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ENTROPY SOLUTIONS FOR SOME ELLIPTIC PROBLEMS INVOLVING THE GENERALIZED $p(u)$-LAPLACIAN OPERATOR

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Abstract. In this paper, we study the existence of entropy solutions for some generalized elliptic $p(u)$-Laplacian problem when $p(u)$ is a local quantity. We get the results by assuming the right-hand side function f to be an integrable function, and by using the regularization approach combined with the theory of Sobolev spaces with variable exponents.

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1. INTRODUCTION

In this paper, we study the existence of entropy solutions for some generalized elliptic equations with exponents p which may depends on the unknown solution u. We consider the case where the dependency of p on u is a local quantity. Namely, we study the following problem

$$\begin{cases} -\operatorname{div}(|\nabla u - \Theta(u)|^{p(u)-2}(\nabla u - \Theta(u))) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N, N \geq 2$, f is a given data, $p\colon \mathbb{R} \to [p_-, p_+]$ is a real continuous function, $1 < p_- \leq p_+ < +\infty$ and $p'(z) = \frac{p(z)}{p(z)-1}$ is the conjugate exponent of $p(z)$, with

$$p_- := \operatorname*{ess\,inf}_{z\in\mathbb{R}} p(z), \quad p_+ := \operatorname*{ess\,sup}_{z\in\mathbb{R}} p(z).$$

The problem (1.1) models several natural phenomena which appear in area of oceanography, turbulent fluid flows, induction heating and electrochemical problems. We cite for example the following parabolic model:

- Fluid flow through porous media. This model is governed by the following equation

  $$\frac{\partial \theta}{\partial t} - \operatorname{div}\left(|\nabla \varphi(\theta) - K(\theta)e|^{p-2}(\nabla \varphi(\theta) - K(\theta)e)\right) = 0,$$

  where θ is the volumetric content of moisture, $K(\theta)$ the hydraulic conductivity, $\varphi(\theta)$ the hydrostatic potential and e is the unit vector in the vertical direction.

In the classical cases, when $p(u) = p(x)$ or $p(u) = p$ many authors have studied the problem (1.1) by proving the existence and the uniqueness of several types of solutions, and by different approaches ([1, 7, 8, 15]).

The novelty of this work is to study some problems involving the generalized p-Laplacian operator in the case when the variable exponents p depend on the unknown solution u. The motivation to study these kind of problems relies in the fact that, in reality the measurements of some physical quantities are not made pointwise but through some local averages. The situation where the variable exponents p depend on the unknown solution u is non-standard as in the classical case (see [2–4, 7, 8, 10, 14, 15, 19]). This kind of problems appear in the applications of some numerical techniques for the total variation image restoration method that have been used in some restoration problems of mathematical image processing and computer vision [11, 12, 18]. Türola, J. in [18] have presented several numerical examples suggesting that the consideration of exponents $p = p(u)$ preserves the edges and reduces the noise of the restored images u. A numerical example suggesting a reduction of noise in the restored images u when the exponent of the regularization term is $p = p(|\nabla u|)$ is presented in [11]. Many authors have considered the problem (1.1) in the case where $\Theta = 0$ and especially the study of questions of existence and uniqueness of weak or entropy solutions to the problem (1.1). In this case, M. Chipot and H. B. de Oliveira in [13] have proved the existence of weak solutions for some $p(u)$-Laplacian problems, the existence proofs of [13] are based on the Schauder fixed-point theorem. C. Allalou, K. Hilal and S. A. Temghart in [5], extended the results established in [13] by proving some existence results for some local and nonlocal problems. Andreianov et al. [6], have studied the following prototype problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(u)-2}\nabla u) + u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

C. Zhang and X. Zhang in [20] have proved the existence of entropy solutions to problem (1.1) in the case where $\Theta = 0$ and they have provided some positive answers for the two questions proposed by Chipot and de Oliveira in [13]. S. Ouaro and N. Sawadogo in [16] and [17] considered the following nonlinear Fourier boundary value problem

$$\begin{cases} b(u) - \operatorname{div} a(x,u,\nabla u) = f & \text{in } \Omega \\ a(x,u,\nabla u)\cdot\eta + \lambda u = g & \text{on } \partial\Omega. \end{cases}$$

The existence and uniqueness results of entropy and weak solutions are established by an approximation method and convergent sequences in terms of Young measure.

In this paper, we first show that the approximated problems admits a sequence of weak solutions by applying the variational method combined with a special type of operators. In the second step, we will prove that the sequence of weak solutions converges to some function u and by using some a priori estimates, we will show that this function u is an entropy solution of elliptic problem (1.1).

