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ON CO-ANNIHILATORS IN EQUALITY ALGEBRAS

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Abstract. In this paper, we introduce the notion of co-annihilator on a lattice equality algebra $\mathbb{E}$ and investigate some basic properties of them. Specially, by using the generated filters in $\mathbb{E}$, we prove that the set of all filters of $\mathbb{E}$ forms a bounded distributive pseudo-complemented lattice, which pseudo-complemented of any filter is co-annihilator of it. Finally, we construct a Boolean algebra by the set of all co-annihilators of $\mathbb{E}$.

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1. INTRODUCTION

A more general algebraic structure in logic without contractions is residuated lattices [14]. Among logical algebras, residuated lattices have received the most attention due to their interesting properties and including two important sub-classes: BL-algebras and MV-algebras. Fuzzy type theory was developed as a higher order fuzzy logic. Novák and De Beats generalized residuated lattices and proposed EQ-algebra [10]. Because by replacing the product operation with a lesser or equal operation, we get an EQ-algebra again, Jenei [7] introduced a new algebra, called equality algebra. Since equality algebra can be a good alternative to possible algebraic semantics for fuzzy type theory, the study of equality algebra is very valuable.

In [3], it was proved that equality algebras and BCK-meet semilattices (under distributivity condition) correspond to each other. Because different filters have natural expressions as diverse sets of provable formulas, filter theory has a significant impact on the study of logical algebras. For this, in [1], Borzooei et al. introduced some types of filters in equality algebras. For more recent studies about equality algebras, you can see [5, 6, 11, 12].

Many works have been done with respect to the co-annihilators and annihilators. Davey studied the relationship between minimal prime ideals conditions and annihilators conditions on distributive lattices [4]. Turunen defined co-annihilator of a non-empty subset of a BL-algebra and proved some of its properties [13]. Leustean introduced the notion of co-annihilator relative to a filter F on pseudo BL-algebras. Then Meng introduced generalized co-annihilators in BL-algebras and gave characterizations of prime and minimal prime filters [9]. Also, Zou et al introduced the notion of annihilators in BL-algebras and investigated some related properties of them in [16].

In this paper, we introduce the notion of co-annihilator in equality algebras. We study basic properties of co-annihilators and investigate relationship between them and filters. Also, we get that the lattice of all filters of an equality algebra forms a pseudo-complemented lattice. Moreover, the collection of co-annihilators of an equality algebra forms a Boolean algebra. The paper is organized as follows: In Section 2, we gather the basic notions and results on equality algebras. In Section 3, we study generated filters in equality algebras and get some interesting results about them. In Section 4, we introduce the co-annihilators on a lattice equality algebra and investigate some related properties of them.

2. Preliminaries

In this section, we gather some basic notions relevant to equality algebra which will be needed in the next sections.

Definition 1 ([7, Definition 1.1]). An algebraic structure $\mathbb{E} = (\mathbb{E}; \wedge, \sim, 1)$ of type $(2,2,0)$ is called an *equality algebra*, if, for all $\alpha, \gamma, \eta \in \mathbb{E}$, it satisfies the following conditions:

$(E1)$ $(\mathbb{E}, \wedge, 1)$ is a commutative idempotent integral monoid,
$(E2)$ $\alpha \sim \gamma = \gamma \sim \alpha$,
$(E3)$ $\alpha \sim \alpha = 1$,
$(E4)$ $\alpha \sim 1 = \alpha$,
$(E5)$ $\alpha \leq \gamma \leq \eta$ implies $\alpha \sim \eta \leq \gamma \sim \eta$ and $\alpha \sim \eta \leq \alpha \sim \gamma$,
$(E6)$ $\alpha \sim \gamma \leq (\alpha \wedge \eta) \sim (\gamma \wedge \eta)$,
$(E7)$ $\alpha \sim \gamma \leq (\alpha \sim \eta) \sim (\gamma \sim \eta)$.

The operation $\wedge$ is called *meet* and $\sim$ is an *equality* operation.

Notation 1. From now on, we let $(\mathbb{E}, \sim, \wedge, 0, 1)$ or $\mathbb{E}$ be an equality algebra, unless otherwise state.

On $\mathcal{E}$ we write $\alpha \leq \gamma$ if and only if $\alpha \wedge \gamma = \alpha$. Clearly, $(\mathbb{E}, \leq)$ is a poset. Also, other two derived operations are defined, as the following, and we call them *implication* and *equivalence*, respectively:

$$\alpha \twoheadrightarrow \gamma = \alpha \sim (\alpha \wedge \gamma) \quad \text{and} \quad \alpha \leftrightarrow \gamma = (\alpha \twoheadrightarrow \gamma) \wedge (\gamma \twoheadrightarrow \alpha).$$

Proposition 1 ([7, Proposition 2], [15, Proposition 3.1]). *The following statements hold on $\mathbb{E}$, for all $\alpha, \gamma, \eta \in \mathbb{E}$:*

(i) $\alpha \twoheadrightarrow \gamma = 1$ *if and only if* $\alpha \leq \gamma$,

(ii) $1 \twoheadrightarrow \alpha = \alpha,\ \alpha \twoheadrightarrow 1 = 1,$ *and* $\alpha \twoheadrightarrow \alpha = 1,$
(iii) $\alpha \leq \gamma \to \alpha,$
(iv) $\alpha \leq (\alpha \twoheadrightarrow \gamma) \twoheadrightarrow \gamma,$
(v) $\alpha \twoheadrightarrow (\gamma \twoheadrightarrow \eta) = \gamma \twoheadrightarrow (\alpha \twoheadrightarrow \eta),$
(vi) $\alpha \leq \gamma$ *implies* $\gamma \twoheadrightarrow \eta \leq \alpha \twoheadrightarrow \eta$ *and* $\eta \to \alpha \leq \eta \to \gamma,$
(vii) $\alpha \twoheadrightarrow \gamma = \alpha \twoheadrightarrow (\alpha \wedge \gamma),$
(viii) $\alpha \to \gamma \leq (\eta \twoheadrightarrow \alpha) \twoheadrightarrow (\eta \twoheadrightarrow \gamma).$

A *bounded* equality algebra $\mathbb{E}$ is an equality algebra with the least element $0 \in \mathbb{E}$ such that $0 \leq \alpha$, for all $\alpha \in \mathbb{E}$. If $\mathbb{E}$ is bounded, then we define the operation negation "$^-$" on $\mathbb{E}$ by, $\alpha^- = \alpha \twoheadrightarrow 0 = \alpha \sim 0$, for all $\alpha \in \mathbb{E}$. If $\alpha^{--} = \alpha$, for all $\alpha \in \mathbb{E}$, then the bounded equality algebra $\mathbb{E}$ is called *involutive*. Also, $\mathbb{E}$ is called prelinear, if 1 is the unique upper bound of the set $\{\alpha \twoheadrightarrow \gamma,\ \gamma \twoheadrightarrow \alpha\}$ for all $\alpha, \gamma \in \mathbb{E}$. A *lattice equality algebra* is an equality algebra which is a lattice.

Proposition 2 ([15, Proposition 3.7])**.** *If* $\mathbb{E}$ *is a lattice, then for all* $\alpha, \gamma, \eta \in \mathbb{E}$*, the following statements hold:*

(i) $(\alpha \vee \gamma) \twoheadrightarrow \eta = (\alpha \twoheadrightarrow \eta) \wedge (\gamma \twoheadrightarrow \eta),$
(ii) $\alpha \twoheadrightarrow \gamma = (\alpha \vee \gamma) \twoheadrightarrow \gamma.$

Definition 2 ([8, Definition 3])**.** Let $\varnothing \neq \mathbb{F} \subseteq \mathbb{E}$. Then $\mathbb{F}$ is called a *deductive system* or *filter* of $\mathbb{E}$, if for all $\alpha, \gamma \in \mathbb{E}$, we have

(i) $\alpha \in \mathbb{F}$ and $\alpha \leq \gamma$ imply $\gamma \in \mathbb{F}$;
(ii) $\alpha \in \mathbb{F}$ and $\alpha \sim \gamma \in \mathbb{F}$ imply $\gamma \in \mathbb{F}$.

Proposition 3 ([8, Proposition 3])**.** *Let* $\mathbb{F} \subseteq \mathbb{E}$*. Then* $\mathbb{F} \in \mathcal{F}(\mathbb{E})$ *if and only if, for all* $\alpha, \gamma \in \mathbb{E}$

(i) $1 \in \mathbb{F}$,
(ii) if $\alpha \in \mathbb{F}$ *and* $\alpha \twoheadrightarrow \gamma \in \mathbb{F}$*, then* $\gamma \in \mathbb{F}$.

The set of all filters of $\mathbb{E}$ is denoted by $\mathcal{F}(\mathbb{E})$. Clearly, $1 \in \mathbb{F}$, for all $\mathbb{F} \in \mathcal{F}(\mathbb{E})$. A filter $\mathbb{F}$ of $\mathbb{E}$ is called *proper* if $\mathbb{F} \neq \mathbb{E}$. Obviously, if $\mathbb{E}$ is bounded, then a filter is proper if and only if $0 \notin \mathbb{F}$.

Definition 3 ([1, Definition 3.1])**.** Let $\mathbb{F} \subseteq \mathbb{E}$ such that $1 \in \mathbb{F}$. Then $\mathbb{F}$ is called a *positive implicative filter* of $\mathbb{E}$ if $\alpha \twoheadrightarrow (\gamma \twoheadrightarrow \eta) \in \mathbb{F}$ and $\alpha \twoheadrightarrow \gamma \in \mathbb{F}$ imply $\alpha \twoheadrightarrow \eta \in \mathbb{F}$, for all $\alpha, \gamma, \eta \in \mathbb{E}$.

Definition 4 ([11, Definition 4.1])**.** Let $(\mathbb{E}; \wedge_{\mathbb{E}}, \sim_{\mathbb{E}}, 1_e)$ and $(\mathbb{Q}; \wedge_{\mathbb{Q}}, \sim_{\mathbb{Q}}, 1_{\mathbb{Q}})$ be two equality algebras. Then a map $f: \mathbb{E} \twoheadrightarrow \mathbb{Q}$ is called an *equality homomorphism*, if for all $\alpha, \gamma \in \mathbb{E}$, the following conditions hold:

$$f(\alpha \wedge_{\mathbb{E}} \gamma) = f(\alpha) \wedge_{\mathbb{Q}} f(\gamma) \qquad \text{and} \qquad f(\alpha \sim_{\mathbb{E}} \gamma) = f(\alpha) \sim_{\mathbb{Q}} f(\gamma).$$

Moreover, if $\mathbb{E}$ and $\mathbb{Q}$ are two bounded, then an equality homomorphism f is called bounded, if $f(0_{\mathbb{E}}) = 0_{\mathbb{Q}}$. An equality homomorphism f is called an equality *(epimorphism) monomorphism*, if f is an (onto)one-to-one mapping and an equality homomorphism f is called an *equality isomorphism*, if f is a one-to-one and onto mapping.

