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ORIGINAL RESEARCH

On the properties of weighted minimum colouring games

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Abstract
A weighted minimum colouring (WMC) game is induced by an undirected graph and a positive weight vector on its vertices. The value of a coalition in a WMC game is determined by the weighted chromatic number of its induced subgraph. A graph G is said to be globally (respectively, locally) WMC totally balanced, submodular, or PMAS-admissible, if for all positive integer weight vectors (respectively, for at least one positive integer weight vector), the corresponding WMC game is totally balanced, submodular or admits a population monotonic allocation scheme (PMAS). We show that a graph G is globally WMC totally balanced if and only if it is perfect, whereas any graph G is locally WMC totally balanced. Furthermore, G is globally (respectively, locally) WMC submodular if and only if it is complete multipartite (respectively, $(2K_2, P_4)$-free). Finally, we show that G is globally PMAS-admissible if and only if it is $(2K_2, P_4)$-free, and we provide a partial characterisation of locally PMAS-admissible graphs.

Keywords Weighted minimum colouring game · Totally balancedness · Submodularity · Population monotonic allocation schemes · Complete multipartite graph · $(2K_2, P_4)$-free graph

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1 Introduction

The weighted minimum colouring problem is a combinatorial optimisation problem defined on a graph G where a positive integer weight associated with each vertex of the graph represents the number of colours required to colour this vertex. The objective is to find the minimum number of colours $\chi(G)$ such that adjacent vertices are coloured with disjoint sets of colours, where $\chi(G)$ is referred to as the weighted chromatic number of the graph G. An application of this problem is the channel assignment in cellular telephone networks (McDiarmid and Reed 2000). This problem consists of assigning sets of frequency bands to transmitters, each of which demands a different number of bands. In case unacceptable interference might occur between two transmitters, they should be assigned disjoint sets of bands. If a conflict graph is constructed by letting each transmitter be represented by a vertex, letting the number of frequency bands required by a transmitter be represented by the positive integer weight of the corresponding vertex, and letting the interference relation between two transmitters be represented by an edge between the corresponding vertices, then the minimum number of frequency bands needed is the weighted chromatic number of this graph. Consider a scenario in which a number of mobile network operators are to provide cell phone service to a geographical area. Assume that all frequency bands have the same cost and that the transmitters are owned by different operators. In order to provide the cell phone service with the minimum number of frequency bands, the operators should cooperate with each other. The allocation of the total cost of the minimum number of frequency bands among the operators involved can in this case be tackled using cooperative game theory.

In this paper, we define a new class of cooperative games modeling this type of cost allocation problem, and we analyse the properties of these games. More specifically, we introduce the class of weighted minimum colouring (WMC) games, where the cost of a subset of players is equal to the weighted chromatic number of the conflict subgraph induced by this subset.

A special case of the weighted colouring problem is when all the vertex weights are equal to 1. This problem is called a minimum colouring problem. The objective is to find the minimum number of colours k such that adjacent vertices are not assigned the same colour, and k is referred to as the chromatic number of the graph. Therefore, the minimum colouring games defined by Deng et al. (1999) can be considered an instance of the WMC games. The cost of a subset of players in a minimum colouring game is equal to the chromatic number of the conflict subgraph induced by this subset. The class of minimum colouring games as well as the WMC games belong to the more general class of combinatorial optimisation games, which are cooperative games where the cost of each subset of players is obtained by solving a combinatorial optimisation problem (Curiel 1997).

There are numerous solution concepts in cooperative game theory defining different approaches to the allocation of cost among the players. The most prominent one among those is the core (Gillies 1959), which consists of all vectors (allocations) that distribute exactly the total cost of all players such that no subset of players can be better off by breaking away from the rest of the players. Core allocations create no disincentive for cooperation and consequently are considered to be stable. If the core of a cooperative game is not empty, then the game is said to be balanced, and if the core of any of the subgames of this game is not empty, then it is said to be totally balanced. If it is furthermore possible to find allocations in the core of every subgame, such that these allocations are monotonic in the sense that no player already present in a subgame is worse off if new players are added, then this allocation scheme is a population monotonic allocation scheme (PMAS), as introduced by Sprumont

(1990). The existence of a PMAS therefore implies a certain dynamic stability. A cooperative game is submodular if the incentive for other players to join a coalition increases as the coalition has more players. Submodularity is a desirable property since submodular cooperative games are totally balanced, the Shapley value is the centre of mass of the core (Shapley 1971), the bargaining set and the core coincide (Maschler et al. 1971), and submodular games have a PMAS.

In general, the core of a minimum colouring game can be empty. Nonetheless, Deng et al. (2000) show that a minimum colouring game is totally balanced if and only if the underlying graph is perfect. A graph is perfect if for all its subgraphs, the chromatic number is equal to the clique number (i.e., the number of vertices in a maximum clique). Furthermore, Okamoto (2003) characterises the submodularity of the minimum colouring games by showing that this property is satisfied if and only if the underlying graph is complete r-partite. A graph is complete r-partite if its vertices can be partitioned into r nonempty partition classes, and two vertices are adjacent if and only if they belong to different partition classes. Later on, we will also refer to the class of complete multipartite graphs. A graph is said to be complete multipartite, if it is complete r-partite for some r.

In this paper, we characterise totally balancedness and submodularity of the WMC games using the properties of the underlying graph. We define a graph G to be globally (respectively, locally) WMC totally balanced if for all positive integer weight vectors w (respectively, for at least one positive integer weight vector w), the corresponding WMC game is totally balanced. Both properties are also relevant from a practical perspective. Consider again the assignment of sets of frequency bands of different size to transmitters. The global setting investigates whether the game theoretical properties hold for any vector of demanded frequency bands, whereas the local setting investigates whether at least one vector of demanded frequency bands exists such that the game theoretical properties hold. At first sight it seems to be more interesting to investigate the global setting only. However, similar results in the local setting can be obtained, establishing a remarkable connection between the local and global setting.

We show that a graph G is globally WMC totally balanced if and only if it is perfect, and that any graph G is locally WMC totally balanced. Furthermore, we define a graph G to be globally (respectively, locally) WMC submodular if for all positive integer weight vectors w (respectively, for at least one positive integer weight vector w), the corresponding WMC game is submodular, and we show that G is globally (respectively, locally) WMC submodular if and only if it is complete multipartite (respectively, $(2K_2, P_4)$-free). A $(2K_2, P_4)$-free graph is a graph that does not have a subgraph isomorphic to the disjoint union of two complete graphs of size 2 (i.e., $2K_2$) or to a line graph of size 4 (i.e., P_4).

Hamers et al. (2014) showed that a minimum colouring game has a population monotonic allocation scheme (PMAS) if and only if the underlying graph is $(2K_2, P_4)$-free. We define a graph to be globally (respectively, locally) WMC PMAS-admissible if for all positive integer weight vectors w (respectively, for at least one positive integer weight vector w), the corresponding WMC game admits a PMAS. We show that a graph G is globally WMC PMAS-admissible if and only if it is $(2K_2, P_4)$-free. In particular, this implies that a graph G is locally WMC submodular if and only if it is globally WMC PMAS-admissible: if there is one weight vector such that the corresponding WMC game is submodular, then for all weight vectors the WMC games at least admit a PMAS. Moreover, we show that if a graph admits a specific *linear ordering*, then it is locally WMC PMAS-admissible, whereas a graph that has an induced subgraph C_n with $n \geq 5$ (i.e., a cycle on 5 or more vertices) will, for any weight vector, induce games that do not have a PMAS.

Our approach to the characterisation of totally balancedness, submodularity and PMAS-admissibility of WMC games is in the same spirit as the characterisation of balancedness, totally balancedness and submodularity of Chinese postman (CP) and travelling salesman (TS) games by Granot and Hamers (2004). In this paper, the authors define a graph to be CP globally (respectively, locally) balanced (respectively, totally balanced and submodular) if for all vertices (respectively, at least one vertex) and any non-negative weight vector defined on the edges, the corresponding CP game is balanced (respectively, totally balanced and submodular), and they study the equivalence between globally and locally CP balanced (respectively, totally balanced and submodular) graphs. Similar results are obtained for the TS case. Moreover, from the existing line of research on characterising game theoretical properties by the properties of the underlying graph, we mention the characterisation of the balancedness (respectively, totally balancedness and the submodularity) of CP games by Granot et al. (1999), the characterisation of the submodularity of the Steiner TS games on undirected graphs by Herer and Penn (1995) and on directed graphs by Granot et al. (2000) and of highway games by Çiftçi et al. (2010).[1]

The rest of the paper is organised as follows. WMC games are formally defined in Sect. 2. In Sects. 3, 4 and 5, we characterise totally balanced, submodular, and PMAS-admissible graphs respectively. Section 6 concludes.

2 Weighted minimum colouring games

This section presents the class of weighted minimum colouring games. We start with some game theoretical and graph theoretical definitions and notation.

A *cooperative (cost) game* is a pair (N, c) where $N = \{1, 2, \ldots, n\}$ is the finite set of players, and $c : 2^N \to \mathbb{R}$ is the characteristic function such that $c(\emptyset) = 0$. Here 2^N is the collection of all subsets of N (also referred to as coalitions). The cooperative game (N, c) is *submodular* if for all $i \in N$ and for all $S \subset T \subseteq N \backslash \{i\}$, its characteristic function satisfies $c(S \cup \{i\}) - c(S) \geq c(T \cup \{i\}) - c(T)$.

