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A NEW OSCILLATION RESULT FOR NONLINEAR DIFFERENTIAL EQUATIONS WITH NONMONOTONE DELAY

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Abstract. In this article, our objective is to study the oscillation of first order nonlinear delay differential equation

$$x'(t)+p(t)f(x(\tau(t)))=0,\ t\geq t_0,$$

where the functions $p(t)$ and $\tau(t)$ are functions of nonnegative real numbers, $\tau(t)$ is not necessarily monotone such that $\tau(t)\leq t$ for $t\geq t_0$, $\lim_{t\to\infty}\tau(t)=\infty$, $f\in C(\mathbb{R},\mathbb{R})$ and $xf(x)>0$ for $x\neq 0$. Also, we establish a new oscillation condition involving both limsup and liminf. Finally, we present two examples to demonstrate the main result.

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1. INTRODUCTION

The paper deals with the first order nonlinear delay differential equation

$$x'(t)+p(t)f(x(\tau(t)))=0,\ t\geq t_0, \tag{1.1}$$

where the functions $p(t)$ and $\tau(t)$ are functions of nonnegative real numbers, $\tau(t)$ is not necessarily monotone such that

$$\tau(t)\leq t\ \text{ for } t\geq t_0,\ \lim_{t\to\infty}\tau(t)=\infty, \tag{1.2}$$

$$f\in C(\mathbb{R},\mathbb{R})\ \text{ and }\ xf(x)>0\ \text{ for }\ x\neq 0 \tag{1.3}$$

and

$$0<\widetilde{N}:=\limsup_{x\to 0}\frac{x}{f(x)}<\infty. \tag{1.4}$$

By a solution of (1.1), we mean a continuously differentiable function defined on $[\tau(T_0),\infty)$ for some $T_0\geq t_0$ such that (1.1) holds for $t\geq T_0$. A solution of (1.1) is called *oscillatory* if it has arbitrarily large zeros. Otherwise, it is called *nonoscillatory*.

The question of obtaining new sufficient conditions for the oscillatory solutions of these equations has attracted many researchers (see the references section). Furthermore, delay differential equations have numerous applications in the field of applied sciences and engineering. See the studies in [1,2,10,22,26,27] for more details. The reader is referred to monograph [11] for the general information about oscillation theory.

When $f(x) = x$, we have the linear form of (1.1)

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0. \tag{1.5}$$

The first study about the oscillation of all solutions of (1.5) was examined by Myshkis in 1950. Later, Ladas et al. [19], Koplatadze and Chanturija [16], Ladas and Stavroulakis [20], Fukagai and Kusano [9], Ladde et al. [21], Győri and Ladas [11] and Erbe et al. [7] studied (1.5) and obtained some well-known oscillation criteria.

Now, let α and β be defined by

$$\alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \tag{1.6}$$

and

$$\beta := \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds. \tag{1.7}$$

In 1972, Ladas et al. [19] proved that if $\tau(t)$ is nondecreasing and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 1, \tag{1.8}$$

then all solutions of (1.5) are oscillatory.

Also, in 1982, Koplatadze and Chanturija [16] established the following result. If $\tau(t)$ is not necessarily monotone and

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}, \tag{1.9}$$

then all solutions of (1.5) are oscillatory.

When the researchers encountered with the equations which do not satisfy the well-known oscillation criteria (1.8) and (1.9), they tried to obtain new conditions including both $\liminf$ and $\limsup$ conditions for the oscillatory solutions of these equations. The first successful attempt was carried out by Erbe and Zhang [8] in 1988. They established the following condition by using the upper bound of the ratio $\frac{x(\tau(t))}{x(t)}$ for the nonoscillatory solutions $x(t)$ of (1.5).

If $0 < \alpha \leq \frac{1}{e}$ and $\tau(t)$ is nondecreasing,

$$\beta > 1 - \left(\frac{\alpha}{2}\right)^2, \tag{1.10}$$

then all solutions of (1.5) are oscillatory.