This paper is organized as follow. In Section 2 we introduce the basic assumptions and we recall some definitions, basic properties of generalised Sobolev spaces that we will be used later. The Section 3 is devoted to show the existence of entropy solutions to the local problem (1.1).

2. Preliminaries

The exponent function p depends on the solution u and therefore it depends on the space variable x. This allows us to look for the entropy solutions to the problem (1.1) in a Sobolev space with variable exponents.

Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geq 2$, we say that a real-valued continuous function $h(.)$ is log-Hölder continuous in Ω if

$$\exists C > 0: \ |h(x) - h(y)| \leq \frac{C}{\ln\left(\frac{1}{|x-y|}\right)} \quad \forall x, y \in \Omega, \ \ |x-y| < \frac{1}{2}. \tag{2.1}$$

For any Lebesgue-measurable function $h\colon \Omega \to [1, \infty)$, we define

$$h_- := \operatorname*{ess\,inf}_{x\in\Omega} h(x) \ , \quad h_+ := \operatorname*{ess\,sup}_{x\in\Omega} h(x),$$

and we introduce the variable exponent Lebesgue space by:

$$L^{h(\cdot)}(\Omega) = \{ \ u\colon \Omega \to \mathbb{R} \ / \ \rho_{h(\cdot)}(u) \ := \int_\Omega |u(x)|^{h(x)} dx < \infty \}.$$

Equipped with the Luxembourg norm

$$\|u\|_{h(\cdot)} \ := \inf\left\{\lambda > 0 \ : \ \rho_{h(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\}, \tag{2.2}$$

$L^{h(\cdot)}(\Omega)$ becomes a Banach space. If

$$1 < h_- \leq h_+ < \infty, \tag{2.3}$$

$L^{h(\cdot)}(\Omega)$ is separable and reflexive. The dual space of $L^{h(\cdot)}(\Omega)$ is $L^{h'(\cdot)}(\Omega)$, where $h'(x)$ is the generalised Hölder conjugate of $h(x)$,

$$\frac{1}{h(x)}+\frac{1}{h'(x)}=1.$$

From the definitions of the modular $\rho_{h(\cdot)}(u)$ and the norm (2.2), it can be proved that if (2.3) holds then

$$\min\left\{\|u\|_{h(\cdot)}^{h_-},\ \|u\|_{h(\cdot)}^{h_+}\right\}\le\rho_{h(\cdot)}(u)\le\max\left\{\|u\|_{h(\cdot)}^{h_-},\ \|u\|_{h(\cdot)}^{h_+}\right\}. \tag{2.4}$$

A very useful consequence very useful of (2.4) is the following:

$$\|u\|_{h(\cdot)}^{h_-}-1\le\rho_{h(\cdot)}(u)\le\|u\|_{h(\cdot)}^{h_+}+1.$$

For any functions $u\in L^{h(\cdot)}(\Omega)$ and $v\in L^{h'(\cdot)}(\Omega)$, the generalized Hölder inequality hold:.

$$\int_\Omega uvdx\left(\frac{1}{h_-}+\frac{1}{h'_-}\right)\|u\|_{h(\cdot)}\|v\|_{h'(\cdot)}\le 2\|u\|_{h(\cdot)}\|v\|_{h'(\cdot)}.$$

We define also the generalized Sobolev space by

$$W^{1,h(\cdot)}(\Omega):=\{u\in L^{h(\cdot)}(\Omega):\nabla u\in L^{h(\cdot)}(\Omega)\},$$

which is a Banach space with the norm

$$\|u\|_{1,h(\cdot)}\ :=\|u\|_{h(\cdot)}+\|\nabla u\|_{h(\cdot)}.$$

The space $W^{1,h(\cdot)}(\Omega)$ is separable and is reflexive when (2.3) is satisfied. We also have

$$W^{1,h(\cdot)}(\Omega)\hookrightarrow W^{1,r(\cdot)}(\Omega)\text{ whenever }h(x)\ge r(x)\quad\text{for a.e. }x\in\Omega.$$

