, go back to step 1.
   (b) If $\mathcal{S} = \{0\}$, solve the resulting $n-$lines GMS-problem with Algorithm 3. The order obtained is the solution.

Without proof we state the following result.

Theorem 5.1. *Let $(N, 0, E, \gamma, \alpha)$ be a tree GMS-problem, and let $\hat{\tau}$ be an order provided by Algorithm 4. Thus, $TC(\hat{\tau}) \leq TC(\tau)$ for all feasible order τ.*

5.2. *Allocating the minimal cost*

To appropiately extend the κ rule to the context of tree GMS-problems, we need to contemplate the local savings generated in each sub-source, in a similar way to how we made the transition from $2-$lines to $n-$lines GMS-problems.

Let $\mathcal{G} = (N, 0, E, \gamma, \alpha)$ be a tree GMS-problem. For $s \in \mathcal{S}$, we define $N_s = F(s) \cup \{s\}$, where $F(s)$ is the set of followers of s with respect to 0 in the graph $(N \cup \{0\}, E)$. For every sub-source $s \in \mathcal{S}$, we consider an induced $n-$lines GMS-problem on N_s, $(N_s, 0, E|_{N_s}, \gamma, \alpha)$, where all initial branches with respect to s in E have been recursively replaced by a line that corresponds to an optimal order with respect to this branch. Naturally, if $\ell(s) = v$, then $(N_s, 0, E|_{N_s}, \gamma, \alpha)$ is already an $n-$lines GMS-problem and we call it a subproblem at the highest level. Also, given $s \in \mathcal{S}$, let $\tau_0^{N_s}$ and $\hat{\tau}^{N_s}$ denote the corresponding reference order and the optimal order provided by Algorithm 3 for $(N_s, 0, E|_{N_s}, \gamma, \alpha)$, respectively. Subsequently, the *stand-alone cost savings $w(s)$ with respect to s* are defined by:

$$w(s) = TC(\tau_0^{N_s}) - TC(\hat{\tau}^{N_s}).$$

Thus, $w(s)$ can be interpreted as local savings made at sub-sources. It should be noted that the sum $\sum_{s\in\mathcal{S}} w(s)$ is not necessarily equal to the total savings $g^N = TC(\tau_0^N) - TC(\hat{\tau}^N)$. As done for $n-$lines GMS-problems, these stand-alone savings help in determining the relative importance of each sub-source to realize g^N.

We will follow a recursive procedure, by first solving the subproblems of the highest level. Once these problems have been solved, the highest level of the tree GMS-problem has been reduced by 1, and we repeat the process. Hence, we will always start from an $n-$lines GMS-problem, which will depend on the specific sub-source we are considering. This is reflected in the notation by writing $n(s), g^k(s), \tau_0^k(s)$, and $\hat{\sigma}^k(s)$. We define the cost allocation rule, κ, by setting:

$$\kappa(\mathcal{G}) = c(\tau_0^N) - g^N \cdot \sum_{s\in\mathcal{S}} \frac{w(s)}{\sum_{t\in\mathcal{S}} w(t)} \cdot \frac{1}{\sum_{k=2}^{n(s)} g^k(s)} \cdot \sum_{k=2}^{n(s)} \text{BSR}(\tau_0^k(s), \hat{\sigma}^k(s)), \tag{8}$$

for a tree GMS-problem $\mathcal{G}$.

Figure 16 displays a flowchart of the allocation procedure for tree GMS-problems.

