Moment formulas for multi-type continuous state and continuous time branching process with immigration

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Abstract

Recursions for moments of multi-type continuous state and continuous time branching process with immigration are derived. It turns out that the k-th (mixed) moments and the k-th (mixed) central moments are polynomials of the initial value of the process, and their degree are at most k and $\lfloor k/2 \rfloor$, respectively.

1 Introduction

Moment formulas and estimations play an important role in the theory of stochastic processes, since they are useful in proving limit theorems for processes and for functionals of processes as well. Branching processes form a distinguished class, since they are frequently used for modelling real data sets describing dynamic behaviour of populations, phenomenas in epidemiology, cell kinetics and genetics, so moment estimation for them is of great importance as well. The main purpose of the present paper is to derive recursions for moments of a multi-type continuous state and continuous time branching process with immigration (CBI process) using the identification of such a process as a pathwise unique strong solution of certain stochastic differential equation with jumps, see (2.12).

2010 Mathematics Subject Classifications: 60J80, 60J75, 60H10.

Key words and phrases: multi-type continuous state and continuous time branching process with immigration, moments.

The research of M. Barczy and G. Pap was realized in the frames of TÁMOP 4.2.4. A/2-11-1-2012-0001 ,,National Excellence Program – Elaborating and operating an inland student and researcher personal support system". The project was subsidized by the European Union and co-financed by the European Social Fund. Z. Li has been partially supported by NSFC under Grant No. 11131003 and 973 Program under Grant No. 2011CB808001.

For a special Dawson–Watanabe superprocess (without immigration) with a special branching mechanism a recursion for the moments has been provided by Dynkin [9] and Konno and Shiga [19, Lemma 2.1], see also Li [20, Example 2.8]. Further, Dynkin [10, Chapter 5, Theorems 1.1 and 1.2] gave recursive moment formulae for Dawson–Watanabe superprocesses. We emphasize that our technique for deriving recursions for moments is completely different from that of Dynkin [10]. Li [20, Propositions 2.27 and 2.38] derived formulas for the first and second moments for such processes. For the class of regular immigration superprocesses, which contains multitype CBI processes, Li [20, Propositions 9.11 and 9.14] derived first and second order moment formulas using an explicit form for the Laplace transform of the transition semigroup of the processes in question.

In Filipović et al. [12, formula (4.4)], one can find a formal representation of polynomial moments of affine processes, which include multitype CBI processes as well. The idea behind this formal representation is that the infinitesimal generator of an affine process formally maps the finite-dimensional linear space of all polynomials of degree less than or equal to k into itself, where $k \in \mathbb{N}$, which suggests that the k-th moment of an affine process is a polynomial of the initial value of degree at most k. Very recently, Filipović and Larsson [11, Lemma 4.12 and Theorem 4.13] provided moment formulas and moment estimations for so-called polynomial preserving diffusion processes.

Yamazato [26] considered time continuous Markov chains on the state space of non-negative integers having the so-called branching property and allowing random immigration whenever the population size is zero (as a special state-dependent immigration). He investigated under which conditions the process in question has finite first and second moments, see [26, Theorem 3], and in the so-called critical case he also pointed out that the first moment is a first order polynomial of the initial value of the process, while the second moment is a second order polynomial, see [26, Theorem 5].

Dareiotis et al. [6, Lemma 2] derived some moment bounds for the pathwise unique strong solution of a stochastic differential equation (SDE) with jumps having coefficients satisfying some local Lipschitz condition. We emphasize that the coefficients of the SDE of a multi-type CBI process given in (2.12) do not satisfy the locally Lipschitz condition A-5 in Dareiotis et al. [6], so their result can not be applied to a multi-type CBI process. However, our technique is somewhat similar to theirs in the sense that they also use Itô's formula and Gronwall's inequality.

For some moment estimates for Lévy processes, see Luschgy and Pagès [21]; for nonlocal SDEs with time-varying delay, see Hu and Huang [17]; for linear SDEs driven by analytic fractional Brownian motion, see Unterberger [25]; for unstable INteger-valued AutoRegressive models of order 2 (INAR(2)), see Barczy et al. [1, Appendix A]; for a super-Brownian motion in one dimension with constant branching rate, see Perkins [23, Lemma III.4.6]; for discrete time multi-type branching random walks, see Gün et al. [14], [15], where the main input comes from the many-to-few lemma due to Harris and Roberts [16, Lemma 3]. Döring and Roberts [7, Lemma 3] provided a recursion for moments for a spatial version of a Galton–Watson process for which a system of branching particles moves in space and particles branch only in the presence of a catalyst.

The paper is organized as follows. In Section 2, for completeness and better readability, we recall from Barczy et al. [4] some notions and statements for multi-type CBI processes such as the form of their infinitesimal generator, their branching and immigration mechanisms, and their representation as pathwise unique strong solutions of certain SDEs with jumps, see Theorem 2.9. In Section 3, we consider an appropriately truncated version (3.1) of the SDE (2.12) of a multi-type CBI process, where we truncate the integrand of the integral with respect to a (non-compensated) Poisson random measure. We show that, under some moment conditions, this truncated SDE has a pathwise unique strong solution which is a multi-type CBI process with explicitly given parameters, especially, the jump measures of the branching and immigration mechanisms are truncated, see Theorem 3.1. Then we prove a comparison theorem with respect to the truncation mentioned above, see Theorem 3.2, and, as a consequence, we show that the truncated CBI process at a time point t converges in L^1 and almost surely to the non-truncated CBI process at the time point t as the level of truncation tends to ∞, see Theorem 3.3. Section 4 is devoted to deriving recursion formulas for moments. First, we rewrite the SDE (2.12) of a multi-type CBI process in a form which is more suitable for calculating moments. Namely, we eliminate integrals with respect to non-compensated Poisson random measures, and then we perform a linear transformation in order to remove randomness from the drift, see Theorem 4.1. In view of Theorem 3.3, for the proof of the recursion formula (4.5) in Theorem 4.3, it is enough to prove a recursion formula for moments of a truncated CBI process. After applying Itô's formula for powers of a truncated CBI process, we would like to take expectations, so we have to check martingale property of some stochastic integrals with respect to certain compensated Poisson random measures. In order to do this, by induction with respect to k, we prove certain estimates for the k-th moments of a truncated CBI process, see (4.7) and (4.8). Truncations of the jump measures of the branching and immigration mechanisms are needed to avoid integrability troubles when showing martingale property of the stopped processes (4.11). It turns out that the k-th (mixed) moments and the k-th (mixed) central moments are polynomials of the initial value of the process, and their degrees are at most k and $\lfloor k/2 \rfloor$, respectively, see Theorems 4.3 and 4.5, and Corollaries 4.4 and 4.7. An explicit formula for the second central moment, i.e., for the variance of a CBI process is given in Proposition 4.8.

In a companion paper, Barczy and Pap [5] used the results of the present paper for studying the asymptotic behavior of critical irreducible multi-type continuous state and continuous time branching processes with immigration. Further, in Barczy et al. [2] moment estimations together with the results in [5] serve as a key tool for studying asymptotic behavior of conditional least squares estimators of some parameters for 2-type doubly symmetric critical irreducible CBI processes.

2 Multi-type CBI processes

Let $\mathbb{Z}_+$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote the set of non-negative integers, positive integers, real numbers, non-negative real numbers and positive real numbers, respectively. For $x, y \in \mathbb{R}$, we will use the notations $x \wedge y := \min\{x, y\}$ and $x^+ := \max\{0, x\}$. By $\|\boldsymbol{x}\|$ and $\|\boldsymbol{A}\|$, we denote the Euclidean norm of a vector $\boldsymbol{x} \in \mathbb{R}^d$ and the induced matrix norm of a matrix $\boldsymbol{A} \in \mathbb{R}^{d\times d}$, respectively. The natural basis in $\mathbb{R}^d$ and the Borel σ-algebras on $\mathbb{R}^d$ and on $\mathbb{R}^d_+$ will be denoted by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$, and by $\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}(\mathbb{R}^d_+)$, respectively. For $\boldsymbol{x} = (x_i)_{i\in\{1,\ldots,d\}} \in \mathbb{R}^d$ and $\boldsymbol{y} = (y_i)_{i\in\{1,\ldots,d\}} \in \mathbb{R}^d$, we will use the notation $\boldsymbol{x} \leqslant \boldsymbol{y}$ indicating that $x_i \leqslant y_i$ for all $i \in \{1, \ldots, d\}$. By $C^2_{\mathrm{c}}(\mathbb{R}^d_+, \mathbb{R})$ we denote the set of twice continuously differentiable real-valued functions on $\mathbb{R}^d_+$ with compact support. Throughout this paper, we make the conventions $\int_a^b := \int_{(a,b]}$ and $\int_a^\infty := \int_{(a,\infty)}$ for any $a, b \in \mathbb{R}$ with $a < b$.

2.1 Definition. *A matrix* $\boldsymbol{A} = (a_{i,j})_{i,j\in\{1,\ldots,d\}} \in \mathbb{R}^{d\times d}$ *is called essentially non-negative if* $a_{i,j} \in \mathbb{R}_+$ *whenever* $i, j \in \{1, \ldots, d\}$ *with* $i \neq j$, *i.e., if* $\boldsymbol{A}$ *has non-negative off-diagonal entries. The set of essentially non-negative* $d \times d$ *matrices will be denoted by* $\mathbb{R}^{d\times d}_{(+)}$.

2.2 Definition. *A tuple* $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ *is called a set of admissible parameters if*

(i) $d \in \mathbb{N}$,

(ii) $\boldsymbol{c} = (c_i)_{i\in\{1,\ldots,d\}} \in \mathbb{R}^d_+$,

(iii) $\boldsymbol{\beta} = (\beta_i)_{i\in\{1,\ldots,d\}} \in \mathbb{R}^d_+$,

(iv) $\boldsymbol{B} = (b_{i,j})_{i,j\in\{1,\ldots,d\}} \in \mathbb{R}^{d\times d}_{(+)}$,

(v) ν *is a Borel measure on* $U_d := \mathbb{R}^d_+ \setminus \{\boldsymbol{0}\}$ *satisfying* $\int_{U_d} (1 \wedge \|\boldsymbol{z}\|)\, \nu(\mathrm{d}\boldsymbol{z}) < \infty$,

(vi) $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d)$, *where, for each* $i \in \{1, \ldots, d\}$, μ_i *is a Borel measure on* U_d *satisfying*

$$\int_{U_d} \left[\|\boldsymbol{z}\| \wedge \|\boldsymbol{z}\|^2 + \sum_{j\in\{1,\ldots,d\}\setminus\{i\}} z_j \right] \mu_i(\mathrm{d}\boldsymbol{z}) < \infty. \tag{2.1}$$

2.3 Remark. Our Definition 2.2 of the set of admissible parameters is a special case of Definition 2.6 in Duffie et al. [8], which is suitable for all affine processes. Further, for all $i \in \{1, \ldots, d\}$, the condition (2.1) is equivalent to

$$\int_{U_d} \left[(1 \wedge z_i)^2 + \sum_{j\in\{1,\ldots,d\}\setminus\{i\}} (1 \wedge z_j) \right] \mu_i(\mathrm{d}\boldsymbol{z}) < \infty \quad \text{and} \quad \int_{U_d} \|\boldsymbol{z}\| \mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}}\, \mu_i(\mathrm{d}\boldsymbol{z}) < \infty, \tag{2.2}$$

see Barczy et al. [4, Remark 2.3]. □

2.4 Theorem. *Let* $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ *be a set of admissible parameters in the sense of Definition 2.2. Then there exists a unique conservative transition semigroup* $(P_t)_{t\in\mathbb{R}_+}$ *acting on the Banach space (endowed with the supremum norm) of real-valued bounded Borel-measurable functions on the state space* $\mathbb{R}_+^d$ *such that its infinitesimal generator is*

$$
\begin{aligned}
(\mathcal{A}f)(\boldsymbol{x}) &= \sum_{i=1}^d c_i x_i f''_{i,i}(\boldsymbol{x}) + \langle \boldsymbol{\beta} + \boldsymbol{B}\boldsymbol{x}, \boldsymbol{f}'(\boldsymbol{x})\rangle + \int_{U_d} \big(f(\boldsymbol{x}+\boldsymbol{z}) - f(\boldsymbol{x})\big)\,\nu(\mathrm{d}\boldsymbol{z}) \\
&\quad + \sum_{i=1}^d x_i \int_{U_d} \big(f(\boldsymbol{x}+\boldsymbol{z}) - f(\boldsymbol{x}) - f'_i(\boldsymbol{x})(1\wedge z_i)\big)\,\mu_i(\mathrm{d}\boldsymbol{z})
\end{aligned}
\tag{2.3}
$$

for $f \in C_\mathrm{c}^2(\mathbb{R}_+^d, \mathbb{R})$ *and* $\boldsymbol{x} \in \mathbb{R}_+^d$, *where* f'_i *and* $f''_{i,i}$, $i \in \{1, \ldots, d\}$, *denote the first and second order partial derivatives of* f *with respect to its* i*-th variable, respectively, and* $\boldsymbol{f}'(\boldsymbol{x}) := (f'_1(\boldsymbol{x}), \ldots, f'_d(\boldsymbol{x}))^\top$. *Moreover, the Laplace transform of the transition semigroup* $(P_t)_{t\in\mathbb{R}_+}$ *has a representation*

$$
\int_{\mathbb{R}_+^d} \mathrm{e}^{-\langle \boldsymbol{\lambda}, \boldsymbol{y}\rangle} P_t(\boldsymbol{x}, \mathrm{d}\boldsymbol{y}) = \mathrm{e}^{-\langle \boldsymbol{x}, \boldsymbol{v}(t,\boldsymbol{\lambda})\rangle - \int_0^t \psi(\boldsymbol{v}(s,\boldsymbol{\lambda}))\,\mathrm{d}s}, \qquad \boldsymbol{x} \in \mathbb{R}_+^d, \quad \boldsymbol{\lambda} \in \mathbb{R}_+^d, \quad t \in \mathbb{R}_+,
$$

where, for any $\boldsymbol{\lambda} \in \mathbb{R}_+^d$, *the continuously differentiable function* $\mathbb{R}_+ \ni t \mapsto \boldsymbol{v}(t, \boldsymbol{\lambda}) = (v_1(t, \boldsymbol{\lambda}), \ldots, v_d(t, \boldsymbol{\lambda}))^\top \in \mathbb{R}_+^d$ *is the unique locally bounded solution to the system of differential equations*

$$
\partial_t v_i(t, \boldsymbol{\lambda}) = -\varphi_i(\boldsymbol{v}(t, \boldsymbol{\lambda})), \qquad v_i(0, \boldsymbol{\lambda}) = \lambda_i, \qquad i \in \{1, \ldots, d\},
\tag{2.4}
$$

with

$$
\varphi_i(\boldsymbol{\lambda}) := c_i \lambda_i^2 - \langle \boldsymbol{B}\boldsymbol{e}_i, \boldsymbol{\lambda}\rangle + \int_{U_d} \big(\mathrm{e}^{-\langle \boldsymbol{\lambda}, \boldsymbol{z}\rangle} - 1 + \lambda_i(1\wedge z_i)\big)\,\mu_i(\mathrm{d}\boldsymbol{z})
$$

for $\boldsymbol{\lambda} \in \mathbb{R}_+^d$ *and* $i \in \{1, \ldots, d\}$, *and*

$$
\psi(\boldsymbol{\lambda}) := \langle \boldsymbol{\beta}, \boldsymbol{\lambda}\rangle - \int_{U_d} \big(\mathrm{e}^{-\langle \boldsymbol{\lambda}, \boldsymbol{z}\rangle} - 1\big)\,\nu(\mathrm{d}\boldsymbol{z}), \qquad \boldsymbol{\lambda} \in \mathbb{R}_+^d.
$$

Further, the function $\mathbb{R}_+ \times \mathbb{R}_+^d \ni (t, \boldsymbol{\lambda}) \mapsto \boldsymbol{v}(t, \boldsymbol{\lambda})$ *is continuous.*

2.5 Remark. This theorem is a special case of Theorem 2.7 of Duffie et al. [8] with $m = d$, $n = 0$ and zero killing rate. □