Now, we introduce the following function space

$$W_0^{1,h(\cdot)}(\Omega):=\{u\in \mathrm{W}_0^{1,1}(\Omega):\nabla u\in L^{h(\cdot)}(\Omega)\},$$

endowed with the following norm

$$\|u\|_{W_0^{1,h(\cdot)}(\Omega)}\ :=\|u\|_1+\|\nabla u\|_{h(\cdot)}.$$

If $h\in C(\overline{\Omega})$, then the norm in $W_0^{1,h(\cdot)}(\Omega)$ is equivalent to $\|\nabla u\|_{h(\cdot)}$. $C_0^\infty(\Omega)$ is dense in $W_0^{1,h(.)}(\Omega)$, when h is log-Hölder continuous. If h is a measurable function in Ω satisfying $1\le h_-\le h_+<d$ and the Log-Hölder continuity property (2.1), then

$$\|u\|_{h^*(\cdot)}\le C\|\nabla u\|_{h(\cdot)}\quad\forall u\in W_0^{1,h(\cdot)}(\Omega),$$

for some positive constant C, where

$$h^*(x):=\begin{cases}\frac{Nh(x)}{N-h(x)} & \text{if }\ h(x)<N,\\ \infty & \text{if }\ h(x)\ge N.\end{cases}$$

On the other hand, if h satisfies (2.1) and $h_- > N$, then

$$\|u\|_\infty \le C\|\nabla u\|_{h(\cdot)} \quad \forall u \in W_0^{1,h(\cdot)}(\Omega)\ ,$$

where C is another positive constant.

Let T_k denote the truncation function at height $k \geqslant 0$:

$$T_k(r) = \min\{k, \max\{r, -k\}\} = \begin{cases} k & \text{if } r \geqslant k \\ r & \text{if } |r| < k, \\ -k & \text{if } r \leqslant -k. \end{cases}$$

Next, we define the very weak gradient of a measurable function u with $T_k(u) \in W_0^{1,p(u)}(\Omega)$. The proof follows from [9, Lemma 2.1] due to the fact that $W_0^{1,p(\cdot)}(\Omega) \subset W_0^{1,p_-}(\Omega)$ (for more details we recommend readers see [6, Proposition 3.5]).

Proposition 1. *For every measurable function u with $T_k(u) \in W_0^{1,p(u)}(\Omega)$, there exists a unique measurable function $v\colon \Omega \to \mathbb{R}^N$, which we call the very weak gradient of u and denote $v = \nabla u$, such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}} \qquad \textit{for a.e. } x \in \Omega \textit{ and for every } k > 0,$$

where χ_E denotes the characteristic function of a measurable set E.

Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v coincides with the weak gradient of u.

Lemma 1 ([7, Lemma 2.1])**.** *For $\xi, \eta \in \mathbb{R}^N$ and $1 < p < \infty$, we have*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \le |\xi|^{p-2}\xi(\xi - \eta).$$

Lemma 2. *For $a \ge 0, b \ge 0$ and $1 \le p < \infty$, we have*

$$(a+b)^p \le 2^{p-1}\left(a^p + b^p\right).$$

3. Main results

In this section, we prove the existence of entropy solutions of problem (1.1). Firstly, we state the following assumptions:

(H_1) $f \in L^1(\Omega)$.

(H_2) $\Theta\colon \mathbb{R} \to \mathbb{R}^N$ is a continuous function such that $\Theta(0) = 0$ and $|\Theta(x) - \Theta(y)| \le \lambda|x - y|$ for all $x, y \in \mathbb{R}$, where λ is a positive constant such that $\lambda < \dfrac{1}{2C_0}$, and C_0 is the constant given by the Poincaré inequality.

We define the set where we are going to look for the solutions to problem (1.1) as

$$W_0^{1,p(u)}(\Omega) := \left\{u \in W_0^{1,1}(\Omega) : \int_\Omega |\nabla u|^{p(u)} \mathrm{d}x < \infty\right\}.$$

Now, we give a definition of entropy solutions for the elliptic problem (1.1).

Definition 1. A measurable function u with $T_k(u) \in W_0^{1,p(u)}(\Omega)$ is said to be an entropy solution for the problem (1.1), if

$$\int_\Omega \Phi(\nabla u - \Theta(u)) \nabla T_k(u-\varphi) dx \leqslant \int_\Omega f T_k(u-\varphi) \mathrm{d}x,$$

for all $\varphi \in C_0^1(\Omega)$ and for every $k > 0$ with

$$\Phi(\xi) = |\xi|^{p(u)-2}\xi \quad \forall \xi \in \mathbb{R}^N.$$

Theorem 1. *Let (H_1) and (H_2) be satisfied. Then, there exists at least one entropy solution of the problem (1.1).*

Proof. The proof of Theorem 1 is divided into several steps.

Part 1: The approximate problem.

