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ON GENERIC FLAG VARIETIES FOR ODD SPIN GROUPS

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Abstract: For the spin group $G = \mathrm{Spin}_{2n+1}$ with arbitrary n, a generic G-torsor E over a field, and a parabolic subgroup $P \subset G$, we consider the generic flag variety E/P and describe its Chow ring modulo torsion. This description determines the index of E/P, completing results of [**3**], where the index has been determined for most P.

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1. Introduction

We consider the split spin group $G = \mathrm{Spin}_{2n+1}$ with arbitrary $n \geq 1$ over an arbitrary field. However, the numerous definitions and statements below are valid for an arbitrary semisimple group G^*. We formulate them for G^* but we need them in the case $G^* = G$ only.

A *generic G^*-torsor E* can be defined as the generic fiber of the quotient map

$$\mathrm{GL}(N) \to \mathrm{GL}(N)/G^*$$

given by any embedding of G^* into a general linear group $\mathrm{GL}(N)$ with some N. Of course, different choices of the embedding produce different E. However, our object of interest – the Chow ring $\mathrm{CH}(E/P)$ for a fixed parabolic subgroup $P \subset G^*$ – is canonic ([**9**, Lemma 2.1]).

Understanding $\mathrm{CH}(E/P)$ allows one, in particular, to compute the index $\mathrm{ind}(E/P)$ – the greatest common divisor of degrees of closed points on the variety E/P. In fact, it is enough to know the quotient $\overline{\mathrm{CH}}(E/P)$ of the ring $\mathrm{CH}(E/P)$ by the ideal of the elements of finite order.

Let us fix an extension field $\bar{F}$ of the base field F of E trivializing E (e.g., an algebraic closure). Since the Chow ring of the cellular variety G^*/P is not affected by base field extensions, the change

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of field homomorphism $\mathrm{CH}(G^*/P) \to \mathrm{CH}(G^*/P)_{\bar F}$ is an isomorphism. Choosing a trivialization of the G^*-torsor $E_{\bar F}$, we identify $\mathrm{CH}(E/P)_{\bar F}$ with $\mathrm{CH}(G^*/P)_{\bar F}$. Since G^* acts trivially on $\mathrm{CH}(G^*/P)$ (see [**7**, Corollary 4.2]), the identification is canonical, i.e., does not depend on the choice of trivialization. Summarizing, we get a homomorphism

$$\mathrm{CH}(E/P) \to \mathrm{CH}(G^*/P).$$

Since its kernel is exactly the ideal of torsion elements, it identifies $\overline{\mathrm{CH}}(E/P)$ with a subring in $\mathrm{CH}(G^*/P)$.

For G^* any split spin group, the indexes $\mathrm{ind}(E/P)$ have been computed in [**3**] for many P. The starting point there was the upper bound on $\overline{\mathrm{CH}}(E/P)$ given by the image of the homomorphism

$$\mathsf{S}(\hat T)^W \to \mathrm{CH}(G/P),$$

defined in [**3**, Remark 2.3] for arbitrary G^*, where $T \subset P$ is a split maximal torus, $\hat T$ is the group of characters of T endowed with the action of the Weyl group W of P, $\mathsf{S}(\hat T)$ is the symmetric ring, and $\mathsf{S}(\hat T)^W$ is its subring of the W-invariant elements.

There is a natural ring homomorphism $\mathrm{CH}(\mathsf{B}P) \to \mathsf{S}(\hat T)^W$ and a natural surjective ring homomorphism $\mathrm{CH}(\mathsf{B}P) \twoheadrightarrow \mathrm{CH}(E/P)$ (see [**3**, Section 2]), both departing from the Chow ring $\mathrm{CH}(\mathsf{B}P)$ of the classifying space $\mathsf{B}P$ of P (see [**14**]). The precise value of $\overline{\mathrm{CH}}(E/P)$ is given by the image of the composition

$$\mathrm{CH}(\mathsf{B}P) \to \mathsf{S}(\hat T)^W \to \mathrm{CH}(G/P)$$

simply because it coincides with the composition

$$\mathrm{CH}(\mathsf{B}P) \twoheadrightarrow \mathrm{CH}(E/P) \to \mathrm{CH}(G/P).$$

Unfortunately, in most cases, we do not understand the Chow ring $\mathrm{CH}(\mathsf{B}P)$ well enough. Its description for G^* a split spin group involves the ring $\mathrm{CH}(\mathsf{B}\,\mathrm{Spin}_l)$ for certain l, which is mysterious and complicated if $l > 8$. (For $l < 7$, $\mathrm{CH}(\mathsf{B}\,\mathrm{Spin}_l)$ is well understood; descriptions for $l = 7$ and $l = 8$ are given in [**6**] and [**13**].) For this reason, a precise determination of $\overline{\mathrm{CH}}(E/P)$ for general P seemed to be out of reach.

