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Glivenko–Cantelli-type theorems for weighted empirical distribution functions based on uniform spacings

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Abstract: Let $U_1, U_2, \ldots$ be a sequence of independent r.v.'s having the uniform distribution on (0, 1). Let $\hat{F}_n$ be the empirical distribution based on the transformed uniform spacings $\mathbf{D}_{i,n} := G(nD_{i,n})$, $i = 1, 2, \ldots, n$, where G is the exp(1) d.f. and $D_{i,n}$ is the ith spacing based on $U_1, U_2, \ldots, U_{n-1}$. The main purpose of this paper is the study of the almost sure behaviour of $\limsup_{n\to\infty} \Delta_{n,\alpha}(q, \tilde{q})$ and $\limsup_{n\to\infty} \Lambda_{n,r}(q, \tilde{q})$, where $\Delta_{n,\alpha}(q, \tilde{q}) = \sup_{0<t<1}[n^\alpha |\hat{F}_n(t) - t|/(q(t)\tilde{q}(1-t))]$ and $\Lambda_{n,r}(q, \tilde{q}) = \int_0^1 (|\hat{F}_n(t) - t|/(q(t)\tilde{q}(1-t)))^r \, dt$ for $\alpha \in [0, \frac{1}{2})$, $r > 0$ and certain weight functions q and $\tilde{q}$. Moreover, the weak behaviour of the statistics will be examined briefly. It turns out that compared with the uniform empirical process (i.i.d. case) the considered weighted Kolmogorov–Smirnov- and Cramér–von Mises-type statistics behave differently in the right tail only as far as almost sure convergence is concerned. There is no difference in the weak sense. The results can be applied to the study of linear combinations of functions of ordered spacings.

Keywords: Glivenko-Cantelli theorems, order statistics, strong convergence, uniform spacings, weak convergence, weighted empirical process.

1. Introduction and main results

Let $\{U_i\}_{i=1}^\infty$ be a sequence of independent uniform (0, 1) distributed random variables (r.v.'s) and for any integer $n \geq 2$ define the transformed uniform spacings by $\mathbf{D}_{i,n} := G(nD_{i,n})$, where

$$D_{i,n} = U_{i:n-1} - U_{i-1:n-1}, \quad i = 1, 2, \ldots, n,$$
$$0 =: U_{0:n-1} \leq U_{1:n-1} \leq \cdots \leq U_{n-1:n-1} \leq U_{n:n-1} := 1$$

are the uniform order statistics at stage $n-1$ and G is the standard exponential distribution function (d.f.), i.e. $G(x) = 1 - e^{-x}$, $x \geq 0$. It easily follows that for $t \in (0, 1 - e^{-n}]$,

$$F_n(t) := P(\mathbf{D}_{i,n} \leq t) = P(nU_{1:n-1} \leq -\log(1-t)) = 1 - \left(1 + \frac{\log(1-t)}{n}\right)^{n-1},$$

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and hence for a fixed $t \in (0, 1)$,

$$\lim_{n\to\infty} F_n(t) = t.$$

The empirical d.f. based on the $D_{i,n}$ is defined by

$$\hat{F}_n(t) = n^{-1} \sum_{i=1}^{n} 1_{(0,t]}(D_{i,n}), \quad t \in (0, 1).$$

Recently the authors obtained a complete characterization of the almost sure behaviour of $\limsup_{n\to\infty} b_n V_{n,\alpha}$ and $\limsup_{n\to\infty} b_n W_{n,\alpha}$, for $\alpha \in [0, \frac{1}{2}]$ and a sequence of positive norming constants $\{b_n\}_{n=1}^{\infty}$, where

$$V_{n,\alpha} = \sup_{0<t<1} \frac{n|\hat{F}_n(t) - t|}{t^{1-\alpha}}, \tag{1.1}$$

$$W_{n,\alpha} = \sup_{0<t<1} \frac{n|\hat{F}_n(t) - t|}{(1-t)^{1-\alpha}}. \tag{1.2}$$

The precise result is as follows:

