On convergence properties of infinitesimal generators of scaled multi-type CBI processes

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Abstract

It is a common method for proving weak convergence of a sequence of timehomogeneous Markov processes towards a time-homogeneous Markov process first to show convergence of the corresponding infinitesimal generators and then to check some additional conditions. The aim of the present paper is to investigate convergence properties of discrete infinitesimal generators of appropriately scaled random step functions formed from a multi-type continuous state and continuous time branching process with immigration. We also present a convergence result for usual infinitesimal generators of the branching processes in question appropriately normalized.

1 Introduction

Studying weak convergence of Markov processes has a long tradition and history. It is a common method for proving weak convergence of a sequence of time-homogeneous Markov processes towards a time-homogeneous Markov process first to show convergence of the corresponding infinitesimal generators and then to check some additional conditions, see, e.g., Ethier and Kurtz [9, Chapter 4, Section 8]. In a recent paper, we proved that, under some fourth order moment assumptions, a sequence of scaled random step functions $(n^{-1}\boldsymbol{X}_{\lfloor nt \rfloor})_{t\geq 0}, n \geq 1$, formed from a critical, irreducible multi-type continuous state and continuous time branching process with immigration (CBI process) \boldsymbol{X} converges weakly towards a squared Bessel process supported by a ray determined by the Perron vector of a matrix related to the branching mechanism of \boldsymbol{X} , see Barczy and Pap [6, Theorem 4.1], and Section 2, as well. This convergence result has been shown not by infinitesimal generators, that is why we consider in Section 3 the sequences of

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discrete infinitesimal generators of $(n^{-1}\boldsymbol{X}_{\lfloor nt \rfloor})_{t \ge 0}$, $n \ge 1$, and of usual infinitesimal generators of $(n^{-1}\boldsymbol{X}_{nt})_{t \ge 0}$, $n \ge 1$, formed from a (not necessarily critical or irreducible) multi-type CBI process \boldsymbol{X} . Adding some additional extra terms to these sequences of infinitesimal generators, under some second order moment assumptions, we show their convergence, see Propositions 3.4 and 3.7. As a consequence, the sequences of infinitesimal generators (without the additional extra terms) do not converge in general. We also apply Proposition 3.4 to irreducible and critical multi-type CBI processes, see Corollary 3.5 and Remark 3.6. In Remark 3.8 we specialize Proposition 3.7 to single-type irreducible and critical CBI processes.

In Section 2, for completeness and better readability, from Barczy et al. [2] and [6], we recall some notions and statements for multi-type CBI processes such as the form of their infinitesimal generator, a formula for their first moment, the definition of irreducible CBI processes and a classification of them, namely we recall the notion of subcritical, critical and supercritical irreducible CBI processes, see Definitions 2.7 and 2.8, respectively.

Finally, we note that our main motivation for studying limit theorems for $(n^{-1}\boldsymbol{X}_{\lfloor nt \rfloor})_{t \ge 0}$, $n \ge 1$, relies on the fact that these limit theorems are well-applicable in describing asymptotic behaviour of conditional least squares estimators of some parameters of multi-type CBI processes, see Barczy et al. [4] and [5].

2 Multi-type CBI processes

Let \mathbb{Z}_+ , \mathbb{N} , \mathbb{R} , \mathbb{R}_+ and \mathbb{R}_{++} denote the set of non-negative integers, positive integers, real numbers, non-negative real numbers and positive real numbers, respectively. For $x, y \in \mathbb{R}$, we will use the notations $x \wedge y := \min\{x, y\}$ and $x^+ := \max\{0, x\}$. By $||\mathbf{x}||$ and $||\mathbf{A}||$, we denote the Euclidean norm of a vector $\mathbf{x} \in \mathbb{R}^d$ and the induced matrix norm of a matrix $\mathbf{A} \in \mathbb{R}^{d \times d}$, respectively. The natural basis in \mathbb{R}^d will be denoted by $\mathbf{e}_1, \ldots, \mathbf{e}_d$. By $C_c^2(\mathbb{R}^d_+, \mathbb{R})$ we denote the set of twice continuously differentiable real-valued functions on \mathbb{R}^d_+ with compact support.

2.1 Definition. A matrix $\mathbf{A} = (a_{i,j})_{i,j \in \{1,...,d\}} \in \mathbb{R}^{d \times d}$ is called essentially non-negative if $a_{i,j} \in \mathbb{R}_+$ whenever $i, j \in \{1,...,d\}$ with $i \neq j$, i.e., if \mathbf{A} has non-negative off-diagonal entries. The set of essentially non-negative $d \times d$ matrices will be denoted by $\mathbb{R}^{d \times d}_{(+)}$.

2.2 Definition. A tuple $(d, c, \beta, B, \nu, \mu)$ is called a set of admissible parameters if

- (i) $d \in \mathbb{N}$,
- (ii) $c = (c_i)_{i \in \{1,...,d\}} \in \mathbb{R}^d_+,$
- (iii) $\boldsymbol{\beta} = (\beta_i)_{i \in \{1, \dots, d\}} \in \mathbb{R}^d_+,$
- (iv) $\boldsymbol{B} = (b_{i,j})_{i,j \in \{1,...,d\}} \in \mathbb{R}^{d \times d}_{(+)},$

(v) ν is a Borel measure on $U_d := \mathbb{R}^d_+ \setminus \{\mathbf{0}\}$ satisfying $\int_{U_d} (1 \wedge \|\boldsymbol{z}\|) \nu(\mathrm{d}\boldsymbol{z}) < \infty$,

(vi) $\boldsymbol{\mu} = (\mu_1, \dots, \mu_d)$, where, for each $i \in \{1, \dots, d\}$, μ_i is a Borel measure on U_d satisfying

(2.1)
$$\int_{U_d} \left[\|\boldsymbol{z}\| \wedge \|\boldsymbol{z}\|^2 + \sum_{j \in \{1, \dots, d\} \setminus \{i\}} z_j \right] \mu_i(\mathrm{d}\boldsymbol{z}) < \infty.$$

2.3 Remark. Our Definition 2.2 of the set of admissible parameters is a special case of Definition 2.6 in Duffie et al. [7], which is suitable for all affine processes, see Barczy et al. [2, Remark 2.3]. Roughly speaking, affine processes are characterized by their characteristic functions which are exponentially affine in the state variable. Note that, for all $i \in \{1, \ldots, d\}$, condition (2.1) is equivalent to

$$\int_{U_d} \left[(1 \wedge z_i)^2 + \sum_{j \in \{1, \dots, d\} \setminus \{i\}} (1 \wedge z_j) \right] \mu_i(\mathrm{d}\boldsymbol{z}) < \infty \quad \text{and} \quad \int_{U_d} \|\boldsymbol{z}\| \mathbb{1}_{\{\|\boldsymbol{z}\| \ge 1\}} \, \mu_i(\mathrm{d}\boldsymbol{z}) < \infty,$$

see Barczy et al. [2, Remark 2.3].

2.4 Theorem. Let $(d, c, \beta, B, \nu, \mu)$ be a set of admissible parameters. Then there exists a unique conservative transition semigroup $(P_t)_{t \in \mathbb{R}_+}$ acting on the Banach space (endowed with the supremum norm) of real-valued bounded Borel-measurable functions on the state space \mathbb{R}^d_+ such that its (usual) infinitesimal generator is

(2.2)

$$(\mathcal{A}f)(\boldsymbol{x}) = \sum_{i=1}^{d} c_{i} x_{i} f_{i,i}''(\boldsymbol{x}) + \langle \boldsymbol{\beta} + \boldsymbol{B}\boldsymbol{x}, \boldsymbol{f}'(\boldsymbol{x}) \rangle + \int_{U_{d}} (f(\boldsymbol{x} + \boldsymbol{z}) - f(\boldsymbol{x})) \nu(\mathrm{d}\boldsymbol{z}) + \sum_{i=1}^{d} x_{i} \int_{U_{d}} (f(\boldsymbol{x} + \boldsymbol{z}) - f(\boldsymbol{x}) - f_{i}'(\boldsymbol{x})(1 \wedge z_{i})) \mu_{i}(\mathrm{d}\boldsymbol{z})$$

for $f \in C_c^2(\mathbb{R}^d_+, \mathbb{R})$ and $\boldsymbol{x} \in \mathbb{R}^d_+$, where f'_i and $f''_{i,i}$, $i \in \{1, \ldots, d\}$, denote the first and second order partial derivatives of f with respect to its *i*-th variable, respectively, and $\boldsymbol{f}'(\boldsymbol{x}) := (f'_1(\boldsymbol{x}), \ldots, f'_d(\boldsymbol{x}))^\top$. Moreover, the Laplace transform of the transition semigroup $(P_t)_{t \in \mathbb{R}_+}$ has a representation

(2.3)
$$\int_{\mathbb{R}^d_+} e^{-\langle \boldsymbol{\lambda}, \boldsymbol{y} \rangle} P_t(\boldsymbol{x}, \mathrm{d}\boldsymbol{y}) = e^{-\langle \boldsymbol{x}, \boldsymbol{v}(t, \boldsymbol{\lambda}) \rangle - \int_0^t \psi(\boldsymbol{v}(s, \boldsymbol{\lambda})) \, \mathrm{d}s}, \qquad \boldsymbol{x} \in \mathbb{R}^d_+, \quad \boldsymbol{\lambda} \in \mathbb{R}^d_+, \quad t \in \mathbb{R}_+,$$

where, for any $\lambda \in \mathbb{R}^d_+$, the continuously differentiable function $\mathbb{R}_+ \ni t \mapsto v(t, \lambda) = (v_1(t, \lambda), \dots, v_d(t, \lambda))^\top \in \mathbb{R}^d_+$ is the unique locally bounded solution to the system of differential equations

$$\partial_t v_i(t, \boldsymbol{\lambda}) = -\varphi_i(\boldsymbol{v}(t, \boldsymbol{\lambda})), \quad v_i(0, \boldsymbol{\lambda}) = \lambda_i, \quad i \in \{1, \dots, d\},$$

with

$$\varphi_i(\boldsymbol{\lambda}) := c_i \lambda_i^2 - \langle \boldsymbol{B} \boldsymbol{e}_i, \boldsymbol{\lambda} \rangle + \int_{U_d} \left(e^{-\langle \boldsymbol{\lambda}, \boldsymbol{z} \rangle} - 1 + \lambda_i (1 \wedge z_i) \right) \mu_i(\mathrm{d}\boldsymbol{z})$$

for $\boldsymbol{\lambda} \in \mathbb{R}^d_+$ and $i \in \{1, \dots, d\}$, and

$$\psi(\boldsymbol{\lambda}) := \langle \boldsymbol{\beta}, \boldsymbol{\lambda} \rangle - \int_{U_d} (e^{-\langle \boldsymbol{\lambda}, \boldsymbol{z} \rangle} - 1) \nu(d\boldsymbol{z}), \qquad \boldsymbol{\lambda} \in \mathbb{R}^d_+.$$

Further, the function $\mathbb{R}_+ \times \mathbb{R}^d_+ \ni (t, \lambda) \mapsto \boldsymbol{v}(t, \lambda)$ is continuous.

