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Bounds for Weighted Multivariate Empirical Distribution Functions

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1. Introduction

Let $X_1, X_2, \ldots$ be a sequence of independent random vectors, each uniformly distributed over $(0,1)^d$, $d \in \mathbb{N}$. The first n random vectors determine the empirical d.f. $\hat{F}_n$ in the usual way:

$$\hat{F}_n(t) = n^{-1} \sum_{i=1}^{n} \prod_{j=1}^{d} 1_{(0,t_j]}(X_{ij}), \qquad t \in (0,1)^d, \tag{1.1}$$

where X_{ij} is the j-th component of X_i and t_j the j-th component of t. Writing $|t| = \prod_{j=1}^{d} t_j$ we define

$$\|V_{n,\nu}\| = \sup_{0<|t|<1} \frac{n^{\nu}|\hat{F}_n(t) - |t||}{(|t|(1-|t|))^{1-\nu}}, \qquad 0 \leqq \nu \leqq \tfrac{1}{2}. \tag{1.2}$$

In the one-dimensional case, much attention has been paid to criteria which determine the almost sure behaviour of $\limsup_{n\to\infty} a_n \|V_{n,\nu}\|$, where $(a_n)_{n\in\mathbb{N}}$ is a sequence of positive norming constants. Csáki (1974, 1975, 1982) investigated the important value $\nu = \frac{1}{2}$ (i.e. in each point the process is divided by its standard deviation), while Shorack and Wellner (1978) considered the other extreme value $\nu = 0$. Mason (1981) connected these two results and derived criteria for every $\nu \in [0, \frac{1}{2}]$. For arbitrary dimension d the case $\nu = 0$ has been also considered by Mason (1982). In this paper we generalize Csáki's theorem (1974, 1975, 1982) and even Mason's generalization (1981) to arbitrary dimension d. An interesting corollary of this result is a law of the iterated logarithm for $\log \|V_{n,\nu}\|$.

2. Main Results

In this section we present our theorem and its corollaries. The proofs of these are deferred to the next section. Observe that we use sequences of positive norming constants $(a_n)_{n\in\mathbb{N}}$ which differ from those in the introduction.

Theorem. *Let* $F(t)=|t|$, $t\in(0,1)^d$, $d\in\mathbb{N}$ *and* $0\leqq v\leqq\frac{1}{2}$.

(i) *If* $\sum_{n=1}^{\infty} a_n\left(\log\frac{1}{a_n}\right)^{d-1}=\infty$, *then*

$$\limsup_{n\to\infty}(na_n)^{1-v}\|V_{n,v}\|=\infty \quad \text{a.s.} \tag{2.1}$$

(ii) *If* $\sum_{n=1}^{\infty} a_n\left(\log\frac{1}{a_n}\right)^{d-1}<\infty$ *and* $na_n\downarrow$, *then*

$$\lim_{n\to\infty}(na_n)^{1-v}\|V_{n,v}\|=0 \quad \text{a.s.} \tag{2.2}$$

Corollary 1. *There exists no sequence of positive real numbers* $(a_n)_{n\in\mathbb{N}}$ *such that* $na_n\downarrow$ *and*

$$\limsup_{n\to\infty}(na_n)^{1-v}\|V_{n,v}\|=1 \quad \text{a.s.} \tag{2.3}$$

Corollary 2.

$$\limsup_{n\to\infty}\frac{\log\|V_{n,v}\|}{\log\log n}=(1-v)\,d \quad \text{a.s.} \tag{2.4}$$

Corollary 3. *For all* $\alpha>0$

$$\lim_{n\to\infty} n^{-\alpha}\|V_{n,v}\|=0 \quad \text{a.s.} \tag{2.5}$$

Corollary 3 is the multidimensional version of Corollary 1 of Mason (1981).

In order to formulate Corollary 4 we have to introduce some notation. The (open) rectangles $(s_1,t_1)\times\ldots\times(s_d,t_d)$ in $(0,1)^d$ will be written as $R=R(s,t)$. Given an arbitrary rectangle $R\subset(0,1)^d$ we define $\hat{F}_n\{R\}=n^{-1}\sum_{i=1}^{n}1_R(X_i)$ and we write $|R|$ for the Lebesgue measure of R.