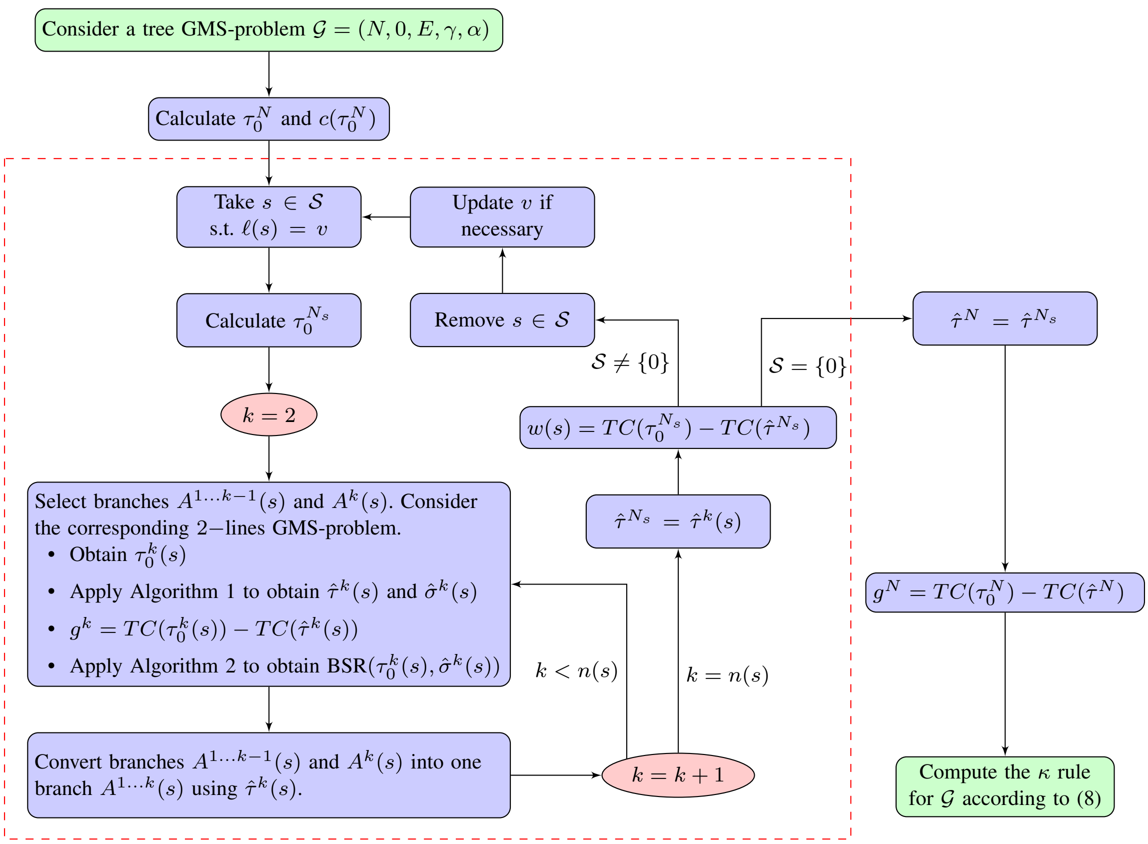


Fig. 16: Flowchart of the allocation procedure for tree GMS-problems. The dashed red box contains the steps of Algorithm 3.

6. Discussion and final remarks

The procedure presented for calculating the κ rule for $2-$lines, $n-$lines, and tree GMS-problems was limited to the case where all players have different urgencies and, consequently, there is a single reference order (also for all local problems). In the following, we will give some general indications on how to proceed when there are ties. First, we consider all possible reference orders. We assume that, at each step, the machine chooses with equal probability between all jobs with highest urgency. Thus, given the set of all possible reference orders, we will also have a probability distribution on this set. Now, given a fixed reference order, we proceed in the same way as explained in Subsections 3.2, 4.2 and 5.2 obtaining a specific allocation proposal (which will now depend on that reference order). The only difference is that, when solving the $n-$lines GMS-problems associated with each sub-source, the (local) reference order in these subproblems will not be recalculated according to possible ties that may exist. Instead, we will take the restriction of the (general) reference order that we are considering to the players involved in that subproblem. Naturally, the final cost allocation vector is defined as the weighted average of all the reference specific allocation proposals, where the weights are the probabilities of each possible reference order.