Quite surprisingly, for $G^* = G$ (our odd split spin group), it turns out that the above upper bound coincides with $\overline{\mathrm{CH}}(E/P)$! We will prove it here by listing certain generators for the ring $\mathsf{S}(\hat T)^W$ and then showing that their images are in $\overline{\mathrm{CH}}(E/P)$ (for a reason unrelated to $\mathrm{CH}(\mathsf{B}P)$: they turn out to be Chern classes of certain elements in the Grothendieck group of E/P). This way we get a very handy system of generators for the ring $\overline{\mathrm{CH}}(E/P)$ and remove the hindrance to computation of $\mathrm{ind}(E/P)$ for arbitrary P.

Note that if G^* is an even spin group Spin_{2n}, the upper bound on $\overline{\mathrm{CH}}(E/P)$ given by $\mathsf{S}(\hat{T})^W$ differs from $\overline{\mathrm{CH}}(E/P)$ for most n and P. This makes the case of even spin groups more complicated and so far unsolved.

Our main result here is Theorem 3.6 describing $\overline{\mathrm{CH}}(E/P)$ in the case of $G^* = G$ and maximal P. (The study of $\mathrm{CH}(E/P)$ and determination of $\mathrm{ind}(E/P)$ for arbitrary P is easily reduced to the case of maximal P; see [**3**].) The description is particularly simple in the situation of Corollary 3.7, explaining and providing a more conceptual proof for [**3**, Theorem 4.2].

Theorem 4.1 is the second main result. It gives a formula and an algorithm for determination of the indexes: in every concrete case the concrete value can then be calculated by computer (having enough computer time and power).

As an example of the application of Theorem 4.1, we do the calculation in some cases. To formulate the answers, let us first recall that the conjugacy classes of maximal parabolic subgroups in G are indexed by the n vertices of the Dynkin diagram of G. Given $m \in \{1, \dots, n\}$, we write P_m for the mth standard maximal parabolic subgroup in the standard realization of $G = \mathrm{Spin}_{2n+1}$ as in [**3**, Section 4] and we write X_m for the variety E/P_m. The G-torsor E yields a non-degenerate $(2n+1)$-dimensional quadratic form q of trivial discriminant and Clifford invariant. The variety X_m is identified with the variety of m-dimensional totally isotropic subspaces of q. In particular, X_1 is the projective quadric.

Let us mention that the index of the highest orthogonal Grassmannian X_n is computed in [**15**]. For all m, the indexes $\mathrm{ind}(X_m)$ have been computed so far for $n \leq 7$ (i.e., $\dim q < 17$) only (see [**8**]). In Sections 5 and 7, this boundary is pushed further away. As a byproduct, we also get some new information on the even spin group Spin_{18} and Spin_{20} (see Sections 6 and 8).

2. Invariants

We continue to consider the odd split spin group $G = \mathrm{Spin}_{2n+1}$ with some $n \geq 1$. We fix some $m \in \{1, \dots, n\}$ and look at the mth standard maximal parabolic subgroup $P = P_m \subset G$. The standard split maximal torus T of G is contained in $P := P_m$. In order to determine $\mathsf{S}(\hat{T})^W$, where W is the Weyl group of P, we need a modification of [**3**, Proposition 3.3].

We consider the polynomial ring $R = \mathbb{Z}[x_1, \dots, x_m, y_1, \dots, y_l]$ over the integers $\mathbb{Z}$ in the variables $x_1, \dots, x_m$ and $y_1, \dots, y_l$, where $m + l = n$. Let $A := (\mathbb{Z}/2\mathbb{Z})^{\times l}$ be the direct product of l copies of the group $\mathbb{Z}/2\mathbb{Z}$ acting on R as follows: for any $i = 1, \dots, l$, the ith copy of $\mathbb{Z}/2\mathbb{Z}$ acts by changing the sign of y_i, and trivially on the remaining variables.

Instead of A, considered in [**3**, Proposition 3.3], we are going to work with larger groups. We start with the Weyl group W' of the spin group Spin_{2l+1}, which is a semidirect product of A and the symmetric group S_l. The action of W' on R we are interested in is the (unique) extension of the action of A, defined above, and the action of S_l by permutation of $y_1, \dots, y_l$. We will also consider the action of S_m by permutation of $x_1, \dots, x_m$ and the resulting action of $W = S_m \times W'$ on R. The latter action extends (uniquely) to an action of W on $R[z]$, where – as in [**3**, Section 3] – $R[z]$ is an R-algebra with a generator z subject to the relation

$$2z = x_1 + \dots + x_m + y_1 + \dots + y_l.$$

The ring $\mathsf{S}(\hat{T})$ is identified with $R[z]$ and the action of the Weyl group W of P on $\mathsf{S}(\hat{T})$ is the action of W on $R[z]$ just defined.

As in [**3**, Section 3], we define an element $\tilde{z} \in R[z]^A$ as the product of all elements in the A-orbit of z. Since the A-orbit of z coincides with its W-orbit, the element $\tilde{z}$ is actually W-invariant.

