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Wrapping corrections for non-diagonal boundaries in AdS/CFT

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Abstract

We consider an open string stretched between a $Y = 0$ brane and a $Y_\theta = 0$ brane. The latter brane is rotated with respect to the former by an angle θ, and is described by a non-diagonal boundary S-matrix. This system interpolates smoothly between the $Y - Y$ ($\theta = 0$) and the $Y - \bar{Y}$ ($\theta = \pi/2$) systems, which are described by diagonal boundary S-matrices. We use integrability to compute the energies of one-particle states at weak coupling up to leading wrapping order (4, 6 loops) as a function of the angle. The results for the diagonal cases exactly match with those obtained previously.

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1 Introduction

The light-cone-gauge worldsheet theory of a free closed type-IIB superstring on $AdS_5 \times S^5$ is integrable [1]. Since this string theory is dual [2] to planar $\mathcal{N} = 4$ super Yang-Mills theory in $3 + 1$ dimensions, we refer to this worldsheet theory as the AdS_5/CFT_4 integrable model. The integrability of this model with periodic boundary conditions can be exploited to compute the energies of multiparticle states of the closed string, which coincide with anomalous dimensions of corresponding single-trace operators in the dual gauge theory. For large volumes L, the energies are determined by the asymptotic Bethe ansatz equations [3], which incorporate all polynomial corrections in the inverse power of the volume. The subsequent exponentially-small finite-size corrections are related [4] to wrapping contributions in the gauge theory. These wrapping corrections are due to the vacuum polarization effects of bound states on multiparticle states in the AdS_5/CFT_4 integrable model, which can be explicitly evaluated [5] and exactly match with perturbative gauge theory results [6]. All higher-order wrapping corrections can be summed up by the excited-states thermodynamic Bethe ansatz (TBA) equations [7, 8], which have a nice reformulation in terms of the quantum spectral curve [9].

It is also possible to consider the AdS_5/CFT_4 integrable model on a strip with integrable boundary conditions. This scenario can be realized by an open string stretched between two maximal giant gravitons (D-branes). Multiparticle states of the open string correspond to so-called determinant-like operators in the dual gauge theory. The $Y - Y$ system, consisting of $Y = 0$ branes [10] at both ends of the string, was studied in [11, 12, 13]. The integrability of this model was exploited in [12, 13] to compute wrapping corrections of one-particle states. The $Y - \bar{Y}$ system, consisting of a $Y = 0$ brane at one end of the string and a $\bar{Y} = 0$ brane at the other end, was subsequently investigated in [14]; and wrapping corrections of one-particle states were again computed.

We consider here the $Y - Y_\theta$ system, consisting of a $Y = 0$ brane at one end of the string and a $Y_\theta = 0$ brane at the other end, where the latter brane is rotated with respect to the former by an angle θ. This system interpolates smoothly between the $Y - Y$ system ($\theta = 0$) and the $Y - \bar{Y}$ system ($\theta = \pi/2$). We exploit the integrability of this model to compute the leading wrapping corrections for one-particle states with $L = 2$, as functions of the angle θ. We verify that these results reduce for $\theta = 0$ and $\theta = \pi/2$ to those obtained previously in [12, 13] and [14], respectively. In principle, it should be possible to confirm these results from 4-loop and 6-loop computations in the dual gauge theory.

In order to carry out this analysis, it is necessary to know the Bethe-ansatz expression for the eigenvalues of transfer matrices constructed with the bulk and boundary worldsheet S-matrices of the AdS_5/CFT_4 integrable model. Since the boundary S-matrix corresponding to the $Y_\theta = 0$ brane is generally not diagonal, the problem of determining the transfer-matrix eigenvalues is nontrivial. Indeed, even for the much simpler problem of the XXX open spin chain with non-diagonal boundary terms, a Bethe ansatz solution was obtained only quite recently using the so-called off-diagonal Bethe ansatz approach [15]. With the help of this approach, the Bethe ansatz solution of the AdS/CFT problem was found in [16]. We use that solution here to formulate the asymptotic Bethe ansatz for the AdS_5/CFT_4 integrable model with non-diagonal boundary conditions, and to compute leading wrapping corrections.

The outline of this paper is as follows. In Section 2, we first collect all the ingredients needed for the computation (S-matrices, transfer matrices, Bethe-ansatz solution, etc.), and then calculate the energies of one-particle states with $L = 2$ at weak coupling up to wrapping order using the asymptotic Bethe ansatz. In Section 3 we compute the leading wrapping corrections for these states, and compare with previous results for the diagonal cases. In Section 4, we give the corresponding results for $L = 1$. We conclude in Section 5 with a brief discussion of our results, and list some related open problems.

2 Asymptotic Bethe ansatz

There are two types of finite-size corrections to the energies of multiparticle states in finite volume. The leading corrections are polynomial in the inverse power of the volume and can be accounted for the momentum quantization of the particles. These corrections can be obtained from the Bethe-Yang equation/asymptotic Bethe ansatz, which implements the periodicity of the wave functions in a very nontrivial way. The other corrections are exponentially small in the volume and have quantum field theoretical origin. Indeed, these corrections come from vacuum polarization effects due to the presence of virtual particles.

In this section, we obtain the finite-size corrections from the asymptotic Bethe ansatz. We first briefly review the scattering theory of the AdS_5/CFT_4 integrable model and formulate the boundary Bethe-Yang equation. We then introduce the relevant transfer matrices, and review the Bethe-ansatz solution for their eigenvalues. Finally, we use the asymptotic Bethe ansatz to compute the energies of one-particle states.

2.1 Fundamental S-matrices

The AdS_5/CFT_4 integrable model is a (1+1)-dimensional non-relativistic quantum field theory with a centrally-extended $SU(2|2) \otimes SU(2|2)$ symmetry. The spectrum of this model includes a set of 16 fundamental particles, which we denote by

$$|(\alpha, \dot{\alpha})\rangle = |\alpha\rangle \otimes |\dot{\alpha}\rangle\,, \qquad \alpha = 1, 2, 3, 4, \qquad \dot{\alpha} = \dot{1}, \dot{2}, \dot{3}, \dot{4}, \tag{2.1}$$

where the $SU(2|2) \otimes SU(2|2)$ labels $1, 2, \dot{1}, \dot{2}$ are bosonic, and $3, 4, \dot{3}, \dot{4}$ are fermionic. These particles all have the same energy-momentum dispersion relation

$$\epsilon(p) = \sqrt{1 + 16g^2 \sin^2 \frac{p}{2}}, \qquad g = \frac{\sqrt{\lambda}}{4\pi}\,, \tag{2.2}$$

where λ is the 't Hooft coupling.

Let us now consider a system of N such particles with momenta p_i $(i = 1, ..., N)$ on a strip of finite length L. (Eventually, we shall restrict to the case $N = 1$.) For large L, this system can be analyzed using the bulk and boundary S-matrices of these fundamental particles. The bulk two-particle S-matrix is given by [17, 18, 19]

$$\mathbb{S}(p_1, p_2) = S_0(p_1, p_2)\, S(p_1, p_2) \otimes \dot{S}(p_1, p_2)\,, \tag{2.3}$$

whose index structure is given by

$$\mathbb{S}^{(\beta,\dot\beta)(\delta,\dot\delta)}_{(\alpha,\dot\alpha)(\gamma,\dot\gamma)} = S_0\, S^{\beta\delta}_{\alpha\gamma}\dot S^{\dot\beta\dot\delta}_{\dot\alpha\dot\gamma}\,.$$

Both $S = S^{\beta\delta}_{\alpha\gamma}$ and $\dot S = S^{\dot\beta\dot\delta}_{\dot\alpha\dot\gamma}$ are given by the graded 16×16 matrix in [19], which is normalized such that $S^{11}_{11} = \dot S^{\dot 1\dot 1}_{\dot 1\dot 1} = 1$, and the scalar factor S_0 is given by

$$S_0(p_1,p_2) = \frac{x_1^+ + \frac{1}{x_1^+} - x_2^- - \frac{1}{x_2^-}}{x_1^- + \frac{1}{x_1^-} - x_2^+ - \frac{1}{x_2^+}}\frac{x_1^-}{x_1^+}\frac{x_2^+}{x_2^-}\sigma^2(p_1,p_2). \tag{2.4}$$