Let (N, c) be a cooperative game. A *subgame* of (N, c) is a game (S, c^S) where $S \subseteq N$, $S \neq \emptyset$ and $c^S(T) = c(T)$ for all $T \subseteq S$. An allocation is a vector $x \in \mathbb{R}^N$. The *core* of (N, c) is defined as

$$Core(c) = \left\{ x \in \mathbb{R}^N | \sum_{i \in N} x_i = c(N) \text{ and } \sum_{i \in S} x_i \leq c(S) \text{ for all } S \subset N \right\}. \tag{1}$$

If the core of (N, c) is not empty, then (N, c) is called *balanced*. Moreover, if the core of any of the subgames (S, c^S) is not empty, then (N, c) is called *totally balanced*.

An allocation scheme $x = (x_{S,i})_{S \in 2^N \backslash \{\emptyset\}, i \in S}$ that assigns an allocation vector to any coalition S is a *population monotonic allocation scheme* (PMAS) as introduced by Sprumont (1990), if x fulfills both an efficiency and a monotonicity requirement. That is, it must be that a) $\sum_{i \in S} x_{S,i} = c(S)$ for all $S \in 2^N \setminus \{\emptyset\}$ and b) $x_{S,i} \geq x_{T,i}$ for all $S, T \in 2^N \setminus \{\emptyset\}$ with $S \subset T$ and all $i \in S$.

Let $G = (N, E)$ be an undirected graph with finite vertex set $N = \{1, 2, \ldots, n\}$ and edge set $E \subseteq \{\{i, j\} : i, j \in N, i \neq j\}$, where each edge represents a connection between an unordered pair of vertices of G. The graph $G^S = (S, E^S)$ is the *subgraph* of G induced by a subset $S \subseteq N$ of its vertices, where $E^S = \{\{i, j\} \in E : i, j \in S\}$. A graph $G = (N, E)$ is *equivalent* to $G' = (N', E')$ if there exists a bijection $v : N \to N'$ such that $\{v(i), v(j)\} \in E'$ if and only if $\{i, j\} \in E$. The *complement* of a graph G is the graph $\bar{G} = (N, \bar{E})$ where

[1] Highway games are cooperative cost allocation games in which the cost to be allocated is associated with the construction of a highway network.

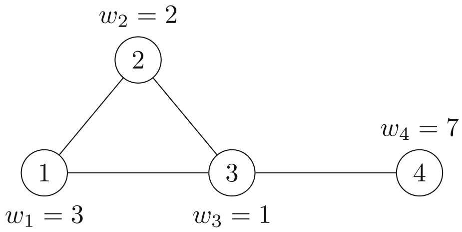

Fig. 1 Graph G and weight vector w

$\bar{E} = \{\{i, j\} : i, j \in N, i \neq j, \{i, j\} \notin E\}$, that is, two vertices in $\bar{G}$ are adjacent if and only if they are not adjacent in G. A graph in which there exists an edge between each pair of distinct vertices is a *complete* graph. A complete graph with n vertices is denoted by K_n. A *clique* in a graph G is a subset $S \subseteq N$ of its vertices such that G^S is complete. A clique is *maximum* if there are no cliques containing more elements, and it is *maximal* if it is not contained within a clique with more elements. Note that maximum cliques are always maximal. The *clique number* of G, denoted by $\omega(G)$, is the number of vertices in a maximum clique in G. Let $w \in \mathbb{Z}_+^N$ be a positive integer weight vector such that w_i is the weight associated with vertex $i \in N$. For a subset $S \subseteq N$ of vertices, the weight of S is defined as the sum of the weights of its elements, that is, $\sum_{i \in S} w_i$. Therefore, we define a *maximum weighted clique* in G with respect to w as a clique $C \subseteq N$ with maximum weight. The corresponding weight is called the *weighted clique number* of G with respect to w and denoted by $\omega_w(G)$. Note that maximum weighted cliques are always maximal. Furthermore, note that a maximum clique in G is not necessarily a maximum weighted clique in G, as we illustrate in the next example.

Example 1 Consider the graph G and the weight vector w displayed in Fig. 1. The maximum clique in G is $\{1, 2, 3\}$ and $\omega(G) = 3$. The maximum weighted clique in G with respect to w is $\{3, 4\}$ and $\omega_w(G) = 8$. $\diamond$

A *proper k-colouring* of G is a map $g : N \to \{1, 2, \ldots, k\}$ such that $g(i) \neq g(j)$ for all $\{i, j\} \in E$, that is, adjacent vertices are not assigned the same colour. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum value of k for which a proper k-colouring of G exists. A *proper weighted k-colouring* of G with respect to w is a function h that assigns a set of w_i different colours to each vertex i such that adjacent vertices i and j receive disjoint sets of colours. Formally, a proper weighted k-colouring of G is a map $h : N \to 2^{\{1,2,\ldots,k\}}$ such that $|h(i)| = w_i$ for all $i \in N$ and $h(i) \cap h(j) = \emptyset$ for all $\{i, j\} \in E$. Accordingly, the *weighted chromatic number* of G with respect to w, denoted by $\chi_w(G)$, is the minimum number k needed for a proper weighted k-colouring of G. Note that the clique number and the weighted clique number are lower bounds for the chromatic number and the weighted chromatic number respectively. Therefore, $\chi(G) \geq \omega(G)$ and $\chi_w(G) \geq \omega_w(G)$. Furthermore, if we let $w_i = 1$ for all $i \in N$, the weighted clique problem and the proper weighted k-colouring problem are equivalent to the clique problem and the proper k-colouring problem respectively.

Example 2 Consider the graph G and the weight vector w displayed in Fig. 1. We have $\chi(G) = 3$. A proper 3-colouring of G is given by $g(1) = 1$, $g(2) = 2$, $g(3) = 3$ and $g(4) = 1$. Furthermore, $\chi_w(G) = 8$, and a proper weighted 8-colouring of G is given by $h(1) = \{1, 2, 3\}$, $h(2) = \{4, 5\}$, $h(3) = \{6\}$ and $h(4) = \{1, 2, 3, 4, 5, 7, 8\}$. $\diamond$

The class of weighted minimum colouring games is formally introduced as follows. Let G be a graph and let $w \in \mathbb{Z}_+^N$ be a positive integer weight vector. Then the *weighted minimum*

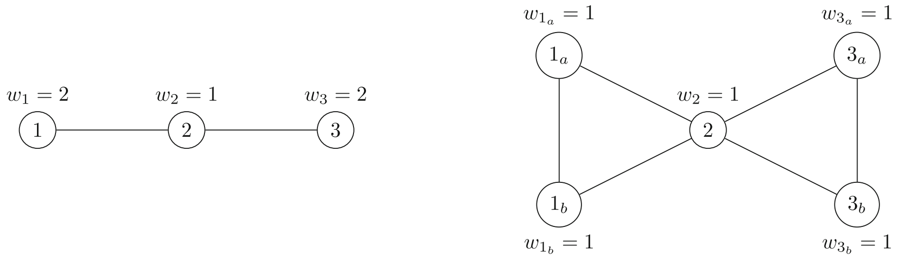

Fig. 2 A weighted graph G and a graph G' obtained from G by replication

colouring (WMC) game $(N, c^{G,w})$ is defined by

$$c^{G,w}(S) = \chi_w(G^S) \quad \text{for all } S \subseteq N, \tag{2}$$

and $c^{G,w}(\emptyset) = 0$. Note that if $w_i = 1$ for all $i \in N$, then the weighted minimum colouring game corresponds to the minimum colouring game, and we denote it simply by c^G.

3 Globally and locally WMC totally balanced graphs

In this section, we establish the equivalence of perfect graphs and globally WMC totally balanced graphs, and we show that any graph is locally WMC totally balanced. A graph G is said to be *globally* (respectively, *locally*) *WMC totally balanced* if for all weight vectors $w \in \mathbb{Z}_+^N$ (respectively, for at least one weight vector $w \in \mathbb{Z}_+^N$), the corresponding WMC game $(N, c^{G,w})$ is totally balanced.

Before moving on to the results, we present a class of graphs that will be important in what follows. A graph G is *perfect* if $\chi(G^S) = \omega(G^S)$ for all induced subgraphs G^S of G, $S \subseteq N$. For a graph $G = (N, E)$, *replication* of a vertex $v \in N$ denotes the act of (repeatedly) adding a new vertex v' to G, such that v' is adjacent to v and to all neighbors of v. Replication of v by a factor k is then the act of duplicating v a total of $k-1$ times, which amounts to replacing the vertex $v \in G$ by a clique of size k. From the replication lemma proved by Lovász (1972), perfectness is preserved by replication.

Lemma 1 (Replication lemma) *Let G be a graph, and let G' be a graph obtained from G by replication of vertices. If G is a perfect graph, then G' is a perfect graph.*

Whereas the player sets of games induced by G and G' will obviously differ, a coalition in G will have the same cost as the corresponding coalition in G'. An example is illustrated in Fig. 2, where we see that for example $c^{G,w}(\{1\}) = c^{G'}(\{1_a, 1_b\})$ and $c^{G,w}(\{1, 2\}) = c^{G'}(\{1_a, 1_b, 2\})$.