Definition 5 ([2, Definition 5.1])**.** Consider X as a non-empty set. A mapping $\phi : \mathcal{P}(X) \to \mathcal{P}(X)$ is called a *closure map* on X, if for all $Y, Z \in \mathcal{P}(X)$, the following statements hold:

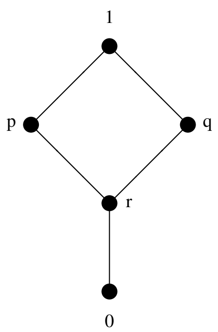
$(C1)$ $Y \subseteq \phi(Y)$,
$(C2)$ $\phi^2(Y) = \phi(Y)$,
$(C3)$ $Y \subseteq Z$ implies $\phi(Y) \subseteq \phi(Z)$.

3. Some results on filters of equality algebras

In this section, we investigate some properties of generated filter on equality algebras, that these results will be used in the next section to prove some theorems related to co-annihilators.

Definition 6. Let $\mathcal{X} \subseteq \mathbb{E}$. The smallest filter of $\mathbb{E}$ containing $\mathcal{X}$ is called *the generated filter by $\mathcal{X}$ in* $\mathbb{E}$ which is denoted by $\langle \mathcal{X} \rangle$. Indeed, $\langle \mathcal{X} \rangle = \bigcap_{\mathcal{X} \subseteq \mathbb{F} \in \mathcal{F}(\mathbb{E})} \mathbb{F}$.

Example 1. Let $\mathbb{E} = \{0, p, q, r, 1\}$ be a poset with the following Hasse diagram. Define operation "$\sim$" on $\mathbb{E}$ as follows:



$\sim$	0	r	p	q	1
0	1	0	0	0	1
r	0	1	q	p	r
p	0	q	1	r	p
q	0	p	r	1	q
1	0	r	p	q	1

$\twoheadrightarrow$	0	r	p	q	1
0	1	1	1	1	1
r	0	1	1	1	1
p	0	q	1	q	1
q	0	p	p	1	1
1	0	r	p	q	1

Then $(\mathbb{E}, \sim, \wedge, 1)$ is an equality algebra. If $\mathcal{X} = \{p, q\}$, then $\langle \mathcal{X} \rangle = \{p, q, r, 1\}$.

Proposition 4. *Let $\varnothing \neq \mathcal{X} \subseteq \mathbb{E}$. Then*

$$\langle \mathcal{X} \rangle = \{\alpha \in \mathbb{E} \mid \text{ for some } n \in \mathbb{N} \text{ and } p_1, \ldots, p_n \in \mathcal{X}, \\ p_1 \to (p_2 \to (\cdots \to (p_n \to \alpha) \ldots)) = 1\}.$$

In particular, for any element $p \in \mathbb{E}$, we have

$$\langle p \rangle = \{\alpha \in \mathbb{E} \mid \text{ for some } n \in \mathbb{N},\ p \to^n \alpha = 1\},$$

where $\alpha \twoheadrightarrow^0 \gamma = \gamma$ *and* $\alpha \twoheadrightarrow^n \gamma = \alpha \twoheadrightarrow (\alpha \twoheadrightarrow^{n-1} \gamma)$.

Proof. Suppose

$$\begin{aligned} B = \{\alpha \in \mathbb{E} \mid \; & \text{for some } n \in \mathbb{N} \text{ and } p_1, \ldots, p_n \in \mathcal{X}, \\ & p_1 \to (p_2 \to (\cdots \to (p_n \to \alpha)\ldots)) = 1\}. \end{aligned}$$

Since $p \twoheadrightarrow 1 = 1$, we have $1 \in B$, and so $B \neq \varnothing$. In addition, since $p \twoheadrightarrow p = 1$, for all $p \in \mathcal{X}$, we get $p \in B$, thus $\mathcal{X} \subseteq B$. Now, we prove $B \in \mathcal{F}(\mathbb{E})$. Let $\alpha, \alpha \twoheadrightarrow \gamma \in B$. Then there exist $m, n \in \mathbb{N}$, $p_1, \ldots, p_n \in \mathcal{X}$ and $q_1, \ldots, q_m \in \mathcal{X}$ such that

$$p_1 \twoheadrightarrow (p_2 \twoheadrightarrow (\ldots \twoheadrightarrow (p_n \to \alpha)\ldots)) = 1.$$

$$q_1 \twoheadrightarrow (q_2 \twoheadrightarrow (\ldots \twoheadrightarrow (q_m \twoheadrightarrow (\alpha \twoheadrightarrow \gamma))\ldots)) = 1.$$

Thus by Proposition 1(viii), we get

$$\begin{aligned} \alpha \twoheadrightarrow \gamma & \leq (p_n \twoheadrightarrow \alpha) \twoheadrightarrow (p_n \twoheadrightarrow \gamma) \\ & \leq (p_{n-1} \twoheadrightarrow (p_n \twoheadrightarrow \alpha)) \twoheadrightarrow (p_{n-1} \twoheadrightarrow (p_n \twoheadrightarrow \gamma)) \\ & \;\; \vdots \\ & \leq \underbrace{[p_1 \twoheadrightarrow (p_2 \twoheadrightarrow (\ldots (p_n \twoheadrightarrow \alpha)\ldots))]}_{1} \twoheadrightarrow \underbrace{[p_1 \twoheadrightarrow (p_2 \twoheadrightarrow (\ldots (p_n \twoheadrightarrow \gamma)\ldots))]}_{Y} \\ & = 1 \twoheadrightarrow Y \\ & = Y. \end{aligned}$$

Since $\alpha \twoheadrightarrow \gamma \leq Y$, by Proposition 1(vi), we get $q_m \twoheadrightarrow (\alpha \twoheadrightarrow \gamma) \leq q_m \twoheadrightarrow Y$. By repeating this method, we conclude that

$$q_1 \twoheadrightarrow (q_2 \twoheadrightarrow (\ldots \twoheadrightarrow (q_m \twoheadrightarrow (\alpha \twoheadrightarrow \gamma))\ldots)) \leq q_1 \twoheadrightarrow (q_2 \twoheadrightarrow (\ldots \twoheadrightarrow (q_m \twoheadrightarrow Y))\ldots).$$

Since $q_1 \twoheadrightarrow (q_2 \twoheadrightarrow (\ldots \twoheadrightarrow (q_m \twoheadrightarrow (\alpha \twoheadrightarrow \gamma))\ldots)) = 1$, we have $q_1 \twoheadrightarrow (q_2 \twoheadrightarrow (\ldots \twoheadrightarrow (q_m \twoheadrightarrow Y)\ldots)) = 1$ and so

$$q_1 \twoheadrightarrow (q_2 \twoheadrightarrow (\ldots \twoheadrightarrow (q_m \twoheadrightarrow (p_1 \twoheadrightarrow (\ldots \twoheadrightarrow (p_n \twoheadrightarrow \gamma))))\ldots)) = 1.$$

Hence $\gamma \in B$. Therefore, $B \in \mathcal{F}(\mathbb{E})$ containing $\mathcal{X}$. It is enough to prove that B is the smallest filter of $\mathbb{E}$ containing $\mathcal{X}$. Suppose $\mathbb{F} \in \mathcal{F}(\mathbb{E})$ containing $\mathcal{X}$ and $\alpha \in B$. Then there exist $p_1, \ldots, p_n \in \mathcal{X}$ such that $p_1 \twoheadrightarrow (p_2 \twoheadrightarrow (\ldots \twoheadrightarrow (p_n \to \alpha)\ldots)) = 1$. Since $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, $1 \in \mathbb{F}$ and $p_i \in \mathcal{X} \subseteq \mathbb{F}$, we get $\alpha \in \mathbb{F}$. Hence $B \subseteq \mathbb{F}$, and so B is the smallest filter of $\mathbb{E}$ containing $\mathcal{X}$. Therefore, $\langle \mathcal{X} \rangle = B$. □

Proposition 5. *If $\mathbb{F} \in \mathcal{F}(\mathbb{E})$ and $p \in \mathbb{E} \setminus \mathbb{F}$, then*

$$\langle \mathbb{F} \cup \{p\} \rangle = \{\alpha \in \mathbb{E} \mid \textit{for some } n \in \mathbb{N},\; p \to^n \alpha \in \mathbb{F}\}.$$

Proof. Let $C = \{\alpha \in \mathbb{E} \mid \text{ for some } n \in \mathbb{N},\; p \to^n \alpha \in \mathbb{F}\}$. Clearly, $1 \in C$. By Propositions 1(iii), for all $f \in \mathbb{F}$, we have $f \leq p \twoheadrightarrow f$ then $p \twoheadrightarrow f \in \mathbb{F}$ i.e. $f \in C$ and so $\mathbb{F} \subseteq C$. Also, since $p \twoheadrightarrow p = 1 \in \mathbb{F}$, we get $p \in C$. Hence $\mathbb{F} \cup \{p\} \subseteq C$.

Now, let $\alpha, \alpha \twoheadrightarrow \gamma \in C$. Then there exist $m,n \in \mathbb{N}$ such that $p \twoheadrightarrow^m \alpha = f \in \mathbb{F}$ and $p \twoheadrightarrow^n (\alpha \twoheadrightarrow \gamma) \in \mathbb{F}$. By m-times using of Proposition 1(viii), we have

$$\begin{aligned}
\alpha \twoheadrightarrow \gamma &\leq (p \twoheadrightarrow \alpha) \twoheadrightarrow (p \twoheadrightarrow \gamma) \\
&\vdots \\
&\leq \underbrace{[f \twoheadrightarrow (p \twoheadrightarrow^m \alpha)]}_{1} \twoheadrightarrow [f \twoheadrightarrow (p \twoheadrightarrow^m \gamma)] \\
&= 1 \twoheadrightarrow (f \twoheadrightarrow (p \twoheadrightarrow^m \gamma)) \quad \text{by Proposition 1(ii)}, \\
&= f \twoheadrightarrow (p \twoheadrightarrow^m \gamma).
\end{aligned}$$

Then $\alpha \twoheadrightarrow \gamma \leq f \twoheadrightarrow (p \twoheadrightarrow^m \gamma)$. Also, by Propositions 1(vi) and (v), we get

$$p \twoheadrightarrow (\alpha \twoheadrightarrow \gamma) \leq p \twoheadrightarrow (f \twoheadrightarrow (p \twoheadrightarrow^m \gamma)) = f \twoheadrightarrow (p \twoheadrightarrow^{m+1} \gamma).$$

By the similar way and n-times using the above argument, we conclude that

$$p \twoheadrightarrow^n (\alpha \twoheadrightarrow \gamma) \leq f \twoheadrightarrow (p \twoheadrightarrow^{m+n} \gamma).$$