Here we define $x^\pm(p)$ by

$$x^\pm(p) = \frac{1}{4g}(\cot\frac{p}{2} \pm i)(1+\epsilon(p))\,, \tag{2.5}$$

and $x_i^\pm = x^\pm(p_i)$. We shall also make use of the rapidity variable u defined by

$$x(u) + \frac{1}{x(u)} = \frac{u}{g}\,. \tag{2.6}$$

The dressing factor is given by [20, 21]

$$\sigma(p_1,p_2) = e^{i\Theta(p_1,p_2)}\,, \qquad \Theta(p_1,p_2) = \chi(x_1^+,x_2^+) + \chi(x_1^-,x_2^-) - \chi(x_1^+,x_2^-) - \chi(x_1^-,x_2^+)\,, \tag{2.7}$$

where

$$\chi(x_1,x_2) = -\sum_{r=2}^{\infty}\sum_{s>r}\frac{c_{r,s}(g)}{(r-1)(s-1)}\left[\frac{1}{x_1^{r-1}x_2^{s-1}} - \frac{1}{x_1^{s-1}x_2^{r-1}}\right] \tag{2.8}$$

with

$$c_{r,s}(g) = (r-1)(s-1)2\cos(\tfrac{\pi}{2}(s-r-1))\int_0^\infty dt\frac{J_{r-1}(2gt)J_{s-1}(2gt)}{t(e^t-1)}\,. \tag{2.9}$$

We assume that the right boundary S-matrix (reflection factor) is given by

$$\mathbb{R}^-(p) = R_0^-(p)R^-(p)\otimes\dot R^-(p) \tag{2.10}$$

with [10, 22]

$$R_0^-(p) = -e^{-ip}\sigma(p,-p)\,, \qquad R^-(p) = \dot R^-(p) = \mathrm{diag}(e^{-ip/2}, -e^{ip/2}, 1, 1)\,. \tag{2.11}$$

This diagonal boundary S-matrix corresponds to a $Y=0$ brane [10]. Let $\mathbb{R}^-_\theta(p)$ denote the boundary S-matrix obtained by an angle θ rotation

$$\mathbb{R}^-_\theta(p) = R_0^-(p)R_\theta^-(p)\otimes\dot R_\theta^-(p) \tag{2.12}$$

where

$$R_\theta^-(p) = O(-\theta)R^-(p)\,O(\theta)\,, \qquad O(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.13}$$

and same for the dotted indices (i.e., with the same angle θ for both undotted and dotted factors). In principle we could have a different angle $\dot{\theta}$ in $\dot{R}$, but for simplicity we assume $\theta = \dot{\theta}$, such that we can easily interpolate between the $Y - Y$ ($\theta = 0$) and the $Y - \bar{Y}$ ($\theta = \pi/2$) cases.

We assume that the left boundary S-matrix is given by [14]

$$\mathbb{R}^{+}(p) = \mathbb{R}^{-}_{\theta}(-p)\,. \tag{2.14}$$

This boundary S-matrix, which corresponds to a $Y_\theta = 0$ brane, is evidently not diagonal for generic angles.

2.2 Boundary Bethe-Yang equation

Our goal is to compute the energies of multiparticle states. For large L, a first approximation to the energy is given by the sum of single-particle energies

$$E = \sum_{i=1}^{N} \epsilon(p_i)\,, \tag{2.15}$$

where $\epsilon(p)$ is defined in (2.2). Hence, it is necessary to determine the particle momenta p_i, which are quantized for finite L. Since the particles have nontrivial scattering, the quantization condition is given by the boundary Bethe-Yang equation (see e.g. [13])

$$e^{-2ip_jL} \prod_{k=j-1}^{1} \mathbb{S}_{jk}(p_j,p_k)\, \mathbb{R}^{-}_{j}(p_j) \prod_{k=1:k\neq j}^{N} \mathbb{S}_{kj}(p_k,-p_j)\, \mathbb{R}^{+}_{j}(-p_j) \prod_{k=N}^{j+1} \mathbb{S}_{jk}(p_j,p_k) = 1\,, \qquad j = 1,...,N\,. \tag{2.16}$$

This condition can be conveniently reformulated in terms of a double-row [23] transfer matrix

$$\mathbb{D}(p,\{p_i\}) = \mathrm{tr}_A\, \mathbb{S}_{AN}(p,p_N)...\mathbb{S}_{A1}(p,p_1)\mathbb{R}^{-}_{A}(p)\mathbb{S}_{1A}(p_1,-p)...\mathbb{S}_{NA}(p_N,-p)\tilde{\mathbb{R}}^{+}_{A}(-p)\,, \tag{2.17}$$

where the trace is over the auxiliary space denoted here by A, which is in the fundamental (16-dimensional) representation of $SU(2|2) \otimes SU(2|2)$. Note that this transfer matrix does not directly depend on the left boundary S-matrix $\mathbb{R}^{+}(-p) = \mathbb{R}^{-}_{\theta}(p)$, but instead depends on $\tilde{\mathbb{R}}^{+}(-p)$, which is defined such that

$$\mathbb{R}^{-}_{\theta}(p)^{(\beta,\dot{\beta})}_{(\gamma,\dot{\gamma})} = \sum_{\alpha,\dot{\alpha}} \mathbb{S}(p,-p)^{(\beta,\dot{\beta})(\delta,\dot{\delta})}_{(\alpha,\dot{\alpha})(\gamma,\dot{\gamma})}\, \tilde{\mathbb{R}}^{+}(-p)^{(\alpha,\dot{\alpha})}_{(\delta,\dot{\delta})}\,. \tag{2.18}$$

The boundary Bethe-Yang equation (2.16) now takes the simpler form

$$e^{-2ip_jL}\mathbb{D}(p_j,\{p_i\}) = -1\,, \qquad j = 1,...,N\,. \tag{2.19}$$

Using the expressions for the bulk (2.3) and boundary (2.10) S-matrices, we can factorize $\mathbb{D}(p,\{p_i\})$ into a tensor product of two "chiral" $SU(2|2)$ transfer matrices

$$\mathbb{D}(p,\{p_i\}) = d(p,\{p_i\})\, D(p,\{p_i\}) \otimes \dot{D}(p,\{p_i\})\,, \tag{2.20}$$

where

$$D(p,\{p_i\}) = \mathrm{tr}_A\, S_{AN}(p,p_N)...S_{A1}(p,p_1)R^-_A(p)S_{1A}(p_1,-p)...S_{NA}(p_N,-p)\tilde{R}^+_A(-p)\,. \quad (2.21)$$

The auxiliary space A is now in the fundamental (4-dimensional) representation of $SU(2|2)$, and similarly for the dotted factor; moreover, the scalar factor is given by

$$d(p,\{p_i\}) = R^-_0(p)\tilde{R}^+_0(-p)\prod_{i=1}^{N} S_0(p,p_i)S_0(p_i,-p)\,. \quad (2.22)$$

We recall [13] that $\tilde{R}^+(-p) \propto (-1)^F R^-_\theta(-p)$, where F is the fermion number, which changes the trace in (2.21) to a supertrace. The transfer matrix (2.17) therefore takes the final form

$$\mathbb{D}(p,\{p_i\}) = \tilde{d}(p,\{p_i\})\,\tilde{D}(p,\{p_i\}) \otimes \dot{\tilde{D}}(p,\{p_i\})\,, \quad (2.23)$$

where the chiral $SU(2|2)$ transfer matrix $\tilde{D}(p,\{p_i\})$ is defined by

$$\tilde{D}(p,\{p_i\}) = \mathrm{str}_A\, S_{AN}(p,p_N)...S_{A1}(p,p_1)R^-_A(p)S_{1A}(p_1,-p)...S_{NA}(p_N,-p)R^-_{\theta\,A}(-p)\,, \quad (2.24)$$

and similarly for the dotted factor. The normalization factor is given by [13] [1]

$$\tilde{d}(p,\{p_i\}) = \frac{e^{-2ip}}{\rho_1^2(p)}\frac{u^-}{u^+}\prod_{i=1}^{N} S_0(p,p_i)S_0(p_i,-p)\,. \quad (2.25)$$

where

$$\rho_1(p) = \frac{(1+(x^-)^2)(x^-+x^+)}{2x^+(1+x^-x^+)}\,. \quad (2.26)$$

For later reference, we recall here that there exists an infinite hierarchy of commuting transfer matrices $\mathbb{D}_{a,s}(p,\{p_i\})$

$$[\mathbb{D}_{a,s}(p,\{p_i\})\,,\mathbb{D}_{a',s'}(p',\{p_i\})] = 0\,, \quad (2.27)$$

defined as in (2.17) except with the auxiliary space in a rectangular representation (a,s) of $SU(2|2)\otimes SU(2|2)$, such that $\mathbb{D}(p,\{p_i\}) \equiv \mathbb{D}_{1,1}(p,\{p_i\})$. These transfer matrices satisfy the Hirota equation

$$\mathbb{D}^+_{a,s}\,\mathbb{D}^-_{a,s} = \mathbb{D}_{a+1,s}\,\mathbb{D}_{a-1,s} + \mathbb{D}_{a,s+1}\,\mathbb{D}_{a,s-1}\,, \quad (2.28)$$

where $f^\pm(u) = f(u\pm\frac{i}{2})$. As in (2.23), we can express $\mathbb{D}_{a,s}$ in terms of corresponding chiral $SU(2|2)$ transfer matrices

$$\mathbb{D}_{a,s}(p,\{p_i\}) = \tilde{d}_{a,s}(p,\{p_i\})\,\tilde{D}_{a,s}(p,\{p_i\}) \otimes \dot{\tilde{D}}_{a,s}(p,\{p_i\})\,. \quad (2.29)$$

[1] $\tilde{d}(p,\{p_i\})$ must be equal to $d(p,\{p_i\})$ up to some scalar function of p. For $N=1$, we see from (2.22) that $\tilde{d}(p,p_1) = g(p)R^-_0(p)\,S_0(p,p_1)\,S_0(p_1,-p)$ for some function $g(p)$. Evaluating this expression at $p=p_1$, and using (2.4), (2.11) and the result $\tilde{d}(p_1,p_1) = -e^{-4ip_1}\sigma^2(p_1,-p_1)/\rho_1^2(p_1)$ which follows from the boundary Bethe-Yang equation for one $(3,\dot{3})$ particle (see (2.42) below), we arrive at (2.25).