Let $w \in \mathbb{Z}_+^N$ be a weight vector. We introduce a property, called *w-perfectness*, which states that a graph G is w-perfect if $\chi_w(G^S) = \omega_w(G^S)$ for all $S \subseteq N$. A graph G is *weighted perfect* if it is w-perfect for all weight vectors $w \in \mathbb{Z}_+^N$. The concept of a weighted perfect graph in graph theory literature can be traced back to the replication lemma, since repeated application of this lemma implies that a perfect graph is weighted perfect (Schrijver 2003).

Corollary 1 *If a graph G is perfect, then it is weighted perfect.*

Note that a graph G that is not perfect can be w-perfect for some $w \in \mathbb{Z}_+^N$ as illustrated in the following example.

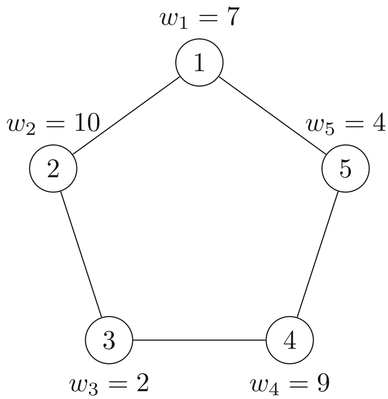

Fig. 3 G is w-perfect but not perfect

Example 3 Consider the graph G and the weight vector w displayed in Fig. 3. Note that G is not perfect since $\chi(G) = 3$ and $\omega(G) = 2$. We have $\chi_w(G) = \omega_w(G) = 17$. Moreover, it is easy to verify that $\chi_w(G^S) = \omega_w(G^S)$ for all $S \subset N$. Hence, G is w-perfect. $\diamond$

The following theorem characterises globally WMC totally balanced graphs.

Theorem 1 *G is perfect if and only if G is globally WMC totally balanced.*

Proof Consider first the 'if' part. From Deng et al. (2000), we know that when all weights are equal to one, a graph G induces a totally balanced minimum colouring game if and only if G is perfect. Therefore, if a graph G is not perfect, then G is not globally WMC totally balanced. For the 'only if' part, let $G = (N, E)$ be a perfect graph, and let $w \in \mathbb{Z}_+^N$ be a weight vector on G. Let G' be the graph obtained from G by replication of each $i \in N$ by a factor w_i. Since G is perfect, it follows from Lemma 1 that G' is perfect. Let $(N, c^{G,w})$ and $(N', c^{G'})$ be the WMC games induced by G and G' respectively. For any vertex $i \in N$, let $C(i)$ denote the clique in G' that consists of i and the $w_i - 1$ 'replicants' of i. Furthermore, for all $S \subseteq N$ let $S' = \cup_{i \in S} C(i)$ and observe that we have $c^{G,w}(S) = c^{G'}(S')$. Furthermore, since G' is perfect it follows from Deng et al. (2000) that the (unweighted) minimum colouring game $(N', c^{G'})$ is totally balanced. In order to show that $(N, c^{G,w})$ is totally balanced, let $S \subseteq N$ and $S' = \cup_{i \in S} C(i)$. There is a core vector x in the game $(S', (c^{G'})^{S'})$. Define the vector y by $y_i = \sum_{j \in C(i)} x_j$ for all $i \in S$. Then $\sum_{i \in S} y_i = \sum_{i \in S} \sum_{j \in C(i)} x_j = \sum_{j \in S'} x_j = (c^{G'})^{S'}(S') = c^{G'}(S') = c^{G,w}(S)$. Now, for all $T \subseteq S$, we have $\sum_{i \in T} y_i = \sum_{i \in T} \sum_{j \in C(i)} x_j = \sum_{j \in T'} x_j \leq (c^{G'})^{S'}(T') = c^{G'}(T') = c^{G,w}(T)$. So y is a core element of $(S, (c^{G,w})^S)$, and $(N, c^{G,w})$ is totally balanced. Now, since the argument holds for all weights, it follows that G is globally WMC totally balanced. $\square$

Before presenting the characterisation of locally WMC totally balanced games, we observe that a w-perfect graph G induces a totally balanced WMC game $(N, c^{G,w})$.

Proposition 1 *Let G be a graph, and let $w \in \mathbb{Z}_+^N$. If G is w-perfect, then G is locally WMC totally balanced.*

Proof Let G be a graph, and let $w \in \mathbb{Z}_+^N$. Let $C \subseteq N$ be a maximum weighted clique in G with respect to w. First, we show that if G is w-perfect, then the allocation $x^C \in \mathbb{R}^N$ defined as:

$$x_i^C = \begin{cases} w_i & \text{if } i \in C \\ 0 & \text{otherwise,} \end{cases}$$

is in the core of $(N, c^{G,w})$.

To do so, we need to show that x^C is in the core of $(N, c^{G,w})$. Since G is w-perfect, we have $c^{G,w}(S) = \chi_w(G^S) = \omega_w(G^S)$ for all $S \subseteq N$. For efficiency, we get

$$\sum_{i \in N} x_i^C = \sum_{i \in C} w_i = \omega_w(G) = c^{G,w}(N).$$

Furthermore, for any subset $S \subset N$, let C_S be a maximum weighted clique in G^S with respect to w. We have

$$\sum_{i \in S} x_i^C = \sum_{i \in (C \cap S)} w_i \leq \sum_{i \in C_S} w_i = \omega_w(G^S) = \chi_w(G^S) = c^{G,w}(S),$$

where: the first equality follows from the definition of x^C; the inequality follows since $C \cap S$ is a clique in G^S, and C_S is a maximum weighted clique in G^S with respect to w; the second equality follows from the definition of a maximum clique; and the third equality follows since G is w-perfect. Hence, $(N, c^{G,w})$ is balanced.

Since every induced subgraph of a w-perfect graph is also w-perfect, it can be shown in a similar way that each subgame of $(N, c^{G,w})$ has a non-empty core. □

Recall that weighted perfect graphs are w-perfect for any weight vector w. Therefore, we can also use the proof of Proposition 1 to generate core elements for WMC games induced by weighted perfect graphs. In the next lemma, we show that for any graph, we can find a w such that the graph is w-perfect.

Lemma 2 *For any graph G, we can find a weight vector w such that G is w-perfect.*

Proof Let G be a graph, and let $w_i = 2^{i-1}$ for all $i \in N$. To show that G is w-perfect for these weights, we must show that $\chi_w(G^S) = \omega_w(G^S)$ for all $S \subseteq N$. Let $C = \{i_1, i_2, \ldots, i_k\}$ be a maximum weighted clique in G with respect to w. We start by stating some observations. Firstly, there is only one maximum weighted clique in G with respect to w. Two different cliques in G cannot have the same weight with respect to w, since the binary representations of their weights, which are in fact the characteristic vectors of these cliques, are always different. Secondly, if we assume without loss of generality that $i_1 > i_2 > \ldots > i_k$, then i_1 is the vertex with the maximum index in N, that is, $i_1 = n$. Assume that $n \notin C$. We have $\sum_{j \in C} w_j \leq 1 + 2 + \cdots + 2^{n-2} < 2^{n-1} = w_n$, which contradicts C being the maximum weighted clique. Therefore, $n \in C$ and $i_1 = n$.

Next, we construct a partition of $N \backslash C$ with k elements in the following way. Let A_1 be the set of vertices in $N \backslash C$ that are not adjacent to i_1. For $l \in \{2, \ldots, k\}$, let A_l be the set of vertices in $N \backslash C$ that are adjacent to vertices $i_1, i_2, \ldots, i_{l-1}$ but not to i_l. Since the maximum weighted clique C is also a maximal clique, there does not exist a vertex in $N \backslash C$ that is adjacent to all the vertices in C. The A_l's are pairwise disjoint, and therefore, they form a partition of $N \setminus C$. It follows that in any partition element A_l, the vertex with the maximum index has at most an index of i_{l-1}. To see this, assume on the contrary that there exists a vertex $i \in A_l$ such that $i > i_{l-1}$. Then, we must have $i > i_l$ as well, since $i_l \in C$ and $A_l \subseteq N \setminus C$, and from the definition of A_l, $C' = \{i_1, i_2, \ldots, i_{l-1}\} \cup \{i\}$ is a clique in G. However, since $w_i > \sum_{m=l}^{k} w_{i_m}$, this contradicts that C is a maximum weighted clique.

Since C is the maximum weighted clique, $\omega_w(G) = 2^{i_1-1} + 2^{i_2-1} + \cdots + 2^{i_k-1}$. We present a colouring of the vertices of G using $\omega_w(G) = \sum_{l=1}^{k} w_{i_l}$ colours. We start by colouring each vertex i_l of C with $w_{i_l} = 2^{i_l-1}$ different colours, therefore using $\omega_w(G)$ different colours in total. Next, we colour the vertices in $N \backslash C$.

