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Sobre um problema que não era interessante para Erdős

On a problem that was not interesting for Erdős

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RESUMO

Estudamos um problema na geometria euclidiana elementar utilizando múltiplas abordagens. Concluímos que o estudo assistido por computador de hoje pode introduzir novos métodos para compreender melhor as relações e conceitos. Nomeadamente, usando as Ferramentas de Raciocínio Automático do GeoGebra, vários detalhes do problema original podem ser colocados em um contexto algébrico e, portanto, sua investigação automatizada é possível, em algum sentido, mecanicamente. Ainda assim, o pensamento criativo e a reformulação do problema em um cenário diferente continua sendo útil.

Palavras-chave: *Ferramentas de Raciocínio Automático; Teorema do Ângulo Inscrito; teorema de Pitágoras.*

ABSTRACT

We study a problem in elementary Euclidean geometry by using multiple approaches. We conclude that today's computer assisted study can introduce new methods to understand relationships and concepts better. Namely, using GeoGebra's Automated Reasoning Tools, several details of the original problem can be put into an algebraic context, and therefore, its automated investigation is possible, in some sense, mechanically. Still, creative thinking and reformulation of the problem in a different setting remains useful.

Keywords: *Automated Reasoning Tools; inscribed angle theorem; Pythagorean theorem.*

Introduction

Paul Erdős is said to be one of the most prolific and influential mathematicians of the 20th century. He wrote about 1500 scientific papers with more than 500 co-authors, mostly in the field of number theory, but he influenced several other areas of mathematics. Despite born in a Hungarian family he worked in several countries in the world and motivated many generations to find new conjectures and work hard to prove them.

I met Erdős twice in person. First, when I was about 15 years old, I attended a mathematics summer camp in Budapest, led by *Lajos Pósa*, a student of Erdős's, his “favorite child”. As a special event, Erdős visited the camp, and the young people

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had the opportunity to ask him questions. At a certain point of the meeting, I could go to the blackboard and show the following geometric problem to Erdős:

Given the triangle *ABC*. Let us erect squares on its sides externally. Choose a point on an external side of one of the squares, say *J* (see Figure 1), join it with *C* and consider the intersection point *K* with the external side of another square. Now repeat this idea and create point *L* by finding the intersection of *KA* and *HI*. Finally, do this again by intersecting *LB* and *DE*, getting the final point *M*.