Let f_n be a sequence of $C_0^\infty(\Omega)$ functions strongly converging to f in L^1 such that $\|f_n\|_{L^1} \leq \|f\|_{L^1}$.

We consider the following problem

$$(\mathcal{P}_n) \begin{cases} -\mathrm{div}(\Phi(\nabla u_n - \Theta(u_n))) = f_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\Phi(\xi) = |\xi|^{p(u_n)-2}\xi \quad \forall \xi \in \mathbb{R}^N.$$

We define the operator A by

$$\langle Au, v\rangle = \int_\Omega \Phi(\nabla u - \Theta(u)) \nabla v dx \quad \text{with} \quad u, v \in W_0^{1,p(\cdot)}(\Omega).$$

We will prove that A is coercive. From Lemma 1, we obtain

$$\begin{aligned} \langle Au, u\rangle &= \int_\Omega \Phi(\nabla u - \Theta(u)) \nabla u dx \\ &= \int_\Omega |\nabla u - \Theta(u)|^{p(u)-2} (\nabla u - \Theta(u)) \nabla u dx \\ &\geq \int_\Omega^1 \frac{1}{p(u)} |\nabla u - \Theta(u)|^{p(u)} dx - \int_\Omega \frac{1}{p(u)} |\Theta(u)|^{p(u)} dx. \end{aligned}$$

Since

$$(a+b)^p \leq 2^{p-1}(|a|^p + |b|^p),$$

we have

$$\frac{1}{2^{p_+-1}} |\nabla u|^{p(u)} = \frac{1}{2^{p_+-1}} |\nabla u - \Theta(u) + \Theta(u)|^{p(u)} \leq |\nabla u - \Theta(u)|^{p(u)} + |\Theta(u)|^{p(u)},$$

then

$$\frac{1}{2^{p_+-1}} |\nabla u|^{p(u)} - |\Theta(u)|^{p(u)} \leq |\nabla u - \Theta(u)|^{p(u)}.$$

Therefore, from Poincaré inequality we get

$$\begin{aligned}\langle Au,u\rangle &\geq \int_\Omega \frac{1}{p(u)}\left[\frac{1}{2^{p_+-1}}|\nabla u|^{p(u)}-|\Theta(u)|^{p(u)}\right]dx-\int_\Omega \frac{1}{p(u)}|\Theta(u)|^{p(u)}dx\\ &\geq \int_\Omega \frac{1}{p(u)}\frac{1}{2^{p_+-1}}|\nabla u|^{p(u)}dx-\int_\Omega \frac{2}{p(u)}|\Theta(u)|^{p(u)}dx\\ &\geq \int_\Omega \frac{1}{p(u)}\frac{1}{2^{p_+-1}}|\nabla u|^{p(u)}dx-\int_\Omega \frac{2}{p(u)}\lambda^{p(u)}|u|^{p(u)}dx\\ &\geq \int_\Omega \frac{1}{p_+}\frac{1}{2^{p_+-1}}|\nabla u|^{p(u)}dx-\int_\Omega \frac{2\lambda^{p(u)}}{p_-}C_0^{p(u)}|\nabla u|^{p(u)}dx.\end{aligned}$$

So the choice of the constant λ in (H_2) implies that

$$\langle Au,u\rangle \geq \left(\frac{1}{p_+}\frac{1}{2^{p_+-1}}-\frac{1}{p_-}\frac{1}{2^{p_--1}}\right)\int_\Omega |\nabla u|^{p(u)}dx.$$

Consequently

$$\frac{\langle Au,u\rangle}{\|u\|_{W_0^{1,p(\cdot)}(\Omega)}}\longrightarrow\infty \quad \text{as} \quad \|u\|_{W_0^{1,p(\cdot)}(\Omega)}\to\infty.$$

We deduce that the operator A is coercive. Besides, that the operator A is bounded and hemi-continuous. Then, the problem $(\mathcal{P}_n)$ admits at least one weak solution $u_n \in W_0^{1,p(\cdot)}(\Omega)$ in the following sense

$$\int_\Omega |\nabla u_n-\Theta(u_n)|^{p(u_n)-2}(\nabla u_n-\Theta(u_n)))\cdot\nabla\varphi dx=\int_\Omega f_n\varphi dx \tag{3.1}$$

for all $\varphi \in W_0^{1,p(\cdot)}(\Omega)$.

Our aim is to prove that a subsequence of these approximate solutions $\{u_n\}$ converges to a measurable function u, which is an entropy solution to (1.1).

Part 1: A priori estimate.