We borrow from [**3**, Section 3] the construction of A-invariant elements $f_k \in R[z]$, $k \geq 0$. We set

$$f_0 := 2z - y_1 - \dots - y_l = x_1 + \dots + x_m \in R^A.$$

Assume that for some $k > 0$ the element f_k is already constructed and has the shape

$$f_k = 2z \cdot g_k + a_1 + \dots + a_s, \tag{2.1}$$

where g_k is a polynomial with integer coefficients in $z, y_1, \dots, y_l$ and where $a_1, \dots, a_s$ for some $s \geq 0$ are monomials in $y_1, \dots, y_l$. Then we define f_{k+1} as one half of the difference

$$f_k^2 - (a_1^2 + \dots + a_s^2) = 2\left(2z(zg_k^2 + (a_1 + \dots + a_s)g_k) + \sum_{i<j} a_i a_j\right). \tag{2.2}$$

Note that the new element f_{k+1} has the shape (2.1), allowing us to continue the procedure.

Lemma 2.3. *For any $k \geq 0$, the element f_k is W-invariant.*

Proof: By construction, the element f_k is in the subring $\mathbb{Z}[z, y_1, \ldots, y_l] \subset R[z]^{S_m}$. Therefore, f_k is S_m-invariant. Since f_k is A-invariant as well, it remains to check that f_k is S_l-invariant.

The element $f_0 = x_1 + \cdots + x_m$ is S_l-invariant. So, let us assume f_k is S_l-invariant for some $k \geq 0$ and let us then check that f_{k+1} is also S_l-invariant. To do this, we view $\mathbb{Z}[z, y_1, \ldots, y_l]$ as a polynomial ring in z over $\mathbb{Z}[y_1, \ldots, y_l]$. Note that z is an independent generator and S_l acts trivially on z. So, a polynomial in $\mathbb{Z}[z, y_1, \ldots, y_l] = \mathbb{Z}[y_1, \ldots, y_l][z]$ is S_l-invariant if and only if all its coefficients are. From the formula (2.1) we see that the sum $a_1 + \cdots + a_s$ is the constant term of the polynomial f_k. Therefore this sum is S_l-invariant. Now it follows by formula (2.2) that f_{k+1} is also S_l-invariant. □

Proposition 2.4. *The R^W-algebra $R[z]^W$ is generated by the elements $f_1, \ldots, f_{l-1}, \tilde{z}$.*

Proof: As a first step, acting as in the proof of [**3**, Proposition 6.1], we prove that the

$$\mathbb{Z}[x_1 + \cdots + x_m, y_1, \ldots, y_l]^{W'}\text{-algebra} \quad \mathbb{Z}[y_1, \ldots, y_l][z]^{W'}$$

is generated by the indicated elements. As a second (and final) step we apply [**11**, Lemma 8.1]. □

3. Images of invariants

We continue using the settings of Section 2. We also let $B \subset G$ be the standard Borel subgroup; we have $T \subset B \subset P$.

We are going to prove that the image in $\mathrm{CH}(G/P)$ of $\mathsf{S}(\hat{T})^W$ lies in

$$\overline{\mathrm{CH}}(X_m) = \overline{\mathrm{CH}}(E/P) \subset \mathrm{CH}(G/P).$$

We start with the easiest part of $\mathsf{S}(\hat{T})^W$, whose image is in the subring $C \subset \overline{\mathrm{CH}}(E/P)$ generated by the Chern classes of the tautological (rank m) vector bundle T on X_m. Note that one can also view or define T as the tautological vector bundle on the split orthogonal Grassmannian G/P.

Proposition 3.1. *The image in $\mathrm{CH}(G/P)$ of $R^W \subset \mathsf{S}(\hat{T})^W$ lies in $C \subset \overline{\mathrm{CH}}(E/P)$.*

Proof: The images of $x_1, \ldots, x_m$ in $\mathrm{CH}(G/B)$ are the roots of the vector bundle T (pulled back to G/B along the projection $G/B \to G/P$). The roots of the vector bundle $\mathsf{T}^\perp$, given by the orthogonal complement, are the images of $x_1, \ldots, x_m$ along with the images of $\pm y_1, \ldots, \pm y_l$ and 0. Finally, the roots of the trivial vector bundle V given by the vector space of definition of q are the images of all $\pm x_1, \dots, \pm x_m$, $\pm y_1, \dots, \pm y_l$, and 0 all together. (Concerning the root 0, see [**4**, proof of Proposition 86.13].)