As future work, it would be of special interest to find characterizing properties that the proposed allocation rule satisfies. Another open direction of research is to study the allocation aspect of a GMS-problem more directly on the basis of an adequately defined cooperative GMS-game.

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Appendix 1

Proof of Proposition 4.1. Let $\sigma^*_{12} = (M_1, \ldots, M_m)$ be the output of Algorithm 1 for $(A^1 \cup A^2, 0, E|_{A^1 \cup A^2}, \gamma, \alpha)$ that has τ^*_{12} as its associated order.

Let τ' be an optimal order for $(N, 0, E, \gamma, \alpha)$. From Lemma 4.1, we know that the elements of $M_k,\ k \in \{1, \ldots, m\}$, are consecutive in τ'. Assume that τ' does not respect the relative order induced by τ^*_{12}, and let $M_k, M_l,\ k, l \in \{1, \ldots, m\}$ be the first merge segments such that $\tau^*_{12}(M_k) > \tau^*_{12}(M_l)$ but $\tau'(M_k) < \tau'(M_l)$. Note that M_k and M_l necessarily belong to different branches. We also have that

$$CAT(M_l) \leq CAT(M_k). \tag{A1}$$

Thus, between M_k and M_l in τ', players from all branches except that of M_l can be present. Let $M_k^1, M_k^2, \ldots, M_k^q$ be the maximal connected segments from the branch of M_k that are between M_k and M_l in τ', and let $Z^1, Z^2, \ldots, Z^q, Z^{q+1}$ be the (potential) maximal connected sets of nodes from $\mathcal{A} \setminus \{A^1 \cup A^2\}$ in τ'. There are four possible cases:

i) $\tau' = (\sim, M_k, Z^1, M_k^1, Z^2, M_k^2, \ldots, Z^q, M_k^q, Z^{q+1}, M_l, \sim)$,

ii) $\tau' = (\sim, M_k, Z^1, M_k^1, Z^2, M_k^2, \ldots, Z^q, M_k^q, M_l, \sim)$,

iii) $\tau' = (\sim, M_k, M_k^1, Z^1, M_k^2, \ldots, Z^{q-1}, M_k^q, Z^q, M_l, \sim)$,

iv) $\tau' = (\sim, M_k, M_k^1, Z^1, M_k^2, \ldots, Z^{q-1}, M_k^q, M_l, \sim)$.

We will only consider i), since the other cases can be treated in an analogous way. We will first show that:

$$\frac{\gamma[Z^{\tilde{q}}]}{\alpha[Z^{\tilde{q}}]} \leq CAT(M_k^{\tilde{q}}) \leq \frac{\gamma[Z^{\tilde{q}+1}]}{\alpha[Z^{\tilde{q}+1}]}, \tag{A2}$$

for all $\tilde{q} \in \{1, \ldots, q+1\}$. Suppose that (A2) does not hold. Then, there exists a $\tilde{q} \in \{1, \ldots, q+1\}$ such that

$$\frac{\gamma[Z^{\tilde{q}}]}{\alpha[Z^{\tilde{q}}]} > AC(M_k^{\tilde{q}}), \tag{A3}$$

or

$$AC(M_k^{\tilde{q}}) > \frac{\gamma[Z^{\tilde{q}+1}]}{\alpha[Z^{\tilde{q}+1}]}. \tag{A4}$$