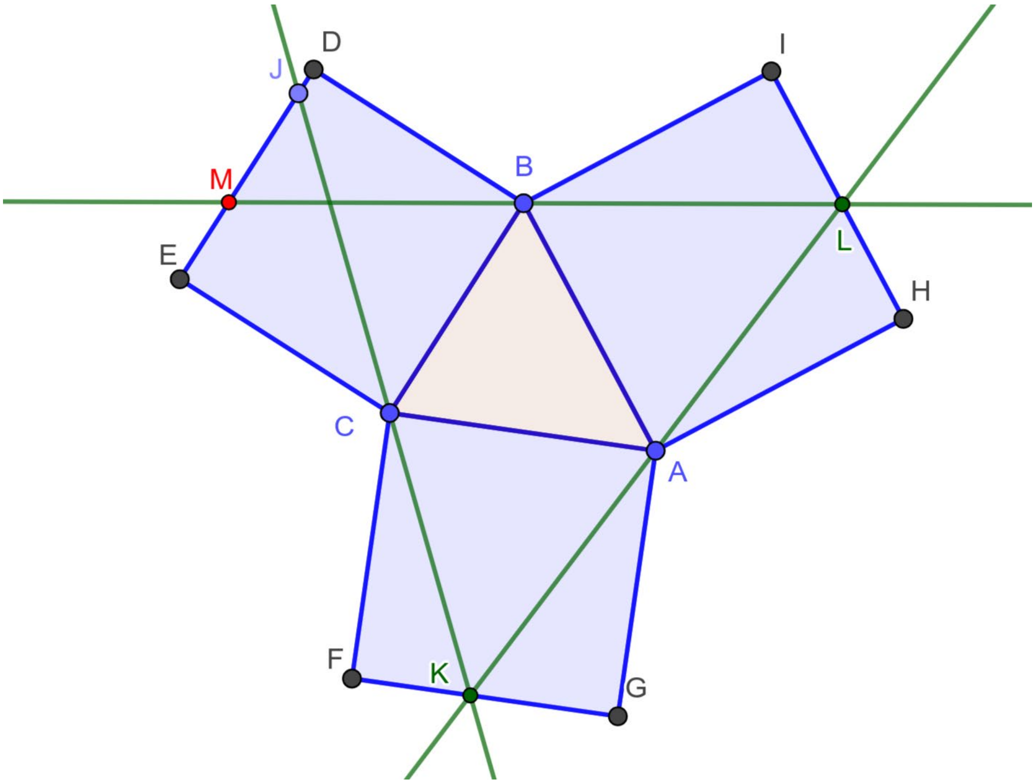

FIGURE 1: The original problem setting, redrawn as a GeoGebra construction
SOURCE: Author's own work

Question: How to choose *J* to get back to this initial point, namely, to have *M=J*?

After thinking a couple of seconds, Erdős wanted to hear the solution so I quickly disclosed a possible approach. Surprisingly, he was not excited, claiming that “I don't think this is very interesting.” At that time, I knew very little of Erdős's life, his attitude and contribution to mathematics, so I was not so disappointed as one may expect. But I still remember that day when I missed a possible connection to higher mathematics by remaining a young man who had a question that was “not selected” by Erdős.

In fact, this problem came from a very positive and accepting working group of young students, led by enthusiastic Hungarian teachers *István Vincze* and *József Kosztolányi*, both influenced by Pósa's experimenting approach to teach mathematics as a discovery adventure. The Hungarian education system, around 1990, was a very open scene for teaching experiments in mathematics, and several new books were published written by experts that were well-accepted and supported by the Hungarian government as well. (See, for example, Reiman (1987) for a book on elementary geometry written for gifted high school students.)

Despite that, this type of geometrical problem was difficult to study at that time. No one could solve it in my class, and in the school either. Besides being active in learning mathematics I was one of the editors of the local newspaper of the school, and I was responsible to create mathematical puzzles for the other students. We, the young editors, printed, copied and sold the newspaper on our own, with very little support from the school's teachers. In fact, the newspaper contained other types of articles like poems and funny stories, but there was a strong bias towards mathematics. We usually collected the solutions of the monthly puzzles in a mailbox settled in the school building and evaluated them right after the submission deadline. For this problem, however, there were not any solutions submitted. So, I assume that this problem was hard: the school's focus was on nature sciences, so it was indeed surprising that no one could solve this puzzle.

After several weeks, the only solution that was finally found was sent to us by the father of one of my classmates, *György Gyurkó*, having doctorate in mathematics. He found an easy and very elegant solution. We discuss his idea in this paper later, but first we focus on the immediate answer that can be obtained with GeoGebra nowadays.

In fact, such questions can be quickly solved by GeoGebra today, unless you are not familiar with some new features. To be honest, the simple way works just for some particular cases, but it is still much more than we had in the nineties! The applet at https://www.GeoGebra.org/m/FZM8J9T8 focuses on the GeoGebra command **LocusEquation(*M* $\stackrel{?}{=}$ *J*, *J*)** and the obtained equation 0=-1 shows that there is no solution, at least for the considered triangle. (If you cannot see the equality 0=-1 immediately when opening the web page, then just drag one of the vertices of the triangle to recompute the equation.) It is important that each step of the construction is drawn with one of the "allowed" tools or commands, namely, *to create a triangle*, *erect squares*, *put a point on a line*, and *intersect lines* or *segments*, in our example. The "allowed" tools have algorithmic representations in GeoGebra's internals that make possible to translate geometry into algebra. Therefore, instead of investigating a pure geometric drawing, the computer can set up a couple of equations in several variables and obtain the consequence purely in an algebraic way.

1. GeoGebra's Automated Reasoning Tools

GeoGebra 5.0 (published in September 2014) came with several new features, but its advanced computer algebra support was something new. The underlying computer algebra system was already changed to *Giac* in version 4.4. Giac is a minimalistic but very robust system that provides fast manipulation on algebraic equation systems. It is developed by the French mathematician *Bernard Parisse*. Since GeoGebra 4.4 fast solution of a system of equations in several variables is performed in a very efficient way because Giac uses an effective method to compute the *Gröbner basis* of a so-called *polynomial ideal*. Also, a related operation called *elimination* was done quickly enough to support real-time animation of symbolic locus visualizations (see KOVÁCS and PARISSE, 2015).