2.6 Definition. *A conservative Markov process with state space* $\mathbb{R}_+^d$ *and with transition semigroup* $(P_t)_{t\in\mathbb{R}_+}$ *given in Theorem 2.4 is called a multi-type CBI process with parameters* $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$. *The function* $\mathbb{R}_+^d \ni \boldsymbol{\lambda} \mapsto (\varphi_1(\boldsymbol{\lambda}), \ldots, \varphi_d(\boldsymbol{\lambda}))^\top \in \mathbb{R}^d$ *is called its branching mechanism, and the function* $\mathbb{R}_+^d \ni \boldsymbol{\lambda} \mapsto \psi(\boldsymbol{\lambda}) \in \mathbb{R}_+$ *is called its immigration mechanism. The measures* μ_i, $i \in \{1, \ldots, d\}$, *and* ν *are the jump measures of the branching and immigration mechanisms, respectively.*

Let $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ be a multi-type CBI process with parameters $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ such that $\mathbb{E}(\|\boldsymbol{X}_0\|) < \infty$ and the moment condition

$$\int_{U_d} \|\boldsymbol{z}\| \mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}} \, \nu(\mathrm{d}\boldsymbol{z}) < \infty \tag{2.5}$$

holds. Then, by Lemma 3.4 in Barczy et al. [4],

$$\mathbb{E}(\boldsymbol{X}_t) = \mathrm{e}^{t\widetilde{\boldsymbol{B}}} \, \mathbb{E}(\boldsymbol{X}_0) + \int_0^t \mathrm{e}^{u\widetilde{\boldsymbol{B}}} \widetilde{\boldsymbol{\beta}} \, \mathrm{d}u, \qquad t \in \mathbb{R}_+, \tag{2.6}$$

where

$$\widetilde{\boldsymbol{B}} := (\widetilde{b}_{i,j})_{i,j\in\{1,\dots,d\}}, \qquad \widetilde{b}_{i,j} := b_{i,j} + \int_{U_d} (z_i - \delta_{i,j})^+ \, \mu_j(\mathrm{d}\boldsymbol{z}), \tag{2.7}$$

$$\widetilde{\boldsymbol{\beta}} := \boldsymbol{\beta} + \int_{U_d} \boldsymbol{z} \, \nu(\mathrm{d}\boldsymbol{z}), \tag{2.8}$$

with $\delta_{i,j} := 1$ if $i = j$, and $\delta_{i,j} := 0$ if $i \neq j$. We also introduce the modified parameters $\boldsymbol{D} := (d_{i,j})_{i,j\in\{1,\dots,d\}}$ given by

$$d_{i,j} := \widetilde{b}_{i,j} - \int_{U_d} z_i \mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}} \, \mu_j(\mathrm{d}\boldsymbol{z}). \tag{2.9}$$

Note that $\widetilde{\boldsymbol{B}} \in \mathbb{R}^{d\times d}_{(+)}$, $\widetilde{\boldsymbol{\beta}} \in \mathbb{R}^d_+$ and $\boldsymbol{D} \in \mathbb{R}^{d\times d}_{(+)}$, since

$$\int_{U_d} \|\boldsymbol{z}\| \, \nu(\mathrm{d}\boldsymbol{z}) < \infty, \qquad \int_{U_d} (z_i - \delta_{i,j})^+ \, \mu_j(\mathrm{d}\boldsymbol{z}) < \infty, \quad i, j \in \{1, \dots, d\}, \tag{2.10}$$

see Barczy et al. [4, Section 2].

Let $\mathcal{R} := \bigcup_{j=0}^d \mathcal{R}_j$, where $\mathcal{R}_j$, $j \in \{0, 1, \dots, d\}$, are disjoint sets given by

$$\mathcal{R}_0 := U_d \times \{(\boldsymbol{0}, 0)\}^d \subset \mathbb{R}^d_+ \times (\mathbb{R}^d_+ \times \mathbb{R}_+)^d,$$

and

$$\mathcal{R}_j := \{\boldsymbol{0}\} \times H_{j,1} \times \cdots \times H_{j,d} \subset \mathbb{R}^d_+ \times (\mathbb{R}^d_+ \times \mathbb{R}_+)^d, \qquad j \in \{1, \dots, d\},$$

where

$$H_{j,i} := \begin{cases} U_d \times U_1 & \text{if } i = j, \\ \{(\boldsymbol{0}, 0)\} & \text{if } i \neq j. \end{cases}$$

(Recall that $U_1 = \mathbb{R}_{++}$.) Let m be the uniquely defined measure on $V := \mathbb{R}^d_+ \times (\mathbb{R}^d_+ \times \mathbb{R}_+)^d$ such that $m(V \setminus \mathcal{R}) = 0$ and its restrictions on $\mathcal{R}_j$, $j \in \{0, 1, \dots, d\}$, are

$$m|_{\mathcal{R}_0}(\mathrm{d}\boldsymbol{r}) = \nu(\mathrm{d}\boldsymbol{r}), \qquad m|_{\mathcal{R}_j}(\mathrm{d}\boldsymbol{z}, \mathrm{d}u) = \mu_j(\mathrm{d}\boldsymbol{z}) \, \mathrm{d}u, \quad j \in \{1, \dots, d\}, \tag{2.11}$$

where we identify $\mathcal{R}_0$ with U_d and $\mathcal{R}_1, \dots, \mathcal{R}_d$ with $U_d \times U_1$ in a natural way. Using again this identification, let $f : \mathbb{R}^d \times V \to \mathbb{R}^d_+$, and $g : \mathbb{R}^d \times V \to \mathbb{R}^d_+$, be defined by

$$f(\boldsymbol{x}, \boldsymbol{r}) := \begin{cases} \boldsymbol{z} \mathbb{1}_{\{\|\boldsymbol{z}\|<1\}} \mathbb{1}_{\{u\leqslant x_j\}}, & \text{if } \boldsymbol{x} = (x_1, \dots, x_d)^\top \in \mathbb{R}^d, \ \boldsymbol{r} = (\boldsymbol{z}, u) \in \mathcal{R}_j, \ j \in \{1, \dots, d\}, \\ \boldsymbol{0}, & \text{otherwise,} \end{cases}$$

$$g(\boldsymbol{x},\boldsymbol{r}) := \begin{cases} \boldsymbol{r}, & \text{if } \ \boldsymbol{x}\in\mathbb{R}^d, \ \boldsymbol{r}\in\mathcal{R}_0, \\ \boldsymbol{z}\mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}}\mathbb{1}_{\{u\leqslant x_j\}}, & \text{if } \ \boldsymbol{x}=(x_1,\dots,x_d)^\top\in\mathbb{R}^d, \ \boldsymbol{r}=(\boldsymbol{z},u)\in\mathcal{R}_j, \ j\in\{1,\dots,d\}, \\ \boldsymbol{0}, & \text{otherwise.} \end{cases}$$

Consider the disjoint decomposition $\mathcal{R} = V_0 \cup V_1$, where $V_0 := \bigcup_{j=1}^d \mathcal{R}_{j,0}$ and $V_1 := \mathcal{R}_0 \cup \big(\bigcup_{j=1}^d \mathcal{R}_{j,1}\big)$ are disjoint decompositions with $\mathcal{R}_{j,k} := \{\boldsymbol{0}\}\times H_{j,1,k}\times\cdots\times H_{j,d,k}$, $j\in\{1,\dots,d\}$, $k\in\{0,1\}$, and

$$H_{j,i,k} := \begin{cases} U_{d,k}\times U_1 & \text{if } \ i=j, \\ \{(\boldsymbol{0},0)\} & \text{if } \ i\neq j, \end{cases} \qquad U_{d,k} := \begin{cases} \{\boldsymbol{z}\in U_d : \|\boldsymbol{z}\|<1\} & \text{if } \ k=0, \\ \{\boldsymbol{z}\in U_d : \|\boldsymbol{z}\|\geqslant 1\} & \text{if } \ k=1. \end{cases}$$

Note that $f(\boldsymbol{x},\boldsymbol{r}) = \boldsymbol{0}$ if $\boldsymbol{r}\in V_1$, $g(\boldsymbol{x},\boldsymbol{r}) = \boldsymbol{0}$ if $\boldsymbol{r}\in V_0$, hence $\boldsymbol{e}_i^\top f(\boldsymbol{x},\boldsymbol{r})g(\boldsymbol{x},\boldsymbol{r})\boldsymbol{e}_j = 0$ for all $(\boldsymbol{x},\boldsymbol{r})\in\mathbb{R}^d\times V$ and $i,j\in\{1,\dots,d\}$.

Consider the following objects:

(E1) a probability space $(\Omega,\mathcal{F},\mathbb{P})$;

(E2) a d-dimensional standard Brownian motion $(\boldsymbol{W}_t)_{t\in\mathbb{R}_+}$;

(E3) a stationary Poisson point process p on V with characteristic measure m;

(E4) a random vector $\boldsymbol{\xi}$ with values in $\mathbb{R}_+^d$, independent of $\boldsymbol{W}$ and p.

2.7 Remark. Note that if objects (E1)–(E4) are given, then $\boldsymbol{\xi}$, $\boldsymbol{W}$ and p are automatically mutually independent according to Remark 3.4 in Barczy et al. [3]. For a short review on point measures and point processes needed for this paper, see, e.g., Barczy et al. [3, Section 2]. □

Provided that the objects (E1)–(E4) are given, let $(\mathcal{F}_t^{\boldsymbol{\xi},\boldsymbol{W},p})_{t\in\mathbb{R}_+}$ denote the augmented filtration generated by $\boldsymbol{\xi}$, $\boldsymbol{W}$ and p, see Barczy et al. [3].

Let us consider the d-dimensional SDE

$$\begin{aligned} \boldsymbol{X}_t = \boldsymbol{X}_0 &+ \int_0^t (\boldsymbol{\beta} + \boldsymbol{D}\boldsymbol{X}_s)\,\mathrm{d}s + \sum_{i=1}^d \boldsymbol{e}_i \int_0^t \sqrt{2c_i X_{s,i}^+}\,\mathrm{d}W_{s,i} \\ &+ \int_0^t\int_{V_0} f(\boldsymbol{X}_{s-},\boldsymbol{r})\,\widetilde{N}(\mathrm{d}s,\mathrm{d}\boldsymbol{r}) + \int_0^t\int_{V_1} g(\boldsymbol{X}_{s-},\boldsymbol{r})\,N(\mathrm{d}s,\mathrm{d}\boldsymbol{r}), \qquad t\in\mathbb{R}_+, \end{aligned} \tag{2.12}$$

where $\boldsymbol{X}_t = (X_{t,1},\dots,X_{t,d})^\top$, $\boldsymbol{D}$ is defined in (2.9), $N(\mathrm{d}s,\mathrm{d}\boldsymbol{r})$ is the counting measure of p on $\mathbb{R}_{++}\times V$, and $\widetilde{N}(\mathrm{d}s,\mathrm{d}\boldsymbol{r}) := N(\mathrm{d}s,\mathrm{d}\boldsymbol{r}) - \mathrm{d}s\,m(\mathrm{d}\boldsymbol{r})$.

2.8 Definition. *Suppose that the objects* (E1)–(E4) *are given. An* $\mathbb{R}_+^d$*-valued strong solution of the SDE* (2.12) *on* $(\Omega,\mathcal{F},\mathbb{P})$ *and with respect to the standard Brownian motion* $\boldsymbol{W}$, *the stationary Poisson point process* p *and initial value* $\boldsymbol{\xi}$, *is an* $\mathbb{R}_+^d$*-valued* $(\mathcal{F}_t^{\boldsymbol{\xi},\boldsymbol{W},p})_{t\in\mathbb{R}_+}$*-adapted càdlàg process* $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ *such that* $\mathbb{P}(\boldsymbol{X}_0=\boldsymbol{\xi})=1$,

$$\mathbb{P}\bigg(\int_0^t\int_{V_0} \|f(\boldsymbol{X}_s,\boldsymbol{r})\|^2\,\mathrm{d}s\,m(\mathrm{d}\boldsymbol{r})<\infty\bigg)=1, \qquad \mathbb{P}\bigg(\int_0^t\int_{V_1} \|g(\boldsymbol{X}_{s-},\boldsymbol{r})\|\,N(\mathrm{d}s,\mathrm{d}\boldsymbol{r})<\infty\bigg)=1$$

for all $t \in \mathbb{R}_+$, *and equation* (2.12) *holds* $\mathbb{P}$*-a.s.*

Note that the integrals $\int_0^t (\boldsymbol{\beta} + \boldsymbol{D}\boldsymbol{X}_s)\,\mathrm{d}s$ and $\int_0^t \sqrt{2c_i X_{s,i}^+}\,\mathrm{d}W_{s,i}$, $i \in \{1, \ldots, d\}$, exist, since $\boldsymbol{X}$ is càdlàg. For the following result see Theorem 4.6 and Remark 3.2 in Barczy et al. [4].

2.9 Theorem. *Let* $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ *be a set of admissible parameters such that the moment condition* (2.5) *holds. Suppose that objects* (E1)–(E4) *are given. If* $\mathbb{E}(\|\boldsymbol{\xi}\|) < \infty$, *then there is a pathwise unique* $\mathbb{R}_+^d$*-valued strong solution to the SDE* (2.12) *with initial value* $\boldsymbol{\xi}$, *and the solution is a CBI process with parameters* $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$. *Moreover, for each* $t \in \mathbb{R}_+$,

$$\mathbb{E}\biggl(\int_0^t \int_{V_0} \|f(\boldsymbol{X}_s, \boldsymbol{r})\|^2 \,\mathrm{d}s\, m(\mathrm{d}\boldsymbol{r})\biggr) < \infty, \qquad \mathbb{E}\biggl(\int_0^t \int_{V_1} \|g(\boldsymbol{X}_s, \boldsymbol{r})\| \,\mathrm{d}s\, m(\mathrm{d}\boldsymbol{r})\biggr) < \infty.$$

3 Approximation of multi-type CBI processes

First we study an appropriately truncated version of the SDE (2.12).

3.1 Theorem. *Let* $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ *be a set of admissible parameters such that the moment condition* (2.5) *holds. Suppose that objects* (E1)–(E4) *are given. Let* $K \in (1, \infty]$. *If* $\mathbb{E}(\|\boldsymbol{\xi}\|) < \infty$, *then there is a pathwise unique* $\mathbb{R}_+^d$*-valued strong solution to the SDE*

$$\begin{aligned} \boldsymbol{X}_t = \boldsymbol{X}_0 &+ \int_0^t (\boldsymbol{\beta} + \boldsymbol{D}\boldsymbol{X}_s)\,\mathrm{d}s + \sum_{i=1}^d \boldsymbol{e}_i \int_0^t \sqrt{2c_i X_{s,i}^+}\,\mathrm{d}W_{s,i} \\ &+ \int_0^t \int_{V_0} f(\boldsymbol{X}_{s-}, \boldsymbol{r})\,\widetilde{N}(\mathrm{d}s, \mathrm{d}\boldsymbol{r}) + \int_0^t \int_{V_1} g_K(\boldsymbol{X}_{s-}, \boldsymbol{r})\,N(\mathrm{d}s, \mathrm{d}\boldsymbol{r}), \qquad t \in \mathbb{R}_+, \end{aligned} \tag{3.1}$$

with initial value $\boldsymbol{\xi}$, *where the function* $g_K : \mathbb{R}^d \times V \to \mathbb{R}_+^d$ *is defined by*

$$g_K(\boldsymbol{x}, \boldsymbol{r}) := \begin{cases} \boldsymbol{r}\mathbb{1}_{\{\|\boldsymbol{r}\|<K\}}, & \text{if } \boldsymbol{x} \in \mathbb{R}^d, \ \boldsymbol{r} \in \mathcal{R}_0, \\ \boldsymbol{z}\mathbb{1}_{\{1\leqslant\|\boldsymbol{z}\|<K\}}\mathbb{1}_{\{u\leqslant x_j\}}, & \text{if } \boldsymbol{x} = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d, \\ & \quad \boldsymbol{r} = (\boldsymbol{z}, u) \in \mathcal{R}_{j,1}, \ j \in \{1, \ldots, d\}, \\ \boldsymbol{0}, & \text{otherwise}, \end{cases}$$

and the solution is a CBI process with parameters $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}_K, \nu_K, \boldsymbol{\mu}_K)$, *where* $\boldsymbol{B}_K = (b_{K,i,j})_{i,j\in\{1,\ldots,d\}}$, ν_K *and* $\boldsymbol{\mu}_K = (\mu_{K,1}, \ldots, \mu_{K,d})$ *are given by*

$$b_{K,i,j} := b_{i,j} - \delta_{i,j} \int_{U_d} (z_i \wedge 1)\mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant K\}}\,\mu_j(\mathrm{d}\boldsymbol{z}), \tag{3.2}$$

$$\nu_K(\mathrm{d}\boldsymbol{r}) := \mathbb{1}_{\{\|\boldsymbol{r}\|<K\}}\,\nu(\mathrm{d}\boldsymbol{r}), \qquad \mu_{K,i}(\mathrm{d}\boldsymbol{z}) := \mathbb{1}_{\{\|\boldsymbol{z}\|<K\}}\,\mu_i(\mathrm{d}\boldsymbol{z}).$$

Proof. In case of $K = \infty$, the SDE (3.1) coincides with the SDE (2.12), since $g_\infty = g$, hence, by Theorem 2.9, the SDE (3.1) with $K = \infty$ admits a pathwise unique $\mathbb{R}_+^d$-valued strong solution with initial value $\boldsymbol{\xi}$, and the solution is a CBI process with parameters $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$.