Corollary 4. *Let* $F(t)=|t|$, $t\in(0,1)^d$, $d\in\mathbb{N}$ *and* $\mu\in(-\infty,1)$. *Then we have*

$$\lim_{n\to\infty}\sup_{|R|\geqq\frac{(\log n)^\mu}{n}}\frac{\log\log n}{(\log n)^{1-\frac{1}{2}\mu}}\frac{n^{\frac{1}{2}}|\hat{F}_n\{R\}-|R||}{(|R|(1-|R|))^{\frac{1}{2}}}=\frac{1}{1-\mu} \quad \text{a.s.} \tag{2.6}$$

however,

$$\limsup_{n\to\infty}\sup_{|R|\geqq\frac{1}{2}}\frac{1}{(\log n)^{\frac{1}{2}}}\frac{n^{\frac{1}{2}}|\hat{F}_n\{R\}-|R||}{(|R|(1-|R|))^{\frac{1}{2}}}=\infty \quad \text{a.s.} \tag{2.7}$$

For the proof of this corollary we will require a result of Alexander (1984).

It is easy to see that the distinction between open, half-open and closed rectangles is inessential for the type of results just stated. Also, all of our results remain true if we replace $F(t)=|t|$ by F, whenever F has a density with respect to Lebesgue measure that is bounded away from 0 and ∞.

3. Proofs

Before beginning the actual proofs we first present two inequalities. The first one can be found in Ruymgaart and Wellner (1982, Corollary 2.4); see also Ruymgaart and Wellner (1984) for related results.

Inequality 1. *There exists* $c_1, c_2, c_3 \in (0, \infty)$, *only depending on d, such that for any* $\theta \in (0, 1)$

$$P\left(\sup_{|t| \geqq \theta} \frac{n^{\frac{1}{2}} |\hat{F}_n(t) - |t||}{|t|^{\frac{1}{2}}} \geqq \lambda\right) \leqq c_1 \left(\log \frac{2}{\theta}\right)^{d} \exp(-c_2 \lambda^2 \psi(c_3 \lambda (n\theta)^{-\frac{1}{2}})), \quad \lambda \geqq 0, \tag{3.1}$$

where $\psi\colon [0, \infty) \to (0, \infty)$ *satisfies* $\psi(\sigma) \sim 2\sigma^{-1} \log \sigma$ *as* $\sigma \to \infty$.

Inequality 2. *For each* $d \in \mathbb{N}$, $0 \leqq \nu \leqq \frac{1}{2}$, $a \in [1, \infty)$ *and* $n \geqq 3$ *we have*

$$P\left(\sup_{0 < |t| \leqq (n a^{\frac{1}{1-\nu}})^{-1}} \frac{n^{\nu} \hat{F}_n(t)}{|t|^{1-\nu}} > 0\right) \leqq d(\log(n a^{\frac{1}{1-\nu}}))^{d-1} a^{\frac{-1}{1-\nu}} \tag{3.2}$$

Proof of Inequality 2. Let $|X|_{1:n} = \min_{1 \leqq i \leqq n} \{|X_i|\}$. Notice that the probability on the left side of (3.2) is equal to

$$\begin{aligned} P(|X|_{1:n} \leqq (n a^{\frac{1}{1-\nu}})^{-1}) &= P(\max_{1 \leqq i \leqq n} (-\log|X_i|) \geqq \log(n a^{\frac{1}{1-\nu}})) \\ &\leqq n P(-\log|X_1| \geqq \log(n a^{\frac{1}{1-\nu}})). \end{aligned}$$

Observe that $-\log|X_1|$ is a gamma random variable with density $f_d(x) = ((d-1)!)^{-1} x^{d-1} e^{-x} 1_{(0,\infty)}(x)$. Thus

$$\begin{aligned} P(-\log|X_1| \geqq \log(n a^{\frac{1}{1-\nu}})) &= \int_{\log(n a^{\frac{1}{1-\nu}})}^{\infty} \frac{x^{d-1} e^{-x}}{(d-1)!} dx \\ &= \sum_{k=0}^{d-1} \{(\log(n a^{\frac{1}{1-\nu}}))^k \cdot (n a^{\frac{1}{1-\nu}} k!)^{-1}\} \\ &\leqq d(\log(n a^{\frac{1}{1-\nu}}))^{d-1} (n a^{\frac{1}{1-\nu}})^{-1}. \end{aligned} \tag{3.3}$$

This completes the proof of Inequality 2.