Since the vertex with the maximum index in A_l has at most an index of i_{l-1}, we have for all $l \in \{1, 2, \ldots, k\}$ that $A_l \subseteq \{1, 2, \ldots, i_{l-1}\}$, which in turn implies $\sum_{j \in A_l} w_j \leq 1 + 2 + \cdots + 2^{i_l-2} < 2^{i_l-1} = w_{i_l}$. Therefore, the w_{i_l} distinct colours that are used to colour vertex $i_l \in C$ are sufficient to colour all the vertices in A_l. Since $\bigcup_{l=1}^{k} A_l = N \backslash C$, all the vertices in $N \backslash C$ are coloured. By construction, this is a proper weighted colouring of G using $\omega_w(G) = \sum_{l=1}^{k} w_{i_l}$ colours. Thus, $\chi_w(G) \leq \omega_w(G)$. Recall that the weighted clique number is a lower bound on the weighted chromatic number, that is, $\chi_w(G) \geq \omega_w(G)$. Therefore, $\chi_w(G) = \omega_w(G)$.

A similar argument holds for all the weighted subgraphs of G, and thus G is w-perfect for $w_i = 2^{i-1}$ for all $i \in N$. □

Theorem 2 *Any graph G is locally WMC totally balanced.*

Proof Consider a graph G. From Lemma 2, there exists at least one positive integer weight vector $w \in \mathbb{Z}_+^N$ such that G is w-perfect. It follows from Proposition 1 that G is locally WMC totally balanced. □

In Theorem 1, we analysed the totally balancedness of WMC games by considering graphs that were obtained from weighted graphs by replication. Let G' be the unweighted graph obtained from a weighted graph G by replication of all $i \in N$ by a factor w_i. Let $(N', c^{G'})$ and $(N, c^{G,w})$ be the WMC games induced by the two graphs. We observed already that for all $S \subseteq N$, we have $c^{G,w}(S) = c^{G'}(S')$, where $S' = \cup_{i \in S} C(i)$, and $C(i)$ is the clique in G' consisting of i and the 'replicants' of i. It is straightforward to check that we have:

(i) If $(N', c^{G'})$ is totally balanced, then $(N, c^{G,w})$ is totally balanced;
(ii) If $(N', c^{G'})$ has a PMAS, then $(N, c^{G,w})$ has a PMAS;
(iii) If $(N', c^{G'})$ is submodular, then $(N, c^{G,w})$ is submodular.

In fact, statement (i) was shown in the proof of Theorem 1, but since the replicated coalitions S' only account for a (small) subset of the coalitions in N', the reverse statements of (i), (ii), and (iii) need not be true. Statement (i) together with (Replication) Lemma 1 were essential in showing Theorem 1 (the collection of perfect graphs coincides with the collection of globally WMC totally balanced graphs). However, we cannot follow the same strategy when characterising globally WMC submodular or PMAS-admissible graphs, because the class of complete multipartite graphs and the class of $(2K_2, P_4)$-free graphs are not closed under replication. As an example, consider the complete multipartite graph G in Fig. 2. Duplicating players 1 and 3 results in the graph G' that is not complete multipartite. From Okamoto (2003), it follows that $(N', c^{G'})$ is not submodular. However, it is easy to check that $(N, c^{G,w})$ is submodular for all weights. We will see in Sect. 4 that the class of complete multipartite graphs coincides with the class of globally WMC submodular graphs. Similarly, duplicating two players who are not adjacent in a $(2K_2, P_4)$-free graph results in a graph G' that is not $(2K_2, P_4)$-free (as it contains a subgraph isomorphic to $2K_2$). From Hamers et al. (2014), it follows that $(N', c^{G'})$ does not have a PMAS. In Sect. 5, we will show that in this case, $(N, c^{G,w})$ nevertheless does admit a PMAS for all weight vectors w.

4 Globally and locally WMC submodular graphs

In this section, we establish the equivalence of complete multipartite and globally WMC submodular graphs and the equivalence of $(2K_2, P_4)$-free graphs and locally WMC submodular graphs. We start by formally defining the graph classes discussed in this section.

A graph G is said to be *globally* (respectively, *locally*) *WMC submodular* if for all weight vectors $w \in \mathbb{Z}_+^N$ (respectively, for at least one weight vector $w \in \mathbb{Z}_+^N$), the corresponding WMC game $(N, c^{G,w})$ is submodular.

A line graph with 4 vertices is denoted by P_4. A complete r-partite graph $G = (N, E)$ is a graph whose vertex set can be partitioned into r nonempty partition classes $N_1, N_2, \ldots, N_r$ such that for $k, l \in \{1, 2, \ldots, r\}$ and any two vertices $i \in N_k$ and $j \in N_l$, $\{i, j\} \in E$ if and only if $k \neq l$. A graph is called complete multipartite, if it is complete r-partite for some r.

The following theorem establishes the equivalence of complete multipartite and globally WMC submodular graphs.

Theorem 3 *G is a complete multipartite graph if and only if G is globally WMC submodular.*

Proof We start with the 'if'-part. Let G be a globally WMC submodular graph. Then $(N, c^{G,w})$ is submodular for all weight vectors $w \in \mathbb{Z}_+^N$. Since Okamoto (2003) showed that in the special case where all weigths are equal to 1, $(N, c^{G,w})$ is submodular if and only if G is a complete multipartite graph, G is complete multipartite.

Next, we prove the 'only-if'-part. Let G be a complete multipartite graph, and let $N_1, N_2, \ldots, N_r$ be the partition classes of the vertex set N. A maximum clique in G consists of exactly one vertex from each of the r partition classes and hence has exactly r elements. Let $w \in \mathbb{Z}_+^N$. A maximum weighted clique in G with respect to w is a maximum clique in G. Moreover, each vertex in a maximum weighted clique in G is a maximum weighted vertex in the partition class where it belongs. Let C be a maximum weighted clique in G with respect to w. Then we have $\sum_{i \in C} w_i = \sum_{k=1}^{r} \max_{i \in N_k} w_i$.

Let $S \subseteq N$. We define $\mathcal{N}_S = \{k \in \{1, 2, \ldots, r\} : S \cap N_k \neq \emptyset\}$ to be the set of indices of the partition classes that have at least one common vertex with S. Now, consider the subgraph G^S and note that G^S is a complete multipartite graph with $|\mathcal{N}_S|$ partition classes, that is, a complete $|\mathcal{N}_S|$-partite graph. Let C^S be a maximum weighted clique in G^S with respect to w. Let $k \in \mathcal{N}_S$. Since G^S is a complete multipartite graph, we know that C^S contains exactly one maximum weighted vertex from $S \cap N_k$. Thus, $\omega_w(G^S) = \sum_{j \in C^S} w_j = \sum_{k \in \mathcal{N}_S} \max_{j \in S \cap N_k} w_j$. Moreover, note that a complete multipartite graph is perfect, and hence, according to Corollary 1, weighted perfect. Therefore, we have $c^{G,w}(S) = \chi_w(G^S) = \omega_w(G^S) = \sum_{k \in \mathcal{N}_S} \max_{j \in S \cap N_k} w_j$.

Now, let $i \in N$ and let $S \subseteq N \backslash \{i\}$. Furthermore, let $p(i) \in \{1, 2, \ldots, r\}$ such that $i \in N_{p(i)}$. We have two cases to consider, namely $p(i) \notin \mathcal{N}_S$ and $p(i) \in \mathcal{N}_S$.
Firstly, if $p(i) \notin \mathcal{N}_S$, then $\omega_w(G^{S \cup \{i\}}) = \omega_w(G^S) + w_i$, and therefore,

$$c^{G,w}(S \cup \{i\}) - c^{G,w}(S) = w_i. \tag{3}$$

Secondly, if $p(i) \in \mathcal{N}_S$, choose $j^*_{S,i} \in S \cap N_{p(i)}$ such that $w_{j^*_{S,i}} = \max_{j \in S \cap N_{p(i)}} w_j$. Then either $w_i \leq w_{j^*_{S,i}}$, in which case

$$c^{G,w}(S \cup \{i\}) = \omega_w(G^{S \cup \{i\}}) = \omega_w(G^S) = c^{G,w}(S),$$

or $w_i > w_{j^*_{S,i}}$, implying that

$$\omega_w(G^{S \cup \{i\}}) = \omega_w(G^S) + w_i - w_{j^*_{S,i}},$$

and hence,

$$c^{G,w}(S \cup \{i\}) = c^{G,w}(S) + w_i - w_{j^*_{S,i}}.$$

For the case of $p(i) \in \mathcal{N}_S$, we therefore have

$$c^{G,w}(S \cup \{i\}) - c^{G,w}(S) = \max\{w_i - w_{j^*_{S,i}}, 0\}. \tag{4}$$

In order to prove submodularity, let $i \in N$ and $S \subset T \subseteq N \backslash \{i\}$ and consider the following cases.

Case 1. $p(i) \notin \mathcal{N}_S$. It follows from (3) and (4) that

$$c^{G,w}(T \cup \{i\}) - c^{G,w}(T) \leq w_i,$$

and therefore,

$$c^{G,w}(S \cup \{i\}) - c^{G,w}(S) = w_i \geq c^{G,w}(T \cup \{i\}) - c^{G,w}(T).$$

Case 2. $p(i) \in \mathcal{N}_S$. This implies $p(i) \in \mathcal{N}_T$, and from (4) we get

$$\begin{aligned} c^{G,w}(S \cup \{i\}) - c^{G,w}(S) &= \max(w_i - w_{j^*_{S,i}}, 0) \\ &\geq \max(w_i - w_{j^*_{T,i}}, 0) \\ &= c^{G,w}(T \cup \{i\}) - c^{G,w}(T), \end{aligned}$$

where the inequality holds since $w_{j^*_{T,i}} \geq w_{j^*_{S,i}}$ for $S \subset T$.