Since $p \twoheadrightarrow^n (\alpha \twoheadrightarrow \gamma) \in \mathbb{F}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, then $f \twoheadrightarrow (p \twoheadrightarrow^{m+n} \gamma) \in \mathbb{F}$. Again, since $f \in \mathbb{F}$, we obtain $p \twoheadrightarrow^{m+n} \gamma \in \mathbb{F}$ and so $\gamma \in C$. Hence, $C \in \mathcal{F}(\mathbb{E})$ containing $\mathbb{F} \cup \{p\}$. Let $\mathbb{G}$ be an arbitrary filter of $\mathbb{E}$ such that containing $\mathbb{F} \cup \{p\}$. If $\alpha \in C$, then there exists $n \in \mathbb{N}$ such that $p \twoheadrightarrow^n \alpha \in \mathbb{F}$. Since $\mathbb{G} \in \mathcal{F}(\mathbb{E})$ and $\mathbb{F} \cup \{p\} \subseteq \mathbb{G}$, we have $\alpha \in \mathbb{G}$. Hence $C \subseteq \mathbb{G}$ and so C is the smallest filter of $\mathbb{E}$ containing $\mathbb{F} \cup \{p\}$. Therefore, $\langle \mathbb{F} \cup \{p\} \rangle = C$. □

Corollary 1. *If $\mathbb{F}$ and $\mathbb{G}$ are two filters of equality algebra $\mathbb{E}$, then*

$$\begin{aligned}
\langle \mathbb{F} \cup \mathbb{G} \rangle &= \{\alpha \in \mathbb{E} \mid \textit{ for some } g \in \mathbb{G},\ g \twoheadrightarrow \alpha \in \mathbb{F}\} \\
&= \{\alpha \in \mathbb{E} \mid \textit{ for some } f \in \mathbb{F},\ f \twoheadrightarrow \alpha \in \mathbb{G}\}.
\end{aligned}$$

Proof. Let $D = \{\alpha \in \mathbb{E} \mid \text{ for some } g \in \mathbb{G},\ g \twoheadrightarrow \alpha \in \mathbb{F}\}$. If $\alpha \in D$, then there exists $g \in \mathbb{G}$ such that $g \twoheadrightarrow \alpha \in \mathbb{F}$. Thus by Proposition 5, we get $\alpha \in \langle \mathbb{F} \cup \{g\} \rangle \subseteq \langle \mathbb{F} \cup \mathbb{G} \rangle$. Thus $D \subseteq \langle \mathbb{F} \cup \mathbb{G} \rangle$. It is enough to show that $\langle \mathbb{F} \cup \mathbb{G} \rangle \subseteq D$. Let $\alpha \in \langle \mathbb{F} \cup \mathbb{G} \rangle$. Then by Proposition 4, there exist $f_1, \ldots, f_m \in \mathbb{F}$ and $g_1, \ldots, g_n \in \mathbb{G}$ such that $f_1 \twoheadrightarrow (\ldots(f_m \twoheadrightarrow (g_1 \twoheadrightarrow (\ldots(g_n \twoheadrightarrow \alpha))))) = 1$. Since $1, f_i \in \mathbb{F}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, then $g_1 \twoheadrightarrow (\ldots(g_n \twoheadrightarrow \alpha)\ldots) \in \mathbb{F}$. Hence there exists $f \in \mathbb{F}$ such that $g_1 \twoheadrightarrow (\ldots(g_n \twoheadrightarrow \alpha)) = f$ and so by Proposition 1(v),

$$1 = f \twoheadrightarrow f = f \twoheadrightarrow (g_1 \twoheadrightarrow (\ldots(g_n \twoheadrightarrow \alpha))\ldots) = g_1 \twoheadrightarrow (\ldots(g_n \twoheadrightarrow (f \twoheadrightarrow \alpha))).$$

Since $1, g_j \in \mathbb{G}$ and $\mathbb{G} \in \mathcal{F}(\mathbb{E})$, we have $f \twoheadrightarrow \alpha \in \mathbb{G}$. Thus there exists $g \in \mathbb{G}$ such that $g = f \twoheadrightarrow \alpha$ and so by Proposition 1(ii) and (v),

$$1 = g \twoheadrightarrow g = g \twoheadrightarrow (f \twoheadrightarrow \alpha) = f \twoheadrightarrow (g \twoheadrightarrow \alpha).$$

In addition, from $1, f \in \mathbb{F}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, we get $g \twoheadrightarrow \alpha \in \mathbb{F}$ i.e. $\alpha \in D$. Therefore, $\langle \mathbb{F} \cup \mathbb{G} \rangle = D$. The proof of other case is similar. □

Proposition 6. *Let $\mathbb{F}$ and $\mathbb{G}$ be two proper filters of $\mathbb{E}$. Then for all $\mathcal{X},\mathcal{Y} \subseteq \mathbb{E}$ and $\alpha, p, q \in \mathbb{E}$, the following statements hold:*

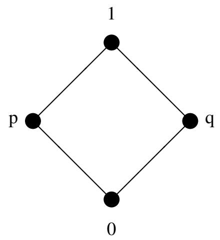
(i) *if $\mathcal{X} \subseteq \mathcal{Y}$, then $\langle \mathcal{X} \rangle \subseteq \langle \mathcal{Y} \rangle$,*
(ii) *if $\mathbb{F} \subseteq \mathbb{G}$, then $\langle \mathbb{F} \cup \{\alpha\} \rangle \subseteq \langle \mathbb{G} \cup \{\alpha\} \rangle$,*
(iii) *if $p \leq q$, then $\langle q \rangle \subseteq \langle p \rangle$,*
(iv) *if $\mathbb{F}$ is a positive implicative filter, then $\langle \mathbb{F} \cup \{p\} \rangle = \{\gamma \in \mathbb{E} \mid\ p \twoheadrightarrow \gamma \in \mathbb{F}\}$,*
(v) *$\langle p \rangle \cup \langle q \rangle \subseteq \langle p \wedge q \rangle$.*

Proof.

(i) Let $\alpha \in \langle \mathcal{X} \rangle$. Then there exist $p_1, \dots, p_n \in \mathcal{X}$ such that $p_1 \twoheadrightarrow (\dots(p_n \twoheadrightarrow \alpha)\dots) = 1$. Since $\mathcal{X} \subseteq \mathcal{Y}$, we get $p_i \in \mathcal{Y}$ for all $1 \leq i \leq n$ and so $\alpha \in \langle \mathcal{Y} \rangle$. Hence $\langle \mathcal{X} \rangle \subseteq \langle \mathcal{Y} \rangle$.
(ii) By (i) the proof is clear.
(iii) If $p \leq q$, then by Proposition 1(ii), $p \twoheadrightarrow q = 1$ and so $q \in \langle p \rangle$. Let $\alpha \in \langle q \rangle$. Then $q \twoheadrightarrow^n \alpha = 1$, for some $n \in \mathbb{N}$ i.e. $q \twoheadrightarrow (q \twoheadrightarrow \cdots (q \twoheadrightarrow \alpha)) = 1 \in \langle p \rangle$. Since $q, 1 \in \langle p \rangle$ and $\langle p \rangle \in \mathcal{F}(\mathbb{E})$, we have $\alpha \in \langle p \rangle$. Hence $\langle q \rangle \subseteq \langle p \rangle$.
(iv) Let $\mathbb{F}$ be a positive implicative filter of $\mathbb{E}$ and $\gamma \in \langle \mathbb{F} \cup \{p\} \rangle$. Then by Proposition 5, $p \twoheadrightarrow^n \gamma \in \mathbb{F}$ for some $n \in \mathbb{N}$ i. e. $p \twoheadrightarrow (p \twoheadrightarrow (\cdots (p \twoheadrightarrow \gamma))) \in \mathbb{F}$. Since $p \twoheadrightarrow p = 1 \in \mathbb{F}$ and $\mathbb{F}$ is a positive implicative filter of $\mathbb{E}$, we get $p \twoheadrightarrow^{n-1} \gamma \in \mathbb{F}$. If we use this argument $(n-1)-$times, then we conclude that $p \twoheadrightarrow \gamma \in \mathbb{F}$. Conversely, if $p \twoheadrightarrow \gamma \in \mathbb{F}$, then by Proposition 5, $\gamma \in \langle \mathbb{F} \cup \{p\} \rangle$.
(v) Since $p \wedge q \leq p, q$, by (iii) we get $\langle p \rangle \cup \langle q \rangle \subseteq \langle p \wedge q \rangle$.

□

Example 2. Let $E = \{0, p, q, 1\}$ be a poset with the following Hasse diagram. Define the operation $\sim$ on $\mathbb{E}$ as follows:



$\sim$	0	p	q	1
0	1	q	p	0
p	q	1	0	p
q	p	0	1	q
1	0	p	q	1

$\rightarrow$	0	p	q	1
0	1	1	1	1
p	q	1	q	1
q	p	p	1	1
1	0	p	q	1

Then $(\mathbb{E}, \wedge, \rightarrow, 1)$ is an equality algebra. Since

$$\langle p \wedge q \rangle = \langle 0 \rangle = \mathbb{E} \nsubseteq \langle p \rangle \cup \langle q \rangle = \{p, 1\} \cup \{q, 1\} = \{p, q, 1\},$$

so the converse of Proposition 6(v), is not true in general.

Theorem 1. *Let $\mathbb{E}$ be a lattice and $p, q, \alpha \in \mathbb{E}$. If $p \twoheadrightarrow^n \alpha = 1 = q \twoheadrightarrow^n \alpha$ for some $n \in \mathbb{N}$, then there exists $k \in \mathbb{N}$ such that $(p \vee q) \twoheadrightarrow^k \alpha = 1$.*

Proof. By using induction on n, we have, if $n = 1$, then $p \twoheadrightarrow \alpha = 1 = q \twoheadrightarrow \alpha$. Thus by Proposition 2(i), $(p \vee q) \twoheadrightarrow \alpha = (p \twoheadrightarrow \alpha) \wedge (q \twoheadrightarrow \alpha) = 1 \wedge 1 = 1$.