Theorem (Einmahl and van Zuijlen, 1988). *For each $\alpha \in [0, \frac{1}{2}]$ we have for sequences of positive constants $\{a_n\}_{n=1}^{\infty}$ the following implications*:

$$\left[\sum a_n = \infty\right] \Rightarrow \left[\limsup_{n\to\infty} a_n^{1-\alpha} V_{n,\alpha} = \infty \ a.s.\right], \tag{1.3}$$

$$\left[\sum a_n < \infty,\ a_n \downarrow \text{ and } na_n \log n \to 0\right] \Rightarrow \left[\lim_{n\to\infty} a_n^{1-\alpha} V_{n,\alpha} = 0 \ a.s.\right], \tag{1.4}$$

$$\left[\sum a_n \log(a_n^{-1}) = \infty\right] \Rightarrow \left[\limsup_{n\to\infty} a_n^{1-\alpha} W_{n,\alpha} = \infty \ a.s.\right] \tag{1.5}$$

and

$$\left[\sum a_n \log(a_n^{-1}) < \infty \text{ and } a_n \downarrow 0\right] \Rightarrow \left[\lim_{n\to\infty} a_n^{1-\alpha} W_{n,\alpha} = 0 \ a.s.\right]. \quad \square \tag{1.6}$$

Note the difference in behaviour between the left tail ($V_{n,\alpha}$) and the right tail ($W_{n,\alpha}$): the result for $V_{n,\alpha}$ is exactly the same as the corresponding result for the one dimensional uniform empirical process (i.i.d. case), whereas the behaviour of $W_{n,\alpha}$ is essentially different.

In the present paper we will use the aforementioned results in the study of weighted Kolmogorov–Smirnov- and Cramér–von Mises-type statistics based on uniform spacings. To be more precise, we study the almost sure behaviour, but also the weak behaviour, of the statistics

$$\Delta_{n,\alpha}(q, \tilde{q}) = \sup_{0<t<1} \frac{n^{\alpha}|\hat{F}_n(t) - t|}{q(t)\tilde{q}(1-t)} \tag{1.7}$$

and

$$\Lambda_{n,r}(q, \tilde{q}) = \int_0^1 \left(\frac{|\hat{F}_n(t) - t|}{q(t)\tilde{q}(1-t)} \right)^r \mathrm{d}t \tag{1.8}$$

for $\alpha \in [0, \frac{1}{2})$, $r > 0$ and certain weight functions q and $\tilde{q}$, which will be specified later.

It is well-known that the unweighted versions of the statistics $\Delta_{n,\alpha}$ and $\Lambda_{n,r}$ ($q \equiv \tilde{q} \equiv 1$) can be used for testing e.g. exponentiality, making use of the fact that the vector of uniform spacings is distributed as a vector of n i.i.d. exponential r.v.'s divided by their sum. Cf. also Pyke (1965, 1972) and Deheuvels (1986). The unweighted versions of $\Delta_{n,\alpha}$ and $\Lambda_{n,r}$ exhibit poor sensitivity to deviations from the exponential distribution that occur in the tails. This drawback can be met by using the just defined weighted versions of the statistics.

In order to state our results (which will be proved in Section 2), let us first introduce the following classes of weight functions. We write

$$Q = \{q:(0,1] \to (0,\infty): q \text{ bounded away from 0 on } [\delta, 1] \text{ for every } \delta > 0 \text{ and } q \text{ bounded}\} \tag{1.9}$$

and for $\tau > 0$,

$$Q_\tau = \{q \in Q: q/I^\tau \text{ is non-increasing}\}, \tag{1.10}$$

where I is the identity function on (0, 1]. The first theorem gives a complete characterization of the almost sure behaviour of $\Delta_{n,\alpha}$:

Theorem 1. *Let* $0 \leqslant \alpha < \frac{1}{2}$ *and* $q, \tilde{q} \in Q_{1-\alpha}$. *If*

$$\int_0^1 (q(t))^{-1/(1-\alpha)}\,dt < \infty \quad \text{and} \quad \int_0^1 \log(1/t)(\tilde{q}(t))^{-1/(1-\alpha)}\,dt < \infty, \tag{1.11}$$

then

$$\lim_{n\to\infty} \Delta_{n,\alpha}(q, \tilde{q}) = 0 \quad a.s. \tag{1.12}$$

If at least one of these integrals is infinite, then

$$\limsup_{n\to\infty} \Delta_{n,\alpha}(q, \tilde{q}) = \infty \quad a.s. \tag{1.13}$$

Remark 1. Note that in case $\alpha = 0$ we have the Glivenko–Cantelli theorem for the weighted empirical process based on uniform spacings.