The ring R^W is easily seen to be generated by the elementary symmetric polynomials in $x_1, \dots, x_m$ together with the elementary symmetric polynomials in $y_1^2, \dots, y_l^2$. The images in $\mathrm{CH}(G/P)$ of the former are the Chern classes of T. The images of the latter are the Chern classes of the quotient $\mathsf{T}^\perp/\mathsf{T}$. The isomorphism $V/\mathsf{T}^\perp = \mathsf{T}^\vee$, where $\mathsf{T}^\vee$ is the dual vector bundle, shows that the Chern classes of $\mathsf{T}^\perp$ are polynomials in the Chern classes of T. □

Proposition 3.2. *For any $i \geq 0$, the images in $\mathrm{CH}(G/P)$ of $f_i \in \mathsf{S}(\hat{T})^W$ also lie in $C \subset \overline{\mathrm{CH}}(E/P)$.*

Proof: The pull-back ring homomorphism $\mathrm{CH}(G/P) \to \mathrm{CH}(G/B)$ is injective and the quotient

$$\mathrm{CH}(G/B)/\,\mathrm{CH}(G/P)$$

is a free abelian group (see [**3**, proof of Lemma 2.2]).

The variety G/B is the variety of complete flags of totally isotropic subspaces of q. Let $C_B \subset \mathrm{CH}(G/B)$ be the subring generated by the Chern classes of all (from rank 1 to rank n) tautological vector bundles on G/B. Then C is a subring of C_B and the quotient C_B/C is also a free abelian group. The claim on the quotient can be shown by identifying respectively C and C_B with the Chow rings of the two varieties: the variety Y_m of m-dimensional totally isotropic subspaces and the variety Y of complete flags of totally isotropic subspaces of a $(2n)$-dimensional non-degenerate alternating bilinear form (see [**10**, Remark 2.6] and Remark 3.3): there is such an identification for which the respective Chern classes of the respective tautological vector bundles correspond to each other. The quotient $\mathrm{CH}(Y)/\,\mathrm{CH}(Y_m)$ is free abelian by the argument of [**3**, proof of Lemma 2.2] once again.

It has been shown in [**3**, Lemma 3.5] that for every $i \geq 0$, the image in $\mathrm{CH}(G/B)$ of f_i is in C_B. Since $2^i f_i \in R$, the image of $2^i f_i$ is in C. It follows that the image of f_i is in C. □

Remark 3.3 (Geometric interpretation of the homomorphism from $\mathrm{CH}(Y_m)$ to C; cf. [**15**, Section 4]). The existence of the isomorphism $\mathrm{CH}(Y_m) = C$, used in the above proof, is justified in [**10**, Remark 2.6] by information about relations on the generators. So, its geometric construction, described below, is not actually needed (but still interesting to look at). Note that both rings are independent of the base field and, in particular, of its characteristic. In characteristic 2, defining Y_m by the associated (alternating) bilinear form b of q on the vector space V

modulo the (1-dimensional) radical $\mathrm{Rad}(b) \subset V$, we get a morphism of varieties $X_m \to Y_m$, mapping every m-dimensional totally isotropic subspace of V (viewed as a point of X_m) to its image in the quotient $V/\mathrm{Rad}(b)$ (which is an m-dimensional totally isotropic subspace giving a point of Y_m). Since T is the pull-back of the tautological vector bundle T' on Y_m and since the Chow ring of Y_m is generated by the Chern classes of T', the pull-back homomorphism $\mathrm{CH}(Y_m) \to \mathrm{CH}(X_m)$ lands in $C \subset \mathrm{CH}(X_m)$ and is the one we are looking for.

Proposition 3.4. *The image in* $\mathrm{CH}(G/P)$ *of the generator* $\tilde{z} \in \mathsf{S}(\hat{T})^W$ *lies in* $\overline{\mathrm{CH}}(E/P)$.

Proof: Since the group G is simply connected, the Grothendieck group $K(E/P)$ coincides with $K(G/P)$ ([**12**]).

The following considerations are valid for a split maximal torus T contained in a Borel subgroup B of any split semisimple group G^* in place of G. We will use them in our case with $G^* = G$.

Let us consider the group ring $\mathbb{Z}[\hat{T}]$. Since the addition in $\hat{T}$ becomes multiplication in $\mathbb{Z}[\hat{T}]$, we use the exponential notation $\chi \in \hat{T} \mapsto \exp(\chi) \in \mathbb{Z}[\hat{T}]$ for the embedding $\hat{T} \hookrightarrow \mathbb{Z}[\hat{T}]$. Any character $\chi \in \hat{T}$ extends uniquely to B and determines a line bundle on the variety G^*/B; see [**2**, Section 1.5]. There is a (surjective) ring homomorphism

$$\mathbb{Z}[\hat{T}] \to K(G^*/B),$$

mapping the exponent $\exp(\chi) \in \mathbb{Z}[\hat{T}]$ of any character $\chi \in \hat{T}$ to the class of the line bundle on G^*/B given by χ. Restricting to the W-invariants, where W is the Weyl group of a parabolic subgroup $P \supset B$, we get a ring homomorphism

$$\mathbb{Z}[\hat{T}]^W \to K(G^*/P) \subset K(G^*/B).$$

Now we return to $G^* = G$. The image in $\mathrm{CH}(G/P)$ of the generator $\tilde{z}$ is the 2^lth Chern class of the image in $K(E/P) = K(G/P)$ of the element

$$\sum_{I \subset \{1,\dots,l\}} \exp\Big(z - \sum_{i \in I} y_i\Big) \in \mathbb{Z}[\hat{T}]^W. \qquad \square$$