Therefore, $c^{G,w}(S \cup \{i\}) - c(S) \geq c^{G,w}(T \cup \{i\}) - c^{G,w}(T)$ for all $i \in N$, and $S \subset T \subseteq N \backslash \{i\}$. Since this result holds for every positive integer weight vector w, G is globally WMC submodular. □

Next, we show the equivalence of $(2K_2, P_4)$-free and locally WMC submodular graphs. Note that a $(2K_2, P_4)$-free graph is a graph that does not have an induced subgraph isomorphic to $2K_2$ or P_4.

Before we are ready to establish the equivalence, we first mention the relationship between a $(2K_2, P_4)$-free graph and a rooted forest.

Let (N, A) be a directed graph where $N = \{1, 2, \ldots, n\}$ is the finite vertex set and $A \subseteq \{(i, j) : i, j \in N, i \neq j\}$ is the set of directed arcs. A *rooted tree* is a directed graph with a special vertex $r \in N$, called the *root*, such that for each vertex $i \in N$ there exists a unique directed path from r to i. The disjoint union of rooted trees is called a *rooted forest*. If $F = (N, A)$ is a rooted forest, then for every $i \in N$ there is a unique directed path from some root to i. Let $P(i)$ denote the collection of vertices on this path. The set of *descendants* of a vertex $i \in N$ is the set $D(i) = \{j \in N : i \in P(j) \text{ and } i \neq j\}$. Let $N^0 = \{j \in N : (i, j) \notin A \text{ for all } i \in N\}$. Then the elements of N^0 are the roots of the rooted trees that constitute F. A root $r \in N^0$ is the root of a vertex i if $i \in D(r)$.

Every rooted forest induces a quasi-threshold graph (or equivalently, a (C_4, P_4)-free graph where C_n is a cycle with n vertices), in the following manner.[2] Let $F = (N, A)$ be a rooted forest. Let $G = (N, E)$ be the graph such that $\{i, j\} \in E$ if and only if $i \in D(j)$ or $j \in D(i)$ in F. From Wölk (1965) and Yan et al. (1996) it follows that a graph G is quasi-threshold if and only if G is induced by a rooted forest F as just described. Then, since a $(2K_2, P_4)$-free graph is the complement of a quasi-threshold graph, it follows that a graph $G = (N, E)$ being $(2K_2, P_4)$-free is equivalent to G being induced by a rooted forest $F = (N, A)$ by

[2] Quasi threshold graphs are also called comparability graphs Wölk (1965) or trivially perfect graphs.

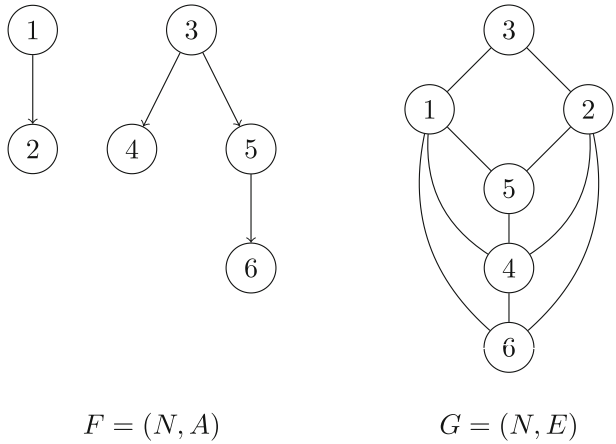

Fig. 4 A rooted forest F and the $(2K_2, P_4)$-free graph G induced by F

letting $\{i, j\} \in E$ if and only if $i \notin D(j)$ and $j \notin D(i)$ in F. We refer to F as a forest representation of G.

Example 4 Consider the rooted forest F and graph G in Fig. 4. The rooted forest F is a forest representation of the $(2K_2, P_4)$-free graph G. $\diamond$

Next, we introduce a special weighing of the vertices of a rooted forest. We start by considering a partition of the vertices of a rooted forest F. First, let us refer to the distance $d(i, j)$ between $i \in N$ and $j \in D(i) \cup \{i\}$ as the number of edges on the path from i to j. Next, recall that $N^0 = \{j \in N : (i, j) \notin A \text{ for all } i \in N\}$ is the set of roots of the rooted trees in F, and let M denote the maximum distance from any of the vertices in N to its root. Let $N^k = \{i \in N : \text{ there exists a root } r \in N^0 \text{ such that } d(r, i) = k\}$. Then $N = \bigcup_{k=0}^{M} N^k$ is a partition of N. Now, consider a permutation of the vertices in N such that all the vertices in N^0 precede all the vertices in N^1, all the vertices in N^1 precede all the vertices in N^2 and so on up to all the vertices in N^{M-1} precede all the vertices in N^M. Formally, such a permutation is a bijection $\sigma : N \to \{1, \dots, n\}$ such that for all $k_1, k_2 \in \{0, \dots, M\}$ with $k_1 < k_2$, all $i \in N^{k_1}$ and all $j \in N^{k_2}$ we have $\sigma(i) < \sigma(j)$. We refer to σ as a *root-first* permutation of the vertices in N. A *root-first* 2*-weighing* of the vertices of a rooted forest $F = (N, A)$ is the corresponding bijection $f : N \to \{1, 2, \dots, 2^{n-1}\}$ such that $f(i) = 2^{n-\sigma(i)}$. We illustrate the concept of a root-first 2-weighing with an example.

Example 5 Consider the rooted forest F in Fig. 4. We have $M = 2$, and $N^0 = \{1, 3\}$, $N^1 = \{2, 4, 5\}$ and $N^2 = \{6\}$. Furthermore, let σ be the root-first permutation of N defined by $\sigma(1) = 1$, $\sigma(3) = 2$, $\sigma(2) = 3$, $\sigma(5) = 4$, $\sigma(4) = 5$, and $\sigma(6) = 6$. Then the corresponding root-first 2-weighing of N is $f(1) = 2^5$, $f(2) = 2^3$, $f(3) = 2^4$, $f(4) = 2$, $f(5) = 2^2$, $f(6) = 1$. $\diamond$

Observe that a root $r \in N^0$ is not adjacent to any of its descendants $D(r)$ on the corresponding $(2K_2, P_4)$-free graph G. Let $w_i = f(i)$ for all $i \in N$. In order to colour a root $r \in N^0$, w_r colours are needed. For the root-first 2-weighing, we have $w_r > \sum_{i \in D(r)} w_i$ for

Table 1 Coalitional costs of the WMC game $(N, c^{G,w})$

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$c^{G,w}(S)$	w_1	w_2	w_3	$w_1 + w_2$	$\max\{w_1, w_3\}$	$\max\{w_2, w_3\}$	$\max\{w_1 + w_2, w_3\}$

all $r \in N^0$, ensuring that the w_r colours are adequate to colour all the vertices in $D(r)$ on G. Therefore, the weighted chromatic number of G with respect to w is $\chi_w(G) = \sum_{r \in N^0} w_r$. In fact, a similar result can be derived for the weighted chromatic number of the subgraph G^S with respect to w in the following manner. Let $S \subset N$. A vertex in S is called an *S-root* if it is not the descendant of any other vertex in S. Let $S^0 = \{j \in S : j \notin D(i) \text{ for all } i \in S\}$ denote the set of S-roots. For a root-first 2-weighing, we have $w_r > \sum_{i \in D(r)} w_i$ for all $r \in S^0$. Therefore, $\chi_w(G^S) = \sum_{r \in S^0} w_r$.

Before establishing the equivalence of $(2K_2, P_4)$-free graphs and locally WMC submodular graphs, we have the following lemma stating that a graph that is isomorphic to $K_1 \cup K_2$ induces a submodular WMC game if and only if the weight of the vertex to which no edge is incident is greater than or equal to the sum of the weights of the vertices that are connected by an edge.

Lemma 3 *Let $G = (N, E)$ be isomorphic to $K_1 \cup K_2$ with $N = \{1, 2, 3\}$ and $E = \{\{1, 2\}\}$. Let $w = (w_1, w_2, w_3)$ be a positive weight vector. Then the corresponding WMC game $(N, c^{G,w})$ is submodular if and only if*

$$w_3 \geq w_1 + w_2. \tag{5}$$

Proof The costs of the coalitions of the WMC game $(N, c^{G,w})$ are displayed in Table 1.

For the 'only-if'-part, let $(N, c^{G,w})$ be submodular and assume, on the contrary, that $w_3 < w_1 + w_2$. Then we have

$$\begin{aligned} c^{G,w}(\{1, 2, 3\}) - c^{G,w}(\{2, 3\}) &= w_1 + w_2 - \max\{w_2, w_3\} \\ &= \min\{w_1, w_1 + w_2 - w_3\} \\ &> \max\{w_1 - w_3, 0\} \\ &= \max\{w_1, w_3\} - w_3 \\ &= c^{G,w}(\{1, 3\}) - c^{G,w}(\{3\}), \end{aligned}$$

and $(N, c^{G,w})$ is not submodular.