Lemma 3. *$(\nabla T_k(u_n))_{n\in\mathbb{N}}$ is bounded in $L^{p_-}(\Omega)$.*

Proof. We take $\varphi = T_k(u_n)$ as a test function in (3.1), we obtain

$$\int_\Omega \Phi(\nabla u_n-\Theta(u_n))\nabla T_k(u_n)dx \leqslant k\|f\|_{L^1(\Omega)}.$$

By the same way as in the proof of the coerciveness, we get

$$\rho_{1,p(u_n)}(T_k(u_n))\leqslant Ck.$$

Therefore

$$\|T_k(u_n)\|_{W_0^{1,p(u_n)}}\leqslant 1+(Ck)^{\frac{1}{p_-}},$$

we deduce that for any $k>0$, the sequence $(T_k(u_n))_{n\in\mathbb{N}}$ is uniformly bounded in $W_0^{1,p(u_n(\cdot))}(\Omega)$ and also in $W_0^{1,p_-}(\Omega)$. Then, up to a subsequence still denoted $T_k(u_n)$, we can assume that for any $k>0$, $T_k(u_n)$ weakly converges to ν_k in $W_0^{1,p_-}(\Omega)$ and also $T_k(u_n)$ strongly converges to ν_k in $L^{p_-}(\Omega)$. $\square$

Lemma 4. *$(u_n)_{n\in\mathbb{N}}$ converges in measure to some measurable function u.*

Proof. Firstly we prove that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. For every fixed $\delta>0$ and every positive integer $k>0$, we know that

$$\begin{aligned}\operatorname{meas}\{|u_n-u_m|>\delta\} \le \operatorname{meas}\{|u_n|>k\}+\operatorname{meas}\{|u_m|>k\}\\ +\operatorname{meas}\{|T_k(u_n)-T_k(u_m)|>\delta\}.\end{aligned}$$

Choosing $T_k(u_n)$ as a test function in (3.1), we get

$$\rho_{1,p(u_n)}(T_k(u_n)) \leqslant k\,\|f_n\|_{L^1(\Omega)} \leqslant k\|f\|_{L^1(\Omega)}.$$

It follows that

$$\int_{\{|u_n|>k\}} k^{p(u_n)}dx \leqslant k\|f\|_{L^1(\Omega)}.$$

Therefore

$$\operatorname{meas}\{|u_n|>k\} \leqslant k^{1-p_-}\|f\|_{L^1(\Omega)}.$$

Let $\varepsilon>0$, we choose $k=k(\varepsilon)$ such that

$$\operatorname{meas}\{|u_n|>k\} \leqslant \frac{\varepsilon}{3} \quad \text{and} \quad \operatorname{meas}\{|u_m|>k\} \leqslant \frac{\varepsilon}{3}.$$

Since $\{T_k(u_n)\}$ converges strongly in $L^{p_-}(\Omega)$, then it is a Cauchy sequence. Thus

$$\operatorname{meas}\{|T_k(u_n)-T_k(u_m)|>\delta\} \leqslant \frac{\varepsilon}{3},$$

for all $n,m \geqslant n_0(\delta,\varepsilon)$.
Finally, we obtain

$$\operatorname{meas}\{|u_n-u_m|>\delta\} \leqslant \varepsilon,$$

for all $n,m \geqslant n_0(\delta,\varepsilon)$. Hence

$$\limsup_{n,m\to\infty} \operatorname{meas}\{|u_n-u_m|>\delta\}=0,$$

which proves that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure and then converges almost everywhere to some measurable function u.

$$u_n \to u \quad \text{a.e in} \quad \Omega.$$

Therefore

$$T_k(u_n) \rightharpoonup T_k(u) \text{ in } W_0^{1,p_-}(\Omega)$$
$$T_k(u_n) \longrightarrow T_k(u) \text{ in } L^{p_-}(\Omega) \text{ and a.e. in } \Omega.$$

$\square$

Lemma 5. *$(\nabla u_n)_{n\in\mathbb{N}}$ converges almost everywhere in Ω to ∇u.*

Proof. We first prove that $\{\nabla u_n\}$ is a Cauchy sequence in measure. Let δ, h, ε are positive real numbers, obviously we have

$$\begin{aligned}\{x\in\Omega:|\nabla u_n-\nabla u_m|>\delta\}\subset\{x\in\Omega:|\nabla u_n|>h\}\cup\{x\in\Omega:|\nabla u_m|>h\}\\ \cup\{x\in\Omega:|u_n-u_m|>1\}\cup E,\end{aligned}$$

where

$$E:=\{x\in\Omega:|\nabla u_n|\leqslant h,|\nabla u_m|\leqslant h,|u_n-u_m|\leqslant 1,|\nabla u_n-\nabla u_m|>\delta\}.$$