Remark 3.5. Propositions 3.1, 3.2, and 3.4 show that the ring $\overline{\mathrm{CH}}(E/P)$ is generated by Chern classes (of elements of $K(E/P)$). Actually, R^W and $\tilde{z}$ are already in the subring of $\mathsf{S}(\hat{T})^W$ generated by Chern classes (of elements of $\mathbb{Z}[\hat{T}]^W$). However, in the process of showing that the images of $f_1, \dots, f_{l-1}$ are in C, certain relations are used which occur only after $\mathsf{S}(\hat{T})^W$ is mapped to $\overline{\mathrm{CH}}(E/P)$.

The ring C (which depends only on n) is well understood. In particular, the relations on its generators – the Chern classes (or rather the Segre classes) of T – are well known (see, e.g., [**10**]). As we just proved,

Theorem 3.6. *The C-algebra $\overline{\mathrm{CH}}(E/P)$ is generated by the image of $\tilde{z}$ in $\overline{\mathrm{CH}}^{2^l}(E/P)$.*

The index $\mathrm{ind}(E/P)$ has been computed in [**3**] in the situation where $2^l > \dim(E/P)$. This situation is simpler for the following reason:

Corollary 3.7. *We have $\overline{\mathrm{CH}}(E/P) = C$ provided that $2^l > \dim(E/P)$.*

4. How to compute $\mathrm{ind}(X_m)$

We keep the notation of the previous section and provide an algorithm computing $\mathrm{ind}(X_m)$.

Since the element $2^{2^l}\tilde{z}$ is in R, it yields an element $\tilde{c} \in C$. The additive group of the ring C is free abelian of finite rank ([**10**, Theorem 2.1]). For every integer $j \geq 0$, let 2^{k_j} be the highest 2-power dividing $\tilde{c}^j$ in C. Here we define $\tilde{c}^0$ to be 1 and therefore $k_0 = 0$. Let k be the maximum of $j2^l - k_j$ over all $j \geq 0$ with $j2^l \leq \dim X_m$.

Theorem 4.1. $\mathrm{ind}(X_m) = 2^{m-k}$.

Example 4.2. If $2^l > \dim X_m$, then $k = 0$ and we recover [**3**, Theorem 4.2].

Example 4.3. In the case of $m = n$, Theorem 4.1 is [**15**, Lemma 4.1].

Proof of Theorem 4.1: Recall that C is the Chow ring of the cellular variety Y_m defined in the proof of Proposition 3.2. Let j be such that $k = j2^l - k_j$. Then $\tilde{c}^j = 2^{k_j}d$ for some $d \in C$ non-divisible by 2. Therefore, by Poincaré duality (see [**15**, Section 4] or [**11**, Remark 5.6]) there exists $d' \in C$ such that dd' has an odd degree e on Y_m. Since the class of a rational point in $\mathrm{CH}(Y_m)$ equals 2^m times the class of a rational point in $\mathrm{CH}(G/P) \supset \overline{\mathrm{CH}}(X_m)$, the product $dd' \in \overline{\mathrm{CH}}(X_m)$ has degree $2^m \cdot e$ on X_m and is divisible by 2^k in $\overline{\mathrm{CH}}(X_m)$. It follows that $\mathrm{ind}(X_m)$ divides 2^{m-k}.

For the opposite, applying Theorem 3.6, write the class in $\overline{\mathrm{CH}}(X_m)$ of a 0-cycle of degree $\mathrm{ind}(X_m)$ on X_m as a polynomial in $2^{-2^l}\tilde{c}$ over C. The polynomial contains a monomial $M = 2^{-j2^l}c\tilde{c}^j$ (with some $c \in C$ and some j) of degree an odd multiple of $\mathrm{ind}(X_m)$. Then $2^{j2^l-k_j}M$ is in C and has degree an odd multiple of $2^{j2^l-k_j-m+i}$ on Y_m, with i such that $2^i = \mathrm{ind}(X_m)$. It follows that $j2^l - k_j - m + i \geq 0$ so that $i \geq m - (j2^l - k_j) \geq m - k$. □

5. $\mathbf{Spin_{17}}$

Note that for any n the index $\operatorname{ind}(X_n)$ is known (due to [**15**]) and coincides with $\operatorname{ind}(X_{n-1})$ and $\operatorname{ind}(X_{n-2})$.

All indexes are known for q of dimension lower than 17 (see [**8**]). For q of dimension 17 we have $n = 8$. Let $n = 8$ and $m = 5$.