Proof of the Theorem. (i) It is a consequence of a result on the almost sure behaviour of the first order statistic in Geffroy (1958/1959) or Kiefer (1972) that $\Sigma a_n \left(\log \frac{1}{a_n}\right)^{d-1} = \infty$ implies $P(|X|_{1:n} < \varepsilon a_n \text{ i.o.}) = 1$ for any $\varepsilon > 0$. It can be easily seen that

$$(n a_n)^{1-\nu} \|V_{n,\nu}\| \geqq \frac{(n a_n)^{1-\nu} n^{\nu} n^{-1}}{2(|X|_{1:n}(1 - |X|_{1:n}))^{1-\nu}}. \tag{3.4}$$

Hence we have

$$\limsup_{n \to \infty} (n a_n)^{1-\nu} \|V_{n,\nu}\| \geqq \frac{1}{2\varepsilon^{1-\nu}} \quad \text{a.s.} \tag{3.5}$$

Letting $\varepsilon \downarrow 0$ proves the first part of our theorem.

(ii) It is easy to see that we may restrict ourselves without loss of generality to sequences $(a_n)_{n \in \mathbb{N}}$ with $\frac{1}{n^2} \leqq a_n \leqq \frac{1}{n}$. We first consider the case $\nu = \frac{1}{2}$. Using

$a_n \leqq \frac{1}{n}$ we see that $\sum_{n=1}^{\infty} a_n(\log n)^{d-1} < \infty$. We now define

$$U_n = \sup_{0<|t|\leqq b_n} \frac{\hat{F}_n(t) - |t|}{|t|^{\frac{1}{2}}}, \tag{3.6}$$

with $b_n = (n(\log n)^{d-2})^{-1}$, and prove that

$$\lim_{n\to\infty} n a_n^{\frac{1}{2}} U_n = 0 \quad \text{a.s.} \tag{3.7}$$

It suffices to prove that $\limsup_{n\to\infty} n a_n^{\frac{1}{2}} U_n \leqq 1$ a.s.

Define the following events:

$$A_n = \left\{ U_n \geqq \frac{1}{n a_n^{\frac{1}{2}}} \right\}; \qquad C_n = A_n A_{n-1}^c. \tag{3.8}$$

According to the Borel-Cantelli lemma we need to prove that $\Sigma P C_n < \infty$ and $P A_n \to 0$ as $n \to \infty$. Define

$$B_{n,k} = \{\forall_{t: x_{k-1} < |t| \leqq x_k \wedge b_n} (n-1) \hat{F}_{n-1}(t) \leqq k-1;\ \exists_{t: x_{k-1} < |t| \leqq x_k \wedge b_n} n \hat{F}_n(t) = k\},$$

where $x_k (= x_{n,k})$ is the solution of the equation

$$n x + a_n^{-\frac{1}{2}} x^{\frac{1}{2}} = k, \tag{3.9}$$

i.e.

$$x_k = ((1 + 4nka_n)^{\frac{1}{2}} - 1)^2 / (4n^2 a_n) = \frac{k}{n}\{1 - 2(1 + (1 + 4nka_n)^{\frac{1}{2}})^{-1}\}, \tag{3.10}$$

and

$$B'_{n,k} = \{ \sup_{|t| \leqq x_k} (n-1) \hat{F}_{n-1}(t) \geqq k-1;\ |X_n| \leqq x_k \}.$$

We see that $B_{n,k} \subset B'_{n,k}$.

For any $\tau > 0$ we have the following inclusions (for large n):

$$\begin{aligned} C_n \subset & \left\{ \forall_{t: 0<|t|\leqq b_n} \hat{F}_{n-1}(t) < |t| + \frac{1}{n-1} \left(\frac{|t|}{a_{n-1}} \right)^{\frac{1}{2}}; \right. \\ & \left. \exists_{t: 0<|t|\leqq b_n} \hat{F}_n(t) \geqq |t| + \frac{1}{n} \left(\frac{|t|}{a_n} \right)^{\frac{1}{2}} \right\} \\ & \subset \bigcup_{k=1}^{k_0} B_{n,k} \subset \bigcup_{k=1}^{k_0} B'_{n,k}, \end{aligned} \tag{3.11}$$

where $k_0 = \left[\frac{\tau}{n a_n (\log n)^{d-1}} \right]$. (We will choose τ later on.) For the verification of the second inclusion we have to show that $x_{k_0} \geqq b_n$ for large n, which follows from an elementary computation using the fact that $\sum_{n=1}^{\infty} a_n (\log n)^{d-1} < \infty$ and