2.5 Remark. This theorem is a special case of Theorem 2.7 of Duffie et al. [7] with m = d, n = 0 and zero killing rate.

2.6 Definition. A conservative Markov process with state space \mathbb{R}^d_+ and with transition semigroup $(P_t)_{t \in \mathbb{R}_+}$ given in Theorem 2.4 is called a multi-type CBI process with parameters $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$. The function $\mathbb{R}^d_+ \ni \boldsymbol{\lambda} \mapsto (\varphi_1(\boldsymbol{\lambda}), \dots, \varphi_d(\boldsymbol{\lambda}))^\top \in \mathbb{R}^d$ is called its branching mechanism, and the function $\mathbb{R}^d_+ \ni \boldsymbol{\lambda} \mapsto \psi(\boldsymbol{\lambda}) \in \mathbb{R}_+$ is called its immigration mechanism.

Let $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ be a multi-type CBI process with parameters $(d, \mathbf{c}, \boldsymbol{\beta}, \mathbf{B}, \nu, \boldsymbol{\mu})$ such that $\mathbb{E}(\|\mathbf{X}_0\|) < \infty$ and the moment condition

(2.4)
$$\int_{U_d} \|\boldsymbol{z}\| \mathbb{1}_{\{\|\boldsymbol{z}\| \ge 1\}} \nu(\mathrm{d}\boldsymbol{z}) < \infty$$

holds. Then, by (3.3) in Barczy et al. [2],

(2.5)
$$\mathbb{E}(\boldsymbol{X}_t \,|\, \boldsymbol{X}_0 = \boldsymbol{x}) = \mathrm{e}^{t\widetilde{\boldsymbol{B}}} \boldsymbol{x} + \int_0^t \mathrm{e}^{u\widetilde{\boldsymbol{B}}} \widetilde{\boldsymbol{\beta}} \,\mathrm{d}\boldsymbol{u}, \qquad \boldsymbol{x} \in \mathbb{R}^d_+, \quad t \in \mathbb{R}_+,$$

where

(2.6)
$$\widetilde{\boldsymbol{B}} := (\widetilde{b}_{i,j})_{i,j \in \{1,\dots,d\}}, \qquad \widetilde{b}_{i,j} := b_{i,j} + \int_{U_d} (z_i - \delta_{i,j})^+ \mu_j (\mathrm{d}\boldsymbol{z}),$$

(2.7)
$$\widetilde{\boldsymbol{\beta}} := \boldsymbol{\beta} + \int_{U_d} \boldsymbol{z} \,\nu(\mathrm{d}\boldsymbol{z}),$$

with $\delta_{i,j} := 1$ if i = j, and $\delta_{i,j} := 0$ if $i \neq j$. Note that $\widetilde{B} \in \mathbb{R}^{d \times d}_{(+)}$ and $\widetilde{\beta} \in \mathbb{R}^d_+$, since

(2.8)
$$\int_{U_d} \|\boldsymbol{z}\| \,\nu(\mathrm{d}\boldsymbol{z}) < \infty, \qquad \int_{U_d} (z_i - \delta_{i,j})^+ \,\mu_j(\mathrm{d}\boldsymbol{z}) < \infty, \quad i, j \in \{1, \dots, d\},$$

see Barczy et al. [2, Section 2].

Next we recall a classification of multi-type CBI processes. For a matrix $A \in \mathbb{R}^{d \times d}$, $\sigma(A)$ will denote the spectrum of A, i.e., the set of the eigenvalues of A. Then $r(A) := \max_{\lambda \in \sigma(A)} |\lambda|$ is the spectral radius of A. Moreover, we will use the notation

$$s(\mathbf{A}) := \max_{\lambda \in \sigma(\mathbf{A})} \operatorname{Re}(\lambda).$$

A matrix $A \in \mathbb{R}^{d \times d}$ is called reducible if there exist a permutation matrix $P \in \mathbb{R}^{d \times d}$ and an integer r with $1 \leq r \leq d-1$ such that

$$oldsymbol{P}^ opoldsymbol{A} oldsymbol{P}^ opoldsymbol{A} oldsymbol{A} = egin{bmatrix} oldsymbol{A}_1 & oldsymbol{A}_2 \ oldsymbol{0} & oldsymbol{A}_3 \end{bmatrix},$$

where $A_1 \in \mathbb{R}^{r \times r}$, $A_3 \in \mathbb{R}^{(d-r) \times (d-r)}$, $A_2 \in \mathbb{R}^{r \times (d-r)}$, and $\mathbf{0} \in \mathbb{R}^{(d-r) \times r}$ is a null matrix. A matrix $A \in \mathbb{R}^{d \times d}$ is called irreducible if it is not reducible, see, e.g., Horn and Johnson [11, Definitions 6.2.21 and 6.2.22]. We do emphasize that no 1-by-1 matrix is reducible.

2.7 Definition. Let $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ be a multi-type CBI process with parameters $(d, c, \beta, B, \nu, \mu)$ such that the moment condition (2.4) holds. Then $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ is called irreducible if \widetilde{B} is irreducible.

2.8 Definition. Let $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ be a multi-type CBI process with parameters $(d, c, \beta, B, \nu, \mu)$ such that $\mathbb{E}(\|\mathbf{X}_0\|) < \infty$ and the moment condition (2.4) holds. Suppose that $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ is irreducible. Then $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ is called

 $\begin{cases} subcritical & if \ s(\widetilde{B}) < 0, \\ critical & if \ s(\widetilde{B}) = 0, \\ supercritical & if \ s(\widetilde{B}) > 0. \end{cases}$

For motivations of Definitions 2.7 and 2.8, see Barczy et al. [6, Section 3]. To shed some light, we note that formula (2.4) in Barczy and Pap [6] shows that the semigroup $(e^{t\widetilde{B}})_{t\in\mathbb{R}_+}$ of matrices plays a crucial role in the asymptotic behavior of the expectations $\mathbb{E}(X_t)$ as $t \to \infty$ described in Proposition B.1 in Barczy and Pap [6]. Namely, under the conditions of Definition 2.8, if $s(\widetilde{B}) < 0$, then $\lim_{t\to\infty} \mathbb{E}(X_t)$; if $s(\widetilde{B}) = 0$, then $\lim_{t\to\infty} t^{-1} \mathbb{E}(X_t)$; and if $s(\widetilde{B}) > 0$, then $\lim_{t\to\infty} e^{-s(\widetilde{B})t} \mathbb{E}(X_t)$ exists, respectively. We point out that the notion of criticality given in Definition 2.8 depends only on the branching mechanism of the CBI process in question, but not on its immigration mechanism.

Next we will recall a convergence result for irreducible and critical multi-type CBI processes.

A function $f: \mathbb{R}_+ \to \mathbb{R}^d$ is called $c\dot{a}dl\dot{a}g$ if it is right continuous with left limits. Let $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ and $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$ denote the space of all \mathbb{R}^d -valued càdlàg and continuous functions on \mathbb{R}_+ , respectively. Let $\mathcal{D}_{\infty}(\mathbb{R}_+, \mathbb{R}^d)$ denote the Borel σ -field in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ for the metric characterized by Jacod and Shiryaev [13, VI.1.15] (with this metric $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ is a complete and separable metric space). For \mathbb{R}^d -valued stochastic processes $(\mathcal{Y}_t)_{t\in\mathbb{R}_+}$ and $(\mathcal{Y}_t^n)_{t\in\mathbb{R}_+}$, $n \in \mathbb{N}$, with càdlàg paths we write $\mathcal{Y}^n \xrightarrow{\mathcal{D}} \mathcal{Y}$ as $n \to \infty$ if the distribution of \mathcal{Y}^n on the space $(\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d), \mathcal{D}_{\infty}(\mathbb{R}_+, \mathbb{R}^d))$ converges weakly to the distribution of \mathcal{Y} on the space $(\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d), \mathcal{D}_{\infty}(\mathbb{R}_+, \mathbb{R}^d))$ as $n \to \infty$.

The proof of the following convergence theorem can be found in Barczy and Pap [6, Theorem 4.1].

2.9 Theorem. Let $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ be a multi-type CBI process with parameters $(d, c, \beta, B, \nu, \mu)$ such that $\mathbb{E}(\|\mathbf{X}_0\|^4) < \infty$ and

(2.9)
$$\int_{U_d} \|\boldsymbol{z}\|^4 \mathbb{1}_{\{\|\boldsymbol{z}\| \ge 1\}} \nu(\mathrm{d}\boldsymbol{z}) < \infty, \qquad \int_{U_d} \|\boldsymbol{z}\|^4 \mathbb{1}_{\{\|\boldsymbol{z}\| \ge 1\}} \mu_i(\mathrm{d}\boldsymbol{z}) < \infty, \quad i \in \{1, \dots, d\}.$$

Suppose that $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ is irreducible and critical. Then

$$(\boldsymbol{\mathcal{X}}_t^{(n)})_{t\in\mathbb{R}_+} := (n^{-1}\boldsymbol{X}_{\lfloor nt \rfloor})_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}} (\boldsymbol{\mathcal{X}}_t)_{t\in\mathbb{R}_+} := (\mathcal{X}_t\boldsymbol{u}_{\mathrm{right}})_{t\in\mathbb{R}_+} \quad as \quad n \to \infty$$

in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$, where $\mathbf{u}_{\text{right}} \in \mathbb{R}^d_{++}$ is the right Perron vector of $e^{\tilde{B}}$ corresponding to the eigenvalue 1 with $\sum_{i=1}^d e_i^{\top} \mathbf{u}_{\text{right}} = 1$ (see Barczy and Pap [6, Lemma A.3]), $(\mathcal{X}_t)_{t \in \mathbb{R}_+}$ is the unique strong solution of the stochastic differential equation (SDE)

$$\mathrm{d}\mathcal{X}_t = \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \,\mathrm{d}t + \sqrt{\langle \overline{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle \mathcal{X}_t^+} \,\mathrm{d}\mathcal{W}_t, \qquad t \in \mathbb{R}_+, \qquad \mathcal{X}_0 = 0,$$

where $\mathbf{u}_{\text{left}} \in \mathbb{R}^{d}_{++}$ is the left Perron vector of $e^{\widetilde{B}}$ corresponding to the eigenvalue 1 with $\mathbf{u}_{\text{left}}^{\top}\mathbf{u}_{\text{right}} = 1$ (see Barczy and Pap [6, Lemma A.3]), $(\mathcal{W}_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion, $\widetilde{\boldsymbol{\beta}}$ is given in (2.7), and

$$\overline{oldsymbol{C}} := \sum_{k=1}^d \langle oldsymbol{e}_k, oldsymbol{u}_{ ext{right}}
angle oldsymbol{C}_k \in \mathbb{R}^{d imes d}_+$$

with

(2.10)
$$\boldsymbol{C}_{k} := 2c_{k}\boldsymbol{e}_{k}\boldsymbol{e}_{k}^{\top} + \int_{\mathcal{U}_{d}}\boldsymbol{z}\boldsymbol{z}^{\top}\boldsymbol{\mu}_{k}(\mathrm{d}\boldsymbol{z}) \in \mathbb{R}^{d \times d}_{+}, \qquad k \in \{1, \dots, d\}$$

For a motivation of studying limit theorems for $(\mathcal{X}_t^{(n)})_{t \in \mathbb{R}_+}, n \in \mathbb{N}$, see the end of Introduction.