For each $K \in (1, \infty)$,

$$
\begin{aligned}
&\int_0^t \int_{V_0} f(\boldsymbol{X}_{s-}, \boldsymbol{r}) \, \widetilde{N}(\mathrm{d}s, \mathrm{d}\boldsymbol{r}) = \int_0^t \int_{V_0} f(\boldsymbol{X}_{s-}, \boldsymbol{r}) \, \widetilde{N}_K(\mathrm{d}s, \mathrm{d}\boldsymbol{r}), \\
&\int_0^t \int_{V_1} g_K(\boldsymbol{X}_{s-}, \boldsymbol{r}) \, N(\mathrm{d}s, \mathrm{d}\boldsymbol{r}) = \int_0^t \int_{V_1} g(\boldsymbol{X}_{s-}, \boldsymbol{r}) \, N_K(\mathrm{d}s, \mathrm{d}\boldsymbol{r}),
\end{aligned} \tag{3.3}
$$

where $N_K(\mathrm{d}s, \mathrm{d}\boldsymbol{r})$ is the counting measure of the stationary Poisson point process p_K, where p_K denotes the thinning of p onto $V_0 \cup \mathcal{R}_{0,K} \cup \big(\cup_{j=1}^d \mathcal{R}_{j,1,K}\big)$ given by

$$
\begin{aligned}
\mathcal{R}_{0,K} &:= \{\boldsymbol{r} \in U_d : \|\boldsymbol{r}\| < K\} \times \{(\boldsymbol{0}, 0)\}^d \subset \mathbb{R}_+^d \times (\mathbb{R}_+^d \times \mathbb{R}_+)^d, \\
\mathcal{R}_{j,1,K} &:= \{\boldsymbol{0}\} \times H_{j,1,1,K} \times \cdots \times H_{j,d,1,K} \subset \mathbb{R}_+^d \times (\mathbb{R}_+^d \times \mathbb{R}_+)^d, \qquad j \in \{1, \ldots, d\},
\end{aligned}
$$

where

$$
H_{j,i,1,K} := \begin{cases} \{\boldsymbol{z} \in U_d : 1 \leqslant \|\boldsymbol{z}\| < K\} \times U_1 & \text{if } i = j, \\ \{(\boldsymbol{0}, 0)\} & \text{if } i \neq j, \end{cases}
$$

and $\widetilde{N}_K(\mathrm{d}s, \mathrm{d}\boldsymbol{r}) := N_K(\mathrm{d}s, \mathrm{d}\boldsymbol{r}) - \mathrm{d}s \, m_K(\boldsymbol{r})$, where m_K denotes the restriction of m onto $V_0 \cup \mathcal{R}_{0,K} \cup \big(\cup_{j=1}^d \mathcal{R}_{j,1,K}\big) = \mathcal{R}_{0,K} \cup \big(\cup_{j=1}^d (\mathcal{R}_{j,0} \cup \mathcal{R}_{j,1,K})\big)$. Note that the characteristic measure of p_K is m_K (this can be checked calculating the corresponding conditional Laplace transforms, see Ikeda and Watanabe [18, page 44]). Moreover, $m_K|_{V_0}(\mathrm{d}\boldsymbol{r}) = m|_{V_0}(\mathrm{d}\boldsymbol{r})$, $m_K|_{\mathcal{R}_{0,K}}(\mathrm{d}\boldsymbol{r}) = \nu_K(\mathrm{d}\boldsymbol{r})$ and

$$
\begin{aligned}
&m_K|_{\mathcal{R}_{j,0} \cup \mathcal{R}_{j,1,K}}(\mathrm{d}\boldsymbol{z}, \mathrm{d}u) = m_K|_{\mathcal{R}_{j,0}}(\mathrm{d}\boldsymbol{z}, \mathrm{d}u) + m_K|_{\mathcal{R}_{j,1,K}}(\mathrm{d}\boldsymbol{z}, \mathrm{d}u) \\
&= \mathbb{1}_{\{\|\boldsymbol{z}\| < 1\}} \mu_j(\mathrm{d}\boldsymbol{z}) \, \mathrm{d}u + \mathbb{1}_{\{1 \leqslant \|\boldsymbol{z}\| < K\}} \mu_j(\mathrm{d}\boldsymbol{z}) \, \mathrm{d}u = \mu_{K,j}(\mathrm{d}\boldsymbol{z}) \, \mathrm{d}u \qquad \text{for } j \in \{1, \ldots, d\}.
\end{aligned}
$$

Consequently, the SDE (3.1) can be rewritten as

$$
\begin{aligned}
\boldsymbol{X}_t = \boldsymbol{X}_0 &+ \int_0^t (\boldsymbol{\beta} + \boldsymbol{D} \boldsymbol{X}_s) \, \mathrm{d}s + \sum_{i=1}^d \boldsymbol{e}_i \int_0^t \sqrt{2 c_i X_{s,i}^+} \, \mathrm{d}W_{s,i} \\
&+ \int_0^t \int_{V_0} f(\boldsymbol{X}_{s-}, \boldsymbol{r}) \, \widetilde{N}_K(\mathrm{d}s, \mathrm{d}\boldsymbol{r}) + \int_0^t \int_{V_1} g(\boldsymbol{X}_{s-}, \boldsymbol{r}) \, N_K(\mathrm{d}s, \mathrm{d}\boldsymbol{r}), \qquad t \in \mathbb{R}_+.
\end{aligned} \tag{3.4}
$$

Further, for each $K \in (1, \infty)$, ν_K and $\boldsymbol{\mu}_K$ satisfy parts (v) and (vi) of Definition 2.2, respectively. Further, $\boldsymbol{B}_K \in \mathbb{R}_{(+)}^{d \times d}$, hence $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}_K, \nu_K, \boldsymbol{\mu}_K)$ is a set of admissible parameters. By Theorem 2.9, the SDE (3.4) admits a pathwise unique $\mathbb{R}_+^d$-valued strong solution with initial value $\boldsymbol{\xi}$, and the solution is a CBI process with parameters $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}_K, \nu_K, \boldsymbol{\mu}_K)$, since, using (2.7) and (2.9),

$$
\begin{aligned}
d_{K,i,j} &:= b_{K,i,j} + \int_{U_d} (z_i - \delta_{i,j})^+ \, \mu_{K,j}(\mathrm{d}\boldsymbol{z}) - \int_{U_d} z_i \mathbb{1}_{\{\|\boldsymbol{z}\| \geqslant 1\}} \, \mu_{K,j}(\mathrm{d}\boldsymbol{z}) \\
&= b_{i,j} - \delta_{i,j} \int_{U_d} (z_i \wedge 1) \mathbb{1}_{\{\|\boldsymbol{z}\| \geqslant K\}} \, \mu_j(\mathrm{d}\boldsymbol{z}) \\
&\quad + \int_{U_d} (z_i - \delta_{i,j})^+ \mathbb{1}_{\{\|\boldsymbol{z}\| < K\}} \, \mu_j(\mathrm{d}\boldsymbol{z}) - \int_{U_d} z_i \mathbb{1}_{\{1 \leqslant \|\boldsymbol{z}\| < K\}} \, \mu_j(\mathrm{d}\boldsymbol{z}) \\
&= b_{i,j} + \int_{U_d} (z_i - \delta_{i,j})^+ \, \mu_j(\mathrm{d}\boldsymbol{z}) - \int_{U_d} z_i \mathbb{1}_{\{\|\boldsymbol{z}\| \geqslant 1\}} \, \mu_j(\mathrm{d}\boldsymbol{z}) - \int_{U_d} (z_i - \delta_{i,j})^+ \mathbb{1}_{\{\|\boldsymbol{z}\| \geqslant K\}} \, \mu_j(\mathrm{d}\boldsymbol{z}) \\
&\quad + \int_{U_d} z_i \mathbb{1}_{\{\|\boldsymbol{z}\| \geqslant K\}} \, \mu_j(\mathrm{d}\boldsymbol{z}) - \delta_{i,j} \int_{U_d} (z_i \wedge 1) \mathbb{1}_{\{\|\boldsymbol{z}\| \geqslant K\}} \, \mu_j(\mathrm{d}\boldsymbol{z})
\end{aligned}
$$

equals $d_{i,j}$ for all $i, j \in \{1, \ldots, d\}$, since the sum of the last three terms is 0. Especially,

$$
\mathbb{E}\biggl(\int_0^t \int_{V_0} \|f(\boldsymbol{X}_s, \boldsymbol{r})\|^2 \, \mathrm{d}s \, m_K(\mathrm{d}\boldsymbol{r}) \biggr) < \infty, \qquad \mathbb{E}\biggl(\int_0^t \int_{V_1} \|g(\boldsymbol{X}_s, \boldsymbol{r})\| \, \mathrm{d}s \, m_K(\mathrm{d}\boldsymbol{r}) \biggr) < \infty
$$

for all $t \in \mathbb{R}_+$. Using (3.3), we conclude

$$
\mathbb{E}\biggl(\int_0^t \int_{V_0} \|f(\boldsymbol{X}_s, \boldsymbol{r})\|^2 \, \mathrm{d}s \, m(\mathrm{d}\boldsymbol{r}) \biggr) < \infty, \qquad \mathbb{E}\biggl(\int_0^t \int_{V_1} \|g_K(\boldsymbol{X}_s, \boldsymbol{r})\| \, \mathrm{d}s \, m(\mathrm{d}\boldsymbol{r}) \biggr) < \infty.
$$

Hence the SDE (3.1) also admits a pathwise unique $\mathbb{R}_+^d$-valued strong solution with initial value $\boldsymbol{\xi}$, and the solution is a CBI process with parameters $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}_K, \nu_K, \boldsymbol{\mu}_K)$. □

Next we prove a comparison theorem for the SDE (3.1) in K.

3.2 Theorem. *Let* $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ *be a set of admissible parameters such that the moment condition* (2.5) *holds. Suppose that objects* (E1)–(E3) *are given. Let* $\boldsymbol{\xi}$ *and* $\boldsymbol{\xi}'$ *be random vectors with values in* $\mathbb{R}_+^d$ *independent of* $\boldsymbol{W}$ *and* p *such that* $\mathbb{E}(\|\boldsymbol{\xi}\|) < \infty$, $\mathbb{E}(\|\boldsymbol{\xi}'\|) < \infty$ *and* $\mathbb{P}(\boldsymbol{\xi} \leqslant \boldsymbol{\xi}') = 1$. *Let* $K, K' \in (1, \infty]$ *with* $K \leqslant K'$. *Let* $(\boldsymbol{X}_t)_{t \in \mathbb{R}_+}$ *be a pathwise unique* $\mathbb{R}_+^d$*-valued strong solution to the SDE* (3.1) *with initial value* $\boldsymbol{\xi}$. *Let* $(\boldsymbol{X}'_t)_{t \in \mathbb{R}_+}$ *be a pathwise unique* $\mathbb{R}_+^d$*-valued strong solution to the SDE* (3.1) *with initial value* $\boldsymbol{\xi}'$ *and with* K *replaced by* K'. *Then* $\mathbb{P}(\boldsymbol{X}_t \leqslant \boldsymbol{X}'_t \text{ for all } t \in \mathbb{R}_+) = 1$.

Proof. We follow the ideas of the proof of Theorem 3.1 of Ma [22], which is an adaptation of that of Theorem 5.5 of Fu and Li [13]. There is a sequence $\phi_k : \mathbb{R} \to \mathbb{R}_+$, $k \in \mathbb{N}$, of twice continuously differentiable functions such that

(i) $\phi_k(z) \uparrow z^+$ as $k \to \infty$ for all $z \in \mathbb{R}$;

(ii) $\phi_k'(z) \in [0, 1]$ for all $z \in \mathbb{R}_+$ and $k \in \mathbb{N}$;

(iii) $\phi_k'(z) = \phi_k(z) = 0$ whenever $-z \in \mathbb{R}_+$ and $k \in \mathbb{N}$;

(iv) $\phi_k''(x-y)(\sqrt{x}-\sqrt{y})^2 \leqslant 2/k$ for all $x, y \in \mathbb{R}_+$ and $k \in \mathbb{N}$.

For a construction of such functions, see, e.g., the proof of Theorem 3.1 of Ma [22]. Let $\boldsymbol{Y}_t = (Y_{t,1}, \ldots, Y_{t,d})^\top := \boldsymbol{X}_t - \boldsymbol{X}'_t$ for all $t \in \mathbb{R}_+$. By the SDE (3.1), we have

$$\begin{aligned} Y_{t,i} = Y_{0,i} &+ \int_0^t \boldsymbol{e}_i^\top \boldsymbol{D} \boldsymbol{Y}_s \, \mathrm{d}s + \int_0^t \sqrt{2c_i}\Big(\sqrt{X_{s,i}} - \sqrt{X'_{s,i}}\Big) \, \mathrm{d}W_{s,i} \\ &+ \int_0^t \int_{V_0} \boldsymbol{e}_i^\top (f(\boldsymbol{X}_{s-}, \boldsymbol{r}) - f(\boldsymbol{X}'_{s-}, \boldsymbol{r})) \, \widetilde{N}(\mathrm{d}s, \mathrm{d}\boldsymbol{r}) \\ &+ \int_0^t \int_{V_1} \boldsymbol{e}_i^\top (g_K(\boldsymbol{X}_{s-}, \boldsymbol{r}) - g_{K'}(\boldsymbol{X}'_{s-}, \boldsymbol{r})) \, N(\mathrm{d}s, \mathrm{d}\boldsymbol{r}) \end{aligned}$$

for all $t \in \mathbb{R}_+$ and $i \in \{1, \ldots, d\}$. For each $m \in \mathbb{N}$, put

$$\tau_m := \inf\Big\{t \in \mathbb{R}_+ : \max_{i \in \{1,\ldots,d\}} \max\{X_{t,i}, X'_{t,i}\} \geqslant m\Big\}.$$