For $k>0$, we can write

$$\{x\in\Omega:|\nabla u_n|\geqslant h\}\subset\{x\in\Omega:|u_n|\geqslant k\}\cup\{x\in\Omega:|\nabla T_k(u_n)|\geqslant h\},$$

then by using the same method as in Lemma 4, we obtain for k sufficiently large

$$\operatorname{meas}\{\{x\in\Omega:|\nabla u_n|>h\}\cup\{x\in\Omega:|\nabla u_m|>h\}\cup\{x\in\Omega:|u_n-u_m|>1\}\}\leqslant\frac{\varepsilon}{2}.$$

Notice that the application

$$\mathcal{G}:(s,t,\xi_1,\xi_2)\mapsto(\Phi(\xi_1-\Theta(s))-\Phi(\xi_2-\Theta(t)))(\xi_1-\xi_2)$$

is continuous and the set

$$\mathcal{H}:=\left\{(s,t,\xi_1,\xi_2)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N,|s|\le h,|t|\le h,|\xi_1|\le h,|\xi_2|\le h,|\xi_1-\xi_2|>\delta\right\}$$

is compact and

$$(\Phi(\xi_1-\Theta(s))-\Phi(\xi_2-\Theta(t)))(\xi_1-\xi_2)>0\quad\forall\xi_1\neq\xi_2.$$

Then the application $\mathcal{G}$ has its minimum on $\mathcal{H}$. Therefore, there exists a real valued function $\beta(h,\delta)>0$ such that

$$\begin{aligned}\beta(h,\delta)\operatorname{meas}(E)\le&\int_E\Big[|\nabla u_n-\Theta(u_n)|^{p(u_n)-2}(\nabla u_n-\Theta(u_n))\\&-|\nabla u_m-\Theta(u_m)|^{p(u_n)-2}(\nabla u_m-\Theta(u_m))\Big][\nabla u_n-\nabla u_m]\,dx,\\=&\int_E\Big[|\nabla u_m-\Theta(u_m)|^{p(u_m)-2}(\nabla u_m-\Theta(u_m))\\&-|\nabla u_m-\Theta(u_m)|^{p(u_n)-2}(\nabla u_m-\Theta(u_m))\Big][\nabla u_n-\nabla u_m]\,dx\\&+\int_E\Big[|\nabla u_n-\Theta(u_n)|^{p(u_n)-2}(\nabla u_n-\Theta(u_n))\\&-|\nabla u_m-\Theta(u_m)|^{p(u_m)-2}(\nabla u_m-\Theta(u_m))\Big][\nabla u_n-\nabla u_m]\,dx.\end{aligned}$$

We take $T_1(u_n-u_m)$ as a test function in (3.1) to get

$$\begin{aligned}\beta(h,\delta)\operatorname{meas}(E)\leqslant&\int_E\Big[|\nabla u_m-\Theta(u_m)|^{p(u_m)-2}(\nabla u_m-\Theta(u_m))\\&-|\nabla u_m-\Theta(u_m)|^{p(u_n)-2}(\nabla u_m-\Theta(u_m))\Big][\nabla u_n-\nabla u_m]\,dx\end{aligned}$$

$$+\int_E [f_n - f_m]\, T_1(u_n - u_m)\, dx.$$

By using the mean value theorem, there exists η taking values between $p(u_n)$ and $p(u_m)$ such that

$$\beta(h,\delta)\,\mathrm{meas}(E) \leqslant \int_E |\nabla u_m - \Theta(u_m)|^{\eta-1} \cdot \left|\log|\nabla u_m - \Theta(u_m)|\right| \cdot |\nabla u_n - \nabla u_m| \\ \cdot |p(u_m) - p(u_n)|\, dx + \|f_n - f_m\|_{L^1(\Omega)}.$$

By using Lemma 2, (H_2), the facts that $h \gg 1$ and the definition of E, we get

$$\beta(h,\delta)\,\mathrm{meas}(E) \leqslant 2^{p^+} h^{p^+}\left(1+\lambda^{\eta-1}\right)\log((1+\lambda)h) \cdot \int_\Omega |p(u_m) - p(u_n)|\, dx \\ + \|f_n - f_m\|_{L^1(\Omega)} := \alpha_{n,m}.$$