Since then, many authors have tried to obtain better results by improving the upper bound for $\frac{x(\tau(t))}{x(t)}$. Also, in 1991, Chao [4] obtained the following condition for (1.5) with nondecreasing argument.

$$\beta > 1 - \frac{\alpha^2}{2(1-\alpha)}. \tag{1.11}$$

In 1992 Yu and Wang [30] and Yu et al. [31] found out the following result. If $0 < \alpha \le \frac{1}{e}$ and $\tau(t)$ is nondecreasing,

$$\beta > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{1.12}$$

then all solutions of (1.5) are oscillatory.

When $0 < \alpha \le \frac{1}{e}$ and $\tau(t)$ is nondecreasing, in 1990, Elbert and Stavroulakis [6] and in 1991, Kwong [18] established the following criteria for the oscillatory solutions of (1.5) by using different techniques, respectively.

$$\beta > 1 - \left(1 - \frac{1}{\sqrt{\lambda_1}}\right)^2 \tag{1.13}$$

and

$$\beta > \frac{\ln\lambda_1 + 1}{\lambda_1}, \tag{1.14}$$

where λ_1 is the smaller root of equation $\lambda = e^{\alpha\lambda}$.

In 1994, Koplatadze and Kvinikadze [17] improved (1.12). Furthermore, in 1998 Philos and Sficas [25], in 1999 Jaros and Stavroulakis [12], in 2000 Kon et al. [15] and in 2003 Sficas and Stavroulakis [28] established the following conditions for oscillatory solutions of (1.5) when $0 < \alpha \le \frac{1}{e}$ and $\tau(t)$ is nondecreasing.

$$\beta > 1 - \frac{\alpha^2}{2(1-\alpha)} - \frac{\alpha^2}{2}\lambda_1, \tag{1.15}$$

$$\beta > \frac{\ln\lambda_1 + 1}{\lambda_1} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{1.16}$$

$$\beta > 2\alpha + \frac{2}{\lambda_1} - 1 \tag{1.17}$$

and

$$\beta > \frac{\ln\lambda_1 - 1 + \sqrt{5 - 2\lambda_1 + 2\alpha\lambda_1}}{\lambda_1}, \tag{1.18}$$

where λ_1 is the smaller root of equation $\lambda = e^{\alpha\lambda}$.

Now, we define the function

$$h(t) = \sup_{s\le t}\{\tau(s)\},\ t \ge 0. \tag{1.19}$$

Clearly, $h(t)$ is nondecreasing and $\tau(t) \le h(t)$ for all $t \ge 0$.

In 2011, Braverman and Karpuz [3] established the following oscillation condition for (1.5). If $\tau(t)$ is not necessarily monotone and

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)ds > 1, \tag{1.20}$$

where $h(t)$ is defined by (1.19), then all solutions of (1.5) are oscillatory.

Moreover, in 2014, Stavroulakis [29] improved the condition (1.20) to the following condition for the oscillatory solutions of (1.5). If $0<\alpha\leq\frac{1}{e}$, $\tau(t)$ is not necessarily monotone and

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)ds > 1-\left(\frac{\alpha}{2}\right)^2, \tag{1.21}$$

where $h(t)$ is defined by (1.19), then all solutions of (1.5) are oscillatory. It can be seen that the right side of (1.21) is smaller than (1.20). Hence, (1.21) improves (1.20).

When the delay argument $\tau(t)$ is not necessarily monotone, the results which were presented by Chatzarakis and Pėics [5] and Kılıç [13] include (1.16) and (1.17), respectively.

On the other hand, in 2017 and 2020, Öcalan et al. [23, 24] obtained the following criteria for the oscillatory solution of (1.1). Assume that (1.2) and (1.3) hold. If $\tau(t)$ is not necessarily monotone and

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)ds > \widetilde{N},\ 0<\widetilde{N}<\infty, \tag{1.22}$$

or

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds > \frac{\widetilde{N}}{e},\ 0\leq\widetilde{N}<\infty, \tag{1.23}$$

where $\widetilde{N}$ and $h(t)$ are defined by (1.4) and (1.19), respectively, then all solutions of (1.1) are oscillatory.