By Itô's formula, we obtain

$$\phi_k(Y_{t\wedge\tau_m,i}) = \phi_k(Y_{0,i}) + \sum_{\ell=1}^{7} I_{i,m,k,\ell}(t)$$

for all $t \in \mathbb{R}_+$, $i \in \{1, \ldots, d\}$ and $k, m \in \mathbb{N}$, where

$$\begin{aligned} I_{i,m,k,1}(t) &:= \int_0^{t\wedge\tau_m} \phi_k'(Y_{s,i})\big(\boldsymbol{e}_i^\top \boldsymbol{D} \boldsymbol{Y}_s\big) \, \mathrm{d}s, \\ I_{i,m,k,2}(t) &:= \int_0^{t\wedge\tau_m} \phi_k'(Y_{s,i})\sqrt{2c_i}\Big(\sqrt{X_{s,i}} - \sqrt{X'_{s,i}}\Big) \, \mathrm{d}W_{s,i}, \\ I_{i,m,k,3}(t) &:= \frac{1}{2}\int_0^{t\wedge\tau_m} \phi_k''(Y_{s,i}) 2c_i\Big(\sqrt{X_{s,i}} - \sqrt{X'_{s,i}}\Big)^2 \, \mathrm{d}s, \\ I_{i,m,k,4}(t) &:= \int_0^{t\wedge\tau_m} \int_{V_0} \Big[\phi_k\big(Y_{s-,i} + \boldsymbol{e}_i^\top (f(\boldsymbol{X}_{s-}, \boldsymbol{r}) - f(\boldsymbol{X}'_{s-}, \boldsymbol{r}))\big) - \phi_k(Y_{s-,i})\Big] \widetilde{N}(\mathrm{d}s, \mathrm{d}\boldsymbol{r}), \\ I_{i,m,k,5}(t) &:= \int_0^{t\wedge\tau_m} \int_{V_0} \Big[\phi_k\big(Y_{s-,i} + \boldsymbol{e}_i^\top (f(\boldsymbol{X}_{s-}, \boldsymbol{r}) - f(\boldsymbol{X}'_{s-}, \boldsymbol{r}))\big) - \phi_k(Y_{s-,i}) \\ &\qquad\qquad - \phi_k'(Y_{s-,i}) \boldsymbol{e}_i^\top (f(\boldsymbol{X}_{s-}, \boldsymbol{r}) - f(\boldsymbol{X}'_{s-}, \boldsymbol{r}))\Big] \mathrm{d}s \, m(\mathrm{d}\boldsymbol{r}), \\ I_{i,m,k,6}(t) &:= \int_0^{t\wedge\tau_m} \int_{V_1} \Big[\phi_k\big(Y_{s-,i} + \boldsymbol{e}_i^\top (g_K(\boldsymbol{X}_{s-}, \boldsymbol{r}) - g_{K'}(\boldsymbol{X}'_{s-}, \boldsymbol{r}))\big) - \phi_k(Y_{s-,i})\Big] N(\mathrm{d}s, \mathrm{d}\boldsymbol{r}). \end{aligned}$$

Using formula (3.8) in Chapter II in Ikeda and Watanabe [18], the last integral can be written as $I_{i,m,k,6}(t) = I_{i,m,k,7}(t) + I_{i,m,k,8}(t)$, where

$$I_{i,m,k,7}(t) := \int_0^{t\wedge\tau_m} \int_{V_1} \Big[\phi_k\big(Y_{s-,i} + \boldsymbol{e}_i^\top (g_K(\boldsymbol{X}_{s-}, \boldsymbol{r}) - g_{K'}(\boldsymbol{X}'_{s-}, \boldsymbol{r}))\big) - \phi_k(Y_{s-,i})\Big] \widetilde{N}(\mathrm{d}s, \mathrm{d}\boldsymbol{r}),$$

$$I_{i,m,k,8}(t) := \int_0^{t\wedge\tau_m} \int_{V_1} \Big[\phi_k\big(Y_{s-,i} + \boldsymbol{e}_i^\top (g_K(\boldsymbol{X}_{s-}, \boldsymbol{r}) - g_{K'}(\boldsymbol{X}'_{s-}, \boldsymbol{r}))\big) - \phi_k(Y_{s-,i})\Big] \mathrm{d}s\, m(\mathrm{d}\boldsymbol{r}),$$

since the function

(3.5)
$$\mathbb{R}_+ \times V \times \Omega \ni (s, \boldsymbol{r}, \omega) \mapsto \phi_k\big(Y_{s-,i}(\omega) + \boldsymbol{e}_i^\top (g_K(\boldsymbol{X}_{s-}(\omega), \boldsymbol{r}) - g_{K'}(\boldsymbol{X}'_{s-}(\omega), \boldsymbol{r}))\big) - \phi_k(Y_{s-,i}(\omega))$$

belongs to the class $\boldsymbol{F}_p^1$ for each $i \in \{1, \ldots, d\}$ defined on page 62 in Ikeda and Watanabe [18]. Indeed, the predictability follows from part (iii) of Lemma A.1 in Barczy et al. [3], and

$$\begin{aligned}
&\mathbb{E}\bigg(\int_0^{t\wedge\tau_m} \int_{V_1} \Big|\phi_k\big(Y_{s-,i} + \boldsymbol{e}_i^\top (g(\boldsymbol{X}_{s-}, \boldsymbol{r}) - g_K(\boldsymbol{X}'_{s-}, \boldsymbol{r}))\big) - \phi_k(Y_{s-,i})\Big| \mathrm{d}s\, m(\mathrm{d}\boldsymbol{r})\bigg) \\
&\qquad \leqslant \mathbb{E}\bigg(\int_0^{t\wedge\tau_m} \int_{V_1} \big|\boldsymbol{e}_i^\top (g_K(\boldsymbol{X}_{s-}, \boldsymbol{r}) - g_{K'}(\boldsymbol{X}'_{s-}, \boldsymbol{r}))\big| \,\mathrm{d}s\, m(\mathrm{d}\boldsymbol{r})\bigg),
\end{aligned}$$

where we used that by properties (ii) and (iii) of the function ϕ_k, we have $\phi_k'(u) \in [0, 1]$ for all $u \in \mathbb{R}$, and hence, by mean value theorem,

$$(3.6) \quad -z \leqslant \phi_k(y - z) - \phi_k(y) \leqslant 0 \leqslant \phi_k(y + z) - \phi_k(y) \leqslant z, \qquad y \in \mathbb{R}, \quad z \in \mathbb{R}_+, \quad k \in \mathbb{N}.$$

We have $\boldsymbol{e}_i^\top (g_K(\boldsymbol{X}_{s-}, \boldsymbol{r}) - g_{K'}(\boldsymbol{X}'_{s-}, \boldsymbol{r})) = r_i(\mathbb{1}_{\{\|\boldsymbol{r}\|<K\}} - \mathbb{1}_{\{\|\boldsymbol{r}\|<K'\}}) = -r_i \mathbb{1}_{\{K\leqslant\|\boldsymbol{r}\|<K'\}}$ for $\boldsymbol{r} \in \mathcal{R}_0$, and

$$\begin{aligned}
&\boldsymbol{e}_i^\top (g_K(\boldsymbol{X}_{s-}, \boldsymbol{r}) - g_{K'}(\boldsymbol{X}'_{s-}, \boldsymbol{r})) = z_i(\mathbb{1}_{\{1\leqslant\|\boldsymbol{z}\|<K\}} \mathbb{1}_{\{u\leqslant X_{s-,j}\}} - \mathbb{1}_{\{1\leqslant\|\boldsymbol{z}\|<K'\}} \mathbb{1}_{\{u\leqslant X'_{s-,j}\}}) \\
&\quad = \begin{cases} z_i & \text{if } Y_{s-,j} > 0, \ X'_{s-,j} < u \leqslant X_{s-,j} \text{ and } 1 \leqslant \|\boldsymbol{z}\| < K, \\ -z_i & \text{if } Y_{s-,j} < 0, \ X_{s-,j} < u \leqslant X'_{s-,j} \text{ and } 1 \leqslant \|\boldsymbol{z}\| < K, \\ & \text{or if } u \leqslant X'_{s-,j} \text{ and } K \leqslant \|\boldsymbol{z}\| < K', \\ 0 & \text{otherwise}, \end{cases} \qquad \mathbb{P}\text{-a.s.}
\end{aligned} \tag{3.7}$$

for $\boldsymbol{r} = (\boldsymbol{z}, u) \in \mathcal{R}_j$, $j \in \{1, \ldots, d\}$. Consequently,

$$\begin{aligned}
&\mathbb{E}\bigg(\int_0^{t\wedge\tau_m} \int_{V_1} \Big|\phi_k\big(Y_{s-,i} + \boldsymbol{e}_i^\top (g_K(\boldsymbol{X}_{s-}, \boldsymbol{r}) - g_{K'}(\boldsymbol{X}'_{s-}, \boldsymbol{r}))\big) - \phi_k(Y_{s-,i})\Big| \,\mathrm{d}s\, m(\mathrm{d}\boldsymbol{r})\bigg) \\
&\leqslant \mathbb{E}\bigg(\int_0^{t\wedge\tau_m} \int_{U_d} r_i \mathbb{1}_{\{K\leqslant\|\boldsymbol{r}\|<K'\}} \,\mathrm{d}s\, \nu(\mathrm{d}\boldsymbol{r})\bigg) \\
&\quad + \sum_{j=1}^d \mathbb{E}\bigg(\int_0^{t\wedge\tau_m} \int_{U_d} \int_{U_1} z_i \mathbb{1}_{\{X'_{s-,j}<u\leqslant X_{s-,j}\}} \mathbb{1}_{\{Y_{s-,j}>0\}} \mathbb{1}_{\{1\leqslant\|\boldsymbol{z}\|<K\}} \,\mathrm{d}s\, \mu_j(\mathrm{d}\boldsymbol{z})\, \mathrm{d}u\bigg)
\end{aligned}$$

$$
+\sum_{j=1}^d \mathbb{E}\bigg(\int_0^{t\wedge\tau_m} \int_{U_d}\int_{U_1} z_i \mathbb{1}_{\{X_{s-,j}<u\leqslant X'_{s-,j}\}} \mathbb{1}_{\{Y_{s-,j}<0\}} \mathbb{1}_{\{1\leqslant\|\boldsymbol{z}\|<K\}} \,\mathrm{d}s\,\mu_j(\mathrm{d}\boldsymbol{z})\,\mathrm{d}u\bigg)
$$

$$
+\sum_{j=1}^d \mathbb{E}\bigg(\int_0^{t\wedge\tau_m} \int_{U_d}\int_{U_1} z_i \mathbb{1}_{\{u\leqslant X'_{s-,j}\}} \mathbb{1}_{\{K\leqslant\|\boldsymbol{z}\|<K'\}} \,\mathrm{d}s\,\mu_j(\mathrm{d}\boldsymbol{z})\,\mathrm{d}u\bigg)
$$

$$
= \mathbb{E}\bigg(\int_0^{t\wedge\tau_m} \int_{U_d} r_i \mathbb{1}_{\{K\leqslant\|\boldsymbol{r}\|<K'\}} \,\mathrm{d}s\,\nu(\mathrm{d}\boldsymbol{r})\bigg)
$$

$$
+\sum_{j=1}^d \mathbb{E}\bigg(\int_0^{t\wedge\tau_m} \int_{U_d} z_i \mathbb{1}_{\{1\leqslant\|\boldsymbol{z}\|<K\}} Y_{s-,j} \mathbb{1}_{\{Y_{s-,j}>0\}} \,\mathrm{d}s\,\mu_j(\mathrm{d}\boldsymbol{z})\bigg)
$$

$$
+\sum_{j=1}^d \mathbb{E}\bigg(\int_0^{t\wedge\tau_m} \int_{U_d} z_i \mathbb{1}_{\{1\leqslant\|\boldsymbol{z}\|<K\}} (-Y_{s-,j}) \mathbb{1}_{\{Y_{s-,j}<0\}} \,\mathrm{d}s\,\mu_j(\mathrm{d}\boldsymbol{z})\bigg)
$$

$$
+\sum_{j=1}^d \mathbb{E}\bigg(\int_0^{t\wedge\tau_m} \int_{U_d} z_i \mathbb{1}_{\{K\leqslant\|\boldsymbol{z}\|<K'\}} X'_{s-,j} \,\mathrm{d}s\,\mu_j(\mathrm{d}\boldsymbol{z})\bigg)
$$

$$
\leqslant t\int_{U_d} \|\boldsymbol{r}\| \mathbb{1}_{\{\|\boldsymbol{r}\|\geqslant 1\}}\,\nu(\mathrm{d}\boldsymbol{r}) + 2mt\sum_{j=1}^d \int_{U_d} \|\boldsymbol{z}\| \mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}}\,\mu_j(\mathrm{d}\boldsymbol{z}) < \infty
$$

by the moment condition (2.5) and (2.1).

As in the proof of Lemma 4.2 in Barczy et al. [4], we obtain that the processes $(I_{i,m,k,2}(t))_{t\in\mathbb{R}_+}$ and $(I_{i,m,k,4}(t))_{t\in\mathbb{R}_+}$ are martingales. Moreover, the process $\big(I_{i,m,k,7}(t)\big)_{t\in\mathbb{R}_+}$ is also a martingale by Ikeda and Watanabe [18, page 62], since the function (3.5) belongs to the class $\boldsymbol{F}_p^1$.

Using that the matrix $\boldsymbol{D}$ has non-negative off-diagonal entries and properties (ii) and (iii) of the function ϕ_k, we obtain

$$
\begin{aligned}
I_{i,m,k,1}(t) &= \int_0^{t\wedge\tau_m} \phi_k'(Y_{s,i})\bigg(d_{i,i}Y_{s,i} + \sum_{j\in\{1,\ldots,d\}\setminus\{i\}} d_{i,j}Y_{s,j}\bigg)\mathbb{1}_{\mathbb{R}_+}(Y_{s,i})\,\mathrm{d}s \\
&= \int_0^{t\wedge\tau_m} \phi_k'(Y_{s,i})\bigg(d_{i,i}Y_{s,i}^+ + \sum_{j\in\{1,\ldots,d\}\setminus\{i\}} d_{i,j}Y_{s,j}\mathbb{1}_{\mathbb{R}_+}(Y_{s,i})\bigg)\mathrm{d}s \\
&\leqslant \int_0^{t\wedge\tau_m} \bigg(|d_{i,i}|Y_{s,i}^+ + \sum_{j\in\{1,\ldots,d\}\setminus\{i\}} d_{i,j}Y_{s,j}^+\bigg)\,\mathrm{d}s = \sum_{j=1}^d |d_{i,j}|\int_0^{t\wedge\tau_m} Y_{s,j}^+\,\mathrm{d}s.
\end{aligned}
$$

By property (iv) of the function ϕ_k,

$$
I_{i,m,k,3}(t) \leqslant (t\wedge\tau_m)c_i\frac{2}{k} \leqslant \frac{2c_it}{k}.
$$

As in the proof of Lemma 4.2 in Barczy et al. [4], by (2.10), we obtain

$$I_{i,m,k,5}(t) \leqslant \frac{t}{k} \int_{U_d} z_i^2 \mathbb{1}_{\{\|\boldsymbol{z}\|<1\}} \, \mu_i(\mathrm{d}\boldsymbol{z}) + \sum_{j \in \{1,\ldots,d\} \setminus \{i\}} \int_0^{t \wedge \tau_m} Y_{s,j}^+ \, \mathrm{d}s \int_{U_d} z_i \, \mu_j(\mathrm{d}\boldsymbol{z}).$$

Using again (3.7) and integrating with respect to the variable u, we get $I_{i,m,k,8}(t) = \sum_{\ell=1}^4 I_{i,m,k,8,\ell}(t)$, where

$$I_{i,m,k,8,1}(t) := \int_0^{t \wedge \tau_m} \int_{U_d} \big[\phi_k(Y_{s-,i} - r_i) - \phi_k(Y_{s-,i})\big] \mathbb{1}_{\{K \leqslant \|\boldsymbol{r}\| < K'\}} \, \mathrm{d}s \, \nu(\mathrm{d}\boldsymbol{r}),$$

$$I_{i,m,k,8,2}(t) := \sum_{j=1}^d \int_0^{t \wedge \tau_m} \int_{U_d} \big[\phi_k(Y_{s-,i} + z_i) - \phi_k(Y_{s-,i})\big] Y_{s-,j} \mathbb{1}_{\{Y_{s-,j} > 0\}} \mathbb{1}_{\{1 \leqslant \|\boldsymbol{z}\| < K\}} \, \mathrm{d}s \, \mu_j(\mathrm{d}\boldsymbol{z}),$$

$$I_{i,m,k,8,3}(t) := \sum_{j=1}^d \int_0^{t \wedge \tau_m} \int_{U_d} \big[\phi_k(Y_{s-,i} - z_i) - \phi_k(Y_{s-,i})\big] (-Y_{s-,j}) \mathbb{1}_{\{Y_{s-,j} < 0\}} \mathbb{1}_{\{1 \leqslant \|\boldsymbol{z}\| < K\}} \, \mathrm{d}s \, \mu_j(\mathrm{d}\boldsymbol{z}),$$

$$I_{i,m,k,8,4}(t) := \sum_{j=1}^d \int_0^{t \wedge \tau_m} \int_{U_d} \big[\phi_k(Y_{s-,i} - z_i) - \phi_k(Y_{s-,i})\big] X'_{s-,j} \mathbb{1}_{\{K \leqslant \|\boldsymbol{z}\| < K'\}} \, \mathrm{d}s \, \mu_j(\mathrm{d}\boldsymbol{z}).$$

By (2.1), $\int_{U_d} z_i \mathbb{1}_{\{\|\boldsymbol{z}\| \geqslant 1\}} \, \mu_j(\mathrm{d}z) < \infty$ for all $i, j \in \{1, \ldots, d\}$, thus applying (3.6), we obtain

$$\begin{aligned} I_{i,m,k,8,2}(t) &= \sum_{j=1}^d \int_0^{t \wedge \tau_m} \int_{U_d} \Big[\phi_k(Y_{s-,i} + z_i) - \phi_k(Y_{s-,i})\Big] Y_{s-,j}^+ \mathbb{1}_{\{1 \leqslant \|\boldsymbol{z}\| < K\}} \, \mathrm{d}s \, \mu_j(\mathrm{d}\boldsymbol{z}) \\ &\leqslant \sum_{j=1}^d \int_0^{t \wedge \tau_m} Y_{s,j}^+ \, \mathrm{d}s \int_{U_d} z_i \mathbb{1}_{\{\|\boldsymbol{z}\| \geqslant 1\}} \, \mu_j(\mathrm{d}\boldsymbol{z}). \end{aligned}$$

By (3.6), we obtain $I_{i,m,k,8,1}(t) \leqslant 0$, $I_{i,m,k,8,3}(t) \leqslant 0$ and $I_{i,m,k,8,4}(t) \leqslant 0$.