Remark 2. The weak analogue of Theorem 1 for $\Delta_{n,\alpha}(q, \tilde{q})$ with $\alpha \in (0, \frac{1}{2})$ and $q, \tilde{q} \in Q$ is almost immediate from the corresponding weak result for $\Delta_{n,\alpha}(I^{1-\alpha}, I^{1-\alpha})$ (see Theorem 3 in Csörgő and Horváth, 1986) and reads as follows. If

$$\lim_{t\downarrow 0} \frac{t^{1-\alpha}}{q(t)} = c_1 \in [0, \infty] \quad \text{and} \quad \lim_{t\downarrow 0} \frac{t^{1-\alpha}}{\tilde{q}(t)} = c_2 \in [0, \infty], \tag{1.14}$$

then we have for $c_1 \in [0, \infty)$ and $c_2 \in [0, \infty)$:

$$\Delta_{n,\alpha} \xrightarrow{D} c_1 X_\alpha \vee c_2 X'_\alpha, \tag{1.15}$$

where X'_α is an independent copy of $X_\alpha = \sup_{t\in[0,\infty)}(|N(t) - t|/t^{1-\alpha})$, with $\{N(t), t \in [0, \infty)\}$ a Poisson process with intensity parameter 1. Moreover, if $c_1 = \infty$ or $c_2 = \infty$ then we have

$$\Delta_{n,\alpha} \xrightarrow{P} \infty. \tag{1.16}$$

Corollary 1. *Let* $q \in Q_1$. *If*

$$\int_0^1 (q(t))^{-1}\, \mathrm{d}t < \infty, \tag{1.17}$$

then

$$\lim_{n\to\infty} \sup_{0<t<1} \frac{\hat{F}_n(t)}{q(t)} = \sup_{0<t<1} \frac{t}{q(t)} \quad a.s. \tag{1.18}$$

If the integral in (1.17) *is infinite, then*

$$\limsup_{n\to\infty} \sup_{0<t<1} \frac{\hat{F}_n(t)}{q(t)} = \infty \quad a.s. \tag{1.19}$$

Our next theorem deals with the statistic $\Lambda_{n,r}$. In this case we give sufficient conditions for the almost sure convergence of $\Lambda_{n,r}$ to zero and show that they are very close to being also necessary.

Theorem 2A. *Let* $r > 0$ *and* $q, \tilde{q} \in Q_{(r+1)/r}$. *If*

$$\int_0^1 (q(t))^{-r/(r+1)}\, \mathrm{d}t < \infty \quad \text{and} \quad \int_0^1 (\log(1/t))^{r/(r+1)} (\tilde{q}(t))^{-r/(r+1)}\, \mathrm{d}t < \infty, \tag{1.20}$$

then

$$\lim_{n\to\infty} \Lambda_{n,r}(q, \tilde{q}) = 0 \quad a.s. \tag{1.21}$$

However, if $q = I^{(r+1)/r}$ *or* $\tilde{q} = I^{(r+1)/r}$, *then*

$$\Lambda_{n,r}(q, \tilde{q}) = \infty \quad a.s. \tag{1.22}$$

The in probability version of the above theorem is as follows:

Theorem 2B. *Let* $r > 0$ *and* $q, \tilde{q} \in Q$. *If*

$$\int_0^1 \left(\frac{t}{q(t)}\right)^r \mathrm{d}t < \infty \quad \text{and} \quad \int_0^1 \left(\frac{t}{\tilde{q}(t)}\right)^r \mathrm{d}t < \infty, \tag{1.23}$$

then

$$\Lambda_{n,r} \xrightarrow{\mathrm{P}} 0. \tag{1.24}$$

However, if at least one of the integrals in (1.23) *is infinite, then* (*almost surely*)

$$\Lambda_{n,r} = \infty. \tag{1.25}$$

Discussion of the results. For the uniform empirical process (abbreviated as 'i.i.d. case'), the convergence half of Theorem 1 with $\alpha = 0$ has been established in Lai (1974), whereas Wellner (1977; 1978) essentially solved the whole characterization problem for $\Delta_{n,0}$ and showed how that result can be used to obtain almost sure nearly linear bounds for the empirical distribution function in that case. These nearly linear bounds can be used in turn as a tool for proving strong laws of large numbers for linear combinations of functions of order statistics in the i.i.d. case. Obviously, our results for $\Delta_{n,0}$ — in the case of uniform spacings — can be used similarly to obtain almost sure nearly linear bounds for the empirical distribution function in the considered situation and to obtain strong laws for linear combinations of functions of ordered spacings. For related results see also Beirlant and van Zuijlen (1985). Recently, Andersen, Giné and Zinn (1988) obtained essentially Theorem 1 for general α in the i.i.d. case. See also Einmahl and Mason (1988) for other related results in the i.i.d. case.