The conditions (1.22) and (1.23) which are established for nonlinear delay differential equations can be considered as equivalent conditions to (1.8) and (1.9) which are obtained for linear delay differential equations.

As seen above, there are few papers dealing with oscillation of (1.1). In 1984, Fukagai and Kusano [9] studied (1.1) with nondecreasing delay. They obtained that if (1.23) holds, then all solutions of (1.1) are oscillatory. Also, see the results in Ladde et al. [21] for some oscillation criteria for the solutions of (1.1).

In view of this, an important question that arises in the case $\tau(t)$ is not necessarily monotone and (1.22) and (1.23) are not satisfied, is whether we can obtain new oscillation condition for (1.1). In the present paper, we will give the positive answer to this question. Also, the main purpose of this paper is to modify the condition (1.21) for the nonlinear delay differential equations. Especially, the present article will be the first study involving both lim inf and lim sup integral conditions in the literature for the nonlinear differential equations with nonmonotone delay by using the ratio $\frac{x(h(t))}{x(t)}$.

2. Main Results

In this section, we present a new sufficient condition for the oscillation of all solutions of (1.1), under the assumption that the delay argument $\tau(t)$ is not necessarily monotone. The following lemmas will be useful to establish our result.

Lemma 1 ([7], Lemma 2.1.1)**.** *Assume that* (1.19) *holds and*

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds > 0.$$

Then

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds = \liminf_{t\to\infty}\int_{h(t)}^{t} p(s)ds. \tag{2.1}$$

Lemma 2 ([14], Lemma 2.2)**.** *Assume that $x(t)$ is an eventually positive solution of* (1.1)*. If*

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)ds > 0, \tag{2.2}$$

then $\lim_{t\to\infty} x(t) = 0$, where $h(t)$ is defined by (1.19)*.*

Also, assume that $x(t)$ is an eventually negative solution of (1.1)*. If* (2.2) *holds, then $\lim_{t\to\infty} x(t) = 0$.*

Lemma 3. *Assume that x(t) is an eventually positive solution of* (1.1) *and*

$$\alpha := \liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds > 0. \tag{2.3}$$

Then, we have

$$\limsup_{t\to\infty}\frac{x(h(t))}{x(t)} \le \left(\frac{2N}{\alpha}\right)^2, \tag{2.4}$$

where $h(t)$ is defined by (1.19) *and N is a constant with $\tilde{N} < N$.*

Also, assume that x(t) is an eventually negative solution of (1.1)*. If* (2.3) *holds, then we get* (2.4)*.*

Proof. Let $x(t)$ be an eventually positive solution of (1.1). Then, there exists $t_1 > t_0$ such that $x(t),\ x(\tau(t)),\ x(h(t)) > 0$ for all $t \geq t_1$. Thus, from (1.1) we have

$$x'(t) = -p(t)f(x(\tau(t))) \leq 0$$

for all $t \geq t_1$, which means that $x(t)$ is an eventually nonincreasing function. Also, with the help of Lemma 1, (2.3) implies (2.2), then from Lemma 2, we know that $\lim_{t\to\infty} x(t) = 0$. Then from (1.4), we can choose $t_2 > t_1$ and there exists N with $\widetilde{N} < N$ such that

$$f(x(\tau(t))) > \frac{1}{N}x(\tau(t)) \text{ for } t \geq t_2. \tag{2.5}$$

Using the fact that $x(t)$ is nonincreasing, $\tau(t) \leq h(t)$ and the inequality (2.5), from (1.1) we obtain

$$x'(t) + \frac{1}{N}p(t)x(h(t)) < 0. \tag{2.6}$$

Moreover, from (2.3) and Lemma 1, we have

$$\int_{h(t)}^{t} p(s)ds \geq \alpha - \varepsilon,\ \varepsilon \in (0,\alpha), \tag{2.7}$$

therefore there exists a $t^* > t$ such that

$$\int_{h(t^*)}^{t} p(s)ds \geq \frac{\alpha-\varepsilon}{2} \text{ and } \int_{t}^{t^*} p(s)ds \geq \frac{\alpha-\varepsilon}{2}. \tag{2.8}$$