The 'if'-part follows readily by checking that all submodularity conditions are satisfied if $w_3 \geq w_1 + w_2$. □

Note that the condition on the weights in Lemma 3 must hold for any induced subgraph isomorphic to $K_1 \cup K_2$, implying that if a graph induces a submodular WMC game, then for any subset of three players with only two players connected, the weight of the singleton player must be larger than or equal to the sum of the weights of the other two players.

The following theorem shows that $(2K_2, P_4)$-freeness and local WMC submodularity of a graph G are equivalent.

Theorem 4 *A graph G is $(2K_2, P_4)$-free if and only if G is locally WMC submodular.*

Proof We start with the 'if'-part. Assume that G is locally WMC submodular but assume, on the contrary, that G has a subgraph isomorphic to $2K_2$ or P_4. Without loss of generality, let G be a graph such that $\{1, 2, 3, 4\} \subseteq N$, $\{1, 2\}, \{3, 4\} \in E$ and $\{1, 3\}, \{2, 3\}, \{2, 4\} \notin E$. Let $S = \{1, 2, 3\}$, and let $T = \{2, 3, 4\}$. Let $w \in \mathbb{Z}_+^N$. From (5) of Lemma 3, (S, c^{G^S,w^S}) is submodular if and only if $w_3 \geq w_1 + w_2$, and (T, c^{G^T,w^T}) is submodular if and only if $w_2 \geq w_3 + w_4$. This, however, leads to a contradiction, since $w_1, w_4 > 0$. Therefore, the WMC game $(N, c^{G,w})$ is not submodular for any $w \in \mathbb{Z}_+^N$. This contradicts our initial assumption that G is locally WMC submodular. Hence, G does not have a subgraph isomorphic to $2K_2$ or P_4, and G is $(2K_2, P_4)$-free.

Next, we prove the 'only-if'-part. Observe that the characteristic function of a WMC game $c^{G,w}$ satisfies $c^{G,w}(T) = \chi_w(G^T) \geq \chi_w(G^S) = c^{G,w}(S)$ for all $S \subseteq T \subseteq N$ and for all $w \in \mathbb{Z}_+^N$ and thus is monotone.

Let $G = (N, E)$ be a $(2K_2, P_4)$-free graph, and let $F = (N, A)$ be a rooted forest representation of G. Let f be a root-first 2-weighing, and let $w_i = f(i)$ for all $i \in N$. Let $i \in N$ and $S \subset T \subseteq N \backslash \{i\}$. We have to show that

$$c^{G,w}(S \cup \{i\}) - c^{G,w}(S) \geq c^{G,w}(T \cup \{i\}) - c^{G,w}(T). \tag{6}$$

We distinguish between two cases. **Case 1.** Assume that i is not a root in $T \cup \{i\}$, that is, $i \notin (T \cup \{i\})^0$. Then since $w_r > \sum_{j \in D(r)} w_j$ for all $r \in (T \cup \{i\})^0$, we have $c^{G,w}(T \cup \{i\}) - c^{G,w}(T) = 0$. Furthermore, we have $c^{G,w}(S \cup \{i\}) - c^{G,w}(S) \geq 0$ from the monotonicity of $c^{G,w}$, and hence (6) holds.

Case 2. Assume that i is a root in $T \cup \{i\}$, that is, $i \in (T \cup \{i\})^0$. Let $S_i^0 = \{s \in S^0 : s \in D(i)\}$ and $T_i^0 = \{t \in T^0 : t \in D(i)\}$ be the set consisting of roots in S and T, respectively, that are descendants of i. Then

$$\begin{aligned} c^{G,w}(S \cup \{i\}) - c^{G,w}(S) &= w_i + \sum_{j \in S^0 \backslash S_i^0} w_j - \sum_{j \in S^0} w_j \\ &= w_i - \sum_{j \in S_i^0} w_j \end{aligned} \tag{7}$$

and similarly

$$c^{G,w}(T \cup \{i\}) - c^{G,w}(T) = w_i - \sum_{j \in T_i^0} w_j. \tag{8}$$

Hence, in order to prove (6), it follows from (7) and (8) that it is sufficient to show

$$\sum_{j \in T_i^0} w_j \geq \sum_{j \in S_i^0} w_j. \tag{9}$$

First, if $S_i^0 = \emptyset$, then (9) holds trivially. Therefore, assume that $S_i^0 \neq \emptyset$. Next, let $R_1 = T_i^0 \cap S_i^0$, $R_2 = T_i^0 \backslash S_i^0$ and $R_3 = S_i^0 \backslash T_i^0$. Obviously, R_1 and R_2 form a partition of T_i^0, and R_1 and R_3 form a partition of S_i^0. Moreover, consider $s \in S_i^0$ and note that since $S \subset T$, either $s \in T_i^0$, which implies $s \in R_1$, or $s \in D(t)$ for some $t \in R_2$. This in turn implies that $R_3 \subseteq D(R_2)$ where $D(R_2) = \{j \in N : j \in D(t) \text{ for some } t \in R_2\}$. Now, we prove (9). We have

$$\begin{aligned}\sum_{j\in T_i^0} w_j &= \sum_{j\in R_1} w_j + \sum_{j\in R_2} w_j \\ &\geq \sum_{j\in R_1} w_j + \sum_{j\in D(R_2)} w_j \\ &\geq \sum_{j\in R_1} w_j + \sum_{j\in R_3} w_j \\ &= \sum_{j\in S_i^0} w_j\end{aligned}$$

where the first inequality follows from the root-first 2-weighing since $w_t > \sum_{j\in D(t)} w_j$ for all $t \in R_2$.

Therefore, $(N, c^{G,w})$ is submodular, and hence G is locally WMC submodular. □

5 Globally and locally WMC PMAS-admissible graphs

We now turn to discuss the existence of a population monotonic allocation scheme (PMAS) for the class of WMC games. Recall that a graph G is said to be globally (respectively, locally) WMC PMAS-admissible if for all weight vectors $w \in \mathbb{Z}_+^N$ (respectively, for at least one weight vector $w \in \mathbb{Z}_+^N$), the corresponding WMC game $(N, c^{G,w})$ admits a PMAS.

In the first part of this section, we establish the equivalence between $(2K_2, P_4)$-free graphs and globally WMC PMAS-admissible graphs, and for that purpose, we again use the relationship between a $(2K_2, P_4)$-free graph and a rooted forest and introduce a special weighing of the vertices of a rooted forest. We start by providing a few definitions and related insights that will be useful in characterising globally PMAS-admissible graphs.

Definition 1 Let $F = (N, A)$ be a rooted forest representation of a $(2K_2, P_4)$-free graph G. Let $S \subseteq N$. For any $j \in S$, define the branch $B_j^S = D(j) \cup \{j\}$.

Observe that B_j^S is the maximal subtree in F that starts at vertex $j \in S$, and that j is the root of B_j^S. A branch B_j^S is *maximal*, if there does not exist a vertex $i \in S$ such that B_j^S is contained in B_i^S, or equivalently, if there does not exist a vertex $i \in S$ such that $j \in D(i)$. Let $B^S = \{B_1^S, \ldots, B_p^S\}$ denote the collection of maximal branches in relation to S and note that $\{(S \cap B_1^S), (S \cap B_2^S), \ldots, (S \cap B_p^S)\}$ forms a partition of S. We state the following proposition:

Proposition 2 *Let $F = (N, A)$ be a rooted forest representation of a $(2K_2, P_4)$-free graph G, let $S \subseteq N$, and let $B^S = \{B_1^S, \ldots, B_p^S\}$ denote the collection of maximal branches in relation to S. Then $c^{G,w}(S) = \sum_{j=1}^p c^{G,w}(S \cap B_j^S)$ for all $w \in \mathbb{Z}_+^N$.*

Proof First, note that for any $i \in S$, there is a unique $j \in \{1, \ldots, p\}$ such that $i \in B_j^S$. Second, let $i, i' \in S$ such that $i \in B_j^S$, $i' \in B_k^S$, and $j \neq k$. We then know that $i \notin D(i')$ and $i' \notin D(i)$, and therefore, the two vertices i, i' are adjacent in G and must be assigned disjoint sets of colours. Since this holds for all vertex pairs $i, i' \in S$ with $i \in B_j^S$, $i' \in B_k^S$

and for all $j, k \in \{1, ..., p\}$ with $j \neq k$, it follows that

$$c^{G,w}(S) = \chi_w(G^S) = \chi_w\Big(G^{\{\bigcup_{j=1}^{p} S\cap B_j^S\}}\Big) = \sum_{j=1}^{p} \chi_w(G^{S\cap B_j^S}) = \sum_{j=1}^{p} c^{G,w}(S \cap B_j^S).$$

$\square$

Next, we define a particular number $W_S(i)$ that can be calculated for all players in a coalition S by adjusting their weights in a specific way.