3 Non-convergence of infinitesimal generators

We will need some differentiability properties of the functions ψ and \boldsymbol{v} introduced in Theorem 2.4.

3.1 Lemma. Let $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ be a multi-type CBI process with parameters $(d, c, \beta, B, \nu, \mu)$ such that the moment condition (2.4) holds. Then

(3.1)
$$\partial_{\lambda_i}\psi(\boldsymbol{\lambda}) = \langle \widetilde{\boldsymbol{\beta}}, \boldsymbol{e}_i \rangle - \int_{U_d} (-z_i) (\mathrm{e}^{-\langle \boldsymbol{\lambda}, \boldsymbol{z} \rangle} - 1) \,\nu(\mathrm{d}\boldsymbol{z}), \qquad \boldsymbol{\lambda} \in \mathbb{R}^d_{++},$$

(3.2)
$$\lim_{\lambda \downarrow 0} \partial_{\lambda_i} \psi(\lambda) = \langle \widetilde{\beta}, e_i \rangle$$

for all $i \in \{1, \ldots, d\}$, where the function $\psi : \mathbb{R}^d_+ \to \mathbb{R}_+$ is defined in Theorem 2.4.

Proof. Under the moment condition (2.4) together with part (v) of Definition 2.2 we can write the function ψ in the form

(3.3)
$$\psi(\boldsymbol{\lambda}) = \langle \widetilde{\boldsymbol{\beta}}, \boldsymbol{\lambda} \rangle - \int_{U_d} \left(e^{-\langle \boldsymbol{\lambda}, \boldsymbol{z} \rangle} - 1 + \langle \boldsymbol{\lambda}, \boldsymbol{z} \rangle \right) \nu(\mathrm{d}\boldsymbol{z}), \qquad \boldsymbol{\lambda} \in \mathbb{R}^d_+.$$

Indeed, by (2.8),

$$\begin{split} \langle \widetilde{\boldsymbol{\beta}}, \boldsymbol{\lambda} \rangle &- \int_{U_d} \left(\mathrm{e}^{-\langle \boldsymbol{\lambda}, \boldsymbol{z} \rangle} - 1 + \langle \boldsymbol{\lambda}, \boldsymbol{z} \rangle \right) \nu(\mathrm{d}\boldsymbol{z}) - \psi(\boldsymbol{\lambda}) \\ &= \langle \widetilde{\boldsymbol{\beta}} - \boldsymbol{\beta}, \boldsymbol{\lambda} \rangle - \int_{U_d} \langle \boldsymbol{\lambda}, \boldsymbol{z} \rangle \, \nu(\mathrm{d}\boldsymbol{z}) = \left\langle \int_{U_d} \boldsymbol{z} \, \nu(\mathrm{d}\boldsymbol{z}), \boldsymbol{\lambda} \right\rangle - \int_{U_d} \langle \boldsymbol{\lambda}, \boldsymbol{z} \rangle \, \nu(\mathrm{d}\boldsymbol{z}) = 0. \end{split}$$

By the dominated convergence theorem one can derive

$$\partial_{\lambda_{i}}\psi(\boldsymbol{\lambda}) = \lim_{h \downarrow 0} h^{-1}(\psi(\boldsymbol{\lambda} + h\boldsymbol{e}_{i}) - \psi(\boldsymbol{\lambda})) = \langle \widetilde{\boldsymbol{\beta}}, \boldsymbol{e}_{i} \rangle - \lim_{h \downarrow 0} \int_{U_{d}} \left(e^{-\langle \boldsymbol{\lambda}, \boldsymbol{z} \rangle} \frac{e^{-hz_{i}} - 1}{h} + z_{i} \right) \nu(\mathrm{d}\boldsymbol{z})$$
$$= \langle \widetilde{\boldsymbol{\beta}}, \boldsymbol{e}_{i} \rangle - \int_{U_{d}} (-z_{i}) (e^{-\langle \boldsymbol{\lambda}, \boldsymbol{z} \rangle} - 1) \nu(\mathrm{d}\boldsymbol{z})$$

for all $\boldsymbol{\lambda} \in \mathbb{R}^d_{++}$ and $i \in \{1, \dots, d\}$, since $|e^{-\langle \boldsymbol{\lambda}, \boldsymbol{z} \rangle} \frac{e^{-hz_i}-1}{h}| \leq z_i \leq ||\boldsymbol{z}||$, $\int_{U_d} ||\boldsymbol{z}|| \nu(\mathrm{d}\boldsymbol{z}) < \infty$ and $\lim_{h\downarrow 0} \frac{e^{-hz_i}-1}{h} = -z_i$. Again by the dominated convergence theorem, we have $\lim_{\lambda\downarrow 0} \int_{U_d} (-z_i)(e^{-\langle \boldsymbol{\lambda}, \boldsymbol{z} \rangle} - 1) \nu(\mathrm{d}\boldsymbol{z}) = 0.$

In order to derive differentiability properties of the function v, we need the following simple observation; for the 1-dimensional case, see, e.g., Feller [10, page 435].

3.2 Lemma. Let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_d)^{\top}$ be a random vector such that $\mathbb{P}(\boldsymbol{\xi} \in \mathbb{R}^d_+) = 1$. Consider its Laplace transform $g : \mathbb{R}^d_+ \to \mathbb{R}_{++}$ defined by $g(\boldsymbol{\lambda}) := \mathbb{E}(e^{-\langle \boldsymbol{\lambda}, \boldsymbol{\xi} \rangle})$ for $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_d)^{\top} \in \mathbb{R}^d_+$. Then g is infinitely differentiable on \mathbb{R}^d_{++} , and for all $(k_1, \dots, k_d)^{\top} \in \mathbb{Z}^d_+$, we have

(3.4)
$$\partial_{\lambda_1}^{k_1} \dots \partial_{\lambda_d}^{k_d} g(\boldsymbol{\lambda}) = (-1)^{k_1 + \dots + k_d} \mathbb{E}(\xi_1^{k_1} \dots \xi_d^{k_d} e^{-\langle \boldsymbol{\lambda}, \boldsymbol{\xi} \rangle}), \qquad \boldsymbol{\lambda} \in \mathbb{R}^d_{++},$$

(3.5)
$$\mathbb{E}(\xi_1^{k_1}\cdots\xi_d^{k_d}) = (-1)^{k_1+\cdots+k_d} \lim_{\lambda \downarrow 0} \partial_{\lambda_1}^{k_1} \dots \partial_{\lambda_d}^{k_d} g(\boldsymbol{\lambda}) \in \mathbb{R}_+ \cup \{\infty\}.$$

Consequently, $\mathbb{E}(\xi_1^{k_1}\cdots\xi_d^{k_d}) < \infty$ if and only if $(-1)^{k_1+\cdots+k_d} \lim_{\lambda \downarrow \mathbf{0}} \partial_{\lambda_1}^{k_1} \dots \partial_{\lambda_d}^{k_d} g(\boldsymbol{\lambda}) < \infty$.

Proof. First we prove (3.4) by induction. If $k_1 = \ldots = k_d = 0$, then (3.4) holds trivially. Suppose that (3.4) holds for $(k_1, \ldots, k_d)^\top \in \mathbb{Z}_+^d$. Then for all $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d)^\top \in \mathbb{R}_{++}^d$, $i \in \{1, \ldots, d\}$ and $h \in \mathbb{R}$ with $h \neq 0$ and $h \geq -\lambda_i/2$ we have

$$\frac{\partial_{\lambda_1}^{k_1} \dots \partial_{\lambda_d}^{k_d} g(\boldsymbol{\lambda} + h\boldsymbol{e}_i) - \partial_{\lambda_1}^{k_1} \dots \partial_{\lambda_d}^{k_d} g(\boldsymbol{\lambda})}{h} = (-1)^{k_1 + \dots + k_d} \mathbb{E}\left(\xi_1^{k_1} \dots \xi_d^{k_d} \left(\frac{\mathrm{e}^{-\langle \boldsymbol{\lambda}, \boldsymbol{\xi} \rangle - h\xi_i} - \mathrm{e}^{-\langle \boldsymbol{\lambda}, \boldsymbol{\xi} \rangle}}{h}\right)\right),$$

where the mean value theorem and $\min\{\lambda_i + h, \lambda_i\} \ge \lambda_i/2$ yields

$$\mathbb{E}\left(\xi_1^{k_1}\cdots\xi_d^{k_d}\left|\frac{\mathrm{e}^{-\langle\boldsymbol{\lambda},\boldsymbol{\xi}\rangle-h\xi_i}-\mathrm{e}^{-\langle\boldsymbol{\lambda},\boldsymbol{\xi}\rangle}}{h}\right|\right)\leqslant\mathbb{E}\left(\xi_1^{k_1}\cdots\xi_{i-1}^{k_{i-1}}\xi_i^{k_i+1}\xi_{i+1}^{k_{i+1}}\cdots\xi_d^{k_d}\,\mathrm{e}^{-\langle\boldsymbol{\lambda},\boldsymbol{\xi}\rangle+\lambda_i\xi_i/2}\right)<\infty,$$

since the random variable $\xi_1^{k_1} \cdots \xi_{i-1}^{k_i-1} \xi_i^{k_i+1} \xi_{i+1}^{k_i+1} \cdots \xi_d^{k_d} e^{-\langle \boldsymbol{\lambda}, \boldsymbol{\xi} \rangle + \lambda_i \xi_i/2}$ is bounded. By the dominated convergence theorem, we obtain (3.4) for $\boldsymbol{\lambda} \in \mathbb{R}^d_{++}$ and $(k_1, \ldots, k_{i-1}, k_i+1, k_{i+1}, \ldots, k_d)^{\top}$. The monotone convergence theorem yields (3.5).

3.3 Lemma. Let $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ be a multi-type CBI process with parameters $(d, c, \beta, B, \nu, \mu)$. Then

(3.6)
$$v(t, \lambda) \downarrow v(t, 0) = 0$$
 as $\lambda \downarrow 0$

for all $t \in \mathbb{R}_+$, where the function $\boldsymbol{v} : \mathbb{R}_+ \times \mathbb{R}^d_+ \to \mathbb{R}^d_+$ is defined in Theorem 2.4.