$na_n\downarrow$ imply

$$\lim_{n\to\infty} na_n(\log n)^d = 0. \tag{3.12}$$

We are now going to derive an upper bound for PC_n. The inclusions in (3.11) yield

$$PC_n \leqq \sum_{k=1}^{k_0} PB'_{n,k}. \tag{3.13}$$

Using (3.3), it is easy to see that

$$P(\sup_{|t|\leqq x_k}(n-1)\hat{F}_{n-1}(t) \geqq k-1) \leqq \binom{n-1}{k-1} x_k^{k-1}\left(\sum_{i=0}^{d-1}\left(\log\frac{1}{x_k}\right)^i\right)^{k-1}. \tag{3.14}$$

Hence we have

$$PB'_{n,k} \leqq \binom{n-1}{k-1} x_k^k \left(\sum_{i=0}^{d-1}(\log\frac{1}{x_k})^i\right)^k \leqq \frac{k}{n}\frac{(nx_k)^k}{k!}\left(\sum_{i=0}^{d-1}\left(\log\frac{1}{x_k}\right)^i\right)^k. \tag{3.15}$$

It can be seen from (3.9) that $x_k \leqq k^2 a_n$. This yields for large n

$$\begin{aligned} PB'_{n,k} &\leqq \frac{k}{n}(cnk^2a_n)^k\frac{1}{k!}(\log n)^{k(d-1)} \\ &= ck^3a_n(cnk^2a_n)^{k-1}\frac{1}{k!}(\log n)^{k(d-1)}, \end{aligned} \tag{3.16}$$

where $c\in(0,\infty)$ is a constant depending on d. Using $k \leqq k_0$ for a *sufficiently small* τ we see that

$$PB'_{n,k} \leqq ck^3(ck\tau)^{k-1}\frac{1}{k!}a_n(\log n)^{d-1} \leqq c(\tfrac{1}{2})^{k-1}a_n(\log n)^{d-1}, \tag{3.17}$$

which entails $PC_n \leqq 2ca_n(\log n)^{d-1}$, hence $\Sigma\ PC_n < \infty$.

For the proof of $PA_n \to 0$ as $n\to\infty$ we need

$$A_{n,k} = \{\sup_{|t|\leqq x_k} n\hat{F}_n(t) \geqq k\}. \tag{3.18}$$

Using $x_{k_0} \geqq b_n$ we see that

$$A_n \subset \bigcup_{k=1}^{k_0} A_{n,k}. \tag{3.19}$$

We have

$$PA_{n,k} \leqq \binom{n}{k} x_k^k\left(\sum_{i=0}^{d-1}\left(\log\frac{1}{x_k}\right)^i\right)^k \leqq \frac{(nx_k)^k}{k!}\left(\sum_{i=0}^{d-1}\left(\log\frac{1}{x_k}\right)^i\right)^k. \tag{3.20}$$

Using (3.15)–(3.17) we see that

$$PA_{n,k} \leqq c\frac{n}{k}\left(\frac{1}{2}\right)^{k-1} a_n(\log n)^{d-1} \quad \text{for } k \leqq k_0; \tag{3.21}$$

hence

$$PA_n \leqq c \sum_{k=1}^{k_0} (\tfrac{1}{2})^{k-1} n a_n (\log n)^{d-1} \leqq 2 c n a_n (\log n)^{d-1} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.22}$$

Our next step is investigating the a.s. behaviour of

$$H_n = \sup_{0<|t|\leqq b_n} \frac{|t| - \hat{F}_n(t)}{|t|^{\frac{1}{2}}}. \tag{3.23}$$