Suppose the statement hold for n. Let $p \twoheadrightarrow^{n+1} \alpha = 1 = q \twoheadrightarrow^{n+1} \alpha$. Then by Proposition 1(ii), $q \twoheadrightarrow^{n} (p \twoheadrightarrow^{n+1} \alpha) = 1 = p \twoheadrightarrow^{n} (q \twoheadrightarrow^{n+1} \alpha)$. Thus by Proposition 1(v), we have

$$p \twoheadrightarrow (q \twoheadrightarrow^{n} (p \twoheadrightarrow^{n} \alpha)) = q \twoheadrightarrow^{n} (p \twoheadrightarrow (p \twoheadrightarrow^{n} \alpha)) = q \twoheadrightarrow^{n} (p \twoheadrightarrow^{n+1} \alpha) = 1, \tag{3.1}$$

$$q \twoheadrightarrow (q \twoheadrightarrow^{n} (p \twoheadrightarrow^{n} \alpha)) = q \twoheadrightarrow^{n+1} (p \twoheadrightarrow^{n} \alpha) = p \twoheadrightarrow^{n} (q \twoheadrightarrow^{n+1} \alpha) = 1. \tag{3.2}$$

By Proposition 2(i), (3.1) and (3.2) we get

$$\begin{aligned}(p \vee q) \twoheadrightarrow [q \twoheadrightarrow^{n} (p \twoheadrightarrow^{n} \alpha)] &= [p \twoheadrightarrow (q \twoheadrightarrow^{n} (p \twoheadrightarrow^{n} \alpha))] \wedge [q \twoheadrightarrow (q \twoheadrightarrow^{n} (p \twoheadrightarrow^{n} \alpha))] \\ &= 1 \wedge 1 = 1. \end{aligned} \tag{3.3}$$

By (3.3) and Proposition 1(v), we have

$$q \twoheadrightarrow [(p \vee q) \twoheadrightarrow (q \twoheadrightarrow^{n-1} (p \twoheadrightarrow^{n} \alpha))] = 1. \tag{3.4}$$

Also, by Propositions 2(i), 1(v) and (ii), we have

$$\begin{aligned} & p \twoheadrightarrow [(p \vee q) \twoheadrightarrow (q \twoheadrightarrow^{n-1} (p \twoheadrightarrow^{n} \alpha))] \\ &= p \twoheadrightarrow [\, [p \twoheadrightarrow (q \twoheadrightarrow^{n-1} (p \twoheadrightarrow^{n} \alpha))] \\ &\quad \wedge [q \twoheadrightarrow (q \twoheadrightarrow^{n-1} (p \twoheadrightarrow^{n} \alpha))]\,] \\ &= p \twoheadrightarrow [\, [q \twoheadrightarrow^{n-1} \underbrace{(p \twoheadrightarrow^{n+1} \alpha)}_{1}] \wedge [q \twoheadrightarrow^{n} (p \twoheadrightarrow^{n} \alpha)]\,] \\ &= p \twoheadrightarrow [\, \underbrace{(q \twoheadrightarrow^{n-1} 1)}_{1} \wedge (q \twoheadrightarrow^{n} (p \twoheadrightarrow^{n} \alpha))\,] \\ &= p \twoheadrightarrow [q \twoheadrightarrow^{n} (p \twoheadrightarrow^{n} \alpha)] \\ &= q \twoheadrightarrow^{n} \underbrace{(p \twoheadrightarrow^{n+1} \alpha)}_{1} \\ &= 1. \end{aligned}$$

So we get

$$p \twoheadrightarrow [(p \vee q) \twoheadrightarrow (q \twoheadrightarrow^{n-1} (p \twoheadrightarrow^{n} \alpha))] = 1. \tag{3.5}$$

Now, by Proposition 2(i), (3.4) and (3.5), we get

$$(p \vee q) \twoheadrightarrow^{2} (q \twoheadrightarrow^{n-1} (p \twoheadrightarrow^{n} \alpha)) = 1. \tag{3.6}$$

So, if we repeat this argument (from (3.1) to (3.6)) $(n+1)-$times, then we conclude

$$(p \vee q) \twoheadrightarrow^{n+1} (p \twoheadrightarrow^{n} \alpha) = 1, \tag{3.7}$$

thus by Proposition 1(v), and (3.7)

$$p \twoheadrightarrow^{n} \underbrace{((p \vee q) \twoheadrightarrow^{n+1} \alpha)}_{\eta} = 1. \tag{3.8}$$

By the similar way,

$$(p \vee q) \twoheadrightarrow^{n+1} (q \twoheadrightarrow^{n} \alpha) = 1. \tag{3.9}$$

By Proposition 1(v), and (3.9), we have

$$q \twoheadrightarrow^{n} \underbrace{((p \vee q) \twoheadrightarrow^{n+1} \alpha)}_{\eta} = 1. \tag{3.10}$$

Hence, by induction hypothesis for (3.8) and (3.10), there exists $t \in \mathbb{N}$ such that $(p \vee q) \twoheadrightarrow^{t} \eta = 1$. Thus

$$1 = (p \vee q) \twoheadrightarrow^{t} \eta = (p \vee q) \twoheadrightarrow^{t} ((p \vee q) \twoheadrightarrow^{n+1} \alpha) = (p \vee q) \twoheadrightarrow^{t+n+1} \alpha.$$

□

Theorem 2. *Let $\mathbb{F} \in \mathcal{F}(\mathbb{E})$. If $p \vee q \in \mathbb{F}$, then*

$$\mathbb{F} = \langle \mathbb{F} \cup \{p\} \rangle \cap \langle \mathbb{F} \cup \{q\} \rangle,$$

for all $p, q \in \mathbb{E}$.

Proof. Let $p \vee q \in \mathbb{F}$. Obviously,$\mathbb{F} \subseteq \langle \mathbb{F} \cup \{p\} \rangle \cap \langle \mathbb{F} \cup \{q\} \rangle$. Conversely, if $\alpha \in \langle \mathbb{F} \cup \{p\} \rangle \cap \langle \mathbb{F} \cup \{q\} \rangle$, then by Proposition 5, there exist $m, n \in \mathbb{N}$ such that $p \twoheadrightarrow^{m} \alpha$, $q \twoheadrightarrow^{n} \alpha \in \mathbb{F}$. Let $l = \max\{m, n\}$. Then by Proposition 1(vi), $p \twoheadrightarrow^{m} \alpha \leq p \twoheadrightarrow^{l} \alpha$ and $q \twoheadrightarrow^{n} \alpha \leq q \twoheadrightarrow^{l} \alpha$. Since $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, we get $p \twoheadrightarrow^{l} \alpha, q \twoheadrightarrow^{l} \alpha \in \mathbb{F}$. Hence there exist $f_1, f_2 \in \mathbb{F}$ such that $f_1 \twoheadrightarrow (p \twoheadrightarrow^{l} \alpha) = 1 = f_2 \twoheadrightarrow (q \twoheadrightarrow^{l} \alpha)$ and so by Proposition 1(viii),

$$p \twoheadrightarrow^{l} (f_1 \twoheadrightarrow \alpha) = 1 \quad \text{and} \quad q \twoheadrightarrow^{l} (f_2 \twoheadrightarrow \alpha) = 1.$$

Also, by Proposition 1(iii),

$$f_2 \twoheadrightarrow (p \twoheadrightarrow^{l} (f_1 \twoheadrightarrow \alpha)) = 1, \quad \text{and} \quad f_1 \twoheadrightarrow (q \twoheadrightarrow^{l} (f_2 \twoheadrightarrow \alpha)) = 1. \tag{3.11}$$

Again, by Proposition 1(viii), and (3.11), we get

$$p \twoheadrightarrow^{l} \underbrace{[f_2 \twoheadrightarrow (f_1 \twoheadrightarrow \alpha)]}_{\eta} = 1 \quad \text{and} \quad q \twoheadrightarrow^{l} \underbrace{[f_2 \twoheadrightarrow (f_1 \twoheadrightarrow \alpha)]}_{\eta} = 1. \tag{3.12}$$

By Theorem 1, and (3.12) there exists $k \in \mathbb{N}$ such that $(p \vee q) \twoheadrightarrow^{k} [f_2 \twoheadrightarrow (f_1 \twoheadrightarrow \alpha)] = 1$. Since $p \vee q, f_1, f_2, 1 \in \mathbb{F}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, we get $\alpha \in \mathbb{F}$. Hence $\langle \mathbb{F} \cup \{p\} \rangle \cap \langle \mathbb{F} \cup \{q\} \rangle \subseteq F$. Therefore, $\mathbb{F} = \langle \mathbb{F} \cup \{p\} \rangle \cap \langle \mathbb{F} \cup \{q\} \rangle$, for all $p, q \in \mathbb{E}$. □

Corollary 2. *If $\mathbb{E}$ is a lattice, then $\langle p \vee q \rangle = \langle p \rangle \cap \langle q \rangle$.*

Proof. Since $p, q \leq p \vee q$, by Proposition 6(iii), $\langle p \vee q \rangle \subseteq \langle p \rangle \cap \langle q \rangle$. Conversely, from $p \in \langle \langle p \vee q \rangle \cup \{p\} \rangle$ we have $\langle p \rangle \subseteq \langle \langle p \vee q \rangle \cup \{p\} \rangle$ and similarly $\langle q \rangle \subseteq \langle \langle p \vee q \rangle \cup \{q\} \rangle$. In Theorem 2, take $\mathbb{F} := \langle p \vee q \rangle$, then

$$\langle p \rangle \cap \langle q \rangle \subseteq \langle \langle p \vee q \rangle \cup \{p\} \rangle \cap \langle \langle p \vee q \rangle \cup \{q\} \rangle = \langle p \vee q \rangle.$$

Therefore, $\langle p \vee q \rangle = \langle p \rangle \cap \langle q \rangle$. □

Definition 7. Let $\mathbb{E}$ be a lattice and $\mathbb{F}$ be a proper filter of $\mathbb{E}$. Then $\mathbb{F}$ is called a $\vee-$irreducible filter of $\mathbb{E}$, if $\alpha \vee \gamma \in \mathbb{F}$ implies $\alpha \in \mathbb{F}$ or $\gamma \in \mathbb{F}$, for all $\alpha, \gamma \in \mathbb{E}$.

Example 3. Let $(\mathbb{E}, \wedge, \sim, 1)$ be the equality algebra as in Example 2, $\mathbb{F} = \{p, 1\}$ and $\mathbb{G} = \{1\}$. Then $\mathbb{F}$ is a $\vee-$irreducible filter. Since $p \vee q = 1 \in \mathbb{G}$ but $p, q \notin \mathbb{G}$, we get $\mathbb{G}$ is not a $\vee-$irreducible filter of $\mathbb{E}$.