2.3 Bethe ansatz

In order to determine the momenta p_i using the boundary Bethe-Yang equation (2.19), it is necessary to first determine the eigenvalues of $\mathbb{D}(p, \{p_i\})$. In view of (2.23), the problem in turn reduces to determining the eigenvalues of the chiral transfer matrix $\tilde{D}(p, \{p_i\})$ (2.24). The latter problem is nontrivial due to the fact that the boundary S-matrix $R_\theta^-(p)$ (2.13) is not diagonal. Nevertheless, with the help of the so-called off-diagonal Bethe ansatz approach [15], this problem was recently solved in [16]. The result for the eigenvalues of $\tilde{D}(p, \{p_i\})$ (which, by abuse of notation, we denote in the same way) is given by [16][2]

$$\begin{aligned}\tilde{D}(p, \{p_i\}) = e^{i(N-M+1)p} \frac{\mathcal{R}^{(+)-}}{\mathcal{R}^{(+)+}} \rho_1 \Bigg\{ &- \frac{\mathcal{R}^{(-)-}}{\mathcal{R}^{(+)-}} \frac{\mathcal{R}_1^+}{\mathcal{R}_1^-} - \frac{u^+}{u^-} \frac{\mathcal{B}^{(+)+}}{\mathcal{B}^{(-)+}} \frac{\mathcal{B}_1^-}{\mathcal{B}_1^+} \\ &+ \frac{1}{2}\left(1 + \frac{u^+}{u^-}\right) \left[\frac{u^-}{u} \frac{\mathcal{R}_1^+}{\mathcal{R}_1^-} \frac{Q_2^{--}}{Q_2} + \frac{u^+}{u} \frac{\mathcal{B}_1^-}{\mathcal{B}_1^+} \frac{Q_2^{++}}{Q_2} - 4\sin^2\theta \frac{Q_1^- \mathcal{R}_1^+}{Q_2 \mathcal{R}_1^-} \right] \Bigg\} ,\end{aligned} \tag{2.30}$$

where

$$\mathcal{R}^{(\pm)}(p) = \prod_{i=1}^{N} \big(x(p) - x^{\mp}(p_i)\big) \big(x(p) + x^{\pm}(p_i)\big) , \quad \mathcal{R}_1(p) = \prod_{j=1}^{M} (x(p) - y_j) (x(p) + y_j) , \tag{2.31}$$

and their $\mathcal{B}$ analogues are obtained by changing $x(p)$ to $1/x(p)$:

$$\mathcal{B}^{(\pm)}(p) = \prod_{i=1}^{N} \left(\frac{1}{x(p)} - x^{\mp}(p_i)\right) \left(\frac{1}{x(p)} + x^{\pm}(p_i)\right) , \quad \mathcal{B}_1(p) = \prod_{j=1}^{M} \left(\frac{1}{x(p)} - y_j\right) \left(\frac{1}{x(p)} + y_j\right) . \tag{2.32}$$

We shall often use the abbreviation $x_i^\pm = x^\pm(p_i)$. The Q-functions are

$$Q_1(u) = \prod_{j=1}^{M} (u - v_j)(u + v_j) , \quad Q_2(u) = \prod_{j=1}^{M} (u - w_j)(u + w_j) , \tag{2.33}$$

where $v_j = g(y_j + \frac{1}{y_j})$. Finally, ρ_1 is given by (2.26).

The corresponding Bethe equations for the Bethe roots $\{y_1, \ldots, y_M\}$ and $\{w_1, \ldots, w_M\}$ are

$$\left. \frac{\mathcal{R}^{(-)}}{\mathcal{R}^{(+)}} \frac{Q_2^+}{Q_2^-} \right|_{x(p)=y_j} = 1 , \qquad j = 1, \ldots, M , \tag{2.34}$$

$$\left. \left[\tfrac{u^-}{u} Q_1^+ Q_2^{--} + \tfrac{u^+}{u} Q_1^- Q_2^{++} - 4\sin^2\theta \, Q_1^+ Q_1^- \right] \right|_{u=w_k} = 0 , \quad k = 1, \cdots, M , \tag{2.35}$$

[2] We compensate for the fact that the definition of g in [16] differs by a factor 2 from the one used here. Indeed, there $g = \sqrt{\lambda}/(2\pi)$, c.f. (2.2). Moreover, we change notation $B_1 R_3 \mapsto \mathcal{R}_1$, $R_1 B_3 \mapsto \mathcal{B}_1$.

For a given value of N, the possible values of M are $0, 1, \ldots N$. Notice the presence of the "inhomogeneous" term in (2.30) that is proportional to $\sin^2\theta$, which is absent for the diagonal ($\theta = 0$) case [13].

In order to compute the Lüscher corrections, we shall also need the corresponding result for all the antisymmetric representations $\tilde{D}_{a,1}(p, \{p_i\})$. A generating functional for these transfer-matrix eigenvalues was proposed in [16], which we now briefly recall. We begin by rewriting the eigenvalue result (2.30) for $\tilde{D} = \tilde{D}_{1,1}$ as

$$\tilde{D}_{1,1} = h\,\hat{D}_{1,1}\,, \qquad \hat{D}_{1,1} = -A - B + G + H + C\,, \tag{2.36}$$

where h is a normalization factor

$$h = \rho_1 \left(\frac{x^+}{x^-}\right)^{N-M+1} \frac{\mathcal{R}^{(+)-}}{\mathcal{R}^{(+)+}}\,. \tag{2.37}$$

Furthermore,

$$A = \frac{\mathcal{R}^{(-)-}}{\mathcal{R}^{(+)-}}\frac{\mathcal{R}_1^+}{\mathcal{R}_1^-}\,, \quad B = \frac{u^+}{u^-}\frac{B^{(+)+}}{B^{(-)+}}\frac{\mathcal{B}_1^-}{\mathcal{B}_1^+}\,, \quad G = \frac{\mathcal{R}_1^+}{\mathcal{R}_1^-}\frac{Q_2^{--}}{Q_2}\,, \quad H = \frac{u^+}{u^-}\frac{\mathcal{B}_1^-}{\mathcal{B}_1^+}\frac{Q_2^{++}}{Q_2}\,, \tag{2.38}$$

and the θ-dependent term is

$$C = -2\sin^2\theta \left(1 + \frac{u^+}{u^-}\right) \frac{Q_1^-\mathcal{R}_1^+}{Q_2\mathcal{R}_1^-}\,. \tag{2.39}$$

The proposed generating functional for antisymmetric representations is given by [16]

$$\begin{aligned} W^{-1} &= (1 - \mathcal{D}A\mathcal{D})^{-1}\left[1 - \mathcal{D}(G + H + C)\mathcal{D} + \mathcal{D}G\mathcal{D}^2H\mathcal{D}\right](1 - \mathcal{D}B\mathcal{D})^{-1} \\ &= \sum_{a=0}^{\infty}(-1)^a\mathcal{D}^a\,\hat{D}_{a,1}\,\mathcal{D}^a\,, \end{aligned} \tag{2.40}$$

where $\mathcal{D} = e^{-\frac{i}{2}\partial_u}$ implying $\mathcal{D}f = f^-\mathcal{D}$, with

$$\tilde{D}_{a,1} = h^{[a-1]}h^{[a-3]}\cdots h^{[3-a]}h^{[1-a]}\,\hat{D}_{a,1}\,, \tag{2.41}$$

where $f^{[\pm n]} = f(u \pm \frac{in}{2})$.