Definition 2 Let $F = (N, A)$ be a forest representation of a $(2K_2, P_4)$-free graph G. Let $w \in \mathbb{Z}_+^N$, and let $i \in S \subseteq N$. Define

$$W_S(i) = [w_i - c^{G,w}(S \cap D(i))]_+. \tag{10}$$

Note that for any player $i \in S$ such that $S \cap D(i) = \emptyset$, we simply have $W_S(i) = w_i$. Since $c^{G,w}(S \cap D(i))$ is the number of colours needed to colour the vertices of $S \cap D(i)$, and since in G, i is not connected to any of its descendants $D(i)$, we may interpret $W_S(i)$ as the additional number of colours needed, when vertex i is added to an existing coalition consisting of players $S \cap D(i)$. Note also that since $W_S(i)$ depends only on the intersection between S and the set of descendants of i, it follows that $W_S(i) = W_T(i)$ for any S, T such that $S \cap D(i) = T \cap D(i)$. So in particular, we have $W_S(i) = W_{S\cap B_j^S}(i)$ for $i \in S$ and $j \in \{1, \ldots, p\}$ with $i \in B_j^S$.

Theorem 5 *Let G be a $(2K_2, P_4)$-free graph and let $(N, c^{G,w})$ be the weighted minimum colouring game defined on G. Then $c^{G,w}(S) = \sum_{i\in S} W_S(i)$ for all $S \subseteq N$.*

Proof We prove this by induction. First, if $|S| = 1$ and $S = \{i\}$, then $c^{G,w}(\{i\}) = w_i = W_{\{i\}}(i)$. Next, let $m \in \mathbb{N}, m \geq 2$ and assume for all S with $|S| < m$ that $c^{G,w}(S) = \sum_{k\in S} W_S(k)$ holds. Let $S \subseteq N$ be such that $|S| = m$. We consider two cases separately depending on the value of p, the number of maximal branches:
Case 1. $p = 1$. The vertices of all players of S belong to the same maximal branch. Let j denote the root of the single maximal branch B_1^S. We then have $W_S(j) = [w_j - c^{G,w}(S \cap D(j))]_+ = [w_j - c^{G,w}(S \setminus \{j\})]_+$. Observe also that $W_{S\setminus\{j\}}(i) = W_S(i)$ for all $i \in S \setminus \{j\}$. This holds since $D(i) \subseteq S \setminus \{j\}$ which implies that $S \cap D(i) = (S \setminus \{j\}) \cap D(i)$. We distinguish between two subcases.
Case 1a. If $W_S(j) = 0$, then $w_j \leq c^{G,w}(S \setminus \{j\})$. However, since players belonging to the same branch are not adjacent in G, this implies that the vertex j can be coloured using the $c^{G,w}(S \setminus \{j\})$ colours needed to colour the vertices of $S \setminus \{j\}$. Thus, $W_S(j) = 0$ implies $c^{G,w}(S) = \chi_w(G^S) = \chi_w(G^{S\setminus\{j\}}) = c^{G,w}(S \setminus \{j\})$, and it therefore holds that

$$c^{G,w}(S) = c^{G,w}(S \setminus \{j\}) = \sum_{i\in S\setminus\{j\}} W_{S\setminus\{j\}}(i) = \sum_{i\in S\setminus\{j\}} W_S(i) = \sum_{i\in S} W_S(i), \tag{11}$$

where the second equality follows from the induction hypothesis.
Case 1b. If $W_S(j) > 0$, then $w_j = W_S(j) + c^{G,w}(S \setminus \{j\})$, and the number of colours needed to colour the descendants of j is less than the w_j colours needed to colour j. Therefore,

$c^{G,w}(S) = w_j$. This in turn implies that

$$\begin{aligned} c^{G,w}(S) = w_j &= W_S(j) + c^{G,w}(S \setminus \{j\}) \\ &= W_S(j) + \sum_{i \in S \setminus \{j\}} W_{S \setminus \{j\}}(i) \\ &= W_S(j) + \sum_{i \in S \setminus \{j\}} W_S(i) \\ &= \sum_{i \in S} W_S(i). \end{aligned}$$

Case 2. $p > 1$. We then have that

$$c^{G,w}(S) = \sum_{j=1}^{p} c^{G,w}(S \cap B_j^S) = \sum_{j=1}^{p} \sum_{k \in S \cap B_j^S} W_{S \cap B_j^S}(k) = \sum_{j=1}^{p} \sum_{k \in S \cap B_j^S} W_S(k) = \sum_{k \in S} W_S(k),$$

where the first equality follows from Proposition 2, the second equality follows from the induction hypothesis, and the third equality follows since $W_S(k) = W_{S \cap B_j^S}(k)$ for all $k \in S$ and $j \in \{1, \ldots, p\}$ with $k \in B_j^S$. □

Theorem 6 *G is $(2K_2, P_4)$-free if and only if G is globally WMC PMAS-admissible.*

Proof The 'if' part follows from Hamers et al. (2014). In fact, by choosing weights equal to one for all players, it is straightforward to show that neither graphs isomorphic to $2K_2$ nor graphs isomorphic to P_4 admit a PMAS. To prove the 'only if' part, let G be a $(2K_2, P_4)$-free graph, and let $(N, c^{G,w})$ be the weighted colouring game defined on G for some weight vector $w \in \mathbb{Z}_+^N$. Let $F(N, A)$ be a rooted forest representation of G and define the 'adjusted' weights $W_S(i)$ accordingly. Now, consider the allocation scheme x that for any $S \subseteq N$ assigns $x_{S,i} = W_S(i) = [w_i - c(S \cap D(i))]_+$ to all $i \in S$. We will prove that this allocation scheme is a PMAS for the game $(N, c^{G,w})$. First, it follows directly from Theorem 5 that the scheme x is efficient, since $\sum_{i \in S} x_{S,i} = \sum_{i \in S} W_S(i) = c^{G,w}(S)$. Next, to prove monotonicity of the allocation scheme, we need to show that $W_S(i) \geq W_{S \cup \{j\}}(i)$ for all $S \subset N$, and all $i \in S$, $j \in N \setminus S$. To see this, let $i \in S \subseteq N \setminus \{j\}$. If $W_{S \cup \{j\}}(i) = 0$, then the inequality is fulfilled, since $W_S(i) \geq 0$ for all i and all $S \subseteq N \setminus \{j\}$. Therefore, assume that $W_{S \cup \{j\}}(i) > 0$. We then have that

$$\begin{aligned} W_{S \cup j}(i) &= w_i - c^{G,w}(S \cup \{j\} \cap D(i)) \\ &\leq w_i - c^{G,w}(S \cap D(i)) \\ &= W_S(i), \end{aligned}$$

where the inequality follows from monotonicity of $c^{G,w}$. □

We now relax the restriction that for a given graph G a PMAS must exist for any weight vector and consider instead the local requirement that at least one $w \in \mathbb{Z}_+^N$ must exist such that the induced WMC game $(N, c^{G,w})$ admits a PMAS. We first note that the class of globally WMC PMAS-admissible graphs is a proper subset of the class of locally WMC PMAS-admissible graphs. By definition, a globally WMC PMAS-admissible graph is also locally WMC PMAS-admissible. Therefore, we only need to show that there exists graphs outside the class of $(2K_2, P_4)$-free graphs for which we can choose a $w \in \mathbb{Z}_+^N$ such that the induced WMC game admits a PMAS. We provide an example:

Example 6 Let $G = (N, E)$ be the graph isomorphic to P_4 with $N = \{1, 2, 3, 4\}$ and $E = \{\{12\}, \{23\}, \{34\}\}$. Let $w_i = 2^{i-1}$. For any $S \subseteq N$, let C^S denote the unique maximum weighted clique in G^S. Then the allocation scheme x that for any $S \subseteq N$ assigns $x_{S,i} = w_i$ to all $i \in C^S$ and $x_{S,i} = 0$ to all $i \in S \setminus C^S$ is a PMAS. First, note that G is a perfect graph and hence, weighted perfect. Thus, x is efficient, since $c^{G,w}(S) = \chi_w(G^S) = \omega_w(G^S) = \sum_{i \in C^S} w_i = \sum_{i \in S} x_i$. Furthermore, by inspecting the cost of coalitions and the associated x_S, it is straightforward to verify that x is monotonic. Therefore, x is a PMAS, and G is locally PMAS-admissible. ◇

In fact, any graph that admits a specific type of linear order (to be defined below) is locally WMC PMAS-admissible and, furthermore, admits a PMAS of the same simple type as in Example 6, where the members of the unique maximum weighted clique pay the whole cost in any coalition.

Definition 3 A graph $G = (N, E)$ has a *linear ordering*, if there exists a bijection $\pi : N \to \{1, \ldots, n\}$, such that for every $i, j, k \in N$ with $\pi(i) < \pi(k) < \pi(j)$, we have that if $ij \in E$ then either $ik \in E$ or $kj \in E$.

Some examples of graphs that have such a linear ordering are cycles with three or four vertices (C_3, C_4), line graphs, and 'caterpillar tree' graphs that are graphs for which every vertex is on a central path or at most one graph edge away from the central path. However, graphs isomorphic to C_n with $n \geq 5$ do not have a linear ordering.