Theorem 3. *Let $\mathbb{F} \in \mathcal{F}(\mathbb{E})$. Then for each $p \notin \mathbb{F}$, there exists an $\vee$-irreducible filter P containing $\mathbb{F}$ such that $p \notin P$.*

Proof. Let $\Sigma = \{\mathbb{G} \in \mathcal{F}(\mathbb{E}) | \ \mathbb{F} \subseteq \mathbb{G}, \ p \notin \mathbb{G}\}$. Since $\mathbb{F} \in \Sigma$, we have $\Sigma \neq \varnothing$. Since any chain of elements in Σ has an upper bound in Σ, by Zorn' Lemma on $(\Sigma, \subseteq)$, there exists a maximal element $\mathbb{P} \in \Sigma$ such that $\mathbb{F} \subseteq \mathbb{P}$ and $p \notin \mathbb{P}$. Now, we prove that $\mathbb{P}$ is $\vee$-irreducible. Let $\alpha \vee \gamma \in \mathbb{P}$ and $\alpha, \gamma \notin \mathbb{P}$. Since $\mathbb{P}$ is maximal, we get $\langle \mathbb{P} \cup \{\alpha\} \rangle, \langle \mathbb{P} \cup \{\gamma\} \rangle \notin \Sigma$ and so $p \in \langle \mathbb{P} \cup \{\alpha\} \rangle \cap \langle \mathbb{P} \cup \{\gamma\} \rangle$. Moreover, by Theorem 2, we have $p \notin \mathbb{P} = \langle \mathbb{P} \cup \{\alpha\} \rangle \cap \langle \mathbb{P} \cup \{\gamma\} \rangle$ which is a contradiction. Therefore, $\mathbb{P}$ is a $\vee-$irreducible filter of $\mathbb{E}$. □

4. CO-ANNIHILATORS IN EQUALITY ALGEBRAS

In this section, we introduce the notion of co-annihilators on a lattice equality algebra $\mathbb{E}$ and investigate some related properties of them. Specially, by using the results of last section, we prove that the set of all filters of E is a bounded distributive pseudo-complemented lattice, which pseudo-complemented of any filter is co-annihilator of it. Finally, we will construct a Boolean algebra by the set of all co-annihilators of $\mathbb{E}$.

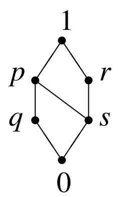
Notation 2. From now on, we let $\mathbb{E} = (\mathbb{E}, \sim, \wedge, 1)$ or $\mathbb{E}$ be a lattice equality algebra (there exists the operation $\vee$ on $\mathbb{E}$), unless otherwise state.

Definition 8. Let $\varnothing \neq \mathcal{X} \subseteq \mathbb{E}$. Then the set

$$\mathcal{X}^{\top} = \{\alpha \in \mathbb{E} \mid p \vee \alpha = 1, \text{ for all } p \in \mathcal{X}\},$$

is called a *co-annihilator* of $\mathcal{X}$. In particular, if $\mathcal{X} = \{\alpha\}$, then $\mathcal{X}^{\top}$ is denoted by $\alpha^{\top}$, for short.

Example 4. Let $(\mathbb{E} = \{0, p, q, r, s, 1\}, \leq)$ be a lattice with the following Hasse diagram. Define the operation "$\sim$" on $\mathbb{E}$ as follows:



$\sim$	0	p	q	r	s	1
0	1	s	r	q	p	0
p	s	1	p	s	r	p
q	r	p	1	0	s	q
r	q	s	0	1	p	r
s	p	r	s	p	1	s
1	0	p	q	r	s	1

$\twoheadrightarrow$	0	p	q	r	s	1
0	1	1	1	1	1	1
p	s	1	p	r	r	1
q	r	1	1	r	r	1
r	q	p	q	1	p	1
s	p	1	p	1	1	1
1	0	p	q	r	s	1

Then $(\mathbb{E},\sim,\wedge,1)$ is a lattice equality algebra. Let $\mathcal{X}=\{q,s\}$. Clearly, $\mathcal{X}^{\top}=\{1\}$ and $q^{\top}=\{r,1\}$.

Proposition 7. *If $\varnothing\neq\mathcal{X}\subseteq\mathbb{E}$, then $\mathcal{X}^{\top}\in\mathcal{F}(\mathbb{E})$.*

Proof. Consider $\varnothing\neq\mathcal{X}\subseteq\mathbb{E}$. Since for any $p\in\mathcal{X}$, $p\vee 1=1$, we have $1\in\mathcal{X}^{\top}$. Suppose $\alpha,\alpha\twoheadrightarrow\gamma\in\mathcal{X}^{\top}$. Then for all $p\in\mathcal{X}$, we have $\alpha\vee p=1$ and $(\alpha\twoheadrightarrow\gamma)\vee p=1$. If $\gamma\vee p:=t$, then $p\leq t$ and so $1=\alpha\vee p\leq t\vee\alpha$. Thus $\alpha\vee t=1$ and by Proposition 2(ii), we get

$$\alpha\twoheadrightarrow t=(\alpha\vee t)\twoheadrightarrow t=1\twoheadrightarrow t=t.$$

Moreover, since $\gamma\leq t$, by Proposition 1(vi), we have $\alpha\twoheadrightarrow\gamma\leq\alpha\twoheadrightarrow t=t$. Thus $1=(\alpha\twoheadrightarrow\gamma)\vee p\leq t\vee p$ and so $t\vee p=1$. On the other hand, $\gamma\vee p=(\gamma\vee p)\vee p=t\vee p=1$. Hence, $\gamma\in\mathcal{X}^{\top}$. Therefore, $\mathcal{X}^{\top}\in\mathcal{F}(\mathbb{E})$. □

Proposition 8. *Suppose $\mathcal{X}\subseteq\mathbb{E}$ and $p,q\in\mathbb{E}$. Then the following statements hold:*

(i) $\mathbb{E}^{\top}=\{1\}$ *and* $1^{\top}=\mathbb{E}$. *If $\mathbb{E}$ is bounded, then* $0^{\top}=\{1\}$,
(ii) *if $p\leq q$, then $p^{\top}\subseteq q^{\top}$,*
(iii) $(p\wedge q)^{\top}\subseteq p^{\top}\cap q^{\top}$,
(iv) $p^{\top}\cup q^{\top}\subseteq(p\vee q)^{\top}$,
(v) *if $\alpha\in p^{\top}$ and $\gamma\in q^{\top}$, then $\alpha\vee\gamma\in(p\vee q)^{\top}$ and $\alpha\vee\gamma\in(p^{\top}\cap q^{\top})$,*
(vi) *If $f:\mathbb{E}_1\to\mathbb{E}_2$ is a lattice-equality isomorphism, then $f(p^{\top})=\{f(p)\}^{\top}$, for any $p\in\mathbb{E}_1$.*

Proof.

(i) By definition of co-annihilator, we have

$$\mathbb{E}^{\top}=\{\alpha\in\mathbb{E}\mid\alpha\vee p=1,\ \text{for any } p\in\mathbb{E}\}=\{\alpha\in\mathbb{E}\mid\alpha\vee\alpha=1\}=\{1\}.$$

Also,

$$1^{\top}=\{\alpha\in\mathbb{E}\mid\alpha\vee 1=1\}=\mathbb{E}.$$

In addition, if $\mathbb{E}$ is bounded, then

$$0^{\top}=\{\alpha\in\mathbb{E}\mid\alpha\vee 0=1\}=\{\alpha\in\mathbb{E}\mid\alpha=1\}=\{1\}.$$

(ii) Let $p\leq q$ and $\alpha\in p^{\top}$. Then $1=\alpha\vee p\leq\alpha\vee q$. Thus $\alpha\vee q=1$ and so $\alpha\in q^{\top}$. Hence $p^{\top}\subseteq q^{\top}$.

(iii) Since $p\wedge q\leq p,q$, by (ii) we get $(p\wedge q)^{\top}\subseteq p^{\top},q^{\top}$. Hence $(p\wedge q)^{\top}\subseteq p^{\top}\cap q^{\top}$.

(iv) From $p,q\leq p\vee q$, by (ii) we have, $p^{\top},q^{\top}\subseteq(p\vee q)^{\top}$. Hence $p^{\top}\cup q^{\top}\subseteq(p\vee q)^{\top}$.

(v) Let $\alpha\in p^{\top}$ and $\gamma\in q^{\top}$. Then

$$(\alpha\vee\gamma)\vee(p\vee q)=(\alpha\vee p)\vee(\gamma\vee q)=1\vee 1=1.$$

Thus $(\alpha\vee\gamma)\in(p\vee q)^{\top}$. Also, since $\alpha,\gamma\leq\alpha\vee\gamma$ and $p^{\top},q^{\top}$ are filters of $\mathbb{E}$, we have $\alpha\vee\gamma\in(p^{\top}\cap q^{\top})$.

(vi) Let $\gamma \in f(p^{\top})$. Then there exists $\alpha \in p^{\top}$ such that $\gamma = f(\alpha)$. Since $\alpha \vee p = 1$, we have

$$\gamma \vee f(p) = f(\alpha) \vee f(p) = f(\alpha \vee p) = f(1) = 1.$$

Thus $\gamma \in (f(p))^{\top}$ and so $f(p^{\top}) \subseteq (f(p))^{\top}$. Conversely, let $\gamma \in f(p)^{\top}$. Then $\gamma \vee f(p) = 1$. Since f is onto, there exists $\alpha \in \mathbb{E}_1$ such that $f(\alpha) = \gamma$ and so

$$f(1) = 1 = \gamma \vee f(p) = f(\alpha) \vee f(p) = f(\alpha \vee p).$$

By f is one-one, we get $\alpha \vee p = 1$. Thus $\alpha \in p^{\top}$ and so $\gamma = f(\alpha) \in f(p^{\top})$. □

Proposition 9. *Let $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{E}$. Then the following statements hold:*

(i) $\mathcal{X}^{\top} = \bigcap_{p \in \mathcal{X}} p^{\top}$,
(ii) $\mathcal{X} \cap \mathcal{X}^{\top} = \{1\}$,
(iii) $\mathcal{X}^{\top} \cap \mathcal{X}^{\top\top} = \{1\}$,
(iv) *if* $\mathcal{X} \subseteq \mathcal{Y}$, *then* $\mathcal{Y}^{\top} \subseteq \mathcal{X}^{\top}$,
(v) $\mathcal{X}^{\top} \cap \mathcal{Y}^{\top} = (\mathcal{X} \cup \mathcal{Y})^{\top}$,
(vi) $\mathcal{X}^{\top} \cup \mathcal{Y}^{\top} \subseteq (\mathcal{X} \cap \mathcal{Y})^{\top}$,
(vii) $\mathcal{X} \subseteq \mathcal{X}^{\top\top}$,
(viii) $\mathcal{X}^{\top} = \mathcal{X}^{\top\top\top}$.

Proof.

(i) By definition, the proof is clear.

(ii) Let $\alpha \in \mathcal{X} \cap \mathcal{X}^{\top}$. Then $\alpha \vee p = 1$, for all $p \in \mathcal{X}$. Since $\alpha \in \mathcal{X}$, we get $\alpha \vee \alpha = 1$ and so $\alpha = 1$.

(iii) By (ii) the proof is straightforward.

(iv) Let $\mathcal{X} \subseteq \mathcal{Y}$ and $\alpha \in \mathcal{Y}^{\top}$. Then for any $p \in \mathcal{Y}$, $\alpha \vee p = 1$. Since $\mathcal{X} \subseteq \mathcal{Y}$, we get for any $p \in \mathcal{X}$, $\alpha \vee p = 1$ and so $\alpha \in \mathcal{X}^{\top}$.