2.4 One-particle states

For simplicity, we henceforth focus on the case $N = 1$. (The case $N = 0$, corresponding to the vacuum state, was considered in [14].) For this case, the boundary Bethe-Yang equation (2.16) reduces to

$$1 \;=\; e^{-2ip_1L}\Lambda(p_1) = e^{-2ip_1L}R_0(p_1)^2\lambda_i(p_1)\dot{\lambda}_j(p_1)\,, \tag{2.42}$$

where $\Lambda(p_1)$ denotes an eigenvalue of $\mathbb{R}^-(p_1)\,\mathbb{R}^+(-p_1) = \mathbb{R}^-(p_1)\,\mathbb{R}_\theta^-(p_1)$. Recalling (2.10) and (2.12), we see that there are 16 such eigenvalues, which are given (up to the factor

$R_0(p_1)^2$) by the products of the 4 eigenvalues of $R^-(p_1)\,R_\theta^-(p_1)$, denoted by $\lambda_i(p_1)$, and the 4 eigenvalues of $\dot{R}^-(p_1)\,\dot{R}_\theta^-(p_1)$, denoted by $\dot{\lambda}_j(p_1)$. We restrict our attention throughout this paper to the 4 symmetric $\dot{\lambda}_j = \lambda_i$ cases. The two eigenvalues corresponding to the bosonic subspace are

$$\lambda_{1,2} = \cos p_1\, \cos^2\theta - \sin^2\theta \pm i\sqrt{1-(\sin^2\theta - \cos p_1\, \cos^2\theta)^2}\,, \tag{2.43}$$

while those in the fermionic subspace are $\lambda_{3,4} = 1$.

Since the Bethe-Yang equation (2.42) can also be written in terms of the transfer matrix as in (2.19), we must have $\Lambda(p_1) = -\mathbb{D}(p_1, p_1)$. Indeed, the eigenvalues (2.43) can be recovered from the Bethe-ansatz result for the transfer-matrix eigenvalue (2.30) as follows: the fermionic eigenvalues are described by $N = 1$ and $M = 0$ as

$$\tilde{D}(p_1, p_1)\big|_{M=0} = e^{ip_1}\rho_1(p_1) \tag{2.44}$$

(note that $\mathcal{R}^{(+)-}(p_1) = 0$), while the bosonic eigenvalues are described by $N = M = 1$ as

$$\tilde{D}(p_1, p_1)\big|_{M=1} = \rho_1(p_1)\frac{\mathcal{R}_1^+(p_1)}{\mathcal{R}_1^-(p_1)}\,. \tag{2.45}$$

Their ratio (which must coincide with $\lambda_{1,2}/\lambda_{3,4} = \lambda_{1,2}$) is therefore given by

$$\frac{\tilde{D}(p_1, p_1)\big|_{M=1}}{\tilde{D}(p_1, p_1)\big|_{M=0}} = e^{-ip_1}\frac{\mathcal{R}_1^+(p_1)}{\mathcal{R}_1^-(p_1)} = e^{-ip_1}\frac{(x_1^+ - y_1)(x_1^+ + y_1)}{(x_1^- - y_1)(x_1^- + y_1)}\,, \tag{2.46}$$

where the "magnonic" Bethe roots y_1 (or $v_1 = g(y_1 + \frac{1}{y_1})$) and w_1 are still to be determined from the Bethe equations (2.34), (2.35) in terms of p_1. The Q-functions (2.33) simplify for $N = M = 1$ to

$$Q_1(u) = (u - v_1)(u + v_1)\,, \qquad Q_2(u) = (u - w_1)(u + w_1)\,. \tag{2.47}$$

The Bethe equation (2.35) expresses w_1^2 in terms of v_1 as

$$w_1^2 = v_1^2 - v_1\cot\theta - \frac{1}{4}\,, \qquad w_1^2 = v_1^2 + v_1\cot\theta - \frac{1}{4}\,. \tag{2.48}$$

As the second equation can be obtained from the first by changing $\theta \to -\theta$ or $v_1 \to -v_1$, we focus on the first equation. It implies that

$$\frac{Q_2^+(v_1)}{Q_2^-(v_1)} = e^{2i\theta}\,, \tag{2.49}$$

which simplifies the other Bethe equation (2.34) to

$$\frac{(y_1 - x_1^+)(y_1 + x_1^-)}{(y_1 - x_1^-)(y_1 + x_1^+)}e^{2i\theta} = 1\,. \tag{2.50}$$

This quadratic equation has two solutions for y_1

$$y_1 = x_1^- \frac{e^{ip_1/2}}{\sin\theta}\left(\cos\theta\sin\frac{p_1}{2} \mp \sqrt{1 - \cos^2\frac{p_1}{2}\cos^2\theta}\right)\,. \tag{2.51}$$

Plugging these two solutions back into (2.46) we recover the two eigenvalues (2.43). Let us note that taking the second solution in eq. (2.48), i.e. changing $\theta \to -\theta$, alters the sign of y_1 but does not change the expression (2.46).

2.5 Energies from the asymptotic Bethe ansatz at weak coupling

The dressing phase appears in the boundary Bethe-Yang equations (2.42) (recall that $R_0^-(p)$ is given by (2.11)), which prevents us from solving these equations explicitly. However, in order to compare with gauge-theory calculations, only the weak-coupling (small g) expansion is needed. We now develop this expansion for $L = 2$, since several results that can be used as checks are already available for this case. Restricting to symmetric states ($\dot{\lambda}_j = \lambda_i$) and taking the square root of the boundary Bethe-Yang equations (2.42), we obtain

$$1 = \gamma e^{-3ip_1} \sigma(p_1, -p_1)\, \lambda_i(p_1)\,. \tag{2.52}$$

We keep track of the square root sign ambiguity by introducing $\gamma = \pm 1$. For the computations that follow, it turns out to be advantageous to work with the rapidity variable u (2.6) instead of the momentum. They are related as

$$u(p) = \frac{1}{2} \cot \frac{p}{2}\, \epsilon(p)\,. \tag{2.53}$$

Our tactic is to first expand u_1 at weak coupling as

$$u_1 = u_{1,0} + g^2\, u_{1,1} + g^4\, u_{1,2} + \dots\,, \tag{2.54}$$

and to substitute these results into the Bethe-Yang equation (2.52), expanding also the dressing factor $\sigma(p_1, -p_1)$ using (2.7)-(2.9) and

$$x(u) = \frac{u}{2g} + \sqrt{\frac{u}{2g} + 1}\sqrt{\frac{u}{2g} - 1}\,. \tag{2.55}$$

We then solve the resulting equation order by order in g. Once u_1 is known up to the required order, we substitute the result into the energy formula

$$\epsilon(u) = 1 + 2ig \left(\frac{1}{x^+} - \frac{1}{x^-} \right)\,, \tag{2.56}$$

which we also expand. Since we are interested in the leading wrapping correction, we expand the energy up to that order. We now summarize our results for all the possible choices of λ_i and γ in (2.52):

1. $\lambda_3 = \lambda_4 = 1$ and $\gamma = +1$: the leading weak-coupling result for u_1 is given by

$$u_{1,0} = \frac{1}{2\sqrt{3}}\,, \tag{2.57}$$

which corresponds to $p_{1,0} = 2\pi/3$, and which gets modified up to g^6 as

$$u_1 = u_{1,0} \left(1 + 6g^2 - 18g^4 + 108g^6 + 24g^6\zeta_3 + \dots \right)\,. \tag{2.58}$$

This leads to the energy

$$E_1 = 1 + 6g^2 - 18g^4 + 108g^6 - 18(45 + 4\zeta_3)g^8\,. \tag{2.59}$$

2. $\lambda_3 = \lambda_4 = 1$ and $\gamma = -1$: we find

$$u_{1,0} = \frac{\sqrt{3}}{2}\,, \tag{2.60}$$

which corresponds to $p_{1,0} = \pi/3$, and which gets modified up to g^6 as

$$u_1 = u_{1,0}\left(1 + 2g^2 - 2g^4 + 4g^6 + \tfrac{8}{3}g^6\zeta_3 + \dots\right)\,. \tag{2.61}$$

This leads to the energy

$$E_2 = 1 + 2g^2 - 2g^4 + 4g^6 - 2(5 + 4\zeta_3)g^8\,. \tag{2.62}$$

3. $\lambda_{1,2}$ and $\gamma = +1$: the leading order gives the relation

$$\cos(2\theta) = \frac{17 - 56u_{1,0}^2 + 16u_{1,0}^4}{(1 + 4u_{1,0}^2)^2}\,. \tag{2.63}$$