Summarizing, we have

$$\begin{aligned} \phi_k(Y_{t \wedge \tau_m, i}) \leqslant \phi_k(Y_{0,i}) + C_i \sum_{j=1}^d \int_0^{t \wedge \tau_m} Y_{s,j}^+ \, \mathrm{d}s + \frac{2c_i t}{k} + \frac{t}{k} \int_{U_d} z_i^2 \mathbb{1}_{\{\|\boldsymbol{z}\|<1\}} \, \mu_i(\mathrm{d}\boldsymbol{z}) \\ + I_{i,m,k,2}(t) + I_{i,m,k,4}(t) + I_{i,m,k,6,1}(t) \end{aligned}$$

for all $t \in \mathbb{R}_+$, where

$$C_i := \max_{j \in \{1,\ldots,d\}} |d_{i,j}| + \max_{j \in \{1,\ldots,d\} \setminus \{i\}} \int_{U_d} z_i \, \mu_j(\mathrm{d}\boldsymbol{z}) + \int_{U_d} z_i \mathbb{1}_{\{\|\boldsymbol{z}\| \geqslant 1\}} \, \mu_i(\mathrm{d}\boldsymbol{z}).$$

The proof can be completed exactly as in the proof of Lemma 4.2 in Barczy et al. [4] using Gronwall's inequality. □

Next we give a useful approximation for a multi-type CBI process.

3.3 Theorem. *Let* $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ *be a set of admissible parameters such that the moment condition* (2.5) *holds. Suppose that objects* (E1)–(E4) *are given with* $\mathbb{E}(\|\boldsymbol{\xi}\|) < \infty$. *Let* $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ *be a pathwise unique* $\mathbb{R}_+^d$*-valued strong solution to the SDE* (2.12) *with initial value* $\boldsymbol{\xi}$. *For each* $K \in (1,\infty)$, *let* $(\boldsymbol{X}_{K,t})_{t\in\mathbb{R}_+}$ *be a pathwise unique* $\mathbb{R}_+^d$*-valued strong solution to SDE* (3.1) *with initial value* $\boldsymbol{\xi}$. *Then* $\mathbb{P}(\boldsymbol{X}_{K,t} \leqslant \boldsymbol{X}_{K',t} \leqslant \boldsymbol{X}_t \text{ for all } t \in \mathbb{R}_+) = 1$ *for all* $K, K' \in (1,\infty)$ *with* $K \leqslant K'$. *Moreover,* $\mathbb{E}(\boldsymbol{X}_t - \boldsymbol{X}_{K,t}) \to \boldsymbol{0}$ *and* $\boldsymbol{X}_{K,t} \uparrow \boldsymbol{X}_t$ $\mathbb{P}$*-a.s. as* $K \to \infty$ *for all* $t \in \mathbb{R}_+$.

Proof. The first statement follows from Theorem 3.2. Further, by (2.12) and (3.1), for each $K \in (1,\infty)$, $t \in \mathbb{R}_+$, and $i \in \{1,\ldots,d\}$, we have

$$
\begin{aligned}
X_{t,i} - X_{K,t,i} &= \int_0^t \boldsymbol{e}_i^\top \boldsymbol{D}(\boldsymbol{X}_s - \boldsymbol{X}_{K,s})\,\mathrm{d}s + \int_0^t \sqrt{2c_i}\Big(\sqrt{X_{s,i}} - \sqrt{X_{K,s,i}}\Big)\mathrm{d}W_{s,i} \\
&\quad + \int_0^t \int_{V_0} \boldsymbol{e}_i^\top (f(\boldsymbol{X}_{s-}, \boldsymbol{r}) - f(\boldsymbol{X}_{K,s-}, \boldsymbol{r}))\,\widetilde{N}(\mathrm{d}s, \mathrm{d}\boldsymbol{r}) \\
&\quad + \int_0^t \int_{V_1} \boldsymbol{e}_i^\top (g(\boldsymbol{X}_{s-}, \boldsymbol{r}) - g_K(\boldsymbol{X}_{K,s-}, \boldsymbol{r}))\,N(\mathrm{d}s, \mathrm{d}\boldsymbol{r}).
\end{aligned}
\tag{3.8}
$$

Here $\int_0^t \sqrt{2c_i}\Big(\sqrt{X_{s,i}} - \sqrt{X_{K,s,i}}\Big)\mathrm{d}W_{s,i}$, $t \in \mathbb{R}_+$, is a martingale, since

$$
\mathbb{E}\left(\int_0^t 2c_i\Big(\sqrt{X_{s,i}} - \sqrt{X_{K,s,i}}\Big)^2 \mathrm{d}s\right) \leqslant 4c_i \int_0^t \mathbb{E}(X_{s,i} + X_{K,s,i})\,\mathrm{d}s \leqslant 8c_i \int_0^t \mathbb{E}(X_{s,i})\,\mathrm{d}s < \infty
$$

due to $\mathbb{P}(\boldsymbol{X}_{K,t} \leqslant \boldsymbol{X}_t \text{ for all } t \in \mathbb{R}_+) = 1$ and (2.6). The process

$$
\int_0^t \int_{V_0} (f(\boldsymbol{X}_{s-}, \boldsymbol{r}) - f(\boldsymbol{X}_{K,s-}, \boldsymbol{r}))\,\widetilde{N}(\mathrm{d}s, \mathrm{d}\boldsymbol{r}), \qquad t \in \mathbb{R}_+,
$$

is a martingale, since the mapping $\mathbb{R}_+ \times V \times \Omega \ni (s, \boldsymbol{r}, \omega) \mapsto f(\boldsymbol{X}_{s-}(\omega), \boldsymbol{r}) - f(\boldsymbol{X}_{K,s-}(\omega), \boldsymbol{r}) \in \mathbb{R}^d$ is in the (multidimensional versions of the) class $\boldsymbol{F}_p^2$ defined on page 62 in Ikeda and Watanabe [18]. The mapping $\mathbb{R}_+ \times V \times \Omega \ni (s, \boldsymbol{r}, \omega) \mapsto g(\boldsymbol{X}_{s-}(\omega), \boldsymbol{r}) - g_K(\boldsymbol{X}_{K,s-}(\omega), \boldsymbol{r}) \in \mathbb{R}^d$ is in the (multidimensional versions of the) class $\boldsymbol{F}_p^1$, hence formula (3.8) in Chapter II in Ikeda and Watanabe [18] yields

$$
\begin{aligned}
&\mathbb{E}\bigg(\int_0^t \int_{V_1} \boldsymbol{e}_i^\top (g(\boldsymbol{X}_{s-}, \boldsymbol{r}) - g_K(\boldsymbol{X}_{K,s-}, \boldsymbol{r}))\,N(\mathrm{d}s, \mathrm{d}\boldsymbol{r})\bigg) \\
&= \mathbb{E}\bigg(\int_0^t \int_{V_1} \boldsymbol{e}_i^\top (g(\boldsymbol{X}_{s-}, \boldsymbol{r}) - g_K(\boldsymbol{X}_{K,s-}, \boldsymbol{r}))\,\mathrm{d}s\,m(\mathrm{d}\boldsymbol{r})\bigg) \\
&= t \int_{U_d} r_i \mathbb{1}_{\{\|\boldsymbol{r}\| \geqslant K\}}\,\nu(\mathrm{d}\boldsymbol{r}) + \sum_{j=1}^d \int_0^t \mathbb{E}(X_{s,j})\,\mathrm{d}s \int_{U_d} z_i \mathbb{1}_{\{\|\boldsymbol{z}\| \geqslant K\}}\,\mu_j(\mathrm{d}\boldsymbol{z}) \\
&\quad + \sum_{j=1}^d \int_0^t \mathbb{E}(X_{s,j} - X_{K,s,j})\,\mathrm{d}s \int_{U_d} z_i \mathbb{1}_{\{1 \leqslant \|\boldsymbol{z}\| < K\}}\,\mu_j(\mathrm{d}\boldsymbol{z}),
\end{aligned}
$$

since $\boldsymbol{e}_i^\top(g(\boldsymbol{X}_{s-},\boldsymbol{r}) - g_K(\boldsymbol{X}_{K,s-},\boldsymbol{r})) = r_i(1 - \mathbb{1}_{\{\|\boldsymbol{r}\|<K\}}) = r_i\mathbb{1}_{\{\|\boldsymbol{r}\|\geqslant K\}}$ for $\boldsymbol{r} \in \mathcal{R}_0$, and

$$\begin{aligned}
&\boldsymbol{e}_i^\top(g(\boldsymbol{X}_{s-},\boldsymbol{r}) - g_K(\boldsymbol{X}_{K,s-},\boldsymbol{r})) = z_i(\mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}}\mathbb{1}_{\{u\leqslant X_{s-,j}\}} - \mathbb{1}_{\{1\leqslant\|\boldsymbol{z}\|<K\}}\mathbb{1}_{\{u\leqslant X_{K,s-,j}\}})\\
&= \begin{cases} z_i & \text{if } \ X_{K,s-,j} < u \leqslant X_{s-,j} \ \text{ and } \ 1 \leqslant \|\boldsymbol{z}\| < K,\\ & \text{or if } \ u \leqslant X_{s-,j} \ \text{ and } \ \|\boldsymbol{z}\| \geqslant K,\\ 0 & \text{otherwise,} \end{cases} \qquad \mathbb{P}\text{-a.s.}
\end{aligned}$$

for $\boldsymbol{r} = (\boldsymbol{z}, u) \in \mathcal{R}_j$, $j \in \{1, \ldots, d\}$ (due to $\mathbb{P}(X_{K,s-,j} \leqslant X_{s-,j}) = 1$).

Hence, by taking the expectations in (3.8), we obtain

$$\begin{aligned}
\mathbb{E}(X_{t,i} - X_{K,t,i}) &= \int_0^t \boldsymbol{e}_i^\top \boldsymbol{D}\,\mathbb{E}(\boldsymbol{X}_s - \boldsymbol{X}_{K,s})\,\mathrm{d}s + \sum_{j=1}^d \int_0^t \mathbb{E}(X_{s,j})\,\mathrm{d}s \int_{U_d} z_i\mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant K\}}\,\mu_j(\mathrm{d}\boldsymbol{z})\\
&\quad + t\int_{U_d} r_i\mathbb{1}_{\{\|\boldsymbol{r}\|\geqslant K\}}\,\nu(\mathrm{d}\boldsymbol{r}) + \sum_{j=1}^d \int_0^t \mathbb{E}(X_{s,j} - X_{K,s,j})\,\mathrm{d}s \int_{U_d} z_i\mathbb{1}_{\{1\leqslant\|\boldsymbol{z}\|<K\}}\,\mu_j(\mathrm{d}\boldsymbol{z}).
\end{aligned}$$

Thus

$$\sum_{i=1}^d \mathbb{E}(X_{t,i} - X_{K,t,i}) \leqslant \alpha_K(t) + C\int_0^t \left(\sum_{j=1}^d \mathbb{E}(X_{s,j} - X_{K,s,j})\right)\mathrm{d}s,$$

where

$$\alpha_K(t) := t\sum_{i=1}^d \int_{U_d} r_i\mathbb{1}_{\{\|\boldsymbol{r}\|\geqslant K\}}\,\nu(\mathrm{d}\boldsymbol{r}) + \sum_{i=1}^d\sum_{j=1}^d \int_0^t \mathbb{E}(X_{s,j})\,\mathrm{d}s \int_{U_d} z_i\mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant K\}}\,\mu_j(\mathrm{d}\boldsymbol{z}),$$

$$C := \max_{j\in\{1,\ldots,d\}} \sum_{i=1}^d \left(|d_{i,j}| + \int_{U_d} z_i\mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}}\,\mu_j(\mathrm{d}\boldsymbol{z})\right).$$

By Gronwall's inequality and using that $\alpha_K(t)$, $t \in \mathbb{R}_+$, is monotone increasing for each $K \in (1, \infty)$, we get

$$0 \leqslant \sum_{i=1}^d \mathbb{E}(X_{t,i} - X_{K,t,i}) \leqslant \alpha_K(t) + C\int_0^t \alpha_K(s)\mathrm{e}^{C(t-s)}\,\mathrm{d}s \leqslant \alpha_K(t) + \alpha_K(t)C\int_0^t \mathrm{e}^{C(t-s)}\,\mathrm{d}s,$$

hence $\mathbb{E}(\boldsymbol{X}_t - \boldsymbol{X}_{K,t}) \to \boldsymbol{0}$ as $K \to \infty$ for all $t \in \mathbb{R}_+$ follows from $\alpha_K(t) \to 0$ as $K \to \infty$ (which holds by dominated convergence theorem). Finally, a non-increasing sequence of random variables converging to 0 in L_1 automatically converges to 0 almost surely, hence $\boldsymbol{X}_{K,t} \uparrow \boldsymbol{X}_t$ $\mathbb{P}$-a.s. as $K \to \infty$ for all $t \in \mathbb{R}_+$. □

4 Recursions for moments of multi-type CBI processes

First we rewrite the SDE (2.12) in a form which does not contain integrals with respect to non-compensated Poisson random measures, and then we perform a linear transformation in order to remove randomness from the drift. This form will be very useful in calculating moments.

4.1 Lemma. *Let* $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ *be a set of admissible parameters such that the moment condition* (2.5) *holds. Suppose that objects* (E1)–(E4) *are given with* $\mathbb{E}(\|\boldsymbol{\xi}\|) < \infty$. *Let* $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ *be a pathwise unique* $\mathbb{R}_+^d$*-valued strong solution to the SDE* (2.12) *with initial value* $\boldsymbol{\xi}$. *Then*

$$
\begin{aligned}
\mathrm{e}^{-t\widetilde{\boldsymbol{B}}}\boldsymbol{X}_t = \boldsymbol{X}_0 + \int_0^t \mathrm{e}^{-s\widetilde{\boldsymbol{B}}}\widetilde{\boldsymbol{\beta}}\,\mathrm{d}s + \sum_{k=1}^d \int_0^t \mathrm{e}^{-s\widetilde{\boldsymbol{B}}}\boldsymbol{e}_k\sqrt{2c_kX_{s,k}}\,\mathrm{d}W_{s,k} \\
+ \int_0^t\int_V \mathrm{e}^{-s\widetilde{\boldsymbol{B}}} h(\boldsymbol{X}_{s-},\boldsymbol{r})\,\widetilde{N}(\mathrm{d}s,\mathrm{d}\boldsymbol{r}), \qquad t\in\mathbb{R}_+,
\end{aligned}
\tag{4.1}
$$

where the function $h : \mathbb{R}^d \times V \to \mathbb{R}^d$ *is defined by* $h := f + g$.