If $\mathbb{E}(\|\boldsymbol{X}_0\|) < \infty$ and the moment condition (2.4) holds, then for all $t \in \mathbb{R}_+$, the function $\mathbb{R}^d_{++} \ni \boldsymbol{\lambda} \mapsto \boldsymbol{v}(t, \boldsymbol{\lambda})$ is infinitely differentiable, and

(3.7)
$$\lim_{\lambda \downarrow 0} \partial_{\lambda_i} v_k(t, \lambda) = \boldsymbol{e}_i^{\mathsf{T}} \mathrm{e}^{t \widetilde{\boldsymbol{B}}} \boldsymbol{e}_k$$

for all $t \in \mathbb{R}_+$ and $i, k \in \{1, \dots, d\}$. Moreover, if $\mathbb{E}(\|\mathbf{X}_0\|^2) < \infty$ and

(3.8)
$$\int_{U_d} \|\boldsymbol{z}\|^2 \mathbb{1}_{\{\|\boldsymbol{z}\| \ge 1\}} \nu(\mathrm{d}\boldsymbol{z}) < \infty, \qquad \int_{U_d} \|\boldsymbol{z}\|^2 \mathbb{1}_{\{\|\boldsymbol{z}\| \ge 1\}} \mu_i(\mathrm{d}\boldsymbol{z}) < \infty, \quad i \in \{1, \dots, d\},$$

then

(3.9)
$$\lim_{\boldsymbol{\lambda} \downarrow \boldsymbol{0}} \partial_{\lambda_i} \partial_{\lambda_j} v_k(t, \boldsymbol{\lambda}) = -\boldsymbol{e}_k^\top \mathrm{e}^{t \widetilde{\boldsymbol{B}}^\top} \int_0^t \mathrm{e}^{-u \widetilde{\boldsymbol{B}}^\top} \sum_{\ell=1}^d \boldsymbol{e}_\ell \boldsymbol{e}_i^\top \mathrm{e}^{u \widetilde{\boldsymbol{B}}} \boldsymbol{C}_\ell \mathrm{e}^{u \widetilde{\boldsymbol{B}}^\top} \boldsymbol{e}_j \,\mathrm{d}u$$

for all $t \in \mathbb{R}_+$, $i, j, k \in \{1, \ldots, d\}$ and $\lambda \in \mathbb{R}^d_+$.

Proof. Let $(\mathbf{Z}_t)_{t \in \mathbb{R}_+}$ be a multi-type CBI process with parameters $(d, \mathbf{c}, \mathbf{0}, \mathbf{B}, 0, \boldsymbol{\mu})$ (which is, in fact, a continuous state and continuous time branching process without immigration). Then, by (2.3), its Laplace transform takes the form

$$g_{t,\boldsymbol{z}}(\boldsymbol{\lambda}) := \mathbb{E}(\mathrm{e}^{-\langle \boldsymbol{\lambda}, \boldsymbol{Z}_t \rangle} \mid \boldsymbol{Z}_0 = \boldsymbol{z}) = \mathrm{e}^{-\langle \boldsymbol{z}, \boldsymbol{v}(t, \boldsymbol{\lambda}) \rangle}, \qquad \boldsymbol{\lambda}, \boldsymbol{z} \in \mathbb{R}^d_+, \quad t \in \mathbb{R}_+.$$

By Lemma 3.2, $g_{t,\boldsymbol{z}}$ is infinitely differentiable on \mathbb{R}_{++}^d for each $t \in \mathbb{R}_+$ and $\boldsymbol{z} \in \mathbb{R}_+^d$, and the limit $\lim_{\boldsymbol{\lambda} \downarrow \mathbf{0}} (-1)^{k_1 + \dots + k_d} \partial_{\lambda_1}^{k_1} \cdots \partial_{\lambda_d}^{k_d} g_{t,\boldsymbol{z}}(\boldsymbol{\lambda}) \in \mathbb{R}_+ \cup \{\infty\}$ exists for all $(k_1, \dots, k_d)^\top \in \mathbb{Z}_+^d$, $t \in \mathbb{R}_+$ and $\boldsymbol{z} \in \mathbb{R}_+^d$. Hence the function $\boldsymbol{\lambda} \mapsto \boldsymbol{v}(t, \boldsymbol{\lambda})$ is also infinitely differentiable on \mathbb{R}_{++}^d for all $t \in \mathbb{R}_+$, and the limit $\lim_{\boldsymbol{\lambda} \downarrow \mathbf{0}} \partial_{\lambda_1}^{k_1} \cdots \partial_{\lambda_d}^{k_d} \boldsymbol{v}(t, \boldsymbol{\lambda}) \in \mathbb{R} \cup \{-\infty, \infty\}$ exists for all $(k_1, \dots, k_d)^\top \in \mathbb{Z}_+^d$ and $t \in \mathbb{R}_+$.

We can express the functions $v_k, k \in \{1, \ldots, d\}$, as

$$v_k(t, \boldsymbol{\lambda}) = -\log g_{t, \boldsymbol{e}_k}(\boldsymbol{\lambda}), \qquad t \in \mathbb{R}_+, \qquad \boldsymbol{\lambda} \in \mathbb{R}^d_+.$$

By monotone convergence theorem, $g_{t,z}(\lambda) \uparrow g_{t,z}(0) = 1$ as $\lambda \downarrow 0$ for all $z \in \mathbb{R}^d_+$ and $t \in \mathbb{R}_+$, hence $v(t, \lambda) \downarrow v(t, 0) = 0$ as $\lambda \downarrow 0$ for all $t \in \mathbb{R}_+$. Clearly,

(3.10)
$$\partial_{\lambda_i} v_k(t, \boldsymbol{\lambda}) = -\frac{\partial_{\lambda_i} g_{t, \boldsymbol{e}_k}(\boldsymbol{\lambda})}{g_{t, \boldsymbol{e}_k}(\boldsymbol{\lambda})}, \quad t \in \mathbb{R}_+, \quad \boldsymbol{\lambda} \in \mathbb{R}^d_{++}, \quad i, k \in \{1, \dots, d\}.$$

With the notation $\mathbf{Z}_t = (Z_{t,1}, \dots, Z_{t,d})^{\top}$, under $\mathbb{E}(\|\mathbf{Z}_0\|) < \infty$ and the moment condition (2.4), formula (2.5) implies $\mathbb{E}(\mathbf{Z}_t | \mathbf{Z}_0 = \mathbf{z}) = e^{t\tilde{\mathbf{B}}} \mathbf{z}$, hence by Lemma 3.2,

$$\lim_{\boldsymbol{\lambda} \downarrow \boldsymbol{0}} \partial_{\lambda_i} v_k(t, \boldsymbol{\lambda}) = -\lim_{\boldsymbol{\lambda} \downarrow \boldsymbol{0}} \partial_{\lambda_i} g_{t, \boldsymbol{e}_k}(\boldsymbol{\lambda}) = \mathbb{E}(Z_{t, i} \mid \boldsymbol{Z}_0 = \boldsymbol{e}_k) = \boldsymbol{e}_i^\top e^{t\boldsymbol{B}} \boldsymbol{e}_k.$$

In a similar way,

$$\partial_{\lambda_i}\partial_{\lambda_j}v_k(t,\boldsymbol{\lambda}) = -\frac{g_{t,\boldsymbol{e}_k}(\boldsymbol{\lambda})\partial_{\lambda_i}\partial_{\lambda_j}g_{t,\boldsymbol{e}_k}(\boldsymbol{\lambda}) - \partial_{\lambda_i}g_{t,\boldsymbol{e}_k}(\boldsymbol{\lambda})\partial_{\lambda_j}g_{t,\boldsymbol{e}_k}(\boldsymbol{\lambda})}{g_{t,\boldsymbol{e}_k}(\boldsymbol{\lambda})^2}, \qquad t \in \mathbb{R}_+, \qquad \boldsymbol{\lambda} \in \mathbb{R}_{++}^d$$

for all $i, j, k \in \{1, \ldots, d\}$. Under $\mathbb{E}(\|\boldsymbol{Z}_0\|^2) < \infty$ and the moment conditions (3.8), Theorem 4.3 and Proposition 4.8 in Barczy et al. [3] implies $\mathbb{E}(\|\boldsymbol{Z}_t\|^2 | \boldsymbol{Z}_0 = \boldsymbol{z}) < \infty$ and

$$\operatorname{Var}(\boldsymbol{Z}_t \mid \boldsymbol{Z}_0 = \boldsymbol{z}) = \sum_{\ell=1}^d \int_0^t (\boldsymbol{e}_\ell^\top e^{(t-u)\tilde{\boldsymbol{B}}} \boldsymbol{z}) e^{u\tilde{\boldsymbol{B}}} \boldsymbol{C}_\ell e^{u\tilde{\boldsymbol{B}}^\top} du,$$

hence, by Lemma 3.2,

$$\begin{split} \lim_{\lambda \downarrow 0} \partial_{\lambda_{i}} \partial_{\lambda_{j}} v_{k}(t, \boldsymbol{\lambda}) &= -\lim_{\lambda \downarrow 0} \left(\partial_{\lambda_{i}} \partial_{\lambda_{j}} g_{t, \boldsymbol{e}_{k}}(\boldsymbol{\lambda}) - \partial_{\lambda_{i}} g_{t, \boldsymbol{e}_{k}}(\boldsymbol{\lambda}) \partial_{\lambda_{j}} g_{t, \boldsymbol{e}_{k}}(\boldsymbol{\lambda}) \right) = -\operatorname{Cov}(Z_{t, i}, Z_{t, j} \mid \boldsymbol{Z}_{0} = \boldsymbol{e}_{k}) \\ &= -\sum_{\ell=1}^{d} \int_{0}^{t} (\boldsymbol{e}_{\ell}^{\top} e^{(t-u)\widetilde{\boldsymbol{B}}} \boldsymbol{e}_{k}) \boldsymbol{e}_{i}^{\top} e^{u\widetilde{\boldsymbol{B}}} \boldsymbol{C}_{\ell} e^{u\widetilde{\boldsymbol{B}}^{\top}} \boldsymbol{e}_{j} \, \mathrm{d}\boldsymbol{u}, \end{split}$$

and the proof is complete.