Andersen, Giné and Zinn (1988) gave for general α in the i.i.d. case a slightly weaker result of our in probability version of Theorem 1 (see Remark 2).

A slight modification of the counterpart of our Theorem 2A in the i.i.d. case can be found in Einmahl and Mason (1988).

Note that the restrictions on the weight functions under which we stated our results are very mild; only some monotonicity-condition is needed. Finally, we remark that we find (as could be expected) a difference between the left and the right tail in case of almost sure results, whereas there is no difference if one considers only 'weak' results.

2. Proofs of the results

In this section we will successively give the proofs of Theorem 1, Corollary 1, Theorem 2A and Theorem 2B. The proof of Remark 2 following Theorem 1 is easy and will be omitted.

Proof of Theorem 1. It is easily seen that it suffices to consider the left and right tail separately, i.e. it is sufficient to prove that

$$\int_0^1 (q(t))^{-1/(1-\alpha)}\,\mathrm{d}t < \infty,$$

implies

$$\lim_{n\to\infty} \sup_{0<t<1} \frac{n^{\alpha}\left|\hat{F}_n(t)-t\right|}{q(t)} = 0 \quad \text{a.s.}$$

and that

$$\int_0^1 (q(t))^{-1/(1-\alpha)}\,\mathrm{d}t = \infty,$$

implies

$$\limsup_{n\to\infty} \sup_{0<t<1} \frac{n^{\alpha}\left|\hat{F}_n(t)-t\right|}{q(t)} = \infty \quad \text{a.s.}$$

and the analogous statements for the right tail. Since the proof for the right tail is slightly more complicated we give the proofs for that tail and omit those for the left one.

So first assume

$$\int_0^1 \log(1/t)(\tilde{q}(t))^{-1/(1-\alpha)}\,\mathrm{d}t < \infty.$$

We begin with proving that

$$\lim_{n\to\infty} \sup_{1-n^{-1/2}\leqslant t<1} \frac{n^{\alpha}\left|\hat{F}_n(t)-t\right|}{\tilde{q}(1-t)} = 0 \quad \text{a.s.} \tag{2.1}$$

Write $\phi(t)=t/(\bar{q}(t))^{1/(1-\alpha)}$, $0<t\leqslant 1$ and observe that $\phi\uparrow$ and that

$$\int_0^1 \log(1/t)t^{-1}\phi(t)\,\mathrm{d}t<\infty. \tag{2.2}$$

By the change of variables $t=s^{-1/2}$ we also see that

$$\int_1^\infty \phi(s^{-1/2})s^{-1}\log s\,\mathrm{d}s<\infty$$

and, since $\phi\uparrow$,

$$\sum_n n^{-1}\phi(n^{-1/2})\log n<\infty. \tag{2.3}$$

Writing $a_n=n^{-1}\phi(n^{-1/2})$ and using some elementary analysis it is immediate from (2.3) that

$$\sum_n a_n\log(1/a_n)<\infty. \tag{2.4}$$

Moreover we have $na_n\downarrow 0$ by (2.4) and the fact that $\phi\uparrow$. Observe that

$$\begin{aligned}\sup_{1-n^{-1/2}\leqslant t<1}\frac{n^\alpha|\hat{F}_n(t)-t|}{\bar{q}(1-t)}&=\sup_{1-n^{-1/2}\leqslant t<1}\frac{n^\alpha|\hat{F}_n(t)-t|}{(1-t)^{1-\alpha}}(\phi(1-t))^{1-\alpha}\\&\leqslant(na_n)^{1-\alpha}\sup_{0<t<1}\frac{n^\alpha|\hat{F}_n(t)-t|}{(1-t)^{1-\alpha}}.\end{aligned} \tag{2.5}$$

Combining (2.5) with (2.4) and (1.6) yields (2.1).