A computation (done in `Maple 2021`), using the Chow ring package (Version 4.0) by S. Nikolenko, V. Petrov, N. Semenov, and K. Zainoulline, shows that the image $\tilde{c}^3$ of $(2^{2^3}\tilde{z})^3 \in R$ in $C \subset \overline{\mathrm{CH}}(X_5)$ is not divisible by $2^{3\cdot 2^3-1}$. It follows by Theorem 4.1 that $\operatorname{ind}(X_5)$ divides 2^3. Since $\operatorname{ind}(X_6) = 2^4$, we conclude that $\operatorname{ind}(X_5) = 2^3$ (see Section 1).

If $\operatorname{ind}(X_3)$ were at most 2^2, we could find a finite extension field L of the base field of degree not divisible by 2^3 such that the anisotropic part of q_L would have dimension at most 11. Then q_L would split completely over a finite field extension of degree dividing 2, a contradiction to $\operatorname{ind}(X_8) = 2^4$. It follows that $\operatorname{ind}(X_3) = 2^3$, implying $\operatorname{ind}(X_m) = 2^m$ for $m < 3$ as well (the latter also being confirmed by [**3**, Theorem 4.2] as well as by [**1**, Theorem 4.2]).

Summarizing, we get the whole list of indexes of $\operatorname{ind}(X_m)$ for Spin_{17}:

$$\begin{aligned} \operatorname{ind}(X_m) &= 2^m && \text{for } m \le 3, \\ \boxed{\operatorname{ind}(X_m) = 2^3} & && \text{for } m \in \{4,5\}, \text{ and} \\ \operatorname{ind}(X_m) &= 2^4 && \text{for } m \ge 6, \end{aligned}$$

where the box marks the values which were not known before.

For more credibility, we provide further details on the computation with the Chow package in Appendix A.

6. $\mathbf{Spin_{18}}$

Let q be a generic quadratic form of dimension 18 of trivial discriminant and Clifford invariant (given by a generic Spin_{18}-torsor). The result of the previous section allows one to determine the index of mth orthogonal Grassmannian X_m (i.e., the variety of totally isotropic m-planes) of q for all m.

Let q' be a 1-codimensional subform of q and let X'_m be the mth orthogonal Grassmannian of q'. Then we have $\operatorname{ind}(X'_m) \ge \operatorname{ind}(X_m)$ for $m = 1, \dots, 8$ and $\operatorname{ind}(X'_m)$ has the upper bound given by the index of Section 5. We also have

$$\operatorname{ind}(X_9) = \operatorname{ind}(X_8) = \operatorname{ind}(X_7) = \operatorname{ind}(X_6) = 2^4.$$

Besides, by the same argument as in the previous section, we have $\operatorname{ind}(X_3) = 2^3$, implying $\operatorname{ind}(X_m) = 2^m$ for $m \le 3$.

Summarizing, we get the whole list of indexes of $\operatorname{ind}(X_m)$ for Spin_{18}:

$$\begin{aligned} \operatorname{ind}(X_m) &= 2^m && \text{for } m \le 3, \\ \boxed{\operatorname{ind}(X_m) = 2^3} & && \text{for } m \in \{4,5\}, \text{ and} \\ \operatorname{ind}(X_m) &= 2^4 && \text{for } m \ge 6. \end{aligned}$$

7. $\mathbf{Spin_{19}}$

Here we start to work out the case of $n = 9$. First of all, we have $\operatorname{ind}(X_m) = 2^m$ for $m = 1, 2, 3$ by [**3**, Theorem 4.2] because the condition $2^{n-m} > \dim X_m$ of [**3**, Theorem 4.2] is satisfied for $m = 3$:

$$2^{n-m} = 2^6 = 64 > \dim X_m = m(m-1)/2 + m(2n-2m+1) = 42.$$

For $m = 4$, [**3**, Theorem 4.2] does not work anymore because

$$2^{n-m} = 2^5 = 32 \le \dim X_4 = 50.$$

A computation with the Chow ring package (see Appendix B) shows that the image $\tilde{c}$ of $2^{2^5}\tilde{z} \in R$ in $C \subset \overline{\mathrm{CH}}(X_4)$ is not divisible by 2^{2^5} inside of C. (It is divisible by 2^{2^5-1} though.) It follows by Theorem 4.1 that $\boxed{\operatorname{ind}(X_4) = 2^3}$.

8. $\mathbf{Spin_{20}}$

We do not expect that knowledge of indexes for Spin_{2n-1} always allows one to determine the indexes for Spin_{2n}. This happens with the highest orthogonal Grassmannians for the very special reason that they are isomorphic to each other. It appears to be a coincidence that in Section 6 we were able to determine all indexes for Spin_{18} using the information on Spin_{17}.

For Spin_{20} and $\operatorname{ind}(X_4)$, the information on Spin_{19} helps again.