Let $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ be a multi-type CBI process with parameters $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ such that $\mathbb{E}(\|\boldsymbol{X}_0\|^2) < \infty$ and the moment conditions (3.8) hold. Note that $(n^{-1}\boldsymbol{X}_k)_{k\in\mathbb{Z}_+}$ is a Markov chain with state space \mathbb{R}^d_+ for all $n \in \mathbb{N}$. The discrete infinitesimal generator of the process $(\boldsymbol{\mathcal{X}}_t^{(n)})_{t\in\mathbb{R}_+} = (n^{-1}\boldsymbol{X}_{\lfloor nt \rfloor})_{t\in\mathbb{R}_+}$ is defined by

(3.11)
$$(\mathcal{A}_{\boldsymbol{\mathcal{X}}^{(n)}}f)(\boldsymbol{x}) := n[\mathbb{E}(f(n^{-1}\boldsymbol{X}_1) \mid n^{-1}\boldsymbol{X}_0 = \boldsymbol{x}) - f(\boldsymbol{x})], \qquad \boldsymbol{x} \in \mathbb{R}^d_+,$$

for any bounded and Borel measurable function $f : \mathbb{R}^d_+ \to \mathbb{R}$, see, e.g., Kato [14, Chapter IX, Section 3, formula (3.1)]. For $\lambda \in \mathbb{R}^d_+$, let us introduce the function

$$e_{\boldsymbol{\lambda}}(\boldsymbol{x}) := \mathrm{e}^{-\langle \boldsymbol{\lambda}, \boldsymbol{x} \rangle}, \qquad \boldsymbol{x} \in \mathbb{R}^d_+.$$

3.4 Proposition. Let $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ be a multi-type CBI process with parameters $(d, \mathbf{c}, \boldsymbol{\beta}, \mathbf{B}, \nu, \boldsymbol{\mu})$ such that $\mathbb{E}(\|\mathbf{X}_0\|^2) < \infty$ and the moment conditions (3.8) hold. Then

$$\begin{split} \lim_{n \to \infty} \left[(\mathcal{A}_{\mathcal{X}^{(n)}} e_{\lambda})(\boldsymbol{x}) + n \left(\mathrm{e}^{-\langle \boldsymbol{\lambda}, \boldsymbol{x} \rangle} - \mathrm{e}^{-\langle \boldsymbol{\lambda}, \mathrm{e}^{\widetilde{\boldsymbol{B}}} \boldsymbol{x} \rangle} \right) \right] \\ &= e_{\lambda} (\mathrm{e}^{\widetilde{\boldsymbol{B}}} \boldsymbol{x}) \left[\frac{1}{2} \sum_{\ell=1}^{d} \int_{0}^{1} (\boldsymbol{e}_{\ell}^{\top} \mathrm{e}^{(1-s)\widetilde{\boldsymbol{B}}} \boldsymbol{x}) \, \boldsymbol{\lambda}^{\top} \mathrm{e}^{s\widetilde{\boldsymbol{B}}} \boldsymbol{C}_{\ell} \mathrm{e}^{s\widetilde{\boldsymbol{B}}^{\top}} \boldsymbol{\lambda} \, \mathrm{d}s - \boldsymbol{\lambda}^{\top} \int_{0}^{1} \mathrm{e}^{s\widetilde{\boldsymbol{B}}} \widetilde{\boldsymbol{\beta}} \, \mathrm{d}s \right] \end{split}$$

for all $\boldsymbol{x} \in \mathbb{R}^d_+$ and $\boldsymbol{\lambda} \in \mathbb{R}^d_+$, where $\boldsymbol{\mathcal{X}}_t^{(n)} = n^{-1}\boldsymbol{X}_{\lfloor nt \rfloor}$, $t \in \mathbb{R}_+$, $n \in \mathbb{N}$. Consequently, given $\boldsymbol{x} \in \mathbb{R}^d_+$ and $\boldsymbol{\lambda} \in \mathbb{R}^d_+$, the sequence $(\mathcal{A}_{\boldsymbol{\mathcal{X}}^{(n)}}e_{\boldsymbol{\lambda}})(\boldsymbol{x})$ converges as $n \to \infty$ if and only if $\langle \boldsymbol{\lambda}, \boldsymbol{x} \rangle = \langle \boldsymbol{\lambda}, e^{\tilde{\boldsymbol{B}}} \boldsymbol{x} \rangle$.

Proof. By (2.3), for each $\lambda \in \mathbb{R}^d_+$ and $x \in \mathbb{R}^d_+$, we obtain

$$\begin{aligned} (\mathcal{A}_{\boldsymbol{\chi}^{(n)}} e_{\boldsymbol{\lambda}})(\boldsymbol{x}) &= n \left[\mathbb{E}(e_{\boldsymbol{\lambda}}(n^{-1}\boldsymbol{X}_{1}) \mid \boldsymbol{X}_{0} = n\boldsymbol{x}) - e_{\boldsymbol{\lambda}}(\boldsymbol{x}) \right] = n \left[\mathbb{E}(e^{-\langle \boldsymbol{\lambda}, n^{-1}\boldsymbol{X}_{1} \rangle} \mid \boldsymbol{X}_{0} = n\boldsymbol{x}) - e^{-\langle \boldsymbol{\lambda}, \boldsymbol{x} \rangle} \right] \\ &= n \left[\exp\left\{ -\langle n\boldsymbol{x}, \boldsymbol{v}(1, n^{-1}\boldsymbol{\lambda}) \rangle - \int_{0}^{1} \psi(\boldsymbol{v}(s, n^{-1}\boldsymbol{\lambda})) \, \mathrm{d}s \right\} - \exp\{-\langle \boldsymbol{\lambda}, \boldsymbol{x} \rangle\} \right]. \end{aligned}$$

Applying (3.7) and L'Hôspital's rule, we obtain

$$\lim_{h \downarrow 0} h^{-1} \langle \boldsymbol{x}, \boldsymbol{v}(1, h\boldsymbol{\lambda}) \rangle = \sum_{k=1}^{d} x_{k} \lim_{h \downarrow 0} h^{-1} v_{k}(1, h\boldsymbol{\lambda}) = \sum_{k=1}^{d} x_{k} \lim_{h \downarrow 0} \partial_{h} v_{k}(1, h\boldsymbol{\lambda})$$
$$= \sum_{k=1}^{d} x_{k} \sum_{i=1}^{d} \lambda_{i} \lim_{h \downarrow 0} \partial_{\lambda_{i}} v_{k}(1, h\boldsymbol{\lambda}) = \sum_{k=1}^{d} x_{k} \sum_{i=1}^{d} \lambda_{i} \boldsymbol{e}_{i}^{\top} e^{\tilde{\boldsymbol{B}}} \boldsymbol{e}_{k} = \boldsymbol{\lambda}^{\top} e^{\tilde{\boldsymbol{B}}} \boldsymbol{x} = \langle \boldsymbol{\lambda}, e^{\tilde{\boldsymbol{B}}} \boldsymbol{x} \rangle.$$

Applying (3.3), we have

$$\int_{0}^{1} \psi(\boldsymbol{v}(s,h\boldsymbol{\lambda})) \,\mathrm{d}s = \int_{0}^{1} \left(\langle \widetilde{\boldsymbol{\beta}}, \boldsymbol{v}(s,h\boldsymbol{\lambda}) \rangle - \int_{U_{d}} \left(\mathrm{e}^{-\langle \boldsymbol{v}(s,h\boldsymbol{\lambda}), \boldsymbol{z} \rangle} - 1 + \langle \boldsymbol{v}(s,h\boldsymbol{\lambda}), \boldsymbol{z} \rangle \right) \nu(\mathrm{d}\boldsymbol{z}) \right) \mathrm{d}s \to 0$$

as $h \downarrow 0$, since, by continuity of $[0,1] \ni s \mapsto \boldsymbol{v}(s,h\boldsymbol{\lambda}) \in \mathbb{R}^d_+$, $h \in \mathbb{R}_+$, by (3.6) and by monotone convergence theorem, we have $\int_0^1 \boldsymbol{v}(s,h\boldsymbol{\lambda}) \, \mathrm{d}s \downarrow \mathbf{0}$ as $h \downarrow 0$, and

$$0 \leq \int_{0}^{1} \left(\int_{U_{d}} \left(e^{-\langle \boldsymbol{v}(s,h\boldsymbol{\lambda}),\boldsymbol{z} \rangle} - 1 + \langle \boldsymbol{v}(s,h\boldsymbol{\lambda}),\boldsymbol{z} \rangle \right) \nu(\mathrm{d}\boldsymbol{z}) \right) \mathrm{d}s$$

$$\leq \frac{1}{2} \int_{0}^{1} \left(\int_{U_{d}} \langle \boldsymbol{v}(s,h\boldsymbol{\lambda}),\boldsymbol{z} \rangle^{2} \nu(\mathrm{d}\boldsymbol{z}) \right) \mathrm{d}s \leq \frac{1}{2} \int_{U_{d}} \|\boldsymbol{z}\|^{2} \nu(\mathrm{d}\boldsymbol{z}) \int_{0}^{1} \|\boldsymbol{v}(s,h\boldsymbol{\lambda})\|^{2} \mathrm{d}s \downarrow 0$$

as $h \downarrow 0$. Consequently,

(3.12)
$$\lim_{h \downarrow 0} \exp\left\{-h^{-1}\langle \boldsymbol{x}, \boldsymbol{v}(1, h\boldsymbol{\lambda})\rangle - \int_{0}^{1} \psi(\boldsymbol{v}(s, h\boldsymbol{\lambda})) \,\mathrm{d}s\right\} = \exp\left\{-\langle \boldsymbol{\lambda}, \mathrm{e}^{\tilde{\boldsymbol{B}}} \boldsymbol{x}\rangle\right\} = e_{\boldsymbol{\lambda}}(\mathrm{e}^{\tilde{\boldsymbol{B}}} \boldsymbol{x}).$$

Hence, applying again L'Hôspital's rule, we obtain

$$\lim_{n \to \infty} \left[(\mathcal{A}_{\mathcal{X}^{(n)}} e_{\boldsymbol{\lambda}})(\boldsymbol{x}) + n(\mathrm{e}^{-\langle \boldsymbol{\lambda}, \boldsymbol{x} \rangle} - \mathrm{e}^{-\langle \boldsymbol{\lambda}, \mathrm{e}^{\tilde{\boldsymbol{B}}} \boldsymbol{x} \rangle}) \right]$$

$$(3.13) \qquad = \lim_{n \to \infty} n \left[\exp\left\{ -\langle n\boldsymbol{x}, \boldsymbol{v}(1, n^{-1}\boldsymbol{\lambda}) \rangle - \int_{0}^{1} \psi(\boldsymbol{v}(s, n^{-1}\boldsymbol{\lambda})) \,\mathrm{d}s \right\} - \exp\{-\langle \boldsymbol{\lambda}, \mathrm{e}^{\tilde{\boldsymbol{B}}} \boldsymbol{x} \rangle\} \right]$$

$$= \lim_{h \downarrow 0} \partial_{h} \exp\left\{ -h^{-1} \langle \boldsymbol{x}, \boldsymbol{v}(1, h\boldsymbol{\lambda}) \rangle - \int_{0}^{1} \psi(\boldsymbol{v}(s, h\boldsymbol{\lambda})) \,\mathrm{d}s \right\}.$$

For each $h \in \mathbb{R}_{++}$ and $\lambda \in \mathbb{R}^d_+$, by dominated convergence theorem, we have