Proof. The SDE (2.12) can be written in the form

$$
\begin{aligned}
X_{t,i} = X_{0,i} + \int_0^t \boldsymbol{e}_i^\top\big(\boldsymbol{\beta} + \boldsymbol{D}\boldsymbol{X}_s\big)\,\mathrm{d}s + \int_0^t \sqrt{2c_iX_{s,i}}\,\mathrm{d}W_{s,i} + \int_0^t\int_{\mathcal{R}_0} r_i\,N(\mathrm{d}s,\mathrm{d}\boldsymbol{r}) \\
+ \sum_{j=1}^d \int_0^t\int_{\mathcal{R}_{j,0}} z_i\mathbb{1}_{\{u\leqslant X_{s-,j}\}}\,\widetilde{N}(\mathrm{d}s,\mathrm{d}\boldsymbol{r}) + \sum_{j=1}^d \int_0^t\int_{\mathcal{R}_{j,1}} z_i\mathbb{1}_{\{u\leqslant X_{s-,j}\}}\,N(\mathrm{d}s,\mathrm{d}\boldsymbol{r})
\end{aligned}
\tag{4.2}
$$

for $t \in \mathbb{R}_+$ and $i \in \{1, \ldots, d\}$. Using formula (3.8) in Chapter II in Ikeda and Watanabe [18], for each $j \in \{1, \ldots, d\}$,

$$
\begin{aligned}
\int_0^t\int_{\mathcal{R}_{j,1}} z_i\mathbb{1}_{\{u\leqslant X_{s-,j}\}}\,N(\mathrm{d}s,\mathrm{d}\boldsymbol{r}) &= \int_0^t\int_{\mathcal{R}_{j,1}} z_i\mathbb{1}_{\{u\leqslant X_{s-,j}\}}\,\widetilde{N}(\mathrm{d}s,\mathrm{d}\boldsymbol{r}) \\
&\quad + \int_0^t\int_{U_d}\int_{U_1} z_i\mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant1\}}\mathbb{1}_{\{u\leqslant X_{s-,j}\}}\,\mathrm{d}s\,\mu_j(\mathrm{d}\boldsymbol{z})\,\mathrm{d}u,
\end{aligned}
$$

since

$$
\int_0^t\int_{U_d}\int_{U_1} z_i\mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant1\}}\mathbb{1}_{\{u\leqslant X_{s-,j}\}}\,\mathrm{d}s\,\mu_j(\mathrm{d}\boldsymbol{z})\,\mathrm{d}u = \int_0^t X_{s,j}\,\mathrm{d}s\int_{U_d} z_i\mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant1\}}\,\mu_j(\mathrm{d}\boldsymbol{z}),
$$

and consequently

$$
\mathbb{E}\left(\int_0^t X_{s,j}\,\mathrm{d}s\int_{U_d} z_i\mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant1\}}\,\mu_j(\mathrm{d}\boldsymbol{z})\right) = \int_0^t \mathbb{E}(X_{s,j})\,\mathrm{d}s\int_{U_d} z_i\mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant1\}}\,\mu_j(\mathrm{d}\boldsymbol{z}) < \infty.
$$

In a similar way,

$$
\int_0^t\int_{\mathcal{R}_0} r_i\,N(\mathrm{d}s,\mathrm{d}\boldsymbol{r}) = \int_0^t\int_{\mathcal{R}_0} r_i\,\widetilde{N}(\mathrm{d}s,\mathrm{d}\boldsymbol{r}) + \int_0^t\int_{U_d} r_i\,\mathrm{d}s\,\nu(\mathrm{d}\boldsymbol{r}),
$$

since

$$
\int_0^t\int_{U_d} r_i\,\mathrm{d}s\,\nu(\mathrm{d}\boldsymbol{r}) = t\int_{U_d} r_i\,\nu(\mathrm{d}\boldsymbol{r}) < \infty.
$$

Consequently, by (2.8),

$$
\begin{aligned}
\boldsymbol{X}_t = \boldsymbol{X}_0 + \int_0^t \big(\widetilde{\boldsymbol{\beta}} + \widetilde{\boldsymbol{B}}\boldsymbol{X}_s\big)\,\mathrm{d}s + \sum_{i=1}^d \boldsymbol{e}_i^\top \int_0^t \sqrt{2c_i X_{s,i}}\,\mathrm{d}W_{s,i} \\
+ \int_0^t \int_V h(\boldsymbol{X}_{s-},\boldsymbol{r})\,\widetilde{N}(\mathrm{d}s,\mathrm{d}\boldsymbol{r})
\end{aligned}
\tag{4.3}
$$

for $t \in \mathbb{R}_+$, since, by (2.9),

$$
\begin{aligned}
&\boldsymbol{e}_i^\top \boldsymbol{D}\boldsymbol{X}_s + \sum_{j=1}^d X_{s,j} \int_{U_d} z_i \mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}}\,\mu_j(\mathrm{d}\boldsymbol{z}) \\
&= \sum_{j=1}^d \left(d_{i,j} + \int_{U_d} z_i \mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}}\,\mu_j(\mathrm{d}\boldsymbol{z})\right) X_{s,j} = \sum_{j=1}^d \widetilde{b}_{i,j} X_{s,j} = \boldsymbol{e}_i^\top \widetilde{\boldsymbol{B}}\boldsymbol{X}_s.
\end{aligned}
$$

The statement of the lemma follows by an application of the multidimensional Itô's formula (see, e.g., Ikeda and Watanabe [18, Chapter II, Theorem 5.1]). Indeed, for each $i \in \{1,\ldots,d\}$, $\boldsymbol{e}_i^\top \mathrm{e}^{-t\widetilde{\boldsymbol{B}}}\boldsymbol{X}_t = F_i(t,\boldsymbol{X}_t)$ with the function $F_i(t,\boldsymbol{x}) := \boldsymbol{e}_i^\top \mathrm{e}^{-t\widetilde{\boldsymbol{B}}}\boldsymbol{x} = \sum_{j=1}^d \boldsymbol{e}_i^\top \mathrm{e}^{-t\widetilde{\boldsymbol{B}}}\boldsymbol{e}_j x_j$ for $t \in \mathbb{R}_+$ and $\boldsymbol{x} = (x_1,\ldots,x_d)^\top \in \mathbb{R}^d$. We have $\partial_t F_i(t,\boldsymbol{x}) = \boldsymbol{e}_i^\top \mathrm{e}^{-t\widetilde{\boldsymbol{B}}}(-\widetilde{\boldsymbol{B}})\boldsymbol{x}$, $\partial_{x_k} F_i(t,\boldsymbol{x}) = \boldsymbol{e}_i^\top \mathrm{e}^{-t\widetilde{\boldsymbol{B}}}\boldsymbol{e}_k$, $\partial_{x_k}\partial_{x_\ell} F_i(t,\boldsymbol{x}) = 0$, $i,k,\ell \in \{1,\ldots,d\}$, hence

$$
\begin{aligned}
\boldsymbol{e}_i^\top \mathrm{e}^{-t\widetilde{\boldsymbol{B}}}\boldsymbol{X}_t &= \boldsymbol{e}_i^\top \boldsymbol{X}_0 + \int_0^t \boldsymbol{e}_i^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}}(-\widetilde{\boldsymbol{B}})\boldsymbol{X}_s\,\mathrm{d}s \\
&\quad + \sum_{k=1}^d \int_0^t \boldsymbol{e}_i^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}}\boldsymbol{e}_k \sqrt{2c_k X_{s,k}}\,\mathrm{d}W_{s,k} + \sum_{k=1}^d \int_0^t \boldsymbol{e}_i^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}}\boldsymbol{e}_k\boldsymbol{e}_k^\top(\widetilde{\boldsymbol{\beta}} + \widetilde{\boldsymbol{B}}\boldsymbol{X}_s)\,\mathrm{d}s \\
&\quad + \int_0^t \int_V \Big[\boldsymbol{e}_i^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}}\big(\boldsymbol{X}_{s-} + h(\boldsymbol{X}_{s-},\boldsymbol{r})\big) - \boldsymbol{e}_i^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}}\boldsymbol{X}_{s-}\Big]\widetilde{N}(\mathrm{d}s,\mathrm{d}\boldsymbol{r}) \\
&\quad + \int_0^t \int_V \Big[\boldsymbol{e}_i^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}}\big(\boldsymbol{X}_s + h(\boldsymbol{X}_s,\boldsymbol{r})\big) - \boldsymbol{e}_i^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}}\boldsymbol{X}_s - \sum_{k=1}^d (\boldsymbol{e}_i^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}}\boldsymbol{e}_k)\boldsymbol{e}_k^\top h(\boldsymbol{X}_s,\boldsymbol{r})\Big]\,\mathrm{d}s\,m(\mathrm{d}\boldsymbol{r}),
\end{aligned}
$$

which yields the statement of the lemma (indeed, the integrand and hence the integral with respect to the measure $\mathrm{d}s\,m(\mathrm{d}\boldsymbol{r})$ is identically zero). □

4.2 Remark. We point out that in the proof of Lemma 4.1 formally we have no right to apply Theorem 5.1 in Ikeda and Watanabe [18, Chapter II] for (4.3), since the integrand of the integral $\int_0^t \int_V h(\boldsymbol{X}_{s-},\boldsymbol{r})\,\widetilde{N}(\mathrm{d}s,\mathrm{d}\boldsymbol{r})$ does not belong to the (multidimensional version of the) space $\boldsymbol{F}_p^{2,loc}$. Instead, we should apply Itô's formula to (4.2) (or equivalently to (2.12)). However, after applying Itô's formula to (4.2), one could rewrite the obtained equation yielding (4.1) under the moment condition (2.5), as desired. We will use this observation in other proofs as well later on. □

4.3 Theorem. *Let* $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ *be a CBI process with parameters* $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ *such that* $\mathbb{E}(\|\boldsymbol{X}_0\|^q) < \infty$,

$$\int_{U_d} \|\boldsymbol{z}\|^q \mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}} \, \nu(\mathrm{d}\boldsymbol{z}) < \infty, \qquad \int_{U_d} \|\boldsymbol{z}\|^q \mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}} \, \mu_i(\mathrm{d}\boldsymbol{z}) < \infty, \quad i \in \{1, \ldots, d\} \tag{4.4}$$

with some $q \in \mathbb{N}$. *Then* $\mathbb{E}(\|\boldsymbol{X}_t\|^q) < \infty$ *for all* $t \in \mathbb{R}_+$, *and we have the recursion*

$$\begin{aligned}
\mathbb{E}\left(X_{t,j}^k\right) &= \mathbb{E}\left[(\boldsymbol{e}_j^\top \mathrm{e}^{t\widetilde{\boldsymbol{B}}} \boldsymbol{X}_0)^k\right] + k \int_0^t (\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}} \widetilde{\boldsymbol{\beta}}) \, \mathbb{E}\left[(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}} \boldsymbol{X}_s)^{k-1}\right] \mathrm{d}s \\
&\quad + k(k-1) \sum_{i=1}^d c_i \int_0^t (\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}} \boldsymbol{e}_i)^2 \, \mathbb{E}\left[(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}} \boldsymbol{X}_s)^{k-2} X_{s,i}\right] \mathrm{d}s \\
&\quad + \sum_{\ell=0}^{k-2} \binom{k}{\ell} \sum_{i=1}^d \int_0^t \int_{U_d} (\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}} \boldsymbol{z})^{k-\ell} \, \mathbb{E}\big[(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}} \boldsymbol{X}_s)^\ell X_{s,i}\big] \mathrm{d}s \, \mu_i(\mathrm{d}\boldsymbol{z}) \\
&\quad + \sum_{\ell=0}^{k-2} \binom{k}{\ell} \int_0^t \int_{U_d} (\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}} \boldsymbol{z})^{k-\ell} \, \mathbb{E}\big[(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}} \boldsymbol{X}_s)^\ell\big] \mathrm{d}s \, \nu(\mathrm{d}\boldsymbol{z})
\end{aligned} \tag{4.5}$$

for all $k \in \{1, \ldots, q\}$, $j \in \{1, \ldots, d\}$ *and* $t \in \mathbb{R}_+$. *Moreover, for each* $t \in \mathbb{R}_+$, $k \in \{1, \ldots, q\}$ *and* $j \in \{1, \ldots, d\}$, *there exists a polynomial* $Q_{t,k,j} : \mathbb{R}^d \to \mathbb{R}$ *having degree at most* k *such that*

$$\mathbb{E}(X_{t,j}^k) = \mathbb{E}[Q_{t,k,j}(\boldsymbol{X}_0)], \qquad t \in \mathbb{R}_+. \tag{4.6}$$

The coefficients of the polynomial $Q_{t,k,j}$ *depend on* d, $\boldsymbol{c}$, $\boldsymbol{\beta}$, $\boldsymbol{B}$, ν, μ_1, $\ldots$, μ_d.

Note that formula (4.5) with $k = 1$ gives back formula (2.6).

Proof of Theorem 4.3. In the Introduction we gave a brief sketch of the present proof. Consider objects (E1)–(E4) with initial value $\boldsymbol{\xi} = \boldsymbol{y} = (y_1, \ldots, y_d)^\top \in \mathbb{R}_+^d$. Let $(\boldsymbol{Y}_t)_{t\in\mathbb{R}_+}$ be a pathwise unique $\mathbb{R}_+^d$-valued strong solution to the SDE (2.12) with initial value $\boldsymbol{y}$. By Theorem 2.9, $\boldsymbol{Y}$ is a CBI process with parameters $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ having càdlàg trajectories. Then the finite dimensional distributions of $\boldsymbol{X}$ conditioned that $\boldsymbol{X}_0 = \boldsymbol{y}$ and $\boldsymbol{Y}$ coincide. Let $K \in (1, \infty)$, and let $(\boldsymbol{Y}_{K,t})_{t\in\mathbb{R}_+}$ be a pathwise unique $\mathbb{R}_+^d$-valued strong solution to SDE (3.1) (or, equivalently, to SDE (3.4)) with initial value $\boldsymbol{y}$. By Theorem 3.1, $(\boldsymbol{Y}_{K,t})_{t\in\mathbb{R}_+}$ is a CBI process with parameters $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}_K, \nu_K, \boldsymbol{\mu}_K)$. Truncation of measures ν and μ_i, $i \in \{1, \ldots, d\}$, will be needed to avoid integrability troubles when showing martingale property of the stopped processes (4.11).