Solving this relation for $u_{1,0}^2$, we obtain

$$u_{1,0}^2 = -\frac{1}{4} + \frac{1}{1 \pm \cos\theta}\,, \tag{2.64}$$

where λ_1 and λ_2 are compatible with the upper and lower signs, respectively. The corrections to the rapidity up to g^{10} can be expressed as

$$\begin{aligned} u_1 =& u_{1,0}\left(1 + 8\tilde{g}^2 - 32\tilde{g}^4 + 256\tilde{g}^6 - 2560\tilde{g}^8 + 28672\tilde{g}^{10}\right) + \\ & u_{1,0}^{-1}(1 - 12u_{1,0}^2)\left(-4\tilde{g}^6(1 + 4u_{1,0}^2)\zeta_3 + \tilde{g}^8\left(16(1 + 20u_{1,0}^2)\zeta_3 + 40(1 + 4u_{1,0}^2)^2\zeta_5\right)\right. \\ & \left.-\tilde{g}^{10}\left(64(1 + 84u_{1,0}^2)\zeta_3 + 224(1 + 16u_{1,0}^2 + 48u_{1,0}^4)\zeta_5 + 420(1 + 4u_{1,0}^2)^3\zeta_7\right)\right)\,, \end{aligned} \tag{2.65}$$

where $\tilde{g}^2 = g^2/(1 + 4u_{1,0}^2)$. The corresponding energy is

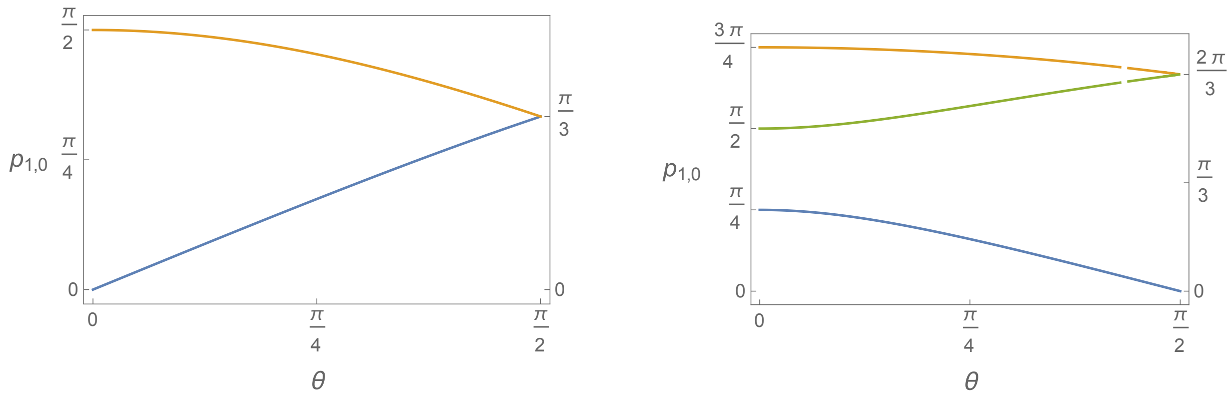
$$\begin{aligned} E_3 =& 1 + 8\tilde{g}^2 - 32\tilde{g}^4 + 256\tilde{g}^6 - 2560\tilde{g}^8 + 28672\tilde{g}^{10} - 344064\tilde{g}^{12} \\ & + (1 - 12u_{1,0}^2)\left[256\tilde{g}^8\zeta_3 - 2560\tilde{g}^{10}(2\zeta_3 + (1 + 4u_{1,0}^2)\zeta_5) + \right. \\ & \left. + 768\tilde{g}^{12}(112\zeta_3 + 8(9 + 28u_{1,0}^2)\zeta_5 + 35(1 + 4u_{1,0}^2)^2\zeta_7\right]\,. \end{aligned} \tag{2.66}$$

4. $\lambda_{1,2}$ and $\gamma = -1$: the leading-order equation gives

$$\cos(2\theta) = \frac{1}{2u_{1,0}^2} - \frac{17 - 56u_{1,0}^2 + 16u_{1,0}^4}{(1 + 4u_{1,0}^2)^2}\,. \tag{2.67}$$

Since the correction to $u_{1,0}$ is quite complicated, we refrain from displaying the result. The corresponding energy correction takes the form

$$\begin{aligned} E_4 =& 1 + 8\tilde{g}^2 - 32\tilde{g}^4 + 256\tilde{g}^6 - 2560\tilde{g}^8 + 28672\tilde{g}^{10} - 344064\tilde{g}^{12} \\ & - \frac{32768u_{1,0}^4(3 - 4u_{1,0}^2)}{1 + 24u_{1,0}^2 - 48u_{1,0}^4}\left(\tilde{g}^8\zeta_3 - 10\tilde{g}^{10}(2\zeta_3 + (1 + 4u_{1,0}^2)\zeta_5)\right. \\ & \left. + \tilde{g}^{12}\left(336\zeta_3 + 24(9 + 28u_{1,0}^2)\zeta_5 + 105(1 + 4u_{1,0}^2)^2\zeta_7\right)\right)\,. \end{aligned} \tag{2.68}$$



(a) (b)

Figure 1: $p_{1,0}$ versus θ for (a) case 3 (b) case 4

For case 3, we see from (2.64) that there are two solutions for $u_{1,0}^2$ in terms of $\cos\theta$. Since

$$u_{1,0} = \frac{1}{2}\cot(\tfrac{p_{1,0}}{2})\,, \tag{2.69}$$

where $p_{1,0}$ is the weak-coupling limit of the momentum p_1, it follows that $p_{1,0}$ and θ are related in a simple manner

$$\cos p_{1,0} = \sin^2\tfrac{\theta}{2}\,, \qquad \cos p_{1,0} = \cos^2\tfrac{\theta}{2}\,. \tag{2.70}$$

These two solutions are plotted in Fig 1(a).

For case 4, it follows from (2.67) that there are three solutions for $u_{1,0}^2$. The corresponding momenta $p_{1,0}$ as functions of angle are plotted in Fig 1(b).

2.6 Weak-coupling expansion of the magnonic Bethe roots

We will calculate in Section 3 the leading wrapping corrections to the energies E_i computed above. For cases 3 and 4, we will need the leading weak-coupling expressions for the Bethe roots v_1 and w_1 in terms of $u_{1,0}$. We set

$$v_1 = v_{1,0} + O(g^2)\,, \qquad w_1 = w_{1,0} + O(g^2)\,, \tag{2.71}$$

and we note that $y_1 \sim \frac{1}{g}v_{1,0}$. Let us now see how the Bethe equations simplify for small g. We begin by introducing the Q-function corresponding to u_1

$$Q(u) = (u-u_1)(u+u_1)\,. \tag{2.72}$$

In the weak-coupling limit, $x^\pm \sim \frac{1}{g}(u \pm \frac{i}{2})$, and therefore $g^2\mathcal{R}^{(\pm)} = Q^\pm$. The first Bethe equation (2.34) therefore simplifies to

$$\left.\frac{Q^- Q_2^+}{Q^+ Q_2^-}\right|_{u=v_{1,0}} = 1\,, \tag{2.73}$$

which implies that at leading order the w_1 root is the same as u_1:

$$w_{1,0} = u_{1,0}\,, \qquad Q_2(u) = Q(u) + O(g^2)\,. \tag{2.74}$$

The result (2.46) from the transfer-matrix eigenvalue also simplifies

$$\lambda_{1,2} = e^{-ip_1} \left.\frac{Q_1^+}{Q_1^-}\right|_{u=u_{1,0}}\,, \tag{2.75}$$

and leads to the following expression for the (square root of the) Bethe-Yang equation (2.52) with $\lambda_{1,2}$ at weak coupling[3]

$$\left.\frac{Q_1^+}{Q_1^-}\right|_{u=u_{1,0}} = \gamma \left(\frac{u_{1,0}+\frac{i}{2}}{u_{1,0}-\frac{i}{2}}\right)^4\,. \tag{2.76}$$

This equation can be used to express $v_{1,0}$ in terms of $u_{1,0}$, which – when plugged back into (2.48) – determines $u_{1,0}$ in terms of θ, or the other way around. We find two solutions for $v_{1,0}^2$ in terms of $u_{1,0}^2$:

$$v_{1,0}^2 = \frac{(1+4u_{1,0}^2)^2}{32u_{1,0}^2-8} \qquad (\gamma=+1)\,, \tag{2.77}$$

$$v_{1,0}^2 = -\frac{(1+4u_{1,0}^2)^2(1-4u_{1,0}^2)}{4-96u_{1,0}^2+64u_{1,0}^4} \qquad (\gamma=-1)\,. \tag{2.78}$$

The former solution corresponds to $\gamma=+1$ and is related to case 3 analyzed above; while the latter solution corresponds to $\gamma=-1$ and case 4.