(3.14)
$$\partial_h \int_0^1 \psi(\boldsymbol{v}(s,h\boldsymbol{\lambda})) \, \mathrm{d}s = \lim_{\Delta \to 0} \int_0^1 \frac{\psi(\boldsymbol{v}(s,(h+\Delta)\boldsymbol{\lambda})) - \psi(\boldsymbol{v}(s,h\boldsymbol{\lambda}))}{\Delta} \, \mathrm{d}s$$
$$= \int_0^1 \partial_h \psi(\boldsymbol{v}(s,h\boldsymbol{\lambda})) \, \mathrm{d}s.$$

Indeed, for all $s, h \in \mathbb{R}_{++}$ and $\Delta \in (-h, h)$ with $\Delta \neq 0$, by mean value theorem,

$$\left|\frac{\psi(\boldsymbol{v}(s,(h+\Delta)\boldsymbol{\lambda}))-\psi(\boldsymbol{v}(s,h\boldsymbol{\lambda}))}{\Delta}\right| \leq \|\boldsymbol{\lambda}\| \sup_{\delta \in [h-|\Delta|,h+|\Delta|]} |\partial_{\delta}\psi(\boldsymbol{v}(s,\delta\boldsymbol{\lambda}))|,$$

where

$$\partial_{\delta}\psi(\boldsymbol{v}(s,\delta\boldsymbol{\lambda})) = \sum_{k=1}^{d} \partial_{\lambda_{k}}\psi(\boldsymbol{v}(s,\delta\boldsymbol{\lambda}))\partial_{\delta}v_{k}(s,\delta\boldsymbol{\lambda}) = \sum_{k=1}^{d} \partial_{\lambda_{k}}\psi(\boldsymbol{v}(s,\delta\boldsymbol{\lambda}))\sum_{i=1}^{d} \lambda_{i}\partial_{\lambda_{i}}v_{k}(s,\delta\boldsymbol{\lambda})$$

for all $\lambda \in \mathbb{R}^d_+$ and $\delta \in \mathbb{R}_{++}$. By (3.1),

(3.15)
$$|\partial_{\lambda_k}\psi(\boldsymbol{\lambda})| \leq \|\widetilde{\boldsymbol{\beta}}\| + \int_{U_d} \|\boldsymbol{z}\| \nu(\mathrm{d}\boldsymbol{z}), \qquad \boldsymbol{\lambda} \in \mathbb{R}^d_+, \quad k \in \{1, \dots, d\}.$$

By (3.10) and Lemma 3.2,

$$0 \leq \partial_{\lambda_{i}} v_{k}(s, \delta \boldsymbol{\lambda}) = -\frac{\partial_{\lambda_{i}} g_{s, \boldsymbol{e}_{k}}(\delta \boldsymbol{\lambda})}{g_{s, \boldsymbol{e}_{k}}(\delta \boldsymbol{\lambda})} = \frac{\mathbb{E}(Z_{s, i} e^{-\delta \langle \boldsymbol{\lambda}, \boldsymbol{Z}_{s} \rangle} \mid \boldsymbol{Z}_{0} = \boldsymbol{e}_{k})}{\mathbb{E}(e^{-\delta \langle \boldsymbol{\lambda}, \boldsymbol{Z}_{s} \rangle} \mid \boldsymbol{Z}_{0} = \boldsymbol{e}_{k})}$$
$$\leq \frac{\mathbb{E}(Z_{s, i} \mid \boldsymbol{Z}_{0} = \boldsymbol{e}_{k})}{\mathbb{E}(e^{-(h + |\Delta|) \langle \boldsymbol{\lambda}, \boldsymbol{Z}_{s} \rangle} \mid \boldsymbol{Z}_{0} = \boldsymbol{e}_{k})} \leq \frac{\mathbb{E}(Z_{s, i} \mid \boldsymbol{Z}_{0} = \boldsymbol{e}_{k})}{\mathbb{E}(e^{-2h \langle \boldsymbol{\lambda}, \boldsymbol{Z}_{s} \rangle} \mid \boldsymbol{Z}_{0} = \boldsymbol{e}_{k})} = \frac{\boldsymbol{e}_{i}^{\top} e^{s \widetilde{\boldsymbol{B}}} \boldsymbol{e}_{k}}{g_{s, \boldsymbol{e}_{k}}(2h\boldsymbol{\lambda})}$$

for all $\delta \in (h - |\Delta|, h + |\Delta|) \subset \mathbb{R}_{++}$, $\lambda \in \mathbb{R}^d_{++}$ and $i, k \in \{1, \ldots, d\}$, where $(\mathbf{Z}_t)_{t \in \mathbb{R}_+}$ is a multi-type CBI process with parameters $(d, \mathbf{c}, \mathbf{0}, \mathbf{B}, 0, \boldsymbol{\mu})$. Consequently,

$$\left|\frac{\psi(\boldsymbol{v}(s,(h+\Delta)\boldsymbol{\lambda})) - \psi(\boldsymbol{v}(s,h\boldsymbol{\lambda}))}{\Delta}\right| \leq \|\boldsymbol{\lambda}\| \left(\|\widetilde{\boldsymbol{\beta}}\| + \int_{U_d} \|\boldsymbol{z}\| \,\nu(\mathrm{d}\boldsymbol{z})\right) \sum_{k=1}^d \sum_{i=1}^d \frac{\lambda_i \boldsymbol{e}_i^\top \mathrm{e}^{s\widetilde{\boldsymbol{B}}} \boldsymbol{e}_k}{g_{s,\boldsymbol{e}_k}(2h\boldsymbol{\lambda})},$$

where the functions $\mathbb{R}_+ \ni s \mapsto e_i^{\top} e^{s\widetilde{B}} e_k \in \mathbb{R}_+$ and $\mathbb{R}_+ \ni s \mapsto g_{s,e_k}(2h\lambda) = e^{-v_k(s,2h\lambda)} \in \mathbb{R}_{++}$ are continuous, hence we conclude (3.14).

Applying (3.13), (3.14) and (3.12), we have

$$\begin{split} \lim_{n \to \infty} \left[(\mathcal{A}_{\mathcal{X}^{(n)}} e_{\lambda})(\boldsymbol{x}) + n \left(\mathrm{e}^{-\langle \boldsymbol{\lambda}, \boldsymbol{x} \rangle} - \mathrm{e}^{-\langle \boldsymbol{\lambda}, \mathrm{e}^{\tilde{\boldsymbol{B}}} \boldsymbol{x} \rangle} \right) \right] \\ &= e_{\boldsymbol{\lambda}} (\mathrm{e}^{\tilde{\boldsymbol{B}}} \boldsymbol{x}) \lim_{h \downarrow 0} \left[h^{-2} \sum_{k=1}^{d} x_{k} \left(v_{k}(1, h\boldsymbol{\lambda}) - h \sum_{i=1}^{d} \lambda_{i} \partial_{\lambda_{i}} v_{k}(1, h\boldsymbol{\lambda}) \right) \right. \\ &\left. - \sum_{k=1}^{d} \sum_{i=1}^{d} \lambda_{i} \int_{0}^{1} \partial_{\lambda_{k}} \psi(\boldsymbol{v}(s, h\boldsymbol{\lambda})) \partial_{\lambda_{i}} v_{k}(s, h\boldsymbol{\lambda}) \, \mathrm{d}s \right], \qquad \boldsymbol{\lambda} \in \mathbb{R}_{++}^{d}. \end{split}$$

By L'Hôspital's rule and by (3.9),

$$\begin{split} \lim_{h \downarrow 0} h^{-2} \sum_{k=1}^{d} x_k \left(v_k(1, h\boldsymbol{\lambda}) - h \sum_{i=1}^{d} \lambda_i \partial_{\lambda_i} v_k(1, h\boldsymbol{\lambda}) \right) \\ &= \sum_{k=1}^{d} x_k \lim_{h \downarrow 0} \frac{\sum_{i=1}^{d} \lambda_i \partial_{\lambda_i} v_k(1, h\boldsymbol{\lambda}) - \sum_{i=1}^{d} \lambda_i \partial_{\lambda_i} v_k(1, h\boldsymbol{\lambda}) - h \sum_{i=1}^{d} \sum_{j=1}^{d} \lambda_i \lambda_j \partial_{\lambda_i} \partial_{\lambda_j} v_k(1, h\boldsymbol{\lambda})}{2h} \\ &= -\sum_{k=1}^{d} x_k \lim_{h \downarrow 0} \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \lambda_i \lambda_j \partial_{\lambda_i} \partial_{\lambda_j} v_k(1, h\boldsymbol{\lambda}) \\ &= \frac{1}{2} \sum_{k=1}^{d} x_k \sum_{i=1}^{d} \sum_{j=1}^{d} \lambda_i \lambda_j \boldsymbol{e}_k^{\mathsf{T}} e^{\tilde{\boldsymbol{B}}^{\mathsf{T}}} \int_0^1 e^{-u\tilde{\boldsymbol{B}}^{\mathsf{T}}} \sum_{\ell=1}^{d} \boldsymbol{e}_\ell \boldsymbol{e}_i^{\mathsf{T}} e^{u\tilde{\boldsymbol{B}}} \boldsymbol{C}_\ell e^{u\tilde{\boldsymbol{B}}^{\mathsf{T}}} \boldsymbol{e}_j \, \mathrm{d}u \\ &= \frac{1}{2} \sum_{\ell=1}^{d} \int_0^1 \boldsymbol{x}^{\mathsf{T}} e^{\tilde{\boldsymbol{B}}^{\mathsf{T}}} e^{-u\tilde{\boldsymbol{B}}^{\mathsf{T}}} \boldsymbol{e}_\ell \lambda^{\mathsf{T}} e^{u\tilde{\boldsymbol{B}}} \boldsymbol{C}_\ell e^{u\tilde{\boldsymbol{B}}^{\mathsf{T}}} \boldsymbol{\lambda} \, \mathrm{d}u, \qquad \boldsymbol{\lambda} \in \mathbb{R}_{++}^d. \end{split}$$

For each $i, k \in \{1, \ldots, d\}$ and $\lambda \in \mathbb{R}^{d}_{++}$, by dominated convergence theorem, we have

(3.16)
$$\lim_{h \downarrow 0} \int_0^1 \partial_{\lambda_k} \psi(\boldsymbol{v}(s,h\boldsymbol{\lambda})) \partial_{\lambda_i} v_k(s,h\boldsymbol{\lambda}) \, \mathrm{d}s = \int_0^1 \lim_{h \downarrow 0} \partial_{\lambda_k} \psi(\boldsymbol{v}(s,h\boldsymbol{\lambda})) \partial_{\lambda_i} v_k(s,h\boldsymbol{\lambda}) \, \mathrm{d}s$$