Consider the order τ_1 as a modification of τ' in which the position of $Z^{\tilde{q}}$ and $M_k^{\tilde{q}}$ is swapped. Then,

$$TC(\tau') - TC(\tau_1) = \alpha[M_k^{\tilde{q}}] \cdot \alpha[Z^{\tilde{q}}] \cdot \left(\frac{\gamma[Z^{\tilde{q}}]}{\alpha[Z^{\tilde{q}}]} - CAT(M_k^{\tilde{q}}) \right) \underset{\text{(A3)}}{>} 0,$$

which contradicts τ' from being optimal. Thus, (A3) cannot hold. Analogously, consider the order τ_2 as a modification of τ' in which the position of $M_k^{\tilde{q}}$ and $Z^{\tilde{q}+1}$ is swapped. Then,

$$TC(\tau') - TC(\tau_2) = \alpha[Z^{\tilde{q}+1}] \cdot \alpha[M_k^{\tilde{q}}] \cdot \left(CAT(M_k^{\tilde{q}}) - \frac{\gamma[Z^{\tilde{q}+1}]}{\alpha[Z^{\tilde{q}+1}]} \right) \underset{\text{(A4)}}{>} 0,$$

which contradicts τ' from being optimal. Thus, (A4) cannot hold. This proves (A2). Using similar arguments, it can be shown that

$$CAT(M_k) \leq \frac{\gamma[Z^1]}{\alpha[Z^1]} \quad \text{and} \quad \frac{\gamma[Z^{q+1}]}{\alpha[Z^{q+1}]} \leq CAT(M_l). \tag{A5}$$

From (A2) and (A5), it follows that

$$CAT(M_k) \leq \frac{\gamma[Z^1]}{\alpha[Z^1]} \leq CAT(M_k^1) \leq \cdots \leq \frac{\gamma[Z^q]}{\alpha[Z^q]} \leq CAT(M_k^q) \leq \frac{\gamma[Z^{q+1}]}{\alpha[Z^{q+1}]} \leq CAT(M_l). \tag{A6}$$

By combining (A1) and (A6), we obtain that $CAT(M_k) = CAT(M_l)$, which in consequence leads to

$$CAT(M_k) = \frac{\gamma[Z^1]}{\alpha[Z^1]} = CAT(M_k^1) = \cdots = \frac{\gamma[Z^q]}{\alpha[Z^q]} = CAT(M_k^q) = \frac{\gamma[Z^{q+1}]}{\alpha[Z^{q+1}]} = CAT(M_l). \tag{A7}$$

Next consider

$$\hat{\tau} = (\sim, M_l, M_k, Z^1, M_k^1, Z^2, M_k^2, \ldots, Z^q, M_k^q, Z^{q+1}, \sim),$$

which results from moving M_l to the front of M_k in τ'. Note that

$$\begin{aligned}
TC(\tau') - TC(\hat{\tau}) = \ & \alpha[M_l] \cdot \left(\alpha[M_k] \cdot CAT(M_k) + \sum_{\tilde{q}=1}^{q} \alpha[M_k^{\tilde{q}}] \cdot CAT(M_k^{\tilde{q}}) + \sum_{\tilde{q}=1}^{q+1} \alpha[Z^{\tilde{q}}] \cdot \frac{\gamma[Z^{\tilde{q}}]}{\alpha[Z^{\tilde{q}}]} \right) \\
& - \alpha[M_l] \cdot \left(\alpha[M_k] + \sum_{\tilde{q}=1}^{q} \alpha[M_k^{\tilde{q}}] + \sum_{\tilde{q}=1}^{q+1} \alpha[Z^{\tilde{q}}] \right) \cdot CAT(M_l) \\
& = \alpha[M_l] \cdot \alpha[M_k] \cdot (CAT(M_k) - CAT(M_l)) \\
& + \alpha[M_l] \cdot \sum_{\tilde{q}=1}^{q} \alpha[M_k^{\tilde{q}}] \cdot (CAT(M_k^{\tilde{q}}) - CAT(M_l)) \\
& + \alpha[M_l] \cdot \sum_{\tilde{q}=1}^{q+1} \alpha[Z^{\tilde{q}}] \cdot \left(\frac{\gamma[Z^{\tilde{q}}]}{\alpha[Z^{\tilde{q}}]} - CAT(M_l) \right) \\
& \underset{(A7)}{=} 0.
\end{aligned}$$

This implies that $\hat{\tau}$ is an optimal order.

□

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