The first relation (2.77), when combined with (2.48), recovers the result (2.64) for $u_{1,0}^2$ in terms of $\cos\theta$; and correspondingly, $v_{1,0} = \pm\csc\theta$. The second relation (2.78), when combined with (2.48), recovers the result (2.67). We prefer to use the variable $u_{1,0}$ instead of θ as both the Bethe-Yang energy and the wrapping correction can be expressed in terms of $u_{1,0}$ in a unified way. That is, we have a single expression for the two cases at $\gamma=1$, and another expression for the three cases at $\gamma=-1$.

We are finally ready to calculate the wrapping corrections to the one-particle states.

3 Leading wrapping corrections

The leading finite-size correction of multiparticle states on the strip has been proposed in [12]. It expresses the energy corrections in terms of the double-row transfer-matrix eigenvalue as:

$$\Delta E = -\sum_{a=1}^{\infty}\int_0^{\infty}\frac{dq}{2\pi}\mathbb{D}_{a,1}(q,p_1)e^{-2\tilde{\epsilon}_a(q)L}\,. \tag{3.1}$$

[3]For $L\neq 2$, the right-hand-side of (2.76) changes to $\gamma\left(\frac{u_{1,0}+\frac{i}{2}}{u_{1,0}-\frac{i}{2}}\right)^{L+2}$.

From (2.29) and (2.36) we have

$$\mathbb{D}_{a,1}(q,p_1) = f_{a,1}(q,p_1)\,\hat{D}_{a,1}(q,p_1)^2\,, \tag{3.2}$$

which must be evaluated for the mirror momenta q. The scalar part can be obtained from fusion

$$f_{a,1} = f^{[a-1]} f^{[a-3]} \dots f^{[3-a]} f^{[1-a]}\,, \tag{3.3}$$

where f is given by

$$f = \tilde{d}(q,p_1)\,h^2 = S_0(q,p_1) S_0(p_1,-q) \frac{u^-}{u^+} \left(\frac{\mathcal{R}^{(+)-}}{\mathcal{R}^{(+)+}}\right)^2 \left(\frac{x^+}{x^-}\right)^{2(N-M)}\,, \tag{3.4}$$

as follows from (2.25) and (2.37). In calculating the fusion of a particles to get the mirror antisymmetric boundstate, we must take the first $a-1$ particles in the "string" kinematics, i.e. use (2.55) with $u = q/2$; and take only the last a^{th} particle in the mirror kinematics [5, 24], where we have

$$e^{-\tilde{\epsilon}_a(q)} = \frac{x^{[-a]}(q)}{x^{[+a]}(q)}\,, \qquad x^{[\pm a]}(q) = \frac{q+ia}{4g}\left(\sqrt{1+\frac{16g^2}{q^2+a^2}} \pm 1\right)\,. \tag{3.5}$$

We calculate the boundstate transfer-matrix eigenvalues from the generating functional (2.40) as

$$(-1)^a \hat{D}_{a,1} = \sum_{j=0}^{a} A^{(j)} B^{(a-j)} - \sum_{j=0}^{a-1} A^{(j)} J^{[a-1-2j]} B^{(a-j-1)} + \sum_{j=0}^{a-2} A^{(j)} G^{[a-1-2j]} H^{[a-3-2j]} B^{(a-j-2)} \tag{3.6}$$

where $J = G + H + C$, and

$$A^{(j)} = A^{[a-1]} A^{[a-3]} \dots A^{[a+1-2j]} = \frac{\mathcal{R}^{(-)[a-2]}}{\mathcal{R}^{(+)[a-2]}} \frac{\mathcal{R}^{(-)[a-4]}}{\mathcal{R}^{(+)[a-4]}} \cdots \frac{\mathcal{R}^{(-)[a-2j]}}{\mathcal{R}^{(+)[a-2j]}} \frac{\mathcal{R}_1^{[a]}}{\mathcal{R}_1^{[a-2j]}}\,, \tag{3.7}$$

together with

$$B^{(k)} = B^{[2k-1-a]} \dots B^{[3-a]} B^{[1-a]} = \frac{u^{[2k-a]}}{u^{[-a]}} \frac{\mathcal{B}^{(+)[2k-a]}}{\mathcal{B}^{(-)[2k-a]}} \cdots \frac{\mathcal{B}^{(+)[4-a]}}{\mathcal{B}^{(-)[4-a]}} \frac{\mathcal{B}^{(+)[2-a]}}{\mathcal{B}^{(-)[2-a]}} \frac{\mathcal{B}_1^{[-a]}}{\mathcal{B}_1^{[2k-a]}}\,. \tag{3.8}$$

In the following we specialize the above expressions for the four cases that we analyzed in Section 2.6, and calculate their weak-coupling limits.

3.1 Wrapping corrections to λ_3

For the $(3,\dot{3})$ particle there are no magnons ($M=0$), thus

$$Q_1 = Q_2 = \mathcal{R}_1 = \mathcal{B}_1 = 1\,. \tag{3.9}$$

In the weak-coupling limit $g^2\mathcal{R}^{(\pm)} = Q^{\pm}$, hence

$$A^{(k)} = \frac{Q^{[a-2k-1]}}{Q^{[a-1]}} \quad \text{for} \quad k < a\,, \qquad A^{(a)} = \frac{Q^{[1-a]}}{Q^{[a-1]}}\,, \qquad B^{(k)} = \frac{u^{[2k-a]}}{u^{[-a]}}\,, \tag{3.10}$$

and

$$H^{[k]} = \frac{u^{[k+1]}}{u^{[k-1]}}\,, \qquad G^{[k]} = 1\,, \qquad G^{[k]}H^{[k-2]} = \frac{u^{[k-1]}}{u^{[k-3]}}\,, \qquad C^{[k]} = -\sin^2\theta \frac{4u^{[k]}}{u^{[k-1]}}\,. \tag{3.11}$$

Performing the sums in (3.6), we obtain the following result for the transfer-matrix part

$$\hat{D}_{a,1}(q, u_{1,0}) = (-1)^{a+1}\,\frac{aq\sin^2\theta(a^2 - 1 - q^2 + 4u_{1,0}^2)}{(q-ia)Q^{[a-1]}}\,, \tag{3.12}$$

where we have used the leading-order rapidity $u_{1,0}$ instead of the momentum p_1. The weak-coupling limit of the scalar part (3.3) gives

$$f_{a,1}(q, u_{1,0}) = \frac{Q^{[a-1]}(u_{1,0}^2 + \frac{1}{4})^2}{Q^{[-a-1]}Q^{[1-a]}Q^{[a+1]}}\frac{q-ia}{q+ia}\,. \tag{3.13}$$

The weak-coupling limit of $\mathbb{D}_{a,1}$ (3.2) is therefore given by

$$\mathbb{D}_{a,1}(q, u_{1,0}) = \frac{a^2q^2\sin^4\theta\left(a^2 - 1 - q^2 + 4u_{1,0}^2\right)^2}{Q^{[-a-1]}Q^{[1-a]}Q^{[a-1]}Q^{[1+a]}}\frac{(u_{1,0}^2 + \frac{1}{4})^2}{(q^2 + a^2)}\,. \tag{3.14}$$

The exponential part is simply

$$e^{-2\tilde{\epsilon}_a L} = \left(\frac{4g^2}{q^2 + a^2}\right)^4\,, \tag{3.15}$$

where we have taken $L = 2$. As the integrand in (3.1) is symmetric in q we extend the integral to the whole line and evaluate it by residues. On the upper half-plane there is a kinematical pole at $q = ia$ and four dynamical poles at $q = i(a \pm 1 + 2u_{1,0})$ and at $q = i(a \pm 1 - 2u_{1,0})$. We find that the contributions from the dynamical poles at $q = i(a + 1 + 2u_{1,0})$ (and similarly for the dynamical poles at $q = i(a + 1 - 2u_{1,0})$) coming from two consecutive values of a cancel provided that $u_{1,0}$ satisfies the Bethe-Yang equations, i.e. it is either $1/(2\sqrt{3})$ (2.57) or $\sqrt{3}/2$ (2.60). The contributions coming from the kinematical pole can be summed up, and we obtain the following results:

1. For $u_{1,0} = \frac{1}{2\sqrt{3}}$ we obtain

$$\Delta E_1 = 2g^8\sin^4\theta(3\zeta_3 - 5\zeta_5)\,. \tag{3.16}$$

2. For $u_{1,0} = \frac{\sqrt{3}}{2}$ we obtain

$$\Delta E_2 = -2g^8\sin^4\theta(\zeta_3 + 5\zeta_5)\,. \tag{3.17}$$

The corresponding energies from the asymptotic Bethe ansatz are given by (2.59) and (2.62), respectively.