Indeed, again by (3.10) and Lemma 3.2,

$$0 \leq \partial_{\lambda_{i}} v_{k}(s, h\boldsymbol{\lambda}) = -\frac{\partial_{\lambda_{i}} g_{s, \boldsymbol{e}_{k}}(h\boldsymbol{\lambda})}{g_{s, \boldsymbol{e}_{k}}(h\boldsymbol{\lambda})} = \frac{\mathbb{E}(Z_{s, i} e^{-h\langle\boldsymbol{\lambda}, \boldsymbol{Z}_{s}\rangle} \mid \boldsymbol{Z}_{0} = \boldsymbol{e}_{k})}{\mathbb{E}(e^{-h\langle\boldsymbol{\lambda}, \boldsymbol{Z}_{s}\rangle} \mid \boldsymbol{Z}_{0} = \boldsymbol{e}_{k})} \\ \leq \frac{\mathbb{E}(Z_{s, i} \mid \boldsymbol{Z}_{0} = \boldsymbol{e}_{k})}{\mathbb{E}(e^{-\langle\boldsymbol{\lambda}, \boldsymbol{Z}_{s}\rangle} \mid \boldsymbol{Z}_{0} = \boldsymbol{e}_{k})} = \frac{\boldsymbol{e}_{i}^{\top} e^{s\tilde{\boldsymbol{B}}} \boldsymbol{e}_{k}}{g_{s, \boldsymbol{e}_{k}}(\boldsymbol{\lambda})}$$

for all $h \in (0,1)$, $\boldsymbol{\lambda} \in \mathbb{R}^d_{++}$, $s \in \mathbb{R}_+$ and $i, k \in \{1, \ldots, d\}$, hence, applying (3.15),

$$|\partial_{\lambda_k}\psi(\boldsymbol{v}(s,h\boldsymbol{\lambda}))\partial_{\lambda_i}v_k(s,h\boldsymbol{\lambda})| \leqslant \left(\|\widetilde{\boldsymbol{\beta}}\| + \int_{U_d} \|\boldsymbol{z}\|\,\nu(\mathrm{d}\boldsymbol{z})\right) \frac{\boldsymbol{e}_i^{\top}\mathrm{e}^{s\widetilde{\boldsymbol{B}}}\boldsymbol{e}_k}{g_{s,\boldsymbol{e}_k}(\boldsymbol{\lambda})},$$

hence we conclude (3.16). Applying (3.2), (3.6) and (3.7), we have

$$\sum_{k=1}^{d} \sum_{i=1}^{d} \lambda_{i} \int_{0}^{1} \lim_{h \downarrow 0} \partial_{\lambda_{k}} \psi(\boldsymbol{v}(s, h\boldsymbol{\lambda})) \partial_{\lambda_{i}} v_{k}(s, h\boldsymbol{\lambda}) \,\mathrm{d}s$$
$$= \sum_{k=1}^{d} \sum_{i=1}^{d} \lambda_{i} \int_{0}^{1} \widetilde{\beta}_{k}(\boldsymbol{e}_{i}^{\top} \mathrm{e}^{s\widetilde{\boldsymbol{B}}} \boldsymbol{e}_{k}) \,\mathrm{d}s = \boldsymbol{\lambda}^{\top} \int_{0}^{1} \mathrm{e}^{u\widetilde{\boldsymbol{B}}} \widetilde{\boldsymbol{\beta}} \,\mathrm{d}u,$$

hence we obtain the statement.

3.5 Corollary. Let $(\mathbf{X}_t)_{t\in\mathbb{R}_+}$ be an irreducible and critical multi-type CBI process with parameters $(d, \mathbf{c}, \boldsymbol{\beta}, \mathbf{B}, \nu, \boldsymbol{\mu})$ such that $\mathbb{E}(\|\mathbf{X}_0\|^4) < \infty$ and the moment conditions (2.9) hold and $\widetilde{\mathbf{B}}$ given in (2.6) is not **0** (implying $d \ge 2$). Then $(\mathbf{X}_t^{(n)})_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}} (\mathbf{X}_t \mathbf{u}_{right})_{t\in\mathbb{R}_+}$ as $n \to \infty$, and, given $\mathbf{x} \in \mathbb{R}^d_+$ and $\mathbf{\lambda} \in \mathbb{R}^d_+$, the sequence $(\mathcal{A}_{\mathbf{X}^{(n)}} e_{\mathbf{\lambda}})(\mathbf{x})$ converges as $n \to \infty$ if and only if $\langle \mathbf{\lambda}, \mathbf{x} \rangle = \langle \mathbf{\lambda}, e^{\widetilde{\mathbf{B}}} \mathbf{x} \rangle$, where $(\mathcal{A}_{\mathbf{X}^{(n)}} e_{\mathbf{\lambda}})(\mathbf{x})$ is defined in (3.11). In particular,

- (i) there exist $\boldsymbol{x} \in \mathbb{R}^d_+$ and $\boldsymbol{\lambda} \in \mathbb{R}^d_+$ such that the sequence $(\mathcal{A}_{\boldsymbol{\mathcal{X}}^{(n)}}e_{\boldsymbol{\lambda}})(\boldsymbol{x})$ does not converge as $n \to \infty$,
- (ii) the sequence $(\mathcal{A}_{\mathcal{X}^{(n)}}e_{\lambda})(\mathbf{x})$ converges as $n \to \infty$ for all $\lambda \in \mathbb{R}^d_+$ if and only if $\mathbf{x} = \delta \mathbf{u}_{right}$ with some $\delta \in \mathbb{R}$.

Proof. First, we note that there exists a multi-type CBI process which satisfies the conditions of the corollary. Namely, every 2-type CBI process with parameters $(2, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ satisfying the moment conditions (2.9) with

$$\widetilde{B} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

serves us as an example. The convergence $(\mathcal{X}_{t}^{(n)})_{t\in\mathbb{R}_{+}} \xrightarrow{\mathcal{D}} (\mathcal{X}_{t}\boldsymbol{u}_{\mathrm{right}})_{t\in\mathbb{R}_{+}}$ as $n \to \infty$ follows by Theorem 2.9. Proposition 3.4 yields that the sequence $(\mathcal{A}_{\mathcal{X}^{(n)}}e_{\lambda})(\boldsymbol{x})$ converges as $n \to \infty$ if and only if $\langle \boldsymbol{\lambda}, \boldsymbol{x} \rangle = \langle \boldsymbol{\lambda}, \mathrm{e}^{\tilde{B}}\boldsymbol{x} \rangle$. Next we prove that there exist $\boldsymbol{x} \in \mathbb{R}_{+}^{d}$ and $\boldsymbol{\lambda} \in \mathbb{R}_{+}^{d}$ such that the sequence $(\mathcal{A}_{\mathcal{X}^{(n)}}e_{\lambda})(\boldsymbol{x})$ does not converge as $n \to \infty$. By Proposition 3.4, if $\langle \boldsymbol{\lambda}, \boldsymbol{x} \rangle \neq \langle \boldsymbol{\lambda}, \mathrm{e}^{\tilde{B}}\boldsymbol{x} \rangle$ with some $\boldsymbol{x} \in \mathbb{R}_{+}^{d}$ and $\boldsymbol{\lambda} \in \mathbb{R}_{+}^{d}$, then the sequence $(\mathcal{A}_{\mathcal{X}^{(n)}}e_{\lambda})(\boldsymbol{x})$ does not converge as $n \to \infty$. Using Dunford and Schwartz [8, Theorem VII.1.8], one can easily check that the following statements are equivalent:

- $\langle \boldsymbol{\lambda}, \mathrm{e}^{\widetilde{\boldsymbol{B}}} \boldsymbol{x} \rangle = \langle \boldsymbol{\lambda}, \boldsymbol{x} \rangle$ for all $\boldsymbol{x} \in \mathbb{R}^d_+$ and $\boldsymbol{\lambda} \in \mathbb{R}^d_+$;
- $e^{\widetilde{B}}x = x$ for all $x \in \mathbb{R}^d_+$;
- $\sigma(\mathbf{e}^{\tilde{B}}) = \{1\};$
- $\sigma(\widetilde{\boldsymbol{B}}) = \{0\};$
- $\widetilde{B} = 0$.

Since $\widetilde{B} \neq 0$, there exist some $x \in \mathbb{R}^d_+$ and $\lambda \in \mathbb{R}^d_+$ such that $\langle \lambda, e^{\widetilde{B}}x \rangle \neq \langle \lambda, x \rangle$, implying (i). Given $x \in \mathbb{R}^d_+$, we have $\langle \lambda, e^{\widetilde{B}}x \rangle = \langle \lambda, x \rangle$ for all $\lambda \in \mathbb{R}^d_+$ if and only if $e^{\widetilde{B}}x = x$, which holds if and only if $x = \delta u_{\text{right}}$ with some $\delta \in \mathbb{R}$, yielding (ii).

3.6 Remark. Rosenkrantz [15], [16] provided an example for a sequence of one-dimensional diffusion processes given by SDEs which converges weakly to a Markov limit process, however the drift coefficients of the corresponding SDEs do not converge, and consequently, the corresponding sequence of (usual) infinitesimal generators does not converge either. He also provided

an example for one-dimensional diffusion processes given by SDEs which converge weakly to a Markov limit process, and the drift and diffusion coefficients of the corresponding SDEs converge, but their limits are not the ones that are expected to appear in the infinitesimal generator of the limit Markov process. On the one hand, Corollary 3.5 can be considered as a non-trivial multi-dimensional example, which resembles the phenomena described by Rosenkrantz. On the other hand, part (ii) of Corollary 3.5 is in accordance with Theorem 2.9, since there the degenerate limit process is concentrated on the ray determined by u_{right} . It is an open question whether Theorem 2.9 might be proved by the help of infinitesimal generators.

It is also interesting to investigate the sequence $\mathcal{Y}_t^{(n)} := n^{-1} \mathbf{X}_{nt}, t \in \mathbb{R}_+, n \in \mathbb{N}$, of scaled CBI processes. Note that both processes $\mathcal{X}^{(n)}$ and $\mathcal{Y}^{(n)}$ have càdlàg sample paths almost surely, however, $\mathcal{Y}^{(n)}$ is no longer a step process, which gives the possibility of studying convergence properties of their usual infinitesimal generators.

3.7 Proposition. Let $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ be a multi-type CBI process with parameters $(d, \mathbf{c}, \boldsymbol{\beta}, \mathbf{B}, \nu, \boldsymbol{\mu})$ such that the moment conditions (3.8) hold. Then

(3.17)
$$\lim_{n \to \infty} \left((\mathcal{A}_{\mathcal{Y}^{(n)}} f)(\boldsymbol{x}) - n \langle \widetilde{\boldsymbol{B}} \boldsymbol{x}, \boldsymbol{f}'(\boldsymbol{x}) \rangle \right) = \frac{1}{2} \sum_{i=1}^{d} x_i \sum_{k=1}^{d} \sum_{\ell=1}^{d} \boldsymbol{e}_k^\top \boldsymbol{C}_i \boldsymbol{e}_\ell f_{k,\ell}''(\boldsymbol{x}) + \langle \widetilde{\boldsymbol{\beta}}, \boldsymbol{f}'(\boldsymbol{x}) \rangle$$

for all $f \in C^2_c(\mathbb{R}^d_+, \mathbb{R})$ and $\boldsymbol{x} \in \mathbb{R}^d_+$, where $\mathcal{A}_{\boldsymbol{\mathcal{Y}}^{(n)}}$ denotes the usual infinitesimal generator of $\boldsymbol{\mathcal{Y}}^{(n)}$. Consequently, given $f \in C^2_c(\mathbb{R}^d_+, \mathbb{R})$ and $\boldsymbol{x} \in \mathbb{R}^d_+$, the sequence $(\mathcal{A}_{\boldsymbol{\mathcal{Y}}^{(n)}}f)(\boldsymbol{x})$ converges as $n \to \infty$ if and only if $\langle \widetilde{\boldsymbol{B}}\boldsymbol{x}, \boldsymbol{f}'(\boldsymbol{x}) \rangle = 0$.