In fact, other systems were also considered to add symbolic computation support to GeoGebra, before Giac. Directly before version 4.2 *Mathpiper* was used to perform such heavy operations, and later *Reduce* in version 4.2. They provided, however, unsatisfactory support in a web browser. Giac was proven to be a very efficient computation engine during the years for a very wide set of applications, including automated reasoning.

At this point we need to give credit to a larger team that contributed in developing GeoGebra's Automated Reasoning Tools, first to the initiator *Tomás Recio* and his research colleagues *Francisco Botana*, *M. Pilar Vélez*, *Miguel Abánades* and *Sergio Arbeo*. This work was started in GeoGebra in the early 2010s, but the theoretical foundations go back to the 1990s or even earlier. The first attempts to translate geometry into algebra and investigate the algebraic counterpart was probably started by *Wen-Tsün Wu* and his student *Shang-Ching Chou*: the latter published a book on proving 512 geometrical statements, completely automatically by using this concept, in 1987 (CHOU, 1987). Several papers (and even several computer software) were published based on the work of these Chinese experts, however, to allow exploiting the benefits of algebra-geometric proofs for a larger audience, an open-sourced dynamic geometry system was required: GeoGebra.

At the time of writing of this paper GeoGebra is at version 5.0.638.0: the 5.0 series is a rolling release, providing the classic Java interface. Similarly, version 6.0.638.0 was published at the same time (on 20 April 2021): the 6.0 series is another rolling release, targeting a modern-looking application for desktop users. In the last years GeoGebra has reached a critical number of users (about 100 million worldwide to date) that made it necessary to keep the development under the control of a professional programmer group. This prevents our research team, led by Recio, to quickly add new features to GeoGebra. Instead, we made a so-called *fork* of

GeoGebra 5.0.591.0 and add our own improvements to this version by publishing the extended program under the name “GeoGebra Discovery”. GeoGebra Discovery contains all important features of GeoGebra, with the following extra features:

- The **Discover** command allows the user to find interesting features of a geometric figure, by selecting a point. The found features will then be proven automatically and reported to the user visually.
- The **Compare** command enables comparing two geometric quantities, by searching for a relationship between them.

There are some additional features that are not covered in this paper. We plan that our improvements will be finally incorporated in the rolling releases of GeoGebra, by the professional help of the programmers of the GeoGebra Team. Here we emphasize the importance of the collaboration between the GeoGebra developers and our research team.

Finally, we point the reader to GeoGebra Discovery's website https://github.com/kovzol/GeoGebra-discovery. Here additional references can be found, and the program can be downloaded for various platforms, including Windows, Mac and Linux. Further information can be obtained in Hohenwarter *et al* (2019) and Kovács *et al* (2018).

2. A proof via the Inscribed Angle Theorem

One drawback of our first approach by getting the solution via the LocusEquation command is that *we know nothing* about the reason why the points J and M never coincide. Even if we know that something is *true*, another challenging question should be raised: why? This should be an essential part in mathematical thinking.

This year I announced this problem for my prospective mathematics teacher students in the frame of a course “How to solve contest problems?” At the University of Linz, Austria, I tried to keep this course as old-fashioned as possible, by pointing on the essence of mathematics many times. Unfortunately, many of my students preferred to use a kind of modern method to solve open questions, namely, to search for an answer with an *Internet search engine*! On one hand, this is usually successful and therefore something positive, but it has nothing to do with mathematical thinking and *mathematical* problem solving. To defend myself I must admit that nowadays it is difficult to give a question that has no answer anywhere on the Internet in some form already. I hoped that my 1990 problem was still unpublished, so I decided to try by showing the Figure 2 as problem setting.