The aim of the following consideration is to show by induction with respect to k that for each $k \in \mathbb{Z}_+$ and $K \in (1, \infty)$ there exists a continuous function $f_{K,k,\boldsymbol{y}} : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\mathbb{E}(\|\boldsymbol{Y}_{K,t}\|^k) \leqslant f_{K,k,\boldsymbol{y}}(t), \qquad t \in \mathbb{R}_+, \tag{4.7}$$

and for each $k \in \{0, 1, \ldots, q\}$, there exists a continuous function $f_{k,\boldsymbol{y}} : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sup_{K \in (1,\infty)} \mathbb{E}(\|\boldsymbol{Y}_{K,t}\|^k) \leqslant f_{k,\boldsymbol{y}}(t), \qquad t \in \mathbb{R}_+. \tag{4.8}$$

For $k = 0$, (4.7) and (4.8) are trivial. By Lemma 4.1,

$$\boldsymbol{w}^\top \mathrm{e}^{-t\widetilde{\boldsymbol{B}}_K} \boldsymbol{Y}_{K,t} = \boldsymbol{w}^\top \boldsymbol{y} + \int_0^t \boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \widetilde{\boldsymbol{\beta}}_K \, \mathrm{d}s + I_{K,\boldsymbol{w},1}(t) + J_{K,\boldsymbol{w},1,0}(t) + J_{K,\boldsymbol{w},1,1}(t) \tag{4.9}$$

for all $t \in \mathbb{R}_+$, $\boldsymbol{w} \in \mathbb{R}^d$ and $K \in (1, \infty)$, where

$$I_{K,\boldsymbol{w},1}(t) := \sum_{i=1}^d \int_0^t (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \boldsymbol{e}_i) \sqrt{2c_i Y_{K,s,i}} \, \mathrm{d}W_{s,i},$$

$$J_{K,\boldsymbol{w},1,i}(t) := \int_0^t \int_{V_i} \boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} h(\boldsymbol{Y}_{K,s-}, \boldsymbol{r}) \, \widetilde{N}_K(\mathrm{d}s, \mathrm{d}\boldsymbol{r}), \qquad i \in \{0, 1\},$$

with $\widetilde{N}_K$ defined in the proof of Theorem 3.1, $\widetilde{\boldsymbol{\beta}}_K = (\widetilde{\beta}_{K,i})_{i \in \{1,\ldots,d\}}$ and $\widetilde{\boldsymbol{B}}_K = (\widetilde{b}_{K,i,j})_{i,j \in \{1,\ldots,d\}}$ are given by

$$\widetilde{\beta}_{K,i} := \beta_i + \int_{U_d} r_i \, \nu_K(\mathrm{d}\boldsymbol{r}) = \beta_i + \int_{U_d} r_i \mathbb{1}_{\{\|\boldsymbol{r}\| < K\}} \, \nu(\mathrm{d}\boldsymbol{r}) = \widetilde{\beta}_i - \int_{U_d} r_i \mathbb{1}_{\{\|\boldsymbol{r}\| \geqslant K\}} \, \nu(\mathrm{d}\boldsymbol{r}),$$

and

$$\begin{aligned}
\widetilde{b}_{K,i,j} &:= b_{K,i,j} + \int_{U_d} (z_i - \delta_{i,j})^+ \, \mu_{K,j}(\mathrm{d}\boldsymbol{z}) \\
&= b_{i,j} - \delta_{i,j} \int_{U_d} (z_i \wedge 1) \mathbb{1}_{\{\|\boldsymbol{z}\| \geqslant K\}} \, \mu_j(\mathrm{d}\boldsymbol{z}) + \int_{U_d} (z_i - \delta_{i,j})^+ \mathbb{1}_{\{\|\boldsymbol{z}\| < K\}} \, \mu_j(\mathrm{d}\boldsymbol{z}) \\
&= b_{i,j} + \int_{U_d} (z_i - \delta_{i,j})^+ \, \mu_j(\mathrm{d}\boldsymbol{z}) - \int_{U_d} z_i \mathbb{1}_{\{\|\boldsymbol{z}\| \geqslant K\}} \, \mu_j(\mathrm{d}\boldsymbol{z}) = \widetilde{b}_{i,j} - \int_{U_d} z_i \mathbb{1}_{\{\|\boldsymbol{z}\| \geqslant K\}} \, \mu_j(\mathrm{d}\boldsymbol{z}),
\end{aligned}$$

with $b_{K,i,j}$ defined in (3.2), where we applied the identity $(z_i \wedge 1) + (z_i - 1)^+ = z_i$ for $z_i \in \mathbb{R}_+$. By Itô's formula, we obtain

$$\begin{aligned}
(\boldsymbol{w}^\top \mathrm{e}^{-t\widetilde{\boldsymbol{B}}_K} \boldsymbol{Y}_{K,t})^k &= (\boldsymbol{w}^\top \boldsymbol{y})^k + I_{K,\boldsymbol{w},k}(t) + J_{K,\boldsymbol{w},k,0}(t) + J_{K,\boldsymbol{w},k,1}(t) \\
&\quad + k \int_0^t (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \boldsymbol{Y}_{K,s})^{k-1} (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \widetilde{\boldsymbol{\beta}}_K) \, \mathrm{d}s \\
&\quad + \frac{1}{2} k(k-1) \int_0^t (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \boldsymbol{Y}_{K,s})^{k-2} \sum_{i=1}^d (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \boldsymbol{e}_i)^2 2c_i Y_{K,s,i} \, \mathrm{d}s \\
&\quad + \int_0^t \int_V \Big[\Big(\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \big(\boldsymbol{Y}_{K,s} + h(\boldsymbol{Y}_{K,s}, \boldsymbol{r}) \big) \Big)^k - (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \boldsymbol{Y}_{K,s})^k \\
&\qquad - k (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \boldsymbol{Y}_{K,s})^{k-1} \big(\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} h(\boldsymbol{Y}_{K,s}, \boldsymbol{r}) \big) \Big] \, \mathrm{d}s \, m_K(\mathrm{d}\boldsymbol{r})
\end{aligned} \tag{4.10}$$

for all $k \in \mathbb{N}$ with $k \geqslant 2$, $t \in \mathbb{R}_+$, $\boldsymbol{w} \in \mathbb{R}^d$ and $K \in (1, \infty)$, where

$$\begin{aligned}
I_{K,\boldsymbol{w},k}(t) &:= k \sum_{i=1}^d \int_0^t (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \boldsymbol{Y}_{K,s})^{k-1} (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \boldsymbol{e}_i) \sqrt{2c_i Y_{K,s,i}} \, \mathrm{d}W_{s,i}, \\
J_{K,\boldsymbol{w},k,i}(t) &:= \int_0^t \int_{V_i} \Big[\Big(\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \big(\boldsymbol{Y}_{K,s-} + h(\boldsymbol{Y}_{K,s-}, \boldsymbol{r}) \big) \Big)^k - (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \boldsymbol{Y}_{K,s-})^k \Big] \widetilde{N}_K(\mathrm{d}s, \mathrm{d}\boldsymbol{r}) \\
&= \sum_{\ell=0}^{k-1} \binom{k}{\ell} \int_0^t \int_{V_i} (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \boldsymbol{Y}_{K,s-})^\ell \big(\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} h(\boldsymbol{Y}_{K,s-}, \boldsymbol{r}) \big)^{k-\ell} \, \widetilde{N}_K(\mathrm{d}s, \mathrm{d}\boldsymbol{r})
\end{aligned}$$

for $i \in \{0, 1\}$. For each $n \in \mathbb{N}$, consider the stopping time $\tau_{K,n} := \inf\{t \in \mathbb{R}_+ : \|\boldsymbol{Y}_{K,t}\| \geqslant n\}$. Clearly, $\tau_{K,n} \xrightarrow{\text{a.s.}} \infty$ as $n \to \infty$, since $(\boldsymbol{Y}_{K,t})_{t \in \mathbb{R}_+}$ has càdlàg trajectories. The stopped processes

$$(I_{K,\boldsymbol{w},k}(t \wedge \tau_{K,n}))_{t \in \mathbb{R}_+} \qquad \text{and} \qquad (J_{K,\boldsymbol{w},k,i}(t \wedge \tau_{K,n}))_{t \in \mathbb{R}_+}, \quad i \in \{0, 1\}, \tag{4.11}$$

are martingales for all $k, n \in \mathbb{N}$, $K \in (1, \infty)$ and $\boldsymbol{w} \in \mathbb{R}^d$. Indeed,

$$\mathbb{E}\left(\int_0^{t \wedge \tau_{K,n}} (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \boldsymbol{Y}_{K,s})^{2k-2} (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \boldsymbol{e}_i)^2 Y_{K,s,i} \, \mathrm{d}s \right) \leqslant n^{2k-1} \|\boldsymbol{w}\|^{2k} t c(t)^{2k} < \infty,$$

since for all $t \in \mathbb{R}_+$ and $s \in [0, t]$, we have

$$\|\mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K}\| \leqslant \mathrm{e}^{s\|\widetilde{\boldsymbol{B}}_K\|} \leqslant \exp\left\{ t \sup_{K \in (1,\infty)} \|\widetilde{\boldsymbol{B}}_K\| \right\} =: c(t) < \infty,$$

because, for all $i, j \in \{1, \ldots, d\}$, by monotone convergence theorem,

$$\widetilde{b}_{K,i,j} = \widetilde{b}_{i,j} - \int_{U_d} z_i \mathbb{1}_{\{\|\boldsymbol{z}\| \geqslant K\}} \, \mu_j(\mathrm{d}\boldsymbol{z}) \uparrow \widetilde{b}_{i,j} \qquad \text{as } K \to \infty.$$

Moreover, for each $\ell \in \{0, 1, \ldots, k-1\}$,

$$\begin{aligned}
&\mathbb{E}\bigg(\int_0^{t \wedge \tau_{K,n}} \int_{V_0} \Big| (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \boldsymbol{Y}_{K,s-})^\ell \big(\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} h(\boldsymbol{Y}_{K,s-}, \boldsymbol{r}) \big)^{k-\ell} \Big|^2 \mathrm{d}s \, m_K(\mathrm{d}\boldsymbol{r}) \bigg) \\
&\leqslant \|\boldsymbol{w}\|^{2k} c(t)^{2k} \sum_{j=1}^d \mathbb{E}\bigg(\int_0^{t \wedge \tau_{K,n}} \int_{U_d} \int_{U_1} \|\boldsymbol{Y}_{K,s-}\|^{2\ell} \|\boldsymbol{z}\|^{2(k-\ell)} \mathbb{1}_{\{\|\boldsymbol{z}\| < 1\}} \mathbb{1}_{\{u \leqslant Y_{K,s-,j}\}} \, \mathrm{d}s \, \mu_{K,j}(\mathrm{d}\boldsymbol{z}) \, \mathrm{d}u \bigg) \\
&\leqslant \|\boldsymbol{w}\|^{2k} t c(t)^{2k} n^{2\ell+1} \sum_{j=1}^d \int_{U_d} \|\boldsymbol{z}\|^{2(k-\ell)} \mathbb{1}_{\{\|\boldsymbol{z}\| < 1\}} \, \mu_j(\mathrm{d}\boldsymbol{z}) < \infty
\end{aligned}$$

and

$$
\begin{aligned}
&\mathbb{E}\bigg(\int_0^{t\wedge\tau_{K,n}}\int_{V_1}\Big|(\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K}\boldsymbol{Y}_{K,s-})^\ell\big(\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K}h(\boldsymbol{Y}_{K,s-},\boldsymbol{r})\big)^{k-\ell}\Big|\,\mathrm{d}s\,m_K(\mathrm{d}\boldsymbol{r})\bigg)\\
&\leqslant \|\boldsymbol{w}\|^k c(t)^k\,\mathbb{E}\bigg(\int_0^{t\wedge\tau_{K,n}}\int_{U_d}\|\boldsymbol{Y}_{K,s-}\|^\ell\|\boldsymbol{r}\|^{k-\ell}\,\mathrm{d}s\,\nu_K(\mathrm{d}\boldsymbol{r})\bigg)\\
&\quad+\|\boldsymbol{w}\|^k c(t)^k\sum_{j=1}^d\mathbb{E}\bigg(\int_0^{t\wedge\tau_{K,n}}\int_{U_d}\int_{U_1}\|\boldsymbol{Y}_{K,s-}\|^\ell\|\boldsymbol{z}\|^{k-\ell}\mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant1\}}\mathbb{1}_{\{u\leqslant Y_{K,s-,j}\}}\,\mathrm{d}s\,\mu_{K,j}(\mathrm{d}\boldsymbol{z})\,\mathrm{d}u\bigg)\\
&\leqslant \|\boldsymbol{w}\|^k t c(t)^k n^\ell\bigg(\int_{U_d}\|\boldsymbol{r}\|^{k-\ell}\mathbb{1}_{\{\|\boldsymbol{r}\|<K\}}\,\nu(\mathrm{d}\boldsymbol{r})+n\sum_{j=1}^d\int_{U_d}\|\boldsymbol{z}\|^{k-\ell}\mathbb{1}_{\{1\leqslant\|\boldsymbol{z}\|<K\}}\,\mu_j(\mathrm{d}\boldsymbol{z})\bigg)<\infty,
\end{aligned}
$$

hence, by Ikeda and Watanabe [18, Chapter II, Proposition 2.2 and page 62], the processes in (4.11) are martingales for all $k,n\in\mathbb{N}$, $K\in(1,\infty)$ and $\boldsymbol{w}\in\mathbb{R}^d$. Here we used that for all $k\in\mathbb{N}$ and $\ell\in\{0,1,\dots,k-1\}$,

$$
\int_{U_d}\|\boldsymbol{r}\|^{k-\ell}\mathbb{1}_{\{\|\boldsymbol{r}\|<K\}}\,\nu(\mathrm{d}\boldsymbol{r})\leqslant\int_{U_d}\|\boldsymbol{r}\|\mathbb{1}_{\{\|\boldsymbol{r}\|<1\}}\,\nu(\mathrm{d}\boldsymbol{r})+K^{k-\ell}\int_{U_d}\mathbb{1}_{\{1\leqslant\|\boldsymbol{r}\|<K\}}\,\nu(\mathrm{d}\boldsymbol{r})<\infty \tag{4.12}
$$

due to part (v) of Definition 2.2,

$$
\begin{aligned}
\int_{U_d}\|\boldsymbol{z}\|^{2(k-\ell)}\mathbb{1}_{\{\|\boldsymbol{z}\|<1\}}\,\mu_i(\mathrm{d}\boldsymbol{z})&\leqslant\int_{U_d}\|\boldsymbol{z}\|^2\mathbb{1}_{\{\|\boldsymbol{z}\|<1\}}\,\mu_i(\mathrm{d}\boldsymbol{z})\\
&\leqslant\int_{U_d}\bigg(z_i^2+\sum_{j\in\{1,\dots,d\}\setminus\{i\}}z_j\bigg)\mathbb{1}_{\{\|\boldsymbol{z}\|<1\}}\,\mu_i(\mathrm{d}\boldsymbol{z})<\infty
\end{aligned}\tag{4.13}
$$

due to part (vi) of Definition 2.2, and

$$
\begin{aligned}
\int_{U_d}\|\boldsymbol{z}\|^{k-\ell}\mathbb{1}_{\{1\leqslant\|\boldsymbol{z}\|<K\}}\,\mu_i(\mathrm{d}\boldsymbol{z})&\leqslant K^{k-\ell}\int_{U_d}\mathbb{1}_{\{1\leqslant\|\boldsymbol{z}\|<K\}}\,\mu_i(\mathrm{d}\boldsymbol{z})\\
&\leqslant K^{k-\ell}\int_{U_d}\|\boldsymbol{z}\|^q\mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant1\}}\,\mu_i(\mathrm{d}\boldsymbol{z})<\infty
\end{aligned}\tag{4.14}
$$

due to assumption (4.4).