From Lebesgue dominated convergence theorem, we obtain

$$\mathrm{meas}(E) \leqslant \frac{\alpha_{n,m}}{\beta(h,\delta)} \leqslant \frac{\varepsilon}{2},$$

for all $n,m \geqslant N_2(\varepsilon,\delta)$. Consequently combining the previous results we get

$$\mathrm{meas}\{x\in\Omega : |\nabla u_n - \nabla u_m| > \delta\} \leqslant \varepsilon \quad \text{for all } n,m \geqslant \max\{N_1,N_2\}.$$

Hence $\{\nabla u_n\}$ is a Cauchy sequence in measure. Then we can choose a subsequence (denoted it by the original sequence) such that

$$\nabla u_n \to v \quad \text{a.e. in } \Omega.$$

Thus, using Proposition 1 and the fact that $\nabla T_k(u_n) \to \nabla T_k(u)$ in $(L^{p_-}(\Omega))^N$, we deduce that v coincides with the very weak gradient of u almost everywhere. Therefore, we have

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega.$$

□

Part 3: Passing to the limit.

We choose $T_k(u_n - \phi)$ as a test function in (3.1) for $\phi \in C_0^1(\Omega)$. Then

$$\int_\Omega |\nabla u_n - \Theta(u_n)|^{p(u_n)-2}(\nabla u_n - \Theta(u_n)) \cdot \nabla T_k(u_n - \phi)\, dx = \\ \int_\Omega f_n T_k(u_n - \phi)\, dx. \quad (3.2)$$

We now focus our attention on the left-hand side of (3.2).
We note that, if $L = k + \|\phi\|_{L^\infty(\Omega)}$

$$\int_\Omega |\nabla u_n - \Theta(u_n)|^{p(u_n)-2}(\nabla u_n - \Theta(u_n)) \cdot \nabla T_k(u_n - \phi)\, dx \\ = \int_\Omega |\nabla T_L(u_n) - \Theta(T_L(u_n))|^{p(u_n)-2}$$

$$
\begin{aligned}
&(\nabla T_L(u_n)-\Theta(T_L(u_n)))\cdot\nabla T_k(T_L(u_n)-\phi)\,dx\\
=&\int_\Omega|\nabla T_L(u_n)-\Theta(T_L(u_n))|^{p(u_n)-2}\\
&(\nabla T_L(u_n)-\Theta(T_L(u_n)))\cdot\nabla T_L(u_n)\,\chi_{\{|T_L(u_n)-\phi|\leqslant k\}}dx\\
&-\int_\Omega|\nabla T_L(u_n)-\Theta(T_L(u_n))|^{p(u_n)-2}\\
&(\nabla T_L(u_n)-\Theta(T_L(u_n)))\cdot\nabla\varphi\chi_{\{|T_L(u_n)-\phi|\leqslant k\}}dx.
\end{aligned}
$$

From (3.2), we have

$$
\begin{aligned}
&\int_\Omega\Big[|\nabla T_L(u_n)-\Theta(T_L(u_n))|^{p(u_n)-2}(\nabla T_L(u_n)-\Theta(T_L(u_n)))\cdot\nabla T_L(u_n)\\
&\qquad+\frac{1}{p_-}|\Theta(T_L(u_n)|^\gamma\Big]\cdot\chi_{\{|T_L(u_n)-\phi|\leqslant k\}}dx\\
&-\int_\Omega|\nabla T_L(u_n)-\Theta(T_L(u_n))|^{p(u_n)-2}(\nabla T_L(u_n)-\Theta(T_L(u_n)))\nabla\varphi\chi_{\{|T_L(u_n)-\phi|\leqslant k\}}dx\\
&=\int_\Omega f_nT_k(u_n-\phi)\,dx+\int_\Omega\frac{1}{p_-}|\Theta(T_L(u_n)|^\gamma\chi_{\{|T_L(u_n)-\phi|\leqslant k\}}dx,
\end{aligned}
\tag{3.3}
$$

where

$$
\gamma=\begin{cases}p_+ & \text{if } |\Theta(T_L(u_n)|\le 1,\\ p_- & \text{if } |\Theta(T_L(u_n)|>1.\end{cases}
$$

Since $\{\nabla T_L(u_n)\}$ is bounded in $\left(L^{p'(u_n)}(\Omega)\right)^N\subset\left(L^{p'_+}(\Omega)\right)^N$, then from the hypothesis (H_3), the sequence $\{\Theta(T_L(u_n)\}$ is also bounded in $\left(L^{p(u_n)}(\Omega)\right)^N\subset(L^{p_-}(\Omega))^N$, which implies that $\left\{|\nabla T_L(u_n)-\Theta(T_L(u_n))|^{p(u_n)-2}(\nabla T_L(u_n)-\Theta(T_L(u_n)))\right\}$ is bounded in $\left(L^{p'(u_n)}(\Omega)\right)^N\subset\left(L^{p'_+}(\Omega)\right)^N$.