3.2 Wrapping correction to $\lambda_{1,2}$

We now consider the more complicated cases. One must be careful in calculating the weak-coupling expansion of the eigenvalues $\hat{D}_{a,1}$ in the mirror kinematics. As already explained, the first $a-1$ particles are in the "string" kinematics, while only the last a^{th} particle is in the mirror kinematics. This implies the following weak-coupling behavior:

$$x^{[j]} = \frac{q+ij}{2g} + \ldots \quad \text{for} \quad j = a, \ldots, 1-a\,, \qquad x^{[-a]} = \frac{2g}{q-ia} + \ldots \tag{3.18}$$

and will introduce a difference in the expansion of the $\mathcal{R}$ and $\mathcal{B}$ functions depending on their shifts:

$$g^2\mathcal{R}^{(\pm)[k]} = Q^{[k\pm1]} + \ldots \quad \text{for} \quad k > -a\,, \qquad g^2\mathcal{R}^{(\pm)[-a]} = -(u_{1,0}^2 + \frac{1}{4})\,, \tag{3.19}$$

and

$$g^2\mathcal{B}^{(\pm)[k]} = -(u_{1,0}^2 + \frac{1}{4}) + \ldots \quad \text{for} \quad k > -a\,, \qquad g^2\mathcal{B}^{(\pm)[-a]} = Q^{[-a\pm1]}\,. \tag{3.20}$$

Similarly for $\mathcal{R}_1$ and $\mathcal{B}_1$ we obtain

$$g^2\mathcal{R}_1^{[k]} = Q_1^{[k]} + \ldots \quad \text{for} \quad k > -a\,, \qquad g^2\mathcal{R}_1^{[-a]} = -v_{1,0}^2\,, \tag{3.21}$$

and

$$g^2\mathcal{B}_1^{[k]} = -v_{1,0}^2 + \ldots \quad \text{for} \quad k > -a\,, \qquad g^2\mathcal{B}_1^{[-a]} = Q_1^{[-a]}\,. \tag{3.22}$$

Substituting these results into the expression for $A^{(k)}$ (3.7), we arrive at

$$A^{(k)} = \frac{Q^{[a-2k-1]}}{Q^{[a-1]}}\frac{Q_1^{[a]}}{Q_1^{[a-2k]}} \quad \text{for} \quad k < a\,, \qquad A^{(a)} = \frac{Q^{[1-a]}}{Q^{[a-1]}}\frac{Q_1^{[a]}}{(-v_{1,0}^2)}\,. \tag{3.23}$$

For $B^{(k)}$ (3.8) we do not have this problem, as it is always evaluated in the last mirror kinematics

$$B^{(k)} = \frac{u^{[2k-a]}}{u^{[-a]}}\frac{Q_1^{[-a]}}{(-v_{1,0}^2)}\,. \tag{3.24}$$

For the other terms in the transfer-matrix eigenvalues we obtain

$$G^{[k]} = \frac{Q_1^{[k+1]}}{Q_1^{[k-1]}}\frac{Q_2^{[k-2]}}{Q_2^{[k]}} \quad \text{for} \quad k > 1-a\,, \qquad G^{[1-a]} = \frac{Q_1^{[2-a]}Q_2^{[-a-1]}}{(-v_{1,0}^2)Q_2^{[1-a]}}\,, \tag{3.25}$$

$$H^{[k]} = \frac{u^{[k+1]}}{u^{[k-1]}}\frac{Q_2^{[k+2]}}{Q_2^{[k]}} \quad \text{for} \quad k > 1-a\,, \qquad H^{[1-a]} = \frac{u^{[2-a]}}{u^{[-a]}}\frac{Q_1^{[-a]}}{(-v_{1,0}^2)}\frac{Q_2^{[3-a]}}{Q_2^{[1-a]}}\,, \tag{3.26}$$

$$C^{[k]} = -2\sin^2\theta\,\frac{2u^{[k]}}{u^{[k-1]}}\frac{Q_1^{[k+1]}}{Q_2^{[k]}} \quad \text{for} \quad k > 1-a\,, \qquad C^{[1-a]} = -2\sin^2\theta\,\frac{2u^{[1-a]}}{u^{[-a]}}\frac{Q_1^{[2-a]}}{Q_2^{[1-a]}}\frac{Q_1^{[-a]}}{(-v_{1,0}^2)}\,. \tag{3.27}$$

Finally,

$$G^{[k]}H^{[k-2]} = \frac{Q_1^{[k+1]}}{Q_1^{[k-1]}}\frac{u^{[k-1]}}{u^{[k-3]}} \quad \text{for} \quad k > 3-a\,, \qquad G^{[3-a]}H^{[1-a]} = \frac{Q_1^{[4-a]}}{Q_1^{[2-a]}}\frac{Q_1^{[-a]}}{(-v_{1,0})^2}\frac{u^{[2-a]}}{u^{[-a]}}\,. \tag{3.28}$$

We recall (2.74) that $Q_2 = Q$ at leading order, which simplifies the sums in (3.6), leading to a remarkably compact expression

$$\hat{D}_{a,1}(q,u_{1,0}) = (-1)^{a+1}\frac{aq(\sin^2\theta\,(q^4+q^2(2a^2-8v_{1,0}^2)+(a^2+4v_{1,0}^2)^2)-16v_{1,0}^2)}{4v_{1,0}^2(q-ia)Q^{[a-1]}}\,. \tag{3.29}$$

Here $v_{1,0}$ is not independent of $u_{1,0}$ as they can be related either by (2.48) or by (2.76). Using (2.48), the explicit θ dependence can be factored out as

$$\hat{D}_{a,1}(q,u_{1,0}) = (-1)^{a+1}\frac{aq\sin^2\theta\,((a^2+q^2)^2+8v_{1,0}^2(a^2-q^2+4u_{1,0}^2-1)-(1+4u_{1,0}^2)^2)}{4v_{1,0}^2(q-ia)Q^{[a-1]}}\,. \tag{3.30}$$

The weak-coupling limit of the scalar part (3.3) is

$$f_{a,1}(q,u_{1,0}) = \frac{Q^{[a-1]}(u_{1,0}^2+\frac{1}{4})^2}{Q^{[-a-1]}Q^{[1-a]}Q^{[a+1]}}\left(\frac{4g^2}{q^2+a^2}\right)^2\frac{q-ia}{q+ia}\,. \tag{3.31}$$

The full contribution of $\mathbb{D}_{a,1}$ (3.2) is therefore

$$\begin{aligned}\mathbb{D}_{a,1} \quad &= \quad \frac{a^2q^2\sin^4\theta\,\left((a^2+q^2)^2+8v_{1,0}^2(a^2-q^2+4u_{1,0}^2-1)-(1+4u_{1,0}^2)^2\right)^2}{Q^{[-a-1]}Q^{[1-a]}Q^{[a-1]}Q^{[1+a]}}\\ &\quad\times\left(\frac{4g^2}{q^2+a^2}\right)^2\frac{(u_{1,0}^2+\frac{1}{4})^2}{16v_{1,0}^4(q^2+a^2)}\,.\end{aligned} \tag{3.32}$$

The exponential part is again given by (3.15).

Since the integrand (3.1) is again symmetric in q, we extend the integral to the whole line and evaluate it by residues. On the upper half-plane we find the same poles that we found in Section 3.1 for the λ_3 case. We also find that the contributions from the dynamical poles at $q = i(a+1+2u_{1,0})$ (and similarly for the dynamical poles at $q = i(a+1-2u_{1,0})$) coming from two consecutive values of a cancel provided $u_{1,0}$ and $v_{1,0}$ are related by the Bethe-Yang equation i.e. (2.77) or (2.78). Summing up only the contributions from the kinematical residues we find the following results:

3. For $\lambda_{1,2}$ with $\gamma = +1$ and (2.77),

$$\Delta E_3 = -49152\tilde{g}^{12}(1-4u_{1,0}^2)^2\zeta_5\,. \tag{3.33}$$

4. For $\lambda_{1,2}$ with $\gamma = -1$ and (2.78),

$$\Delta E_4 = 3\tilde{g}^{12}(1-24u_{1,0}^2+16u_{1,0}^4)^2u_{1,0}^{-4}(256u_{1,0}^2\zeta_5-7(1+4u_{1,0}^2)^4\zeta_9)\,. \tag{3.34}$$

The corresponding energies from the asymptotic Bethe ansatz are given by (2.66) and (2.68), respectively.