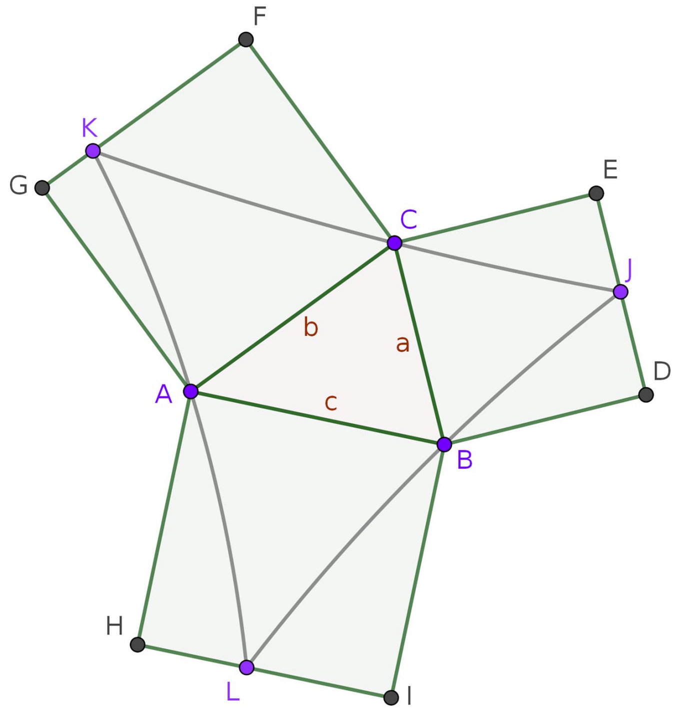

FIGURE 2: The problem setting in 2021
SOURCE: Author's own work

Of course, this figure is a kind of cheating. By using the technical trick that the "lines" *JCK*, *KAL* and *LBJ* can be drawn as circle arcs with a very large radius, we can give the optical illusion that the construction is achievable for some well-chosen inputs. Therefore, the problem setting was to find suitable positions for *J*, *K* and *L*, instead of asking for a proof of impossibility.

This assignment was not easy for the students. Among 25 participants there were only three who submitted a solution, and only two were correct. Both used Gyurkó's concept: If one assumes that a suitable triangle *JKL* can be constructed, then the sum of its interior angles must be less than 180 degrees, and this is a contradiction. If someone is a bit familiar with hyperbolic geometry, Figure 2 can point towards this concept, because "triangle" *JKL* looks like a triangle in the Poincaré model of the hyperbolic geometry (see Figure 3).

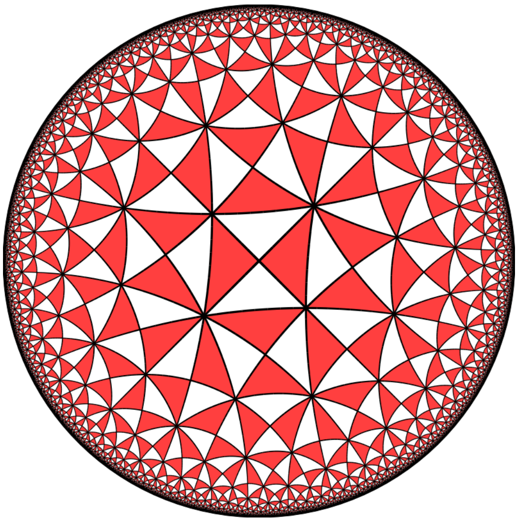

FIGURE 3: A tessellation of the hyperbolic plane with triangles
SOURCE: Claudio Rocchini: Hyperbolic Order-4 bisected pentagonal tiling, Wikipedia, https://it.wikipedia.org/wiki/Geometria_iperbolica, 15 February 2007

We only must show that *all* angles ALB, BJC and CKA are less than 60 degrees. This is, again, not completely trivial. First, the same statement is true for 55 degrees, but not for 50. To learn the reason behind this, one requires some more thinking and therefore more time. One possible way to show these properties is to use trigonometry, but I rather prefer pure Euclidean methods that do not require a calculator and thus numerical approximations. Why am I against calculators when a geometric problem is discussed? Well, using a calculator for solving mathematical problems is a kind of bad habit, similarly to the concept that doing mathematics is nothing more than a clever combination of several formulas. Geometry teaches us to think differently about mathematics.

So, when I disclosed the solution during my lecture, I started to draw a couple of different triangles ACK' inside the square $ACFG$. Sometimes K' was lying on side FG, but sometimes inside the square. This was a kind of introduction to explain the concept of the *inscribed angle theorem*, a generalization of Thales' circle theorem. The question was: How to draw a large set of triangles ACK' with fixed side AC where the angle at vertex K' is fixed. Usually, I do not assume that students have a deeper knowledge on facts in Euclidean geometry, but Thales' circle theorem is mostly well-known and interiorized enough to be a basis for a generalization. In addition, most students usually have no experience in using the LocusEquation command, so an input like **LocusEquation($\boldsymbol{\alpha \stackrel{?}{=} \beta}$, $\boldsymbol{K'}$)** has little to do with their former knowledge.