As $u_n\to u$ a.e. in Ω and $\nabla u_n\to\nabla u$ a.e. in Ω, it follows that

$$
\Theta(T_L(u_n))\longrightarrow\Theta(T_L(u))\text{ a.e. in }\Omega \tag{3.4}
$$

and

$$
\nabla T_L(u_n)\longrightarrow\nabla T_L(u)\text{ a.e. in }\Omega.
$$

Hence,

$$
\begin{aligned}
&|\nabla T_L(u_n)-\Theta(T_L(u_n))|^{p(u_n)-2}(\nabla T_L(u_n)-\Theta(T_L(u_n)))\\
&\rightharpoonup|\nabla T_L(u)-\Theta(T_L(u))|^{p(u)-2}(\nabla T_L(u)-\Theta(T_L(u)))\quad\text{ in }\left(L^{p'_+}(\Omega)\right)^N.
\end{aligned}
$$

As $\phi\in C_0^1(\Omega)$, we get that

$$
\int_\Omega|\nabla T_L(u_n)-\Theta(T_L(u_n))|^{p(u_n)-2}(\nabla T_L(u_n)-\Theta(T_L(u_n)))\cdot\nabla\varphi\chi_{\{|T_L(u_n)-\phi|\leqslant k\}}dx
$$

$$\underset{n\to+\infty}{\longrightarrow} \int_\Omega |\nabla T_L(u) - \Theta(T_L(u))|^{p(u)-2} (\nabla T_L(u) - \Theta(T_L(u))) \cdot \nabla\varphi \chi_{\{|T_L(u)-\phi|\leqslant k\}} dx.$$

From (3.4) and the Lebesgue dominated convergence theorem, we obtain

$$\int_\Omega \frac{1}{p_-} |\Theta(T_L(u_n))|^\gamma \chi_{\{|T_L(u_n)-\phi|\leqslant k\}} dx \to \int_\Omega \frac{1}{p_-} |\Theta(T_L(u))|^\gamma \chi_{\{|T_L(u)-\phi|\leqslant k\}} dx.$$

On the other hand, by using Lemma 1, we have

$$\Bigg[|\nabla T_L(u_n) - \Theta(T_L(u_n))|^{p(u_n)-2} (\nabla T_L(u_n) - \Theta(T_L(u_n))) \cdot \nabla T_L(u_n) \\ + \frac{1}{p_-} |\Theta(T_L(u_n))|^\gamma \Bigg] \cdot \chi_{\{|T_L(u_n)-\phi|\leqslant k\}} \geqslant 0 \quad \text{a.e. in } \Omega.$$

By using Fatou's Lemma, we get

$$\begin{aligned} &\int_\Omega \Bigg[|\nabla T_L(u) - \Theta(T_L(u))|^{p(u)-2} (\nabla T_L(u) - \Theta(T_L(u))) \cdot \nabla T_L(u) \\ &\qquad + \frac{1}{p_-} |\Theta(T_L(u))|^\gamma \Bigg] \chi_{\{|T_L(u)-\phi|\leqslant k\}} dx \\ &\leq \liminf_{n\to\infty} \int_\Omega \Bigg[|\nabla T_L(u_n) - \Theta(T_L(u_n))|^{p(u_n)-2} (\nabla T_L(u_n) - \Theta(T_L(u_n))) \cdot \nabla T_L(u_n) \\ &\qquad\qquad + \frac{1}{p_-} |\Theta(T_L(u_n))|^\gamma \Bigg] \cdot \chi_{\{|T_L(u_n)-\phi|\leqslant k\}} dx. \end{aligned}$$

Now, we consider the first term in the right hand side of (3.3). Since $f_n \to f$ in $L^1(\Omega)$, then

$$\lim_{n\to\infty} \int_\Omega f_n T_k(u_n - \phi)\, dx = \int_\Omega f T_k(u - \phi)\, dx.$$

Finally, by using the above results we can pass to the limit as $n \to \infty$ in the equality (3.3) to concludes that u is an entropy solution to the problem (1.1). □

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