Then, integrating (2.6) from $h(t^*)$ to t and using the fact that $x(t)$ is nonincreasing, $h(t)$ is nondecreasing and (2.8), we obtain

$$x(t) - x(h(t^*)) + \frac{1}{N}\int_{h(t^*)}^{t} p(s)x(h(s))\,ds < 0$$

or

$$x(t) - x(h(t^*)) + \frac{1}{N}x(h(t))\int_{h(t^*)}^{t} p(s)ds < 0$$

and

$$x(h(t^*)) > \frac{1}{N}x(h(t))\,\frac{\alpha-\varepsilon}{2}. \tag{2.9}$$

By using the same facts as above, integrating (2.6) from t to t^*, we have

$$x(t^*) - x(t) + \frac{1}{N}\int_{t}^{t^*} p(s)x(h(s))\,ds < 0$$

or

$$x(t^*) - x(t) + \frac{1}{N} x(h(t^*)) \int_t^{t^*} p(s)ds < 0$$

and

$$x(t) > \frac{1}{N} x(h(t^*)) \frac{\alpha - \varepsilon}{2}. \tag{2.10}$$

Finally, combining (2.9) and (2.10), we get

$$x(t) > \frac{\alpha - \varepsilon}{2N} x(h(t^*)) > \frac{(\alpha - \varepsilon)}{2N} \frac{(\alpha - \varepsilon)}{2N} x(h(t))$$

or

$$\frac{x(h(t))}{x(t)} < \left(\frac{2N}{(\alpha - \varepsilon)} \right)^2.$$

Hence

$$\limsup_{t \to \infty} \frac{x(h(t))}{x(t)} \leq \left(\frac{2N}{(\alpha - \varepsilon)} \right)^2,$$

because of ε is arbitrary, by letting $\varepsilon \to 0$, we obtain (2.4) and this completes the proof. □

Theorem 1. *Assume that* (1.2) *and* (1.3) *hold. If* $\tau(t)$ *is not necessarily monotone,* $0 < \alpha \leq \frac{\widetilde{N}}{e}$ *and*

$$\beta := \limsup_{t \to \infty} \int_{h(t)}^{t} p(s)ds > N \left[1 - \left(\frac{\alpha}{2N} \right)^2 \right], \tag{2.11}$$

then all solutions of (1.1) *are oscillatory, where* $h(t)$ *is defined by* (1.19) *and* N *is a constant with* $\widetilde{N} < N$.

Proof. Assume, for the sake of contradiction, that there exists an eventually positive solution $x(t)$ of (1.1). If $x(t)$ is an eventually negative solution of (1.1), the proof of the theorem can be done similarly. Then, there exists $t_1 > t_0$ such that $x(t), x(\tau(t)), x(h(t)) > 0$ for all $t \geq t_1$. Thus, from (1.1) we have

$$x'(t) = -p(t) f(x(\tau(t))) \leq 0$$

for all $t \geq t_1$, which means, that $x(t)$ is an eventually nonincreasing function. Lemma 2 and condition (2.11) imply that $\lim_{t \to \infty} x(t) = 0$. Then from (1.4), we can choose $t_2 > t_1$ and there exists N with $\widetilde{N} < N$ such that

$$f(x(\tau(t))) > \frac{1}{N} x(\tau(t)) \text{ for } t \geq t_2. \tag{2.12}$$

Using inequality (2.12), the fact that $x(t)$ is nonincreasing and $\tau(t) \leq h(t)$, from (1.1) we obtain