(v) Since $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{X} \cup \mathcal{Y}$, by (iv) we get $(\mathcal{X} \cup \mathcal{Y})^{\top} \subseteq \mathcal{X}^{\top}, \mathcal{Y}^{\top}$ and so $(\mathcal{X} \cup \mathcal{Y})^{\top} \subseteq \mathcal{X}^{\top} \cap \mathcal{Y}^{\top}$. Conversely, let $\alpha \in \mathcal{X}^{\top} \cap \mathcal{Y}^{\top}$. Then for all $p \in \mathcal{X} \cup \mathcal{Y}$, we have $p \in \mathcal{X}$ or $p \in \mathcal{Y}$ and so $\alpha \vee p = 1$. Hence $\alpha \in (\mathcal{X} \cup \mathcal{Y})^{\top}$.

(vi) Since $\mathcal{X} \cap \mathcal{Y} \subseteq \mathcal{X}, \mathcal{Y}$, by (iv) we get $\mathcal{X}^{\top}, \mathcal{Y}^{\top} \subseteq (\mathcal{X} \cap \mathcal{Y})^{\top}$ and so $\mathcal{X}^{\top} \cup \mathcal{Y}^{\top} \subseteq (\mathcal{X} \cap \mathcal{Y})^{\top}$.

(vii) By definition of co-annihilator for $\mathcal{X}^{\top}$,

$$\mathcal{X}^{\top\top} = \{\alpha \in \mathbb{E} \mid \ \gamma \vee \alpha = 1, \text{ for all } \gamma \in \mathcal{X}^{\top}\}.$$

If $p \in \mathcal{X}$, then $p \vee \gamma = 1$, for all $\gamma \in \mathcal{X}^{\top}$. Thus $p \in \mathcal{X}^{\top\top}$ and so $\mathcal{X} \subseteq \mathcal{X}^{\top\top}$.

(viii) By (vii) $\mathcal{X} \subseteq \mathcal{X}^{\top\top}$ and so by (iv) we get $\mathcal{X}^{\top\top\top} \subseteq \mathcal{X}^{\top}$. Conversely, let $\mathcal{X}^{\top} = \mathcal{Y}$. Then by (vii), $\mathcal{Y} \subseteq \mathcal{Y}^{\top\top}$ and so $\mathcal{X}^{\top} \subseteq \mathcal{X}^{\top\top\top}$. Hence, $\mathcal{X}^{\top} = \mathcal{X}^{\top\top\top}$. □

In the following example we show that the converse relation of (vi) and (vii) of Proposition 9, may not be true, in general.

Example 5. Let $\mathbb{E}$ be lattice equality algebra as in Example 1. If $\mathcal{X} = \{p,q\}$ and $\mathcal{Y} = \{q,r\}$, then $\mathcal{X}^\top = \{1\} = \mathcal{Y}^\top$, but

$$(\mathcal{X} \cap \mathcal{Y})^\top = \{q\}^\top = \{p,1\} \not\subseteq \{1\} = \mathcal{X}^\top \cup \mathcal{Y}^\top.$$

Also, since $\mathcal{X}^{\top\top} = \{1\}^\top = \mathbb{E}$, we get $\mathcal{X}^{\top\top} \not\subseteq \mathcal{X}$.

Proposition 10. *If $\varnothing \neq \mathcal{X} \subseteq \mathbb{E}$, then*

$$\mathcal{X}^\top \subseteq \{\alpha \in \mathbb{E} \mid \alpha \to p = p, \text{ for all } p \in \mathcal{X}\} \cap \{\alpha \in \mathbb{E} \mid p \to \alpha = \alpha, \text{ for all } p \in \mathcal{X}\}.$$

Proof. Let $\mathcal{Y} := \{\alpha \in \mathbb{E} \mid \alpha \to p = p, \text{ for all } p \in \mathcal{X}\}$ and $\mathcal{Z} := \{\alpha \in \mathbb{E} \mid p \to \alpha = \alpha, \text{ for all } p \in \mathcal{X}\}$. If $\alpha \in \mathcal{X}^\top$, then $\alpha \vee p = 1$, for all $p \in \mathcal{X}$. Thus by Propositions 2(ii) and 1(iii), we have

$$\alpha \twoheadrightarrow p = (\alpha \vee p) \twoheadrightarrow p = 1 \twoheadrightarrow p = p.$$

Hence $\alpha \twoheadrightarrow p = p$ and so $\alpha \in \mathcal{Y}$. Moreover, by Propositions 1(iii) and 2(i), we get,

$$\alpha = 1 \twoheadrightarrow \alpha = (\alpha \vee p) \twoheadrightarrow \alpha = (\alpha \twoheadrightarrow \alpha) \wedge (p \twoheadrightarrow \alpha) = 1 \wedge (p \twoheadrightarrow \alpha) = p \twoheadrightarrow \alpha.$$

Thus $p \twoheadrightarrow \alpha = \alpha$ and so $\alpha \in \mathcal{Z}$. Therefore, $\mathcal{X}^\top \subseteq \mathcal{Y} \cap \mathcal{Z}$. □

In the following example we show that the converse of Proposition 10 does not hold, in general.

Example 6. Let $\mathbb{E}$ be equality algebra as in Example 4, and $\mathcal{X} = \{r\}$. Then $\mathcal{X}^\top = \{p,1\}$ and

$$\begin{aligned}\{\alpha \in \mathbb{E} \mid \alpha \to p = p, \text{ for all } p \in \mathcal{X}\} \cap \{\alpha \in \mathbb{E} \mid p \to \alpha = \alpha, \text{ for all } p \in \mathcal{X}\}\\ = \{p,q,1\} \cap \{p,q,1\}.\end{aligned}$$

So $\{p,q,1\} \not\subseteq \{p,1\} = \mathcal{X}^\top$.

Proposition 11. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two non-empty subsets of $\mathbb{E}$ and $\mathbb{F}, \mathbb{G} \in \mathcal{F}(\mathbb{E})$. Then*

(i) $\langle \mathcal{X} \rangle \cap \mathcal{X}^\top = \{1\}$,
(ii) $\mathbb{F} \cap \mathbb{G} = \{1\}$ *if and only if* $\mathbb{F} \subseteq \mathbb{G}^\top$ *and* $\mathbb{G} \subseteq \mathbb{F}^\top$,
(iii) $\mathcal{X}^\top \cap \mathcal{Y}^\top = \{1\}$ *if and only if* $\mathcal{X}^\top \subseteq \mathcal{Y}^{\top\top}$ *and* $\mathcal{Y}^\top \subseteq \mathcal{X}^{\top\top}$.

Proof.

(i) Let $\alpha \in \langle \mathcal{X} \rangle \cap \mathcal{X}^\top$. Since $\alpha \in \langle \mathcal{X} \rangle$, there exist $p_1, \dots, p_n \in \mathcal{X}$ such that $p_1 \twoheadrightarrow (p_2 \twoheadrightarrow (\dots (p_n \twoheadrightarrow \alpha) \dots)) = 1$. Also, since $\alpha \in \mathcal{X}^\top$, by Proposition 10, for any $1 \leq i \leq n$, $p_i \twoheadrightarrow \alpha = \alpha$. So

$$\begin{aligned}1 = p_1 \twoheadrightarrow (p_2 \twoheadrightarrow (\dots \twoheadrightarrow \underbrace{(p_n \twoheadrightarrow \alpha)}_{\alpha} \dots)) &= p_1 \twoheadrightarrow (p_2 \twoheadrightarrow (\dots \underbrace{(p_{n-1} \twoheadrightarrow \alpha)}_{\alpha} \dots))\\ &= \quad \dots \quad = p_1 \twoheadrightarrow \alpha = \alpha.\end{aligned}$$

Thus $\alpha = 1$ and so $\langle \mathcal{X} \rangle \cap \mathcal{X}^\top = \{1\}$.

(ii) Let $\mathbb{F}\cap\mathbb{G}=\{1\}$. Then for any $f\in\mathbb{F}$ and $g\in\mathbb{G}$, $f,g\leq f\vee g$. Since $\mathbb{F},\mathbb{G}$ are filters of $\mathbb{E}$, we have $f\vee g\in\mathbb{F}\cap\mathbb{G}=\{1\}$. Thus $f\in\mathbb{G}^{\top}$ and $g\in\mathbb{F}^{\top}$. Hence $\mathbb{F}\subseteq\mathbb{G}^{\top}$ and $\mathbb{G}\subseteq\mathbb{F}^{\top}$. Conversely, if $\mathbb{F}\subseteq\mathbb{G}^{\top}$, then $\mathbb{F}\cap\mathbb{G}\subseteq\mathbb{G}^{\top}\cap\mathbb{G}=\{1\}$.

(iii) By Proposition 7, $\mathcal{X}^{\top},\mathcal{Y}^{\top}\in\mathcal{F}(\mathbb{E})$ then by (ii), the proof is clear.

□

Proposition 12. *Let $\varnothing\neq\mathbb{F}\subseteq\mathbb{E}$. Then there exists $\varnothing\neq\mathcal{X}\subseteq\mathbb{E}$ such that $\mathbb{F}=\mathcal{X}^{\top}$ if and only if $\mathbb{F}$ is upset, $\mathbb{F}^{\top\top}=\mathbb{F}$, $\mathbb{F}\cap\mathcal{X}^{\top\top}=\{1\}$ and $\mathcal{X}^{\top}\cap\mathbb{F}^{\top}=\{1\}$.*

Proof. If there exists $\varnothing\neq\mathcal{X}\subseteq\mathbb{E}$ such that $\mathbb{F}=\mathcal{X}^{\top}$, then by Proposition 7, $\mathbb{F}=\mathcal{X}^{\top}\in\mathcal{F}(\mathbb{E})$, then $\mathbb{F}$ is upset. Also, by Proposition 9(viii), $\mathbb{F}^{\top\top}=\mathcal{X}^{\top\top\top}=\mathcal{X}^{\top}=\mathbb{F}$. Moreover, by Proposition 9(iii) and (vi), we have $\mathbb{F}\cap\mathcal{X}^{\top\top}=\mathcal{X}^{\top}\cap\mathcal{X}^{\top\top}=\{1\}$ and finally, $\mathcal{X}^{\top}\cap\mathbb{F}^{\top}=\mathcal{X}^{\top}\cap\mathcal{X}^{\top\top}=\{1\}$.