Figure 4 shows how this concept can be sketched up in seconds. First the square $ACFG$ is created. Then the triangle ACK' is created where K' is arbitrary.

Next, an arbitrary but fixed angle *PQR* is constructed. We denote this angle by α, and the angle *AK'C* will be denoted by β. The above mentioned GeoGebra command delivers the equation *eq1* which is drawn as a magenta set. Here we need to explain that the obtained set contains all points *K''* such that the angle *AK''C* equals to the angle α. By dragging the points *P*, *Q* and *R* the user can learn that the set *eq1* is changing dynamically, but it is mostly a set of two circles.

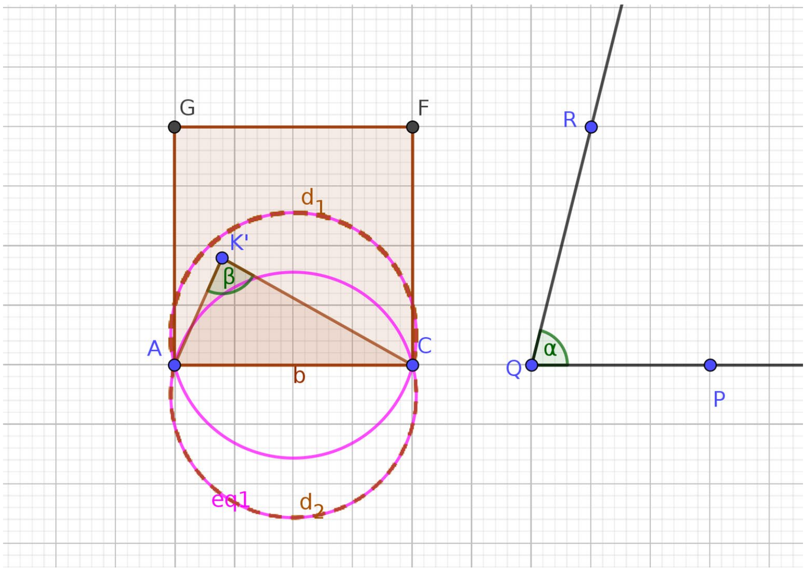

FIGURE 4: An algebro-geometric introduction to the inscribed angle theorem
SOURCE: Author's own work

In fact, GeoGebra here shows a larger set than expected. In a strict geometrical meaning the sought set should not contain the inner parts, only the arcs d_1 and d_2 (shown as dashed lines in Figure 4). This is more than confusing, especially for those users who do not have any deeper knowledge in *algebraic geometry*. Algebraic geometry explains this issue to us, namely, that there is no way to use only equation systems and no other techniques to exclude the inner parts. GeoGebra internally uses Gröbner bases to compute the locus equation, and they are unable to handle inequalities that may be required here to restrict the output to the outer arcs. (For a very detailed explanation see COX *et al*, 2007, Chapter 4 on algebraic *closures*. In a nutshell, the set of two circles is the algebraic closure of the two circular arcs.)

It is surprisingly easy to create an imperfect output, based on the LocusEquation command. Some more work is required if we want to get an accurate picture, by explicitly constructing the two arcs. We note that the locus equation still helps us a lot in the imagination of a first concept of the set of locus curves. Figure 5 shows a possible way by coloring the various double arcs with different colors, based on the angle they belong to. (See LOSADA, 2014, for more on dynamic coloring.) We learn by experimenting that a continuous change of the angle α results in a continuous change of the locus. This implies that the various loci, corresponding to different α values, are non-intersecting. Finally, we obtain a kind monotonicity of the locus curves: The greater radius a pair of arcs has, the greater value of α corresponds to it. Of course, we could find a numerical formula to express this connection, but, for the solution at least, we do *not need* an exact expression, only the property of monotonicity. This enables finding a *purely geometric solution* without involving unnecessary algebraic magick.