We immediately see that

$$H_n \leqq \sup_{0<|t|\leqq b_n} |t|^{\frac{1}{2}} = \frac{1}{n^{\frac{1}{2}} (\log n)^{\frac{d-2}{2}}}, \tag{3.24}$$

yielding that

$$\lim_{n\to\infty} n a_n^{\frac{1}{2}} H_n = 0 \quad \text{a.s.} \tag{3.25}$$

Now consider $\sup_{b_n<|t|<1} \frac{|\hat{F}_n(t) - |t||}{|t|^{\frac{1}{2}}}$; using Inequality 1 we find for large n, with $p_n = \frac{1}{n a_n (\log n)^d}$,

$$\begin{aligned} &P\left(\sup_{b_n<|t|<1} \frac{|\hat{F}_n(t) - |t||}{|t|^{\frac{1}{2}}} \geqq \frac{1}{n a_n^{\frac{1}{2}}}\right) \\ &\leqq c_4 (\log n)^d \exp(-c_2 (\log n)^d p_n \psi(c_3 (\log n)^{d-1} p_n^{\frac{1}{2}})) \\ &\leqq c_4 (\log n)^d \exp(-c_5 p_n^{\frac{1}{2}} \log n) \\ &\leqq c_4 (\log n)^d n^{-c_5 p_n^{\frac{1}{2}}} \leqq \frac{1}{n^2}, \end{aligned} \tag{3.26}$$

where c_4, $c_5 \in (0, \infty)$ are constants depending on d. Applying the Borel-Cantelli lemma shows that

$$\lim_{n\to\infty} n a_n^{\frac{1}{2}} \sup_{b_n<|t|<1} \frac{|\hat{F}_n(t) - |t||}{|t|^{\frac{1}{2}}} = 0 \quad \text{a.s.} \tag{3.27}$$

Summarizing (3.8), (3.25) and (3.27) yields

$$\lim_{n\to\infty} n a_n^{\frac{1}{2}} \sup_{0<|t|<1} \frac{|\hat{F}_n(t) - |t||}{|t|^{\frac{1}{2}}} = 0 \quad \text{a.s.} \tag{3.28}$$

We are now going to consider arbitrary $\nu \in [0, \frac{1}{2})$ and we shall prove

$$\lim_{n\to\infty} (n a_n)^{1-\nu} \sup_{0<|t|<1} \frac{n^\nu |\hat{F}_n(t) - |t||}{|t|^{1-\nu}} = 0 \quad \text{a.s.} \tag{3.29}$$

It suffices to prove that

$$P\left(n a_n^{1-\nu} \sup_{0<|t|<1} \frac{|\hat{F}_n(t) - |t||}{|t|^{1-\nu}} \geqq 1 \quad \text{i.o.}\right) = 0. \tag{3.30}$$

Set $c_n = (n a_n)^{\nu-1}$ and define

$$D = \left\{ \sup_{0<|t|<1} \frac{n^\nu |\hat{F}_n(t) - |t||}{|t|^{1-\nu}} \geqq c_n \quad \text{i.o.} \right\},$$

$$E = \left\{ \sup_{0<|t| \leqq \frac{1}{n c_n^{\frac{1}{1-\nu}}}} \frac{n^\nu |\hat{F}_n(t) - |t||}{|t|^{1-\nu}} \geqq c_n \quad \text{i.o.} \right\},$$

$$F = \left\{ \sup_{\frac{1}{n c_n^{\frac{1}{1-\nu}}} \leqq |t| < 1} \frac{n^\nu |\hat{F}_n(t) - |t||}{|t|^{1-\nu}} \geqq c_n \quad \text{i.o.} \right\}.$$

We see that $P(D) \leqq P(E) + P(F)$.

We will first show that $P(E) = 0$. Note that for all $t \in (0,1)^d$ such that $|t| \leqq \left(n c_n^{\frac{1}{1-\nu}}\right)^{-1}$, we have $n^\nu |t|^\nu \leqq c_n^{\frac{-\nu}{1-\nu}}$. Since necessarily $c_n \to \infty$, $c_n^{\frac{-\nu}{1-\nu}} \leqq \frac{1}{2} c_n$ for large enough n. Hence $E \subset E'$, where

$$E' = \left\{ \sup_{0<|t| \leqq \frac{1}{n c_n^{\frac{1}{1-\nu}}}} \frac{n^\nu \hat{F}_n(t)}{|t|^{1-\nu}} \geqq \frac{1}{2} c_n \quad \text{i.o.} \right\}.$$