Conversely, if $\alpha\in\mathbb{F}$, then for all $p\in\mathcal{X}$, $\alpha,p\leq\alpha\vee p$. Since $\mathbb{F}$ and $\mathcal{X}^{\top\top}$ are upset, we have $p\vee\alpha\in\mathbb{F}\cap\mathcal{X}^{\top\top}=\{1\}$. Thus $p\vee\alpha=1$ and so $\alpha\in\mathcal{X}^{\top}$. Hence $\mathbb{F}\subseteq\mathcal{X}^{\top}$. On the other hand, since $\mathbb{F}^{\top}\cap\mathcal{X}^{\top}=\{1\}$, by Proposition 11(ii), we have $\mathcal{X}^{\top}\subseteq\mathbb{F}^{\top\top}=\mathbb{F}$. Therefore, $\mathcal{X}^{\top}=\mathbb{F}$. □

Theorem 4. *Let $\mathcal{X}\subseteq\mathbb{E}$. Then $\langle\mathcal{X}\rangle^{\top}=\mathcal{X}^{\top}$.*

Proof. Since $\mathcal{X}\subseteq\langle\mathcal{X}\rangle$, by Proposition 9(iv), we have $\langle\mathcal{X}\rangle^{\top}\subseteq\mathcal{X}^{\top}$. Conversely, we show that $\mathcal{X}^{\top}\subseteq\langle\mathcal{X}\rangle^{\top}$: Let $\alpha\in\mathcal{X}^{\top}$ and $\gamma\in\langle\mathcal{X}\rangle$. Then $\alpha\vee p=1$, for all $p\in\mathcal{X}$ and there exist $p_1,\ldots,p_n\in\mathcal{X}$ such that $p_1\twoheadrightarrow(p_2\twoheadrightarrow(\ldots(p_n\twoheadrightarrow\gamma)\ldots))=1$. If $\alpha\vee\gamma\neq1$, then since $\mathbb{F}=\{1\}$ and by Theorem 3, we get there exists an $\vee$-irreducible filter of $\mathbb{E}$ as $\mathbb{P}\in\mathcal{F}(\mathbb{E})$ such that $\gamma\vee\alpha\notin\mathbb{P}$. Moreover, since $\alpha\vee p_i=1\in\mathbb{P}$, for all $1\leq i\leq n$, and definition of $\vee$-irreducible filter, we get $\alpha\in\mathbb{P}$ or $p_i\in\mathbb{P}$. If $\alpha\in\mathbb{P}$, then from $\alpha\leq\alpha\vee\gamma$ and $\mathbb{P}\in\mathcal{F}(\mathbb{E})$, we get $\alpha\vee\gamma\in\mathbb{P}$, which is a contradiction. Thus $p_i\in\mathbb{P}$ for all $1\leq i\leq n$ and so $p_1\twoheadrightarrow(p_2\twoheadrightarrow(\ldots(p_n\twoheadrightarrow\gamma)\ldots))=1\in\mathbb{P}$ implies $\gamma\in\mathbb{P}$, which is a contradiction. Therefore, $\alpha\vee\gamma=1$ and $\alpha\in\langle\mathcal{X}\rangle^{\top}$. □

Definition 9 ([16, Proposition3.17])**.** In a lattice L with bottom element 0, an element $\alpha\in L$ is said to have a pseudo-complement element if there exists the greatest element $\alpha^*\in L$, disjoint from α, with the property that $\alpha\wedge\alpha^*=0$. More formally, $\alpha^*=\max\{\gamma\in L\mid\alpha\wedge\gamma=0\}$. The lattice L itself is called a *pseudo-complemented lattice* if every element of L has a pseudo-complement element.

Theorem 5. *The algebraic structure $(\mathcal{F}(\mathbb{E}),\subseteq,\wedge,\vee,\{1\},\mathbb{E})$ is a bounded distributive pseudo complemented lattice, where for any $\mathbb{F},\mathbb{G}\in\mathcal{F}(\mathbb{E})$, $\mathbb{F}\wedge\mathbb{G}=\mathbb{F}\cap\mathbb{G}$, $\mathbb{F}\vee\mathbb{G}=\langle\mathbb{F}\cup\mathbb{G}\rangle$, and $\mathbb{F}^{\top}$ is pseudo-complement of $\mathbb{F}$.*

Proof. Clearly, $\mathcal{F}(\mathbb{E})$ is closed under $\wedge$ and $\vee$. So, $(\mathcal{F}(\mathbb{E}),\subseteq,\wedge,\vee,\{1\},\mathbb{E})$ is a bounded lattice. Now, we prove distribution i.e., for any $\mathbb{F},\mathbb{G},\mathbb{H}\in\mathcal{F}(\mathbb{E})$,

$$\mathbb{F}\wedge(\mathbb{G}\vee\mathbb{H})=(\mathbb{F}\wedge\mathbb{G})\vee(\mathbb{F}\wedge\mathbb{H}).$$

Since $\mathbb{G},\mathbb{H} \subseteq \mathbb{G}\vee\mathbb{H}$, we have $\mathbb{F}\cap\mathbb{G},\mathbb{F}\cap\mathbb{H} \subseteq \mathbb{F}\cap(\mathbb{G}\vee\mathbb{H})$ and so

$$(\mathbb{F}\cap\mathbb{G})\vee(\mathbb{F}\cap\mathbb{H}) = (\mathbb{F}\wedge\mathbb{G})\vee(\mathbb{F}\wedge\mathbb{H}) \subseteq \mathbb{F}\cap(\mathbb{G}\vee\mathbb{H}) = \mathbb{F}\wedge(\mathbb{G}\vee\mathbb{H}). \tag{4.1}$$

Conversely, if $\alpha\in\mathbb{F}\wedge(\mathbb{G}\vee\mathbb{H})$, then $\alpha\in\mathbb{F}$ and $\alpha\in\mathbb{G}\vee\mathbb{H} = \langle\mathbb{G}\cup\mathbb{H}\rangle$. By Corollary 1, there exists $g\in\mathbb{G}$ such that $g\twoheadrightarrow\alpha\in\mathbb{H}$. Thus there exists $h\in\mathbb{H}$ such that $g\twoheadrightarrow\alpha = h$. Since $\alpha\le g\twoheadrightarrow\alpha$ and $\mathbb{F}$ is upset, we get $h = g\twoheadrightarrow\alpha\in\mathbb{F}\cap\mathbb{H}$. Moreover, by Proposition 1(iv), $g\le(g\twoheadrightarrow\alpha)\twoheadrightarrow\alpha = h\twoheadrightarrow\alpha$ and from $\mathbb{G}$ is upset, we have $h\twoheadrightarrow\alpha\in\mathbb{G}$. Also, by Proposition 1(iii), $\alpha\le h\twoheadrightarrow\alpha$, and $\mathbb{F}$ is upset, we get $h\twoheadrightarrow\alpha\in\mathbb{F}\cap\mathbb{G}$. Hence $\alpha\in\langle(\mathbb{F}\cap\mathbb{H})\cup(\mathbb{F}\cap\mathbb{G})\rangle$. Thus

$$\mathbb{F}\wedge(\mathbb{G}\vee\mathbb{H}) \subseteq \langle(\mathbb{F}\cap\mathbb{H})\cup(\mathbb{F}\cap\mathbb{G})\rangle = (\mathbb{F}\wedge\mathbb{H})\vee(\mathbb{F}\wedge\mathbb{G}). \tag{4.2}$$

By (4.1) and (4.2), we get

$$\mathbb{F}\wedge(\mathbb{G}\vee\mathbb{H}) = (\mathbb{F}\wedge\mathbb{H})\vee(\mathbb{F}\wedge\mathbb{G}).$$

Hence $(\mathcal{F}(\mathbb{E}),\subseteq,\wedge,\vee,\{1\},\mathbb{E})$ is a bounded distributive lattice.

Now, by Propositions 9(ii), $\mathbb{F}\cap\mathbb{F}^{\top} = \{1\}$, for any $\mathbb{F}\in\mathcal{F}(\mathcal{X})$. If $\mathbb{G}\in\mathcal{F}(\mathbb{E})$ such that $\mathbb{G}\cap\mathbb{F} = \{1\}$, then by Proposition 11(ii), $\mathbb{G}\subseteq\mathbb{F}^{\top}$. Hence $\mathbb{F}^{\top}$ is the largest filter of $\mathbb{E}$ such that $\mathbb{G}\cap\mathbb{F} = \{1\}$. Then $\mathbb{F}^{\top}$ is a pseudo-complement of $\mathbb{F}$. Therefore, $(\mathcal{F}(\mathbb{E}),\subseteq,\wedge,\vee,\{1\},\mathbb{E})$ is a bounded distributive pseudo-complemented lattice. $\square$

Notation 3. Let $Co_{ann}(\mathbb{E}) = \{\mathcal{X}^{\top} \mid \varnothing\neq\mathcal{X}\subseteq\mathbb{E}\}$ be the set of all co-annihilators of $\mathbb{E}$. Since $\mathcal{X}^{\top} = \langle\mathcal{X}\rangle^{\top}$, by Proposition 4, then $Co_{ann}(\mathbb{E}) = \{\mathbb{F}^{\top} \mid \mathbb{F}\in\mathcal{F}(\mathbb{E})\}$. Thus $Co_{ann}(\mathbb{E})$ is the set of all pseudo-complements of the pseudo-complement lattice $\mathcal{F}(\mathbb{E})$.

Proposition 13. *For any $\varnothing\neq\mathcal{X}\subseteq\mathbb{E}$ and $\mathbb{F},\mathbb{G}\in\mathcal{F}(\mathbb{E})$, the following statements hold:*

(i) $\{1\},\mathbb{E}\in Co_{ann}(\mathbb{E})$,
(ii) $\mathbb{F}\in Co_{ann}(\mathbb{E})$ *if and only if* $\mathbb{F}^{\top\top} = \mathbb{F}$,
(iii) *a map* $\top\top : \mathcal{F}(\mathbb{E}) \twoheadrightarrow \mathcal{F}(\mathbb{E})$ *defined by* $\mathbb{F}\to\mathbb{F}^{\top\top}$ *is a closure map,*
(iv) $(\mathbb{F}\cap\mathbb{G})^{\top\top} = \mathbb{F}^{\top\top}\cap\mathbb{G}^{\top\top}$,
(v) $(\mathbb{F}\vee\mathbb{G})^{\top} = \mathbb{F}^{\top}\cap\mathbb{G}^{\top}$,
(vi) *if* $\mathbb{F},\mathbb{G}\in Co_{ann}(\mathbb{E})$, *then* $\mathbb{F}\bigvee_{Co_{ann}(\mathbb{E})}\mathbb{G} = (\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top}$.

Proof.

(i) By Proposition 8(i), $\{1\}^{\top} = \mathbb{E}$ and $\mathbb{E}^{\top} = \{1\}$. Thus $\{1\},\mathbb{E}\in Co_{ann}(\mathbb{E})$.

(ii) Let $\mathbb{F}\in Co_{ann}(\mathbb{E})$. Then there exists $\varnothing\neq\mathcal{X}\subseteq\mathbb{E}$ such that $\mathbb{F} = \mathcal{X}^{\top}$. Thus, by Proposition 9(viii),

$$\mathbb{F}^{\top\top} = \mathcal{X}^{\top\top\top} = \mathcal{X}^{\top} = \mathbb{F}$$

Conversely, if $\mathbb{F}^{\top\top} = \mathbb{F}$, then $\mathbb{F} = (\mathbb{F}^{\top})^{\top}\in Co_{ann}(\mathbb{E})$.