First of all, $\operatorname{ind}(X_m) = 2^m$ for Spin_{20} and $m = 1, 2, 3$ by [**3**, Theorem 7.2] because for $m = 3$ we have the inequality

$$2^{n-m-1} = 2^{10-3-1} = 64 > \dim X_m = m(m-1)/2 + 2m(n-m) = 45.$$

It follows that $\operatorname{ind}(X_4)$ is 2^3 or 2^4, but for precise determination, [**3**, Theorem 7.2] does not help anymore since

$$2^{n-m-1} = 2^5 = 32 \le \dim X_m = 54$$

for $m = 4$. We are going to use the result of Section 7 instead.

Let q' be a 1-codimensional subform of q and let X'_4 be the fourth orthogonal Grassmannian of q'. Since $\dim q' = 19$, we know from Section 7 that $\operatorname{ind}(X'_4) \le 2^3$. Since $\operatorname{ind}(X'_4) \ge \operatorname{ind}(X_4)$, we conclude $\boxed{\operatorname{ind}(X_4) = 2^3}$.

Appendix A. Programming Spin_{17}

Anyone with access to Maple can download the Chow package from https://www.mathematik.uni-muenchen.de/~semenov/software/chowring5.txt and verify the computation of Section 5. The algorithm used in the package is described in [**5**, Section 5].

Open a Maple worksheet and load the package with

```
read("C:/Packages/chowring5.txt");
```

indicating your way to the package file. You should receive the message

```
Chow ring package v. 4.0 loaded
```

In Maple 2021, there will be a warning about an implicitly local variable t, which can be ignored. To get rid of the warning, it suffices to add t to the list of local variables in the second line of the definition of the procedure fundam_invariant in "chowring5.txt". Thanks to Nikita Semenov for this information.

Run the following definitions:

```
x1:=omega[8];             x2:=omega[7]-omega[8];
x3:=omega[6]-omega[7];  x4:=omega[5]-omega[6];
x5:=omega[4]-omega[5];  y1:=omega[3]-omega[4];
y2:=omega[2]-omega[3];  y3:=omega[1]-omega[2];
```

This defines our elements $x_1, \ldots, x_5, y_1, y_2, y_3$ in the ring

$$R = \mathbb{Z}[x_1, \ldots, x_5, y_1, y_2, y_3]$$

of Section 2, which we view as the symmetric ring of the group of characters of the standard split maximal torus of the symplectic group $\mathrm{Sp}(16)$ (of type $\mathbf{C}_8$). The simple roots are numbered backwards in the Chow package and omega[i] is the notation for the ith fundamental weight, used in the package.

The next step is the construction of the element $2^3\tilde{z} \in R$, denoted a here:

```
x:=x1+x2+x3+x4+x5;
a:=(x+y1+y2+y3)*(x-y1+y2+y3)*(x+y1-y2+y3)*(x+y1+y2-y3)*
    (x-y1-y2+y3)*(x-y1+y2-y3)*(x+y1-y2-y3)*(x-y1-y2-y3);
```

(The Maple warning about multi-line expression can be ignored; to avoid it, put the definition of a in a single line.)

Now we compute the image $\tilde{c}$ of a in $\mathrm{CH}(Y_5)$. This is done with the procedure c_func of the Chow package. The element $\tilde{c}$ is denoted just c for simplicity:

```
c:=c_func([1,2,3,5,6,7,8],C8,a);
```

The first argument [1, 2, 3, 5, 6, 7, 8] of the procedure c_func indicates the parabolic subgroup we are interested in. (Recall that the simple roots are numbered backwards. In the usual numbering, our maximal parabolic subgroup is obtained by erasing the fifth root, not the fourth.) The second argument is the Dynkin type and the third argument can be any W-invariant element of the ring R. We take a for the third argument. The Maple output, coming almost immediately, is:

```
c:=128Z[5,4,3,2,1,2,3,4]+128Z[4,3,2,1,2,3,5,4]+
   128Z[3,2,1,2,4,3,5,4]+128Z[2,1,2,4,3,6,5,4]+
   128Z[2,1,3,2,4,3,5,4]+128Z[1,3,2,4,3,6,5,4]+
   128Z[1,2,5,4,3,6,5,4]+128Z[1,2,4,3,7,6,5,4]
```

where Z[...] stand for certain Schubert classes in $\mathrm{CH}(Y_5)$ constituting its $\mathbb{Z}$-basis.