Let $n_r = 2^r$ for $r \in \mathbb{N}$. Obviously

$$P(E') \leqq P\left(\max_{n_r < n \leqq n_{r+1}} \sup_{0<|t| \leqq \frac{1}{n c_n^{\frac{1}{1-\nu}}}} \frac{n^\nu \hat{F}_n(t)}{|t|^{1-\nu}} \geqq \frac{1}{2} c_{n_r} \quad \text{i.o.} \right), \tag{3.31}$$

which, since $c_n \uparrow$ and $n\hat{F}_n(t) \uparrow$ as a function of n for fixed t, is less than or equal to

$$P\left(\sup_{0<|t| < \frac{1}{n_r (c_{n_r})^{\frac{1}{1-\nu}}}} \frac{2 n_{r+1}^\nu \hat{F}_{n_{r+1}}(t)}{|t|^{1-\nu}} \geqq \tfrac{1}{2} c_{n_r} \quad \text{i.o.} \right).$$

Now

$$\sum_{r=1}^{\infty} P\left(\sup_{0<|t| \leqq \frac{1}{n_r (c_{n_r})^{\frac{1}{1-\nu}}}} \frac{n_{r+1}^\nu \hat{F}_{n_{r+1}}(t)}{|t|^{1-\nu}} \geqq 4^{-1} c_{n_r} \right)$$

$$\leqq \sum_{r=1}^{\infty} P\left(\sup_{0<|t| \leqq \frac{1}{n_{r+1} (c'_{n_r})^{\frac{1}{1-\nu}}}} \frac{n_{r+1}^\nu F_{n_{r+1}}(t)}{|t|^{1-\nu}} > 0 \right),$$

where $c'_{n_r} = 2^{\nu-1} c_{n_r}$.

Application of Inequality 2 gives that this last series is less than or equal to

$$2d \sum_{r=1}^{\infty} \left(\log\left(n_r c_{n_r}^{\frac{1}{1-\nu}}\right)\right)^{d-1} / c_{n_r}^{\frac{1}{1-\nu}}. \tag{3.32}$$

Since for large enough x $(\log x)^{d-1}/x \downarrow$, we have for $r_0 \in \mathbb{N}$ large enough:

$$\sum_{r=r_0}^{\infty} \sum_{n_{r-1}<n\leq n_r} (\log(n c_n^{\frac{1}{1-\nu}}))^{d-1}/(n c_n^{\frac{1}{1-\nu}})$$

$$\geq \sum_{r=r_0}^{\infty} (n_r - n_{r-1})\{(\log(n_r c_{n_r}^{\frac{1}{1-\nu}}))^{d-1}/n_r c_{n_r}^{\frac{1}{1-\nu}}\}$$

$$=\tfrac{1}{2} \sum_{r=r_0}^{\infty} (\log(n_r c_{n_r}^{\frac{1}{1-\nu}}))^{d-1}/c_{n_r}^{\frac{1}{1-\nu}}. \tag{3.33}$$

We see immediately now that finiteness of the series in (ii) implies that the series in (3.32) is finite. Therefore by the Borel-Cantelli lemma $P(E')=0$, which in turn implies that $P(E)=0$.

We will now show that $P(F)=0$. Notice that

$$\sup_{\frac{1}{n c_n^{\frac{1}{1-\nu}}} \leq |t|<1} \frac{n^\nu |\hat{F}_n(t)-|t||}{|t|^{1-\nu}}$$

$$\leq \sup_{0<|t|<1} \frac{n^\nu |\hat{F}_n(t)-|t||}{|t|^{\frac{1}{2}} \left(\dfrac{1}{n c_n^{\frac{1}{1-\nu}}}\right)^{\frac{1}{2}-\nu}}$$

$$= c_n^{\frac{\frac{1}{2}-\nu}{1-\nu}} \sup_{0<|t|<1} \frac{n^{\frac{1}{2}} |\hat{F}_n(t)-|t||}{|t|^{\frac{1}{2}}}. \tag{3.34}$$

Hence $P(F) \leq P(F')$, where

$$F' = \left\{ \sup_{0<|t|<1} \frac{n^{\frac{1}{2}} |\hat{F}_n(t)-|t||}{|t|^{\frac{1}{2}}} \geq c_n^{\frac{1}{2(1-\nu)}} \quad \text{i.o.} \right\},$$

but we now can use (3.28), i.e. the case $\nu=\frac{1}{2}$, which gives that $P(F')=0$, which in turn implies $P(F)=0$. Thus we have shown that $P(D)=0$. This completes the proof of (3.29).