(iii) By applying Definition 5, Proposition 9(vii), (viii) and (iv) the proof is clear.

(iv) Since $\mathbb{F}\cap\mathbb{G}\subseteq\mathbb{F},\mathbb{G}$, by two times using of Proposition 9(iv), $(\mathbb{F}\cap\mathbb{G})^{\top\top}\subseteq\mathbb{F}^{\top\top}\cap\mathbb{G}^{\top\top}$. On the other hand, by Proposition 9(ii), $(\mathbb{F}\cap\mathbb{G})\cap(\mathbb{F}\cap\mathbb{G})^{\top}=\{1\}$. Then by Propositions 11(ii) and 9(viii), we get

$$\mathbb{F}\cap(\mathbb{F}\cap\mathbb{G})^{\top}\subseteq\mathbb{G}^{\top}=\mathbb{G}^{\top\top\top}$$

So $\mathbb{F}\cap\mathbb{G}^{\top\top}\cap(\mathbb{F}\cap\mathbb{G})^{\top}=\{1\}$. By the similar way, we can see that $\mathbb{G}^{\top\top}\cap(\mathbb{F}\cap\mathbb{G})^{\top}\subseteq\mathbb{F}^{\top}=\mathbb{F}^{\top\top\top}$, and so $\mathbb{F}^{\top\top}\cap\mathbb{G}^{\top\top}\cap(\mathbb{F}\cap\mathbb{G})^{\top}=\{1\}$. Thus, $\mathbb{F}^{\top\top}\cap\mathbb{G}^{\top\top}\subseteq(\mathbb{F}\cap\mathbb{G})^{\top\top}$. Therefore,

$$(\mathbb{F}\cap\mathbb{G})^{\top\top}=\mathbb{F}^{\top\top}\cap\mathbb{G}^{\top\top}.$$

(v) Since $\mathbb{F},\mathbb{G}\subseteq\mathbb{F}\vee\mathbb{G}$, we ahve

$$\begin{aligned}(\mathbb{F}\vee\mathbb{G})^{\top}&\subseteq\mathbb{F}^{\top}\cap\mathbb{G}^{\top} && \text{by Proposition 9(iv),}\\ &=\mathbb{F}^{\top\top\top}\cap\mathbb{G}^{\top\top\top} && \text{by Proposition 9(viii),}\\ &=(\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top\top} && \text{by (iv).}\end{aligned}$$

So $(\mathbb{F}\vee\mathbb{G})^{\top}\subseteq(\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top\top}$. Moreover, $\mathbb{F}^{\top}\cap\mathbb{G}^{\top}\subseteq\mathbb{F}^{\top}$ and by Proposition 9(vii), (iv) $\mathbb{F}\subseteq\mathbb{F}^{\top\top}\subseteq(\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top}$. Similarly, $\mathbb{G}\subseteq(\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top}$, and by definition of "$\vee$", we get $\mathbb{F}\vee\mathbb{G}\subseteq(\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top}$. Thus

$$(\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top\top}\subseteq(\mathbb{F}\vee\mathbb{G})^{\top}\subseteq(\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top\top},$$

and so, $(\mathbb{F}\vee\mathbb{G})^{\top}=(\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top\top}$. Hence, by (iv)

$$(\mathbb{F}\vee\mathbb{G})^{\top}=(\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top\top}=\mathbb{F}^{\top\top\top}\cap\mathbb{G}^{\top\top\top}=\mathbb{F}^{\top}\cap\mathbb{G}^{\top}.$$

(vi) By (ii) and the proof of (v), $\mathbb{F}\bigvee_{Co_{ann}(\mathbb{E})}\mathbb{G}=(\mathbb{F}\vee\mathbb{G})^{\top\top}=(\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top}$.

□

Theorem 6. *The structure* $(Co_{ann}(\mathbb{E}),\wedge,\vee,\top,\{1\},\mathbb{E})$ *is a Boolean algebra.*

Proof. By Proposition 13(i), $Co_{ann}(\mathbb{E})\neq\varnothing$ and it is bounded. By Proposition 13(iv), (v) and (vi), for any $\mathbb{F},\mathbb{G}\in Co_{ann}(\mathbb{E})$,

$$\mathbb{F}\wedge_{Co_{ann}(\mathbb{E})}\mathbb{G}=\mathbb{F}\cap\mathbb{G},\quad \mathbb{F}\vee_{Co_{ann}(\mathbb{E})}\mathbb{G}=(\mathbb{F}^{\top}\cap\mathbb{G}^{\top})^{\top}.$$

Thus, $(Co_{ann}(\mathbb{E}),\wedge,\vee,\{1\},\mathbb{E})$ is a bounded lattice. For distribution, let $\mathbb{F},\mathbb{G},\mathbb{H}\in Co_{ann}(\mathbb{E})$. Then by Proposition 13(ii), we have

$$\mathbb{F}\wedge\mathbb{G}\subseteq(\mathbb{F}\wedge\mathbb{G})\vee(\mathbb{F}\wedge\mathbb{H})=[\underbrace{(\mathbb{F}\cap\mathbb{G})\vee(\mathbb{F}\cap\mathbb{H})}_{\mathbb{K}}]^{\top\top}=\mathbb{K}^{\top\top}.\tag{4.3}$$

By (4.3) and Proposition 11(ii), we get $(\mathbb{F}\cap\mathbb{G})\cap\mathbb{K}^{\top}=\{1\}$. By using one more time of Proposition 11(ii), $\mathbb{F}\cap\mathbb{K}^{\top}\subseteq\mathbb{G}^{\top}$. Similarly, we get $\mathbb{F}\cap\mathbb{K}^{\top}\subseteq\mathbb{H}^{\top}$ and so by Proposition 13(ii),

$$\mathbb{F}\cap\mathbb{K}^{\top}\subseteq\mathbb{G}^{\top}\cap\mathbb{H}^{\top}=(\mathbb{G}^{\top}\cap\mathbb{H}^{\top})^{\top\top}.\tag{4.4}$$

Then by (4.4) and Proposition 11(ii), we get $(\mathbb{F}\cap\mathbb{K}^{\top})\cap(\mathbb{G}^{\top}\cap\mathbb{H}^{\top})=\{1\}$ which implies that

$$\mathbb{F}\cap(\mathbb{G}^{\top}\cap\mathbb{H}^{\top})^{\top}\subseteq\mathbb{K}^{\top\top}=\mathbb{K}. \tag{4.5}$$

Hence, by (4.5) and Proposition 13(vi)

$$\mathbb{F}\wedge(\mathbb{G}\vee\mathbb{H})=\mathbb{F}\cap(\mathbb{G}^{\top}\cap\mathbb{H}^{\top})^{\top}\subseteq\mathbb{K}=(\mathbb{F}\wedge\mathbb{G})\vee(\mathbb{F}\wedge\mathbb{H}). \tag{4.6}$$

On the other hand, since $\mathbb{F}\cap\mathbb{G}\subseteq\mathbb{G}$ and $\mathbb{F}\cap\mathbb{H}\subseteq\mathbb{H}$, by Proposition 9(iv), we have, $\mathbb{G}^{\top}\subseteq(\mathbb{F}\cap\mathbb{G})^{\top}$ and $\mathbb{H}^{\top}\subseteq(\mathbb{F}\cap\mathbb{H})^{\top}$. Thus

$$\mathbb{G}^{\top}\cap\mathbb{H}^{\top}\subseteq(\mathbb{F}\cap\mathbb{G})^{\top}\cap(\mathbb{F}\cap\mathbb{H})^{\top}$$

and so by Proposition 9(iv)

$$[(\mathbb{F}\cap\mathbb{G})^{\top}\cap(\mathbb{F}\cap\mathbb{H})^{\top}]^{\top}\subseteq(\mathbb{G}^{\top}\cap\mathbb{H}^{\top})^{\top}. \tag{4.7}$$

Also, since $\mathbb{F}\cap\mathbb{G},\mathbb{F}\cap\mathbb{H}\subseteq\mathbb{F}$, by two times using of Propositions 9(iv) and 13(ii) we get

$$[(\mathbb{F}\cap\mathbb{G})^{\top}\cap(\mathbb{F}\cap\mathbb{H})^{\top}]^{\top}\subseteq\mathbb{F}^{\top\top}=\mathbb{F}. \tag{4.8}$$

So by (4.7) and (4.8),

$$[(\mathbb{F}\cap\mathbb{G})^{\top}\cap(\mathbb{F}\cap\mathbb{H})^{\top}]^{\top}\subseteq\mathbb{F}\cap(\mathbb{G}^{\top}\cap\mathbb{H}^{\top})^{\top}.$$

Hence, by Proposition 13(iv),

$$(\mathbb{F}\wedge\mathbb{G})\vee(\mathbb{F}\wedge\mathbb{H})=[(\mathbb{F}\cap\mathbb{G})^{\top}\cap(\mathbb{F}\cap\mathbb{H})^{\top}]^{\top}\subseteq\mathbb{F}\cap(\mathbb{G}^{\top}\cap\mathbb{H}^{\top})^{\top}=\mathbb{F}\wedge(\mathbb{G}\vee\mathbb{H}). \tag{4.9}$$

Now, by (4.6) and (4.9), we get

$$\mathbb{F}\wedge(\mathbb{G}\vee\mathbb{H})=(\mathbb{F}\wedge\mathbb{G})\vee(\mathbb{F}\wedge\mathbb{H}).$$

Thus $Co_{ann}(\mathbb{E})$ is a distributive lattice. Since for all $\mathbb{F}\in Co_{ann}(\mathbb{E})$,

$$\mathbb{F}\cap\mathbb{F}^{\top}=\{1\}\ \text{ and }\ \mathbb{F}\vee\mathbb{F}^{\top}=(\mathbb{F}^{\top}\cap\mathbb{F}^{\top\top})^{\top}=(\mathbb{F}^{\top}\cap\mathbb{F})^{\top}=\{1\}^{\top}=\mathbb{E},$$

we get $\mathbb{F}^{\top}$ is the complement of $\mathbb{F}$. Therefore, $(Co_{ann}(\mathbb{E}),\wedge,\vee,\top,\{1\},\mathbb{E})$ is a Boolean algebra. □

5. Conclusions and future works

In this paper we introduced the notion of co-annihilator of a subset X in an equality algebra $\mathbb{E}$. We investigated many important properties of the co-annihilators. Finally, we define the $Co_{ann}(\mathbb{E})$ to be the set of all co-annihilators of $\mathbb{E}$, then we show that it can be made as a Boolean algebra.

In our future work, we will continue our study of algebraic properties of this special sets and define the notion of minimal prime filter and ideals on equality algebras, and we investigate the relation between these new notions with co-annihilators in equality algebras.

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