To simplify, we divide by 128

```
c:=c/128;
```

and compute the cube in $\mathrm{CH}(Y_5)$ of the result, using the procedure chow_expand of the Chow package:

```
c3:=chow_expand([1,2,3,5,6,7,8],C8,c^3);
```

Finally, we divide by 2 and reduce modulo 2:

```
c3/2 mod 2;
```

The output is

```
Z[2,1,2,3,2,1,2,4,3,7,6,5,4,8,7,6,5,4,3,2,1,2,3,4]+
Z[1,2,4,3,2,1,2,5,4,3,6,5,4,8,7,6,5,4,3,2,1,2,3,4]+
Z[1,2,3,2,1,2,5,4,3,7,6,5,4,8,7,6,5,4,3,2,1,2,3,4]+
Z[1,3,2,1,4,3,2,5,4,3,6,5,4,8,7,6,5,4,3,2,1,2,3,4]+
Z[1,2,1,4,3,2,5,4,3,7,6,5,4,8,7,6,5,4,3,2,1,2,3,4]+
Z[1,2,1,3,2,6,5,4,3,7,6,5,4,8,7,6,5,4,3,2,1,2,3,4]+
Z[2,1,2,3,4,5,4,3,2,1,2,3,4,8,7,6,5,4,3,2,1,2,3,4]+
Z[1,2,3,4,6,5,4,3,2,1,2,3,4,8,7,6,5,4,3,2,1,2,3,4]+
Z[3,2,1,2,3,4,3,2,1,2,3,5,4,8,7,6,5,4,3,2,1,2,3,4]+
Z[2,1,2,3,4,3,2,1,2,3,6,5,4,8,7,6,5,4,3,2,1,2,3,4]+
Z[1,2,3,5,4,3,2,1,2,3,6,5,4,8,7,6,5,4,3,2,1,2,3,4]
```

All computations run almost immediately on my small laptop with an exception of the last one, taking a bit longer.

Appendix B. Programming Spin_{19}

Here is the input used in Section 7. We switch to the notation $x_{m+1}, \dots, x_n$ for $y_1, \dots, y_l$ so that the polynomial ring R is simply $\mathbb{Z}[x_1, \dots, x_n]$. We are computing $\mathrm{ind}(X_4)$ for Spin_{19} so that we have $n = 9$, $m = 4$, and $l = n - m = 5$.

We are defining $x_1, \dots, x_9$ in terms of the fundamental weights, next defining $a = 2^{2^5}\tilde{z} \in R$ in terms of $x_1, \dots, x_9$, and finally computing $c = \tilde{c} \in C = \mathrm{CH}(Y_4)$:

```
x[1]:=omega[9];
for i from 2 to 9 do x[i]:=omega[10-i]-omega[11-i]  od;

a:=1:    for s5 from -1 by 2 to 1 do
         for s6 from -1 by 2 to 1 do
         for s7 from -1 by 2 to 1 do
         for s8 from -1 by 2 to 1 do
         for s9 from -1 by 2 to 1 do
a:=a*(x[1]+x[2]+x[3]+x[4]+
s5*x[5]+s6*x[6]+s7*x[7]+s8*x[8]+s9*x[9])
od; od; od; od; od; a;

c:=c_func([1,2,3,4,5,7,8,9],C9,a);
```

The computation of the last line takes about 30 minutes.

Below is the value of `c/2^31 mod 2;`

```
Z[2,1,2,3,5,4,3,2,1,2,3,6,5,4,7,6,5,4,3,2,1,2,3,4,8,7,6,5,9,8,7,6]+
Z[1,2,4,3,5,4,3,2,1,2,3,6,5,4,7,6,5,4,3,2,1,2,3,4,8,7,6,5,9,8,7,6]+
Z[2,1,3,2,4,3,2,1,2,5,4,3,6,5,4,3,2,1,2,3,7,6,5,4,8,7,6,5,9,8,7,6]+
Z[5,4,6,5,7,6,5,4,3,2,1,2,3,4,5,8,7,6,9,8,7,6,5,4,3,2,1,2,3,4,5,6]+
Z[2,1,2,3,4,3,2,1,2,3,5,4,7,6,5,8,7,6,9,8,7,6,5,4,3,2,1,2,3,4,5,6]+
Z[1,2,3,4,3,2,1,2,3,6,5,4,7,6,5,8,7,6,9,8,7,6,5,4,3,2,1,2,3,4,5,6]+
Z[2,3,5,4,3,2,1,2,3,6,5,4,7,6,5,8,7,6,9,8,7,6,5,4,3,2,1,2,3,4,5,6]+
Z[4,3,5,4,3,2,1,2,3,6,5,4,7,6,5,8,7,6,9,8,7,6,5,4,3,2,1,2,3,4,5,6]+
Z[2,1,2,3,2,1,2,4,3,6,5,4,7,6,5,8,7,6,9,8,7,6,5,4,3,2,1,2,3,4,5,6]+
Z[1,2,3,2,1,2,5,4,3,6,5,4,7,6,5,8,7,6,9,8,7,6,5,4,3,2,1,2,3,4,5,6]+
Z[4,3,5,4,6,5,4,3,2,1,2,3,4,7,6,5,8,7,6,5,4,3,2,1,2,3,4,5,9,8,7,6]+
Z[1,2,4,3,2,1,2,5,4,3,6,5,4,7,6,5,8,7,6,5,4,3,2,1,2,3,4,5,9,8,7,6]+
Z[3,2,4,3,2,1,2,5,4,3,6,5,4,7,6,5,8,7,6,5,4,3,2,1,2,3,4,5,9,8,7,6]
```

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