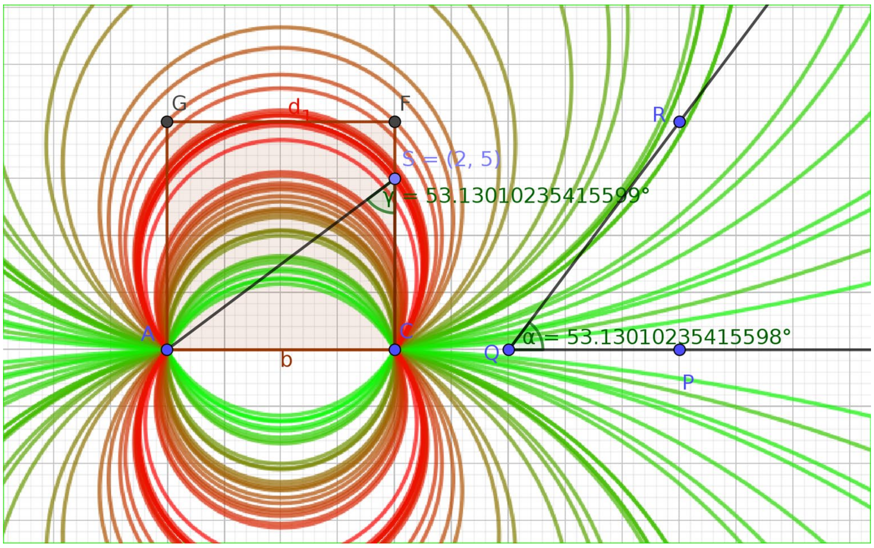

FIGURE 5: Dynamic coloring helps understanding the inscribed angle theorem
SOURCE: Author's own work

Now we can go back to the original problem setting, and to the subproblem which asks about the angles, namely, when K is on segment FG. Both successful students found the magical angle 53.130102… degrees, and it turned out, that this border situation occurs if K is the midpoint of FG. On the other hand, after some experimenting it seems plausible that this angle can also be constructed by dragging points P, Q and R to a simple situation, namely, to make a Pythagorean triangle with sides 3, 4 and 5. For me, this experiment resulted in a conjecture, and I was very interested in giving a simple proof, of course, without any help of trigonometry. Before having a deeper look on this, we could summarize how the inscribed angle theorem helps us showing the impossibility to choose a suitable point K on FG:

- If K is the midpoint of FG, we have an angle less than 54 degrees at K for the target triangle JKL,
- if K is different from the midpoint of FG, then the angle is even less, for example, by choosing F or G, the angle will be just 45 degrees.

So, at the end of the day, the sum of the interior angles of "triangle" *JKL* cannot be more than 3 times 54 degrees, so *JKL* is *not* a triangle. This finishes the proof.

3. Continue with further challenges

The "magical angle" 53.130102… can inspire us towards another challenge. A simple reformulation of the challenge can be seen in Figure 6. We consider an arbitrary square *ABCD*, the midpoints *E* and *F* on two adjacent sides, and another midpoint *G*. In fact, *G* is a quartering point of side *BC*.

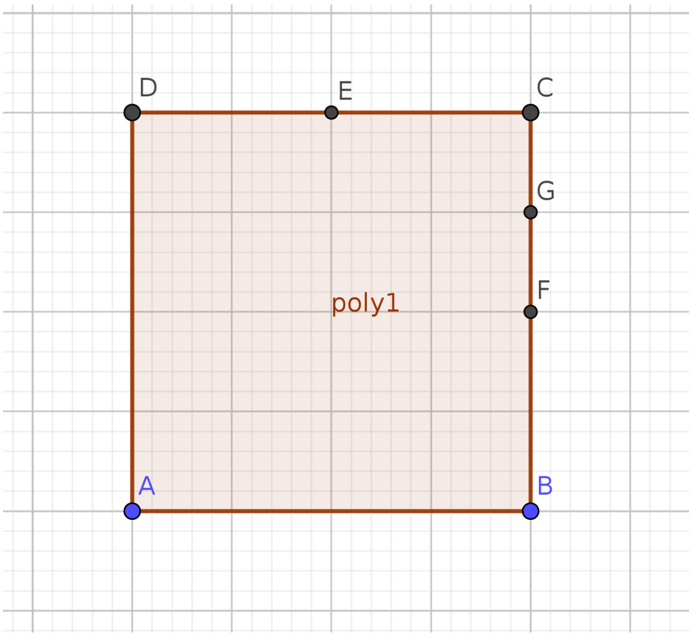

FIGURE 6: Reformulation of the angle equality to a concyclicity problem
SOURCE: Author's own work

It is obvious that triangle *BGA* in Figure 6 plays the same role as triangle *PQR* in Figure 5. They are similar, since both have the same Pythagorean ratios, 3:4:5. Now we are about to show that angles *BGA* and *BEA* are equal. Instead of operating with angles we use the converse of the inscribed angle theorem: If points *A*, *B*, *G* and *E* are concyclic, then the angles *BGA* and *BEA* are equal.