Proposition 3 *Let $G = (N, E)$ be a graph with a linear ordering. Then G is locally WMC PMAS-admissible. In particular, we can choose a weight vector $w \in \mathbb{Z}_+^N$ such that for every $S \subseteq N$, the graph G^S has a unique maximum weighted clique C^S, and for this weight vector, a PMAS x of $(N, c^{G,w})$ can be described as follows: $x_{S,i} = w_i$ if $i \in C^S$, $x_{S,i} = 0$ if $i \notin C^S$.*

Proof Let π be the corresponding bijection and define $w_i = 2^{n-\pi(i)}$ for every $i \in N$. Let $S \subseteq N$. Similarly to the proof of Lemma 2, we can show that G is w-perfect, implying that $c^{G,w}(S) = \chi_w(G^S) = \omega_w(G^S)$ for all $S \subseteq N$, which ensures efficiency of x. Moreover, due to the choice of weights, it is again obvious that for every $S \subseteq N$, the graph G^S has a unique maximum weighted clique C^S. For monotonicity of x, it is sufficient to show that for all $S, T \in 2^N \setminus \emptyset$ with $S \subset T$ and all $i \in S$, we have: if $i \in C^T$, then $i \in C^S$. Therefore, let $S, T \in 2^N \setminus \emptyset$ be such that $S \subset T$, and let $i \in S$. Assume that $i \in C^T$. Write $C^T = \{i_1, \ldots, i_k\}$ with $\pi(i_1) < \pi(i_2), \ldots, \pi(i_k)$, so $w_{i_1} > w_{i_2}, \ldots, w_{i_k}$. Note that i_1 is the player in T with the smallest label (highest weight). If $i = i_1$, then clearly, $i \in C^S$, so from now on assume that $i = i_j$ with $j \in \{2, \ldots, k\}$. Note that $i_1 i = i_1 i_j \in E$, since $i_1 \in C^T$ and $i \in C^T$. We are going to show that i is connected to every player in T with a smaller label. As a consequence, i is then connected to every player in S with a smaller label, and hence, $i \in C^S$. Suppose to the contrary that there is a player $l \in T$ with $\pi(l) < \pi(i)$ and $il \notin E$. Let l^* be such a player with minimum label, i.e., $\pi(l^*) = \min\{\pi(l) | l \in T, \pi(l) < \pi(i), il \notin E\}$. For every $m \in T$ with $\pi(m) < \pi(l^*)$, we have $\pi(m) < \pi(l^*) < \pi(i)$ and $im \in E$. Now, from the definition of the linear ordering and since $il^* \notin E$, we have $ml^* \in E$. So l^* is connected to all players in T with a smaller label, and l^* is therefore also connected to the player in T with the highest weight, implying $l^* \in C^T$, and because $i \in C^T$ as well, $il^* \in E$, a contradiction. □

Next, we provide an example of a graph that does not have a linear ordering, but still is locally WMC PMAS admissible.

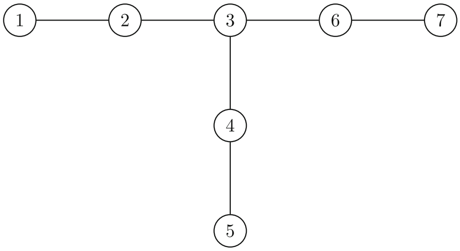

Fig. 5 A graph G that does not have a linear ordering

Example 7 Consider the graph G depicted in Fig. 5, which does not have a linear ordering.

G is, nevertheless, locally PMAS-admissible. To see this, let $w_j = 2^{j-1}$ for all j. Then a PMAS x can be constructed as follows: for all $S \subseteq N \setminus \{6, 7\}$ such that $\{3, 5\} \subset S$, let $x_{S,3} = w_3, x_{S,5} = w_5 - w_3, x_{S,j} = w_j$ for $j \in C^S \setminus \{3, 5\}$, and let $x_{S,j} = 0$ for the remaining players. For all other $S \subseteq N$, let $x_{S,j} = w_j$ for all $j \in C^S$, and let $x_{S,j} = 0$ otherwise. It is straightforward to check that $\sum_{i \in S} x_{S,i} = \sum_{i \in C^S} w_i$, so x is efficient. Furthermore, to see that x satisfies the monotonicity requirement, note the following: $x_{S,1} = w_1$ for $S = \{1\}$ and $S = \{1, 2\}$, and $x_{S,1} = 0$ otherwise. $x_{S,2} = w_2$ for $S \subseteq \{1, 2, 3\}$ containing 2, and $x_{S,2} = 0$ otherwise. $x_{S,3} = w_3$ for all $S \subseteq N \setminus \{7\}$ with $3 \in S$, and $x_{S,3} = 0$ otherwise. $x_{S,4} = w_4$ for all $S \subseteq \{1, 2, 3, 4, 5\}$ that contains 4, and $x_{S,4} = 0$ otherwise. $x_{S,5} = w_5$ for $S \subseteq \{1, 2, 4, 5\}$ that contains 5, $x_{S,5} = w_5 - w_3$ for $S \subseteq \{1, 2, 3, 4, 5\}$ that contains $\{3, 5\}$, and $x_{S,5} = 0$ otherwise. $x_{S,6} = w_6$ for all $S \subseteq N$ containing 6, and $x_{S,7} = w_7$ for all $S \subseteq N$ containing 7. Therefore, a player is never worse off when the coalition increases, and the allocation scheme is a PMAS. $\diamond$

Not all graphs are, however, locally PMAS-admissible. In particular, graphs that contain as an induced subgraph a cycle with 5 vertices or more will for no choice of weight vector induce a game that has a PMAS.

Proposition 4 *Let $G = (N, E)$ be a connected graph. If G has an induced subgraph C_n with $n \geq 5$, then G is not locally WMC PMAS-admissible.*

Proof Let $S \subseteq N$ be such that subgraph G^S is a cycle with at least 5 vertices. Suppose $w \in \mathbb{Z}_+^N$ is a weight vector such that $C^{G,w}$ has a PMAS x. Let $i \in S$ be such that $w_i = \max_{j \in S} w_j$.Without loss of generality assume $i = 1$. Also without loss of generality assume that the other $n - 1$ players in S are labeled $2, \ldots, n$ in such a way that $E^S = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \ldots, \{n - 1, n\}, \{n, 1\}\}$. We then have:

$$\begin{aligned}
&w_1 + \max\{w_3, w_n\} + \max\{w_2, w_n\} \\
&= c^{G,w}(\{1, 3\}) + c^{G,w}(\{3, n\}) + c^{G,w}(\{2, n\}) \\
&= x_{\{1,3\},1} + x_{\{1,3\},3} + x_{\{3,n\},3} + x_{\{3,n\},n} + x_{\{2,n\},2} + x_{\{2,n\},n} \\
&\geq x_{\{1,3,n\},1} + x_{\{1,3,n\},3} + x_{\{1,3,n\},n} + x_{\{2,3,n\},2} + x_{\{2,3,n\},3} + x_{\{2,3,n\},n} \\
&= c^{G,w}(\{1, 3, n\}) + c^{G,w}(\{2, 3, n\}) \\
&= w_1 + w_n + \max\{w_2 + w_3, w_n\}.
\end{aligned}$$

This inequality can be rewritten as

$$\begin{aligned}
&\max\{w_1 + w_3 + w_2, w_1 + w_3 + w_n, w_1 + w_n + w_2, w_1 + 2w_n\} \\
&\geq \max\{w_1 + w_n + w_2 + w_3, w_1 + 2w_n\},
\end{aligned}$$

Table 2 Overview of characterisations

	MC (unweighted)	Globally WMC	Locally WMC
Totally balanced graphs	Perfect	Perfect	Any
PMAS - admissible graphs	$(2K_2, P_4)$-free	$(2K_2, P_4)$-free	Linear order[a]
Submodular graphs	Complete multipartite	Complete multipartite	$(2K_2, P_4)$-free

[a] In contrast to the other cells, this cell does not show a complete characterisation. The set of graphs with linear orders is a proper subset of the set of locally WMC PMAS-admissible graphs. They induce games with a certain simple type of PMAS

from which we derive $w_1 + 2w_n \geq w_1 + w_n + w_2 + w_3$, so $w_n \geq w_2 + w_3$. A similar argument with player 2 replaced by player n, player 3 replaced by player $n-1$ and player n by player 2, leads to the inequality $w_2 \geq w_{n-1} + w_n$. Now $w_2 \geq w_{n-1} + w_n > w_n \geq w_2 + w_3 > w_2$, a contradiction. □

6 Concluding remarks

We have analysed the properties of weighted minimum colouring games and characterised classes of graphs that induce WMC games that are totally balanced, submodular or admit a PMAS. In Table 2 below, we sum up the results on both MC and WMC games.

Since the properties of the weighted minimum colouring games are characterised in terms of specific graph classes, the complexity of recognising each of the particular graph classes may be of interest, especially in applications. The three classes of graphs that characterize global and local game theoretical properties can all be recognized in polynomial time. Perfect graphs can be recognized in polynomial time of order $O(V^9)$, Chudnovsky et al. (2005). Both quasi-threshold and complete multipartite graphs can be recognized in linear time, Chu (2008), Corneil et al. (1985). Chu (2008) provides a linear time algorithm for recognizing quasi-threshold graphs and shows that this algorithm can be adapted to also recognise the complements of trivially perfect graphs. Therefore, $(2K_2, P_4)$-free graphs can be recognised in linear time.

The full characterisation of locally WMC PMAS admissible graphs is an area for further research. In the current paper, we have assumed that costs are proportional to the number of colours, so another area for further research is to allow for more general cost functions in the underlying optimisation problem. This would, for example, be relevant for applications where costs are likely to increase more than proportional to the number of colours demanded. Also other types of cost functions could be considered. Another generalisation relevant for applications would be to allow more than one node to be associated with the same player.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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