Next we show

$$\lim_{n\to\infty}\sup_{0<t\leqslant 1-n^{-1/2}}\frac{n^\alpha|\hat{F}_n(t)-t|}{\bar{q}(1-t)}=0\quad\text{a.s.} \tag{2.6}$$

Observe that, since $I^{1-\alpha}/\bar{q}\uparrow$,

$$\sup_{0<t\leqslant 1-n^{-1/2}}\frac{n^\alpha|\hat{F}_n(t)-t|}{\bar{q}(1-t)}\leqslant\frac{1}{\bar{q}(1)}\sup_{0<t\leqslant 1-n^{-1/2}}\frac{n^\alpha|\hat{F}_n(t)-t|}{(1-t)^{1-\alpha}}. \tag{2.7}$$

Hence for a proof of (2.6) it suffices to prove

$$\lim_{n\to\infty}\sup_{0<t\leqslant 1-n^{-1/2}}\frac{n^\alpha|\hat{F}_n(t)-t|}{(1-t)^{1-\alpha}}=0\quad\text{a.s.} \tag{2.8}$$

Note that

$$\sup_{0<t\leqslant 1-n^{-1/2}}\frac{n^\alpha|\hat{F}_n(t)-t|}{(1-t)^{1-\alpha}}\leqslant\sup_{0<t\leqslant 1-n^{-1/2}}\frac{n^{1/4+\alpha/2}|\hat{F}_n(t)-t|}{(1-t)^{1/2}}. \tag{2.9}$$

Using formula (3.3) in Einmahl and van Zuijlen (1988) yields that for every $\varepsilon>0$,

$$P\left(\sup_{0<t\leqslant 1-n^{-1/2}}\frac{n^{1/4+\alpha/2}|\hat{F}_n(t)-t|}{(1-t)^{1/2}}\geqslant\varepsilon\right)\leqslant K_1\log n\exp\left(-K_2n^{1/2-\alpha}\right) \tag{2.10}$$

for some K_1, $K_2 \in (0, \infty)$ only depending on ε. This, in combination with the Borel–Cantelli lemma, completes the proof of (2.8) and hence of the first part of Theorem 1.

Finally we must show that

$$\int_0^1 \log(1/t)(\tilde{q}(t))^{-1/(1-\alpha)}\,\mathrm{d}t = \infty \tag{2.11}$$

implies

$$\limsup_{n\to\infty} \sup_{0<t<1} \frac{n^\alpha |\hat{F}_n(t) - t|}{\tilde{q}(1-t)} = \infty \quad \text{a.s.} \tag{2.12}$$

Again write $\phi = I/\tilde{q}^{1/(1-\alpha)}$ and note that (2.11) implies

$$\int_0^1 \log(1/t) t^{-1}\phi(t)\,\mathrm{d}t = \infty,$$

which, in turn by the change of variables $t = s^{-2}$, gives

$$\int_1^\infty \phi(s^{-2}) s^{-1} \log s\,\mathrm{d}s = \infty.$$

Since $\phi \uparrow$, we have

$$\sum_n n^{-1}\phi(n^{-2}) \log n = \infty.$$

Writing $a_n = n^{-1}\phi(n^{-2})$ it is easily seen that

$$\sum_n a_n \log(1/a_n) = \infty.$$

Now from (1.3) and (1.10) in Einmahl and van Zuijlen (1988) we have for every $\varepsilon > 0$,

$$P\big(1 - \varepsilon a_n \leqslant D_{n:n} \leqslant 1 - n^{-2} \text{ i.o.}\big) = 1. \tag{2.13}$$

Note that

$$\sup_{0<t<1} \frac{n^\alpha |\hat{F}_n(t) - t|}{\tilde{q}(1-t)} \geqslant (na_n)^{1-\alpha} \sup_{0<t\leqslant 1-n^{-2}} \frac{n^\alpha |\hat{F}_n(t) - t|}{(1-t)^{1-\alpha}}. \tag{2.14}$$

Combination of (2.13) and (2.14) easily yields

$$\limsup_{n\to\infty} \sup_{0<t<1} \frac{n^\alpha |\hat{F}_n(t) - t|}{\tilde{q}(1-t)} \geqslant \frac{1}{2\varepsilon^{1-\alpha}} \quad \text{a.s.}$$

Letting $\varepsilon \downarrow 0$ completes the proof of Theorem 1. □

Proof of Corollary 1. The first part of the corollary follows immediately from Theorem 1, with $\alpha = 0$ and $\tilde{q} \equiv 1$, and an application of the triangle inequality. The second part can be proved along similar lines as the second part of Theorem 1. □

Proof of Theorem 2A. For similar reasons as in the proof of Theorem 1 we restrict ourselves to the proof for the right tail. First assume

$$\int_0^1 (\log(1/t))^{r/(r+1)}(\tilde{q}(t))^{-r/(r+1)}\,\mathrm{d}t < \infty. \tag{2.15}$$