$$x'(t) + \frac{1}{N}p(t)x(h(t)) < 0. \tag{2.13}$$

Integrating (2.13) from $h(t)$ to t, we have

$$x(t) - x(h(t)) + \frac{1}{N}\int_{h(t)}^{t} p(s)x(h(s))ds < 0,$$

$$x(t) - x(h(t)) + \frac{1}{N}x(h(t))\int_{h(t)}^{t} p(s)ds < 0$$

and

$$x(t) + x(h(t))\left[\frac{1}{N}\int_{h(t)}^{t} p(s)ds - 1\right] < 0$$

or

$$\frac{1}{N}\int_{h(t)}^{t} p(s)ds < 1 - \frac{x(t)}{x(h(t))}$$

and by Lemma 3,

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)ds \leq N\left[1 - \left(\frac{\alpha}{2N}\right)^2\right], \tag{2.14}$$

which contradicts to (2.11), so this completes the proof. □

Example 1. We consider the nonlinear delay differential equation

$$x'(t) + \frac{1}{t}x\left(\frac{t}{2.7}\right)\ln\left(\left|x\left(\frac{t}{2.7}\right)\right| + e\right) = 0. \tag{2.15}$$

Then,

$$\widetilde{N} = \limsup_{x\to 0}\frac{x}{f(x)} = \limsup_{x\to 0}\frac{x}{x\ln(|x|+e)} = 1 < N = 1.01$$

and using this, we have

$$\beta = \limsup_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds = \limsup_{t\to\infty}\int_{\frac{t}{2.7}}^{t}\frac{1}{s}ds = \ln(2.7) \approx 0.99325 < 1 = \widetilde{N},$$

so condition (1.22) doesn't hold. But,

$$\beta \approx 0.99325 > N\left[1 - \left(\frac{\alpha}{2N}\right)^2\right] \approx 0.76580.$$

We see that condition (2.11) is satisfied and therefore, all solutions of (2.15) are oscillatory.

Example 2. We consider the nonlinear delay differential equation

$$x'(t)+(0.09)x(\tau(t))\ln(|x(\tau(t))|+19.02)=0, \tag{2.16}$$

where

$$\tau(t)=\begin{cases} -t+12k-2, & t\in[6k,6k+1], \\ 4t-18k-7, & t\in[6k+1,6k+2], \\ 0.5t+3k, & t\in[6k+2,6k+4], \\ -6t+42k+26, & t\in[6k+4,6k+5], \\ 8t-42k-44, & t\in[6k+5,6k+6], \end{cases} \quad k\in\mathbb{N}_0,$$

and with the help of (1.19), we obtain

$$h(t):=\sup_{s\le t}\{\tau(s)\}=\begin{cases} 6k-2, & t\in[6k,6k+1.25], \\ 4t-18k-7, & t\in[6k+1.25,6k+2], \\ 0.5t+3k, & t\in[6k+2,6k+4], \\ 6k+2, & t\in[6k+4,6k+5.75], \\ 8t-42k-44, & t\in[6k+5.75,6k+6], \end{cases} \quad k\in\mathbb{N}_0.$$

Then, from (1.4), we have

$$\widetilde{N}=\limsup_{x\to 0}\frac{x}{f(x)}=\limsup_{x\to 0}\frac{x}{x\ln(|x|+19.02)}\approx 0.3395<N=0.3396.$$

Also,

$$\alpha=\liminf_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds=\liminf_{t\to\infty}\int_{6k+1}^{6k+2}0.09ds=0.09<\frac{\widetilde{N}}{e}\approx 0.12489$$

and

$$\beta=\limsup_{t\to\infty}\int_{h(t)}^{t}p(s)ds=\limsup_{t\to\infty}\int_{6k+2}^{6k+5.75}0.09ds=0.3375<\widetilde{N}\approx 0.3395,$$

so, the conditions (1.22) and (1.23) don't hold.

However,

$$\beta=0.3375>N\left[1-\left(\frac{\alpha}{2N}\right)^2\right]\approx 0.33363,$$

which means, the conditions of Theorem 1 hold. So, all solutions of (2.16) are oscillatory.

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