Proof. First note that, under the moment conditions (3.8), the infinitesimal generator (2.2) of the process $(\mathbf{X}_t)_{t \in \mathbb{R}_+}$ can also be written in the form

$$\begin{aligned} (\mathcal{A}_{\boldsymbol{X}}f)(\boldsymbol{x}) &= \frac{1}{2} \sum_{i=1}^{d} x_{i} \sum_{k=1}^{d} \sum_{\ell=1}^{d} f_{k,\ell}''(\boldsymbol{x}) \langle \boldsymbol{C}_{i} \boldsymbol{e}_{\ell}, \boldsymbol{e}_{k} \rangle + \langle \boldsymbol{\beta} + \widetilde{\boldsymbol{B}} \boldsymbol{x}, \boldsymbol{f}'(\boldsymbol{x}) \rangle + \int_{U_{d}} (f(\boldsymbol{x} + \boldsymbol{z}) - f(\boldsymbol{x})) \, \nu(\mathrm{d}\boldsymbol{z}) \\ &+ \sum_{i=1}^{d} x_{i} \int_{U_{d}} \left(f(\boldsymbol{x} + \boldsymbol{z}) - f(\boldsymbol{x}) - \langle \boldsymbol{z}, \boldsymbol{f}'(\boldsymbol{x}) \rangle - \frac{1}{2} \langle \boldsymbol{z}, \boldsymbol{f}''(\boldsymbol{x}) \boldsymbol{z} \rangle \right) \mu_{i}(\mathrm{d}\boldsymbol{z}) \end{aligned}$$

for $f \in C^2_{\rm c}(\mathbb{R}^d_+,\mathbb{R})$ and $\boldsymbol{x} \in \mathbb{R}^d_+$. Indeed, by Remark 4.3 in Barczy et al. [6], $\int_{U_d} \|\boldsymbol{z}\|^2 \mu_i(\mathrm{d}\boldsymbol{z}) < \infty, \ i \in \{1,\ldots,d\},$ and using (2.10),

$$(\mathcal{A}_{\boldsymbol{X}}f)(\boldsymbol{x}) - \frac{1}{2}\sum_{i=1}^{d} x_i \sum_{k=1}^{d} \sum_{\ell=1}^{d} f_{k,\ell}''(\boldsymbol{x}) \langle \boldsymbol{C}_i \boldsymbol{e}_{\ell}, \boldsymbol{e}_k \rangle - \langle \boldsymbol{\beta} + \widetilde{\boldsymbol{B}}\boldsymbol{x}, \boldsymbol{f}'(\boldsymbol{x}) \rangle - \int_{U_d} (f(\boldsymbol{x} + \boldsymbol{z}) - f(\boldsymbol{x})) \nu(\mathrm{d}\boldsymbol{z}) \\ - \sum_{i=1}^{d} x_i \int_{U_d} \left(f(\boldsymbol{x} + \boldsymbol{z}) - f(\boldsymbol{x}) - \langle \boldsymbol{z}, \boldsymbol{f}'(\boldsymbol{x}) \rangle - \frac{1}{2} \langle \boldsymbol{z}, \boldsymbol{f}''(\boldsymbol{x}) \boldsymbol{z} \rangle \right) \mu_i(\mathrm{d}\boldsymbol{z}) = D_1 + D_2,$$

where

$$D_{1} := \sum_{i=1}^{d} c_{i} x_{i} f_{i,i}''(x) + \frac{1}{2} \sum_{i=1}^{d} x_{i} \sum_{k=1}^{d} \sum_{\ell=1}^{d} f_{k,\ell}''(x) \int_{U_{d}} z_{k} z_{\ell} \mu_{i}(\mathrm{d}\boldsymbol{z}) - \frac{1}{2} \sum_{i=1}^{d} x_{i} \sum_{k=1}^{d} \sum_{\ell=1}^{d} f_{k,\ell}''(x) \boldsymbol{e}_{k}^{\top} \boldsymbol{C}_{i} \boldsymbol{e}_{\ell} = 0$$

and

$$D_{2} := \sum_{i=1}^{d} x_{i} \int_{U_{d}} \left(\langle \boldsymbol{z}, \boldsymbol{f}'(\boldsymbol{x}) \rangle - f_{i}'(\boldsymbol{x})(1 \wedge z_{i}) \right) \mu_{i}(\mathrm{d}\boldsymbol{z}) - \langle (\widetilde{\boldsymbol{B}} - \boldsymbol{B})\boldsymbol{x}, \boldsymbol{f}'(\boldsymbol{x}) \rangle$$
$$= \sum_{i=1}^{d} x_{i} \int_{U_{d}} \left(f_{i}'(\boldsymbol{x})(z_{i} - (1 \wedge z_{i})) + \sum_{j \in \{1, \dots, d\} \setminus \{i\}} z_{j} f_{j}'(\boldsymbol{x}) \right) \mu_{i}(\mathrm{d}\boldsymbol{z})$$
$$- \sum_{i=1}^{d} \sum_{j=1}^{d} x_{j} f_{i}'(\boldsymbol{x}) \int_{U_{d}} (z_{i} - \delta_{i,j})^{+} \mu_{j}(\mathrm{d}\boldsymbol{z}) = 0.$$

For each $n \in \mathbb{N}$, the infinitesimal generator of the process $(\boldsymbol{\mathcal{Y}}_{t}^{(n)})_{t \in \mathbb{R}_{+}}$ is

$$(\mathcal{A}_{\mathcal{Y}^{(n)}}f)(\boldsymbol{x}) = n(\mathcal{A}_{\boldsymbol{X}}f_n)(n\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^d_+,$$

where $f_n(\boldsymbol{x}) := f(n^{-1}\boldsymbol{x}), \ \boldsymbol{x} \in \mathbb{R}^d_+$, for all $f \in C^2_c(\mathbb{R}^d_+, \mathbb{R})$, see, e.g., Barczy et al. [1, Lemma 2.1]. Consequently, by (2.7),

$$\begin{aligned} (\mathcal{A}_{\boldsymbol{\mathcal{Y}}^{(n)}}f)(\boldsymbol{x}) &= \frac{1}{2} \sum_{i=1}^{d} x_{i} \sum_{k=1}^{d} \sum_{\ell=1}^{d} f_{k,\ell}''(\boldsymbol{x}) \boldsymbol{e}_{k}^{\top} \boldsymbol{C}_{i} \boldsymbol{e}_{\ell} + \langle \widetilde{\boldsymbol{\beta}} + n \widetilde{\boldsymbol{B}} \boldsymbol{x}, \boldsymbol{f}'(\boldsymbol{x}) \rangle \\ &+ n \int_{U_{d}} \left(f(\boldsymbol{x} + n^{-1}\boldsymbol{z}) - f(\boldsymbol{x}) - \langle n^{-1}\boldsymbol{z}, \boldsymbol{f}'(\boldsymbol{x}) \rangle \right) \nu(\mathrm{d}\boldsymbol{z}) \\ &+ n^{2} \sum_{i=1}^{d} x_{i} \int_{U_{d}} \left(f(\boldsymbol{x} + n^{-1}\boldsymbol{z}) - f(\boldsymbol{x}) - \langle n^{-1}\boldsymbol{z}, \boldsymbol{f}'(\boldsymbol{x}) \rangle - \frac{1}{2} \langle n^{-1}\boldsymbol{z}, \boldsymbol{f}''(\boldsymbol{x}) n^{-1}\boldsymbol{z} \rangle \right) \mu_{i}(\mathrm{d}\boldsymbol{z}). \end{aligned}$$

One can show

$$\lim_{n \to \infty} \sup_{\boldsymbol{x} \in \mathbb{R}^d_+} \left| n \int_{U_d} \left(f(\boldsymbol{x} + n^{-1}\boldsymbol{z}) - f(\boldsymbol{x}) - \langle n^{-1}\boldsymbol{z}, \boldsymbol{f}'(\boldsymbol{x}) \rangle \right) \, \nu(\mathrm{d}\boldsymbol{z}) \right| = 0,$$
$$\lim_{n \to \infty} \sup_{\boldsymbol{x} \in \mathbb{R}^d_+} \left| n^2 x_i \int_{U_d} \left(f(\boldsymbol{x} + n^{-1}\boldsymbol{z}) - f(\boldsymbol{x}) - \langle n^{-1}\boldsymbol{z}, \boldsymbol{f}'(\boldsymbol{x}) \rangle - \frac{1}{2} \langle n^{-1}\boldsymbol{z}, \boldsymbol{f}''(\boldsymbol{x}) n^{-1}\boldsymbol{z} \rangle \right) \mu_i(\mathrm{d}\boldsymbol{z}) \right| = 0$$

for all $i \in \{1, \ldots, d\}$, see the method of the proof of formulas (2.6) and (2.7) in Barczy et al. [1]. Consequently, for each $f \in C_c^2(\mathbb{R}^d_+, \mathbb{R})$, we obtain (3.17).

3.8 Remark. If we consider a single-type (hence irreducible) and critical (hence $\tilde{B} = 0$) CBI process with parameters $(1, c, \beta, B, \nu, \mu)$ such that the moment conditions (3.8) hold, then, by Proposition 3.7,

$$\lim_{n \to \infty} (\mathcal{A}_{\mathcal{Y}^{(n)}} f)(x) = \frac{1}{2} x C_1 f_{1,1}''(x) + \widetilde{\beta} f'(x), \qquad f \in C_c^2(\mathbb{R}_+, \mathbb{R}), \qquad x \in \mathbb{R}_+$$

Here the limit is nothing else but the infinitesimal generator of a squared Bessel process, which is in accordance with the result of Huang et al. [12, Theorem 2.3]. In fact, Huang et al. [12] proved that for a critical single-type CBI process $(X_t)_{t\in\mathbb{R}_+}$ satisfying the moment conditions (3.8), the sequence of scaled processes $(n^{-1}X_{nt})_{t\in\mathbb{R}_+}$, $n \in \mathbb{N}$, converges weakly to a squared Bessel process. Finally, we note that, to the best knowledge of the authors, it is not known, whether the sequence of scaled processes $(n^{-1}X_{nt})_{t\in\mathbb{R}_+}$, $n \in \mathbb{N}$, is convergent for an irreducible and critical d-type CBI process with $d \ge 2$.

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