GeoGebra's Automated Reasoning Tools provide multiple ways to check concyclicity in a symbolic way. We can use the Input Bar by entering **Relation({*A*, *B*, *G*, *E*})** to get a numerical check first, and then, by clicking "More…" we can obtain a complete statement. Mathematically the same result can be performed when typing **Prove(AreConcyclic(*A*, *B*, *G*, *E*))** or **ProveDetails(AreConcyclic(*A*, *B*, *G*,

***E*))**, but these commands are designed for advanced users or programmers. Here we consider yet another method, namely, the Discover tool. It does not require any keyboard interaction. The user only needs to click on the Discover tool (being available only in GeoGebra Discovery at the time of writing this paper) and selecting one point in the figure as the target of interest. Any of our four points will succeed, so let us simply choose point *A*.

Figure 7, left, shows a communication window in GeoGebra: it contains some remarkable facts that are obtained in the figure, in connection to point *A*. All these facts are found in a mechanical way, using the algebro-geometric framework in GeoGebra's internals. In fact, some of the discovered properties are not surprising. For example, concyclicity of “*ABCD*” is somewhat trivial, but concyclicity of “*ABEG*” is exactly what we expect! Other relationships like parallelism of lines *AB* and *CDE*, furthermore lines *AD* and *BCFG*, and their perpendicularity, can also be considered trivial. On the other hand, showing, say, the perpendicularity of segments *AE* and *DF* may be some non-trivial assignment at introductory level for learners of geometry. Figure 7, right, shows the found properties in a visualized form: equal long segments are colored uniquely, and parallel/perpendicular line sets have the same color.

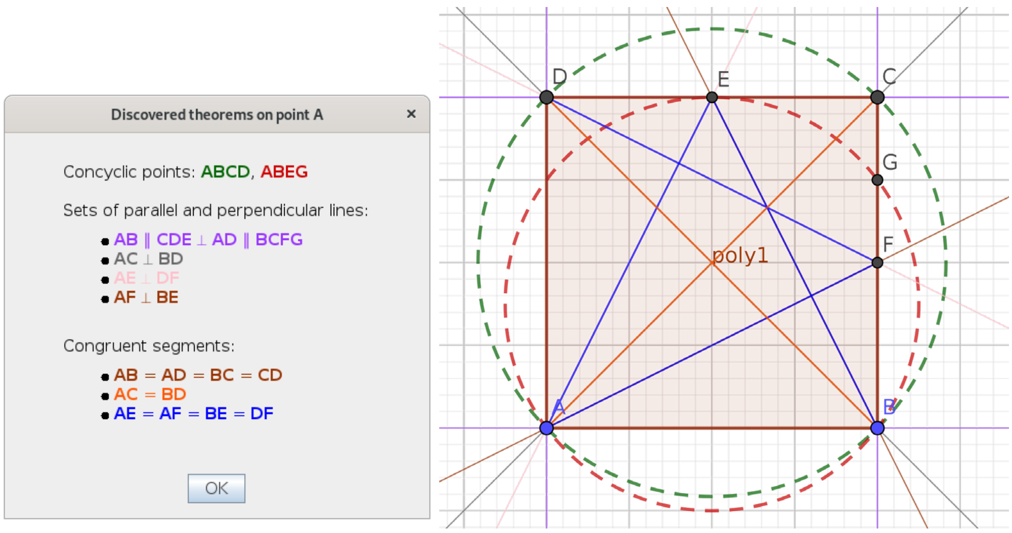

FIGURE 7: Finding concyclicity by using the Discover tool in GeoGebra Discovery
SOURCE: Author's own work

I leave the reader to play with this simple construction a bit more. (One can, for example, discover parallelism of *DF* and *EG*, which is a special case of the *midline theorem*.) At last, Figure 8 is shown to identify additional grid points on circle *ABEG*, namely *H*, *I*, *J*, *K*, *L*, *M*, *N* and *O*, a total of 12 points! But after having a deeper look on the figure, and creating point *P* as center of the circle, it is quite clear that 7 additional copies of the Pythagorean triplet appear, and this explains the reason quickly.