Let us now compare these results with those obtained previously for the two diagonal cases:

$\theta = 0$**:** For the $Y - Y$ case with $\gamma = +1$, (2.64) implies that there are two solutions $u_{1,0} = \infty\,, 1/2$, corresponding to $p_{1,0} = 0, \pi/2$, as can be seen from Fig 1(a). From (3.33) it follows that, up to g^{12}, there are no wrapping corrections, $\Delta E_3 = 0$. For the $Y-Y$ case with $\gamma = -1$, we find from (2.67) that there are three solutions $u_{1,0} = (\sqrt{2}\pm 1)/2\,, 1/2$, corresponding to $p_{1,0} = \pi/4\,, 3\pi/4\,, \pi/2$, as can be seen from Fig 1(b). From (3.34) it follows that, up to g^{12}, there are no wrapping corrections for $p_{1,0} = \pi/4\,, 3\pi/4$, and

$$\Delta E_4\Big|_{p_{1,0}=\pi/2} = 768 g^{12}\zeta_5 - 1344 g^{12}\zeta_9\,, \tag{3.35}$$

which agrees with (4.26) in [13].

$\theta = \pi/2$**:** For the $Y - \bar{Y}$ case with $\gamma = +1$, (2.64) implies that there is just one solution $u_{1,0} = \sqrt{3}/2$, corresponding to $p_{1,0} = \pi/3$, as can be seen from Fig 1(a). From (3.33) we obtain the wrapping correction

$$\Delta E_3\Big|_{p_{1,0}=\pi/3} = -48 g^{12}\zeta_5\,, \tag{3.36}$$

which agrees with (D.15) in [14]. For the $Y - \bar{Y}$ case with $\gamma = -1$, we find from (2.67) that there are two solutions $u_{1,0} = \infty\,, 1/(2\sqrt{3})$, corresponding to $p_{1,0} = 0\,, 2\pi/3$, as can be seen from Fig 1(b). From (3.34) we obtain the wrapping correction

$$\Delta E_4\Big|_{p_{1,0}=2\pi/3} = 1296 g^{12}\zeta_5 - 1344 g^{12}\zeta_9\,, \tag{3.37}$$

which agrees with (D.16) in [14].

In short, the results (3.33) and (3.34) for the wrapping corrections are in complete agreement with those obtained previously for $\theta = 0$ and $\theta = \pi/2$. While the boundary S-matrix $R^-_\theta(p)$ (2.13) is diagonal for both of these angles, the "extra" term in the Bethe-ansatz solution (2.30) does *not* vanish for $\theta = \pi/2$. [4] Hence, the agreement at $\theta = \pi/2$ provides strong support for the Bethe-ansatz solution (2.30) and for the corresponding generating functional (2.40).

4 Results for $L = 1$

In this section we analyze the energies of the states related to the $\lambda_{1,2}$ eigenvalues for $L = 1$ up to the leading wrapping correction. Although the wrapping correction for the vacuum

[4] For a discussion of this point, see [16].

state at $L = 1$ seems to be divergent [14], our calculation formally makes sense also for this case.

The boundary Bethe-Yang equation for $L = 1$ symmetric states ($\lambda_i = \dot{\lambda}_i$) takes the form (c.f. (2.52))

$$1 = \gamma e^{-2ip_1}\sigma(p_1, -p_1)\lambda_i(p_1)\,, \qquad \gamma = \pm 1\,. \tag{4.1}$$

Depending on the sign of γ, we find the following results.

$\gamma = 1$: The Bethe-Yang equation (4.1) at leading order gives the following relation between the angle θ and the rapidity $u_{1,0}$:

$$\cos(2\theta) = 1 + \frac{1}{2u_{1,0}^2} - \frac{8}{1 + 4u_{1,0}^2}\,. \tag{4.2}$$

The higher-order corrections are

$$u_1 = u_{1,0}\left(1 + 8\tilde{g}^2 - 32\tilde{g}^4 + 256\tilde{g}^6\right) + 1024\tilde{g}^6\frac{u_{1,0}^3(1 + 4u_{1,0}^2)}{1 + 12u_{1,0}^2}\zeta_3\,, \tag{4.3}$$

such that the energy up to order g^8 is

$$E = 1 + 8\tilde{g}^2 - 32\tilde{g}^4 + 256\tilde{g}^6 - 2560\tilde{g}^8 - 65536\tilde{g}^8\frac{u_{1,0}^4}{1 + 12u_{1,0}^2}\zeta_3\,. \tag{4.4}$$

We recall that $\tilde{g}^2 = g^2/(1 + 4u_{1,0}^2)$. At this order, wrapping starts to contribute as

$$\Delta E = 32\tilde{g}^8 u_{1,0}^{-2}(1 - 12u_{1,0}^2)^2\zeta_3 - \frac{5}{2}\tilde{g}^8 u_{1,0}^{-4}(1 - 8u_{1,0}^2 - 48u_{1,0}^4)^2\zeta_5\,. \tag{4.5}$$

For this case, we have $v_1^2 = (1 + 4u_{1,0}^2)^2/(48u_{1,0}^2 - 4)$.

$\gamma = -1$: The leading-order rapidity $u_{1,0}$ can be expressed in terms of θ as

$$\cos(2\theta) = -1 + \frac{8}{1 + 4u_{1,0}^2}\,, \tag{4.6}$$

which can be easily inverted to give $u_{1,0}^2 = \tan^2\theta + \frac{3}{4}$. Its corrections are given by

$$u_1 = u_{1,0}\left(1 + 8\tilde{g}^2 - 32\tilde{g}^4 + 256\tilde{g}^6\right) - 16\tilde{g}^6 u_{1,0}^{-1}(1 - 16u_{1,0}^4)\zeta_3\,, \tag{4.7}$$

which lead to the energy

$$E = 1 + 8\tilde{g}^2 - 32\tilde{g}^4 + 256\tilde{g}^6 - 2560\tilde{g}^8 + 1024\tilde{g}^8(1 - 4u_{1,0}^2)\zeta_3\,. \tag{4.8}$$

The leading wrapping correction to this state is

$$\Delta E = -128\tilde{g}^8(3 - 4u_{1,0}^2)^2\zeta_3 - 40\tilde{g}^8(3 + 8u_{1,0}^2 - 16u_{1,0}^4)^2\zeta_5\,. \tag{4.9}$$

For this case, we have $v_1^2 = \frac{5}{4} + u_{1,0}^2 + 4/(4u_{1,0}^2 - 3)$.

5 Discussion

We have analyzed the leading wrapping corrections for one-particle states in the AdS_5/CFT_4 integrable model on the strip with non-diagonal boundary conditions at one end. This boundary system describes the excitations of an open string stretched between a $Y = 0$ brane and a rotated $Y_\theta = 0$ brane, which interpolates smoothly between the $Y - Y$ ($\theta = 0$) and the $Y - \bar{Y}$ ($\theta = \pi/2$) systems. Our analysis has two novel features: the use of a Bethe ansatz solution with an "inhomogeneous" boundary-dependent term (2.30), (2.35); and the presence of a pair of momentum-dependent magnonic rapidities[5] (v_1 and w_1, which depend on the momentum through a continuous parameter θ) to determine the boundstate transfer-matrix eigenvalues $\mathbb{D}_{a,1}$, which are needed to obtain the leading exponential finite-size corrections. Due to the unusual generating functional (2.40) and the presence of the magnonic Bethe roots, the intermediate expressions are quite complicated. Nevertheless, the final expression (3.32) for $\mathbb{D}_{a,1}$ is remarkably compact. Our results provide the evolution of the energies of all excitations for sizes $L = 1$ and $L = 2$ up to the leading wrapping order from $\theta = 0$ to $\theta = \pi/2$, and reproduce the available limiting cases. Interestingly, the energies exhibit a smooth behavior, even though the ground state develops a tachyonic instability [14].

The AdS_5/CFT_4 integrable model admits other interesting non-diagonal boundary conditions, for which vacuum wrapping corrections were calculated in [26, 27, 28, 29]. It would be interesting to extend those analyses for one-particle states and compare the structure of the results with our findings.

In calculating the leading wrapping correction, it was enough to take into account the effect of vacuum polarization on the energy. This is due to the fact that the dispersion relation (2.2), (2.56) contains the coupling constant. At the next-to leading order wrapping correction, the effect of vacuum polarization on the boundary Bethe-Yang equation should also be taken into account [12]. It would be very interesting to derive these corrections for non-diagonal boundaries.

In order to sum up all (leading as well as sub-leading) wrapping corrections, one should derive the corresponding TBA equations. These equations could also shed some light on the tachyonic instability, as by changing the angle smoothly one could switch from the stable $Y - Y$ system to the unstable $Y - \bar{Y}$ system. The TBA equations would be the first step towards deriving a more compact formulation of exact finite-size energies, and could lead to a non-diagonal boundary generalization of the quantum spectral curve [9].

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[5] In the work [25] the authors also considered the wrapping correction of magnonic states, but only with y-type roots.

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