Write $Lx = 1 \vee \log x$, $x > 0$, and note that

$$\int_0^1 \left(\frac{|\hat{F}_n(t) - t|}{\tilde{q}(1-t)} \right)^r \mathrm{d}t \leqslant \left(\sup_{0<t<1} \frac{|\hat{F}_n(t) - t|}{(\tilde{q}(1-t))^{r/(r+1)} (L(1/(1-t)))^{1/(r+1)}} \right)^r \cdot \int_0^1 \left(L\left(\frac{1}{1-t} \right) \right)^{r/(r+1)} (\tilde{q}(1-t))^{-r/(r+1)} \,\mathrm{dt}. \tag{2.16}$$

It is now immediate from (2.15) and Theorem 1, with $\alpha = 0$, that the right side of (2.16) converges to zero almost surely. (Observe that, strictly speaking, $\tilde{q}^{r/(r+1)}(L(1/I))^{1/(r+1)}$ has to be bounded for the application of Theorem 1. An inspection of the proof of Theorem 1, however, shows that the boundedness of (the present) $\tilde{q}^{r/(r+1)}$, or equivalently of $\tilde{q}$, is in fact sufficient.) This finishes the proof of the first part of Theorem 2A.

Now take $\tilde{q} = I^{(r+1)/r}$. We have

$$\int_0^1 \left(\frac{|\hat{F}_n(t) - t|}{(1-t)^{(r+1)/r}} \right)^r \mathrm{d}t \geqslant \int_{D_{n:n}}^1 \left(\frac{|\hat{F}_n(t) - t|}{(1-t)^{(r+1)/r}} \right)^r \mathrm{d}t = \int_{D_{n:n}}^1 1/(1-t) \,\mathrm{d}t = \infty \quad \text{a.s.} \qquad \square \tag{2.17}$$

Proof of Theorem 2B. Again it suffices to consider both tails separately. Moreover, both proofs are similar again. Therefore we restrict ourselves to the proof for the left tail.

First we show that

$$\int_0^1 \left(\frac{t}{q(t)} \right)^r \mathrm{d}t < \infty \tag{2.18}$$

implies

$$\int_0^1 \left(\frac{|\hat{F}_n(t) - t|}{q(t)} \right)^r \xrightarrow{\mathrm{P}} 0. \tag{2.19}$$

Since $q \in Q$, it is easily seen that for arbitrary $\delta > 0$,

$$\int_\delta^1 \left(\frac{|\hat{F}_n(t) - t|}{q(t)} \right)^r \mathrm{d}t \xrightarrow{\mathrm{P}} 0. \tag{2.20}$$

Therefore it suffices to show that for every $\varepsilon > 0$ there exists a $\delta > 0$ and $N \in \mathbb{N}$ such that

$$P\left(\int_0^\delta \left(\frac{|\hat{F}_n(t) - t|}{q(t)} \right)^r \mathrm{d}t \geqslant \varepsilon \right) \leqslant \varepsilon \quad \text{for } n \geqslant N. \tag{2.21}$$

But note that

$$\int_0^\delta \left(\frac{|\hat{F}_n(t) - t|}{q(t)} \right)^r \mathrm{d}t \leqslant \left(\sup_{0<t<1} \frac{|\hat{F}_n(t) - t|}{t} \right)^r \cdot \int_0^\delta \left(\frac{t}{q(t)} \right)^r \mathrm{d}t. \tag{2.22}$$

It readily follows from Theorem 3 in Csörgő and Horváth (1986), with e.g. their $\nu = \frac{3}{4}$, that the first term on the right side of (2.22) is $\mathscr{O}_{\mathrm{p}}(1)$. This, in combination with (2.18) and (2.22), yields (2.21) and hence (2.19).

Finally we have to show that

$$\int_0^1 \left(\frac{t}{q(t)} \right)^r \mathrm{d}t = \infty \tag{2.23}$$

implies

$$\int_0^1 \left(\frac{|\hat{F}_n(t) - t|}{q(t)} \right)^r \mathrm{d}t = \infty.$$

Note that

$$\int_0^1 \left(\frac{|\hat{F}_n(t) - t|}{q(t)} \right)^r \mathrm{d}t \geqslant \int_0^{D_{1:n}} \left(\frac{|\hat{F}_n(t) - t|}{q(t)} \right)^r \mathrm{d}t = \int_0^{D_{1:n}} \left(\frac{t}{q(t)} \right)^r \mathrm{d}t. \tag{2.24}$$

The right side of (2.24) is equal to infinity because of $q \in Q$ and (2.23). This completes the proof of Theorem 2B. □

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