One may investigate the question further how usual it is to have so many grid points on a circle. The reader can do further experiments in this direction, playing more with number theory than geometry, by studying the topic of the sum of two squares theorem (UNDERWOOD, 1978, Section 18).

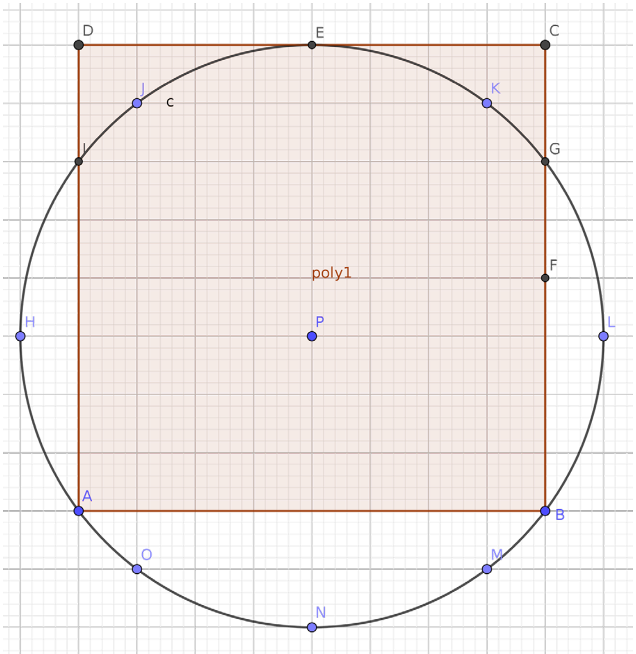

FIGURE 8: A circle with 12 grid points
SOURCE: Author's own work

As a final comment it can be highlighted that the equality of two angles was shown without using any numerical values like the “magical angle” 53.130102… To be precise, the two approximations of this angle do not match in Figure 5: the last digits differ! In the strict mathematical sense, a numerical computation that leads to an infinite process cannot be accepted if equality is to be shown. This means that the pure geometric reasoning, based on concyclicity, gives obvious evidence.

One may say that we do not need to explicitly compute the “magical angle” to solve the original problem. Indeed, it is enough to show that it is below 60 degrees, and this is clear if we erect a regular triangle on side AC inside the square $ACFG$. The same explanation was found in Gyurkó's solution, back in 1990.

4. Erdős's memory in the Euclidean geometry

Honestly, I needed several decades to have a deeper understanding on what a genius Erdős was. Maybe the first step on this journey was Reiman's introductory book for young mathematicians, mentioned above: it recalls Erdős's conjecture on a simple inequality that holds in triangles (ERDŐS, 1935). It was proven by L.J. Mordell and D.F. Barrow in 1937, and it is usually called the *Erdős-Mordell inequality*. It states that for any triangle *ABC* and point *P* inside *ABC*, the sum of distances from *P* to the sides is less than or equal to half of the sum of the distances from *P* to the vertices. Even if the statement sounds simple, its proof is far from trivial.

I can still recall Erdős's words on some simple conjectures that are easy to understand even by babies, "csecsemők" in Hungarian. The sound of this word depicts the shocking fact that a conjecture can be extremely simple, but its solution may exceed the competence of the best researchers. I met Erdős in 1995 for the second time. I was already a university student in Szeged, Hungary. A huge audience, consisting of many great local mathematicians and other interested people listened to him in a full lecture room. A couple of months later he died, only hours after he solved a *geometry problem* in a conference in Warsaw.

Can GeoGebra Discovery prove the Erdős-Mordell inequality by using these new technical means? To date, no. A brand new feature of GeoGebra Automated Reasoning Tools is the ability to prove certain inequalities. But the Erdős-Mordell inequality, as of today, cannot be proven with the most sophisticated new computer algebra methods, either, even if the problem is specified to an equilateral triangle, an isosceles triangle, or a right triangle. Paul Erdős remains with us, by challenging us with *his own* questions.

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