Noting that $0<y\leq\frac{1}{2}$ implies $1<(1-y)^{\nu-1}\leq 2$, we see that it remains to prove for $0\leq\nu\leq\frac{1}{2}$

$$\lim_{n\to\infty} (n a_n)^{1-\nu} \sup_{\frac{1}{2}<|t|<1} \frac{n^\nu |\hat{F}_n(t)-|t||}{(1-|t|)^{1-\nu}} = 0 \quad \text{a.s.} \tag{3.35}$$

With the same approach as in Einmahl et al. (1984, proof of Theorem 2.2) we can prove, using results of Ruymgaart and Wellner (1982, Corollary 2.3) or Alexander (1982, Corollary 6.2) that "large d-dimensional points" behave as "small (or large) 1-dimensional points", i.e.

$$\sum_{n=1}^{\infty} a'_n < \infty \text{ and } n a'_n \downarrow \text{ imply}$$

$$\lim_{n\to\infty} (n a'_n)^{1-\nu} \sup_{0<|t|<1} \frac{n^\nu |\hat{F}_n(t)-|t||}{(1-|t|)^{1-\nu}} = 0 \quad \text{a.s.} \tag{3.36}$$

We omit the proof of (3.36), because it is straightforward though tedious. Q.E.D.

Proof of Corollary 2. Applying the theorem for $a_n=(n(\log n)^d)^{-1}$ and for $a_n=(n(\log n)^{d+\varepsilon})^{-1}$, $\varepsilon>0$, gives the desired result. Q.E.D.

Proof of Corollary 4. "Large d-dimensional rectangles" also have the same behaviour as "small 1-dimensional points". That means that (3.36) holds true with t replaced by R (cf. Einmahl et al. (1984, proof of Theorem 3.2)). Taking $\nu=\frac{1}{2}$ and combining this with Corollary 3.9 of Alexander (1984) proves this corollary. Q.E.D.

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References

1. Alexander, K.S.: Some limit theorems for weighted and non-identically distributed empirical processes. Ph. D. dissertation, M.I.T. (1982)
2. Alexander, K.S.: Rates of growth and sample moduli for weighted empirical processes indexed by sets. Technical report, University of Washington, Seattle (1984)
3. Csáki, E.: Studies on the empirical d.f. MTA III. Oszt. Közl. **23**, 239–327 (in Hungarian) (1974)
4. Csáki, E.: Some notes on the law of the iterated logarithm for empirical distribution function. Coll. Math. Soc. János Bolyai: Limit Theorems of Probability Theory, 47–58 (1975)
5. Csáki, E.: On the standardized empirical distribution function. Coll. Math. Soc. János Bolyai: Nonparametric Statistical Inference, 123–138 (1982)
6. Einmahl, J.H.J., Ruymgaart, F.H., Wellner, J.A.: A characterization of weak convergence of weighted multivariate empirical processes. To appear in Acta Sci. Math. (Szeged) (1984)
7. Geffroy, J.: Contributions à la théorie des valeurs extrêmes. Publ. Inst. Statist. Univ. Paris 7/8, 37–185 (1958/1959)
8. Kiefer, J.: Iterated logarithm analogues for sample quantiles when $p_n \downarrow 0$. Proc. Sixth Berkeley Sympos. Math. Statist. Probab. **1**, 227–244. Univ. of California Press (1972)
9. Mason, D.M.: Bounds for weighted empirical distribution functions. Ann. Probab. **9**, 881–884 (1981)
10. Mason, D.M.: Some characterizations of almost sure bounds for weighted multidimensional empirical distributions and a Glivenko-Cantelli theorem for sample quantiles. Z. Wahrscheinlichkeitstheor. Verw. Geb. **59**, 505–513 (1982)
11. Ruymgaart, F.H., Wellner, J.A.: Growth properties of multivariate empirical processes. Report 8202, Math. Inst., Kath. Un., Nijmegen (1982)
12. Ruymgaart, F.H., Wellner, J.A.: Some properties of weighted multivariate empirical processes. Statist. Decisions **2**, 199–223 (1984)
13. Shorack, G.R., Wellner, J.A.: Linear bounds on the empirical distribution function. Ann. Probab. **6**, 349–353 (1978)

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