By replacing t by $t\wedge\tau_{K,n}$ in (4.9) and (4.10), and then taking expectations on both sides of these equations, we conclude

$$
\mathbb{E}\Big[\boldsymbol{w}^\top\mathrm{e}^{-(t\wedge\tau_{K,n})\widetilde{\boldsymbol{B}}_K}\boldsymbol{Y}_{K,t\wedge\tau_{K,n}}\Big]=\boldsymbol{w}^\top\boldsymbol{y}+\mathbb{E}\bigg(\int_0^{t\wedge\tau_{K,n}}\boldsymbol{w}^\top\mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K}\widetilde{\boldsymbol{\beta}}_K\,\mathrm{d}s\bigg)
$$

and

$$\mathbb{E}\left[(\boldsymbol{w}^\top \mathrm{e}^{-(t\wedge\tau_{K,n})\widetilde{\boldsymbol{B}}_K}\boldsymbol{Y}_{K,t\wedge\tau_{K,n}})^k\right]$$

$$= (\boldsymbol{w}^\top\boldsymbol{y})^k + k\,\mathbb{E}\left(\int_0^{t\wedge\tau_{K,n}} (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K}\widetilde{\boldsymbol{\beta}}_K)(\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K}\boldsymbol{Y}_{K,s})^{k-1}\,\mathrm{d}s\right)$$

$$+ k(k-1)\sum_{i=1}^d c_i\,\mathbb{E}\left(\int_0^{t\wedge\tau_{K,n}} (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K}\boldsymbol{e}_i)^2(\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K}\boldsymbol{Y}_{K,s})^{k-2}\,Y_{K,s,i}\,\mathrm{d}s\right)$$

$$+ \sum_{\ell=0}^{k-2}\binom{k}{\ell}\mathbb{E}\left(\int_0^{t\wedge\tau_{K,n}}\int_V (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K}\boldsymbol{Y}_{K,s})^\ell\big(\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K}h(\boldsymbol{Y}_{K,s},\boldsymbol{r})\big)^{k-\ell}\,\mathrm{d}s\,m_K(\mathrm{d}\boldsymbol{r})\right)$$

for all $k,n\in\mathbb{N}$ with $k\geqslant 2$, $K\in(1,\infty)$ and $\boldsymbol{w}\in\mathbb{R}^d$. By Fatou's lemma,

$$\begin{aligned}\mathbb{E}\left[(\boldsymbol{w}^\top \mathrm{e}^{-t\widetilde{\boldsymbol{B}}_K}\boldsymbol{Y}_{K,t})^k\right] &= \mathbb{E}\left[\lim_{n\to\infty}(\boldsymbol{w}^\top \mathrm{e}^{-(t\wedge\tau_{K,n})\widetilde{\boldsymbol{B}}_K}\boldsymbol{Y}_{K,t\wedge\tau_{K,n}})^k\right]\\ &\leqslant \liminf_{n\to\infty}\mathbb{E}\left[(\boldsymbol{w}^\top \mathrm{e}^{-(t\wedge\tau_{K,n})\widetilde{\boldsymbol{B}}_K}\boldsymbol{Y}_{K,t\wedge\tau_{K,n}})^k\right] \leqslant \|\boldsymbol{w}\|^k\big(\|\boldsymbol{y}\|^k + g_{K,k,\boldsymbol{y}}(t)\big)\end{aligned}\tag{4.15}$$

with

$$g_{K,k,\boldsymbol{y}}(t) := k\|\widetilde{\boldsymbol{\beta}}\|c(t)^k\int_0^t \mathbb{E}(\|\boldsymbol{Y}_{K,s}\|^{k-1})\,\mathrm{d}s + k(k-1)c(t)^k\sum_{i=1}^d c_i\int_0^t \mathbb{E}(\|\boldsymbol{Y}_{K,s}\|^{k-1})\,\mathrm{d}s$$

$$+ c(t)^k\sum_{\ell=0}^{k-2}\binom{k}{\ell}\left[\int_0^t \mathbb{E}(\|\boldsymbol{Y}_{K,s}\|^\ell)\,\mathrm{d}s\int_{U_d}\|\boldsymbol{z}\|^{k-\ell}\,\nu_K(\mathrm{d}\boldsymbol{z})\right.$$

$$\left.+\sum_{j=1}^d\int_0^t \mathbb{E}(\|\boldsymbol{Y}_{K,s}\|^{\ell+1})\,\mathrm{d}s\int_{U_d}\|\boldsymbol{z}\|^{k-\ell}\,\mu_{K,j}(\mathrm{d}\boldsymbol{z})\right].$$

Here we used that $\boldsymbol{0}\leqslant\widetilde{\boldsymbol{\beta}}_K\leqslant\widetilde{\boldsymbol{\beta}}$ for all $K\in(1,\infty)$,

$$h(\boldsymbol{x},\boldsymbol{r}) := \begin{cases}\boldsymbol{r}, & \text{if } \boldsymbol{x}\in\mathbb{R}_+^d,\ \boldsymbol{r}\in\mathcal{R}_0,\\ \boldsymbol{z}\mathbb{1}_{\{u\leqslant x_j\}}, & \text{if } \boldsymbol{x}=(x_1,\ldots,x_d)^\top\in\mathbb{R}_+^d,\ \boldsymbol{r}=(\boldsymbol{z},u)\in\mathcal{R}_j,\ j\in\{1,\ldots,d\},\\ \boldsymbol{0}, & \text{otherwise,}\end{cases}$$

and hence

$$\mathbb{E}\left(\int_0^{t\wedge\tau_{K,n}}\int_V \|\boldsymbol{Y}_{K,s}\|^\ell\|h(\boldsymbol{Y}_{K,s},\boldsymbol{r})\|^{k-\ell}\,\mathrm{d}s\,m_K(\mathrm{d}\boldsymbol{r})\right)$$

$$= \mathbb{E}\left(\int_0^{t\wedge\tau_{K,n}}\int_{U_d} \|\boldsymbol{Y}_{K,s}\|^\ell\|\boldsymbol{r}\|^{k-\ell}\,\mathrm{d}s\,\nu_K(\mathrm{d}\boldsymbol{r})\right)$$

$$+ \sum_{j=1}^d\mathbb{E}\left(\int_0^{t\wedge\tau_{K,n}}\int_{U_d}\int_{U_1} \|\boldsymbol{Y}_{K,s}\|^\ell\|\boldsymbol{z}\|^{k-\ell}\mathbb{1}_{\{u\leqslant Y_{K,s,j}\}}\,\mathrm{d}s\,\mu_{K,j}(\mathrm{d}\boldsymbol{z})\,\mathrm{d}u\right)$$

$$\leqslant \int_0^t \mathbb{E}(\|\boldsymbol{Y}_{K,s}\|^\ell)\,\mathrm{d}s \int_{U_d} \|\boldsymbol{r}\|^{k-\ell}\,\nu_K(\mathrm{d}\boldsymbol{r}) + \sum_{j=1}^d \int_0^t \mathbb{E}(\|\boldsymbol{Y}_{K,s}\|^{\ell+1})\,\mathrm{d}s \int_{U_d} \|\boldsymbol{z}\|^{k-\ell}\,\mu_{K,j}(\mathrm{d}\boldsymbol{z}).$$

If we suppose that (4.7) holds for $0, 1, \ldots, k-1$ with $k \in \mathbb{N}$ and for some $K \in (1, \infty)$, then $g_{K,k,\boldsymbol{y}}$ is a continuous function on $\mathbb{R}_+$. Note that, for each $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}_+^d$ and $k \in \mathbb{N}$, we have

$$\|\boldsymbol{x}\|^k \leqslant d^{k/2} \max_{i\in\{1,\ldots,d\}} x_i^k. \tag{4.16}$$

For $k \geqslant 2$, this is a consequence of the power mean inequality, for $k = 1$, this is trivial. Choosing $\boldsymbol{w} := \boldsymbol{e}_i$, $i \in \{1, \ldots, d\}$, by (4.16) and (4.15), we have

$$\mathbb{E}(\|\mathrm{e}^{-t\widetilde{\boldsymbol{B}}_K}\boldsymbol{Y}_{K,t}\|^k) \leqslant d^{k/2}(\|\boldsymbol{y}\|^k + g_{K,k,\boldsymbol{y}}(t)), \qquad t \in \mathbb{R}_+, \quad k \in \mathbb{N}, \quad K \in (1, \infty).$$

Consequently,

$$\begin{aligned}\mathbb{E}\left[(Y_{K,t,i})^k\right] &= \mathbb{E}\left[(\boldsymbol{e}_i^\top \boldsymbol{Y}_{K,t})^k\right] = \mathbb{E}\left[(\boldsymbol{e}_i^\top \mathrm{e}^{t\widetilde{\boldsymbol{B}}_K}\mathrm{e}^{-t\widetilde{\boldsymbol{B}}_K}\boldsymbol{Y}_{K,t})^k\right]\\ &\leqslant d^{k/2}\|\boldsymbol{e}_i^\top \mathrm{e}^{t\widetilde{\boldsymbol{B}}_K}\|^k(\|\boldsymbol{y}\|^k + g_{K,k,\boldsymbol{y}}(t)) \leqslant d^{k/2}c(t)^k(\|\boldsymbol{y}\|^k + g_{K,k,\boldsymbol{y}}(t))\end{aligned}$$

for each $i \in \{1, \ldots, d\}$, and whence, again by (4.16),

$$\mathbb{E}(\|\boldsymbol{Y}_{K,t}\|^k) \leqslant d^{k/2} \max_{i\in\{1,\ldots,d\}} \mathbb{E}\left[(Y_{K,t,i})^k\right] \leqslant d^k c(t)^k(\|\boldsymbol{y}\|^k + g_{K,k,\boldsymbol{y}}(t)) =: f_{K,k,\boldsymbol{y}}(t),$$

where $f_{K,k,\boldsymbol{y}} : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, hence we obtain (4.7) for k and K.

If we suppose that (4.8) holds for $0, 1, \ldots, k-1$ with $k \in \{1, \ldots, q\}$, then the continuity of the function c and condition (4.4) imply the existence of a continuous function $g_{k,\boldsymbol{y}} : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sup_{K\in(1,\infty)} g_{K,k,\boldsymbol{y}}(t) \leqslant g_{k,\boldsymbol{y}}(t), \qquad t \in \mathbb{R}_+. \tag{4.17}$$

Namely, one can choose

$$\begin{aligned}g_{k,\boldsymbol{y}}(t) &:= k\|\widetilde{\boldsymbol{\beta}}\|c(t)^k \int_0^t f_{k-1,\boldsymbol{y}}(s)\,\mathrm{d}s + k(k-1)c(t)^k \sum_{i=1}^d c_i \int_0^t f_{k-1,\boldsymbol{y}}(s)\,\mathrm{d}s\\ &\quad + c(t)^k \sum_{\ell=0}^{k-2}\binom{k}{\ell}\left[\sum_{j=1}^d \int_0^t f_{\ell+1,\boldsymbol{y}}(s)\,\mathrm{d}s \int_{U_d}\|\boldsymbol{z}\|^{k-\ell}\,\mu_j(\mathrm{d}\boldsymbol{z}) + \int_0^t f_{\ell,\boldsymbol{y}}(s)\,\mathrm{d}s \int_{U_d}\|\boldsymbol{r}\|^{k-\ell}\,\nu(\mathrm{d}\boldsymbol{r})\right],\end{aligned}$$

for $t \in \mathbb{R}_+$, and the continuity of $g_{k,\boldsymbol{y}}$ is obvious, since

$$\begin{aligned}&\sup_{K\in(1,\infty)} \int_{U_d}\|\boldsymbol{z}\|^{k-\ell}\,\mu_{K,j}(\mathrm{d}\boldsymbol{z}) = \int_{U_d}\|\boldsymbol{z}\|^{k-\ell}\,\mu_j(\mathrm{d}\boldsymbol{z}), \qquad j \in \{1, \ldots, d\},\\ &\sup_{K\in(1,\infty)} \int_{U_d}\|\boldsymbol{r}\|^{k-\ell}\,\nu_K(\mathrm{d}\boldsymbol{r}) = \int_{U_d}\|\boldsymbol{r}\|^{k-\ell}\,\nu(\mathrm{d}\boldsymbol{r}).\end{aligned} \tag{4.18}$$

We have $g_{k,\boldsymbol{y}}(t) < \infty$, since for all $k \in \{1, \ldots, q\}$ and $\ell \in \{0, 1, \ldots, k-2\}$,

$$\int_{U_d} \|\boldsymbol{r}\|^{k-\ell} \, \nu(\mathrm{d}\boldsymbol{r}) \leqslant \int_{U_d} \|\boldsymbol{r}\| \mathbb{1}_{\{\|\boldsymbol{r}\|<1\}} \, \nu(\mathrm{d}\boldsymbol{r}) + \int_{U_d} \|\boldsymbol{r}\|^q \mathbb{1}_{\{\|\boldsymbol{r}\|\geqslant 1\}} \, \nu(\mathrm{d}\boldsymbol{r}) < \infty \tag{4.19}$$

due to part (v) of Definition 2.2 and assumption (4.4), $\int_{U_d} \|\boldsymbol{z}\|^{k-\ell} \mathbb{1}_{\{\|\boldsymbol{z}\|<1\}} \, \mu_i(\mathrm{d}\boldsymbol{z}) < \infty$ can be derived as in (4.13), and

$$\int_{U_d} \|\boldsymbol{z}\|^{k-\ell} \mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}} \, \mu_i(\mathrm{d}\boldsymbol{z}) \leqslant \int_{U_d} \|\boldsymbol{z}\|^q \mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}} \, \mu_i(\mathrm{d}\boldsymbol{z}) < \infty \tag{4.20}$$

due to assumption (4.4). Thus (4.8) holds for k with the continuous function $f_{k,\boldsymbol{y}}(t) := d^k c(t)^k \big(\|\boldsymbol{y}\|^k + g_{k,\boldsymbol{y}}(t)\big)$, $t \in \mathbb{R}_+$. Note that $f_{k,\boldsymbol{y}}(t)$ and $g_{k,\boldsymbol{y}}(t)$ are polynomials of $\|\boldsymbol{y}\|$ having degree k and $k-1$, respectively.

Here we point out that (4.8) may not hold for any $k \in \mathbb{N}$, but only for $k \in \{0, 1, \ldots, q\}$. Indeed, the integrals in (4.18) are not necessarily finite, thus our constructions for $f_{k,\boldsymbol{y}}$ and $g_{k,\boldsymbol{y}}$ do not necessarily work.

By Theorem 3.3, $\boldsymbol{Y}_{K,t} \uparrow \boldsymbol{Y}_t$ a.s. as $K \to \infty$. Hence $Y_{K,t,j}^k \uparrow Y_{t,j}^k$ a.s. as $K \to \infty$ for all $j \in \{1, \ldots, d\}$, $k \in \mathbb{N}$ and $t \in \mathbb{R}_+$, which yields $\lim_{K\to\infty} \mathbb{E}(Y_{K,t,j}^k) = \mathbb{E}(Y_{t,j}^k) \in [0, \infty]$ by monotone convergence theorem. Using (4.8) with $k = q$, we obtain $\mathbb{E}(Y_{t,j}^q) \in [0, \infty)$, $t \in \mathbb{R}_+$, $j \in \{1, \ldots, d\}$, implying $\mathbb{E}(\|\boldsymbol{Y}_t\|^q) \leqslant f_{q,\boldsymbol{y}}(t) < \infty$ for all $t \in \mathbb{R}_+$. By the tower rule for conditional expectations (i.e., the law of iterated expectations), it suffices to show

$$\mathbb{E}(\|\boldsymbol{X}_t\|^q \,|\, \boldsymbol{X}_0) \leqslant f_{q,\boldsymbol{X}_0}(t) \quad \mathbb{P}\text{-a.s.}, \qquad t \in \mathbb{R}_+, \tag{4.21}$$

since $f_{q,\boldsymbol{X}_0}(t)$ is a polynomial of $\|\boldsymbol{X}_0\|$ having degree q, where the conditional expectation $\mathbb{E}(\|\boldsymbol{X}_t\|^q \,|\, \boldsymbol{X}_0) \in [0, \infty]$ is meant in the generalized sense, see, e.g., Stroock [24, Theorem 5.1.6]. In order to show (4.21), let $\phi_n : \mathbb{R}_+ \to \mathbb{R}_+$, $n \in \mathbb{N}$, be simple functions such that $\phi_n(y) \uparrow y$ as $n \to \infty$ for all $y \in \mathbb{R}_+$. Then, by the monotone convergence theorem for (generalized) conditional expectations, see, e.g., Stroock [24, Theorem 5.1.6], we obtain $\mathbb{E}(\phi_n(\|\boldsymbol{X}_t\|^q) \,|\, \boldsymbol{X}_0) \uparrow \mathbb{E}(\|\boldsymbol{X}_t\|^q \,|\, \boldsymbol{X}_0)$ as $n \to \infty$ $\mathbb{P}$-almost surely. For each $B \in \mathcal{B}(\mathbb{R}^d)$, we have

$$\mathbb{E}(\mathbb{1}_B(\boldsymbol{X}_t) \,|\, \boldsymbol{X}_0) = \mathbb{P}(\boldsymbol{X}_t \in B \,|\, \boldsymbol{X}_0) = \int_{\mathbb{R}_+^d} \mathbb{1}_B(\boldsymbol{y}) \, P_t(\boldsymbol{X}_0, \mathrm{d}\boldsymbol{y}) \quad \mathbb{P}\text{-a.s.},$$

hence $\mathbb{E}(\phi_n(\|\boldsymbol{X}_t\|^q) \,|\, \boldsymbol{X}_0) = \int_{\mathbb{R}_+^d} \phi_n(\|\boldsymbol{y}\|^q) \, P_t(\boldsymbol{X}_0, \mathrm{d}\boldsymbol{y})$ $\mathbb{P}$-almost surely. By the monotone convergence theorem, $\int_{\mathbb{R}_+^d} \phi_n(\|\boldsymbol{y}\|^q) \, P_t(\boldsymbol{X}_0, \mathrm{d}\boldsymbol{y}) \uparrow \int_{\mathbb{R}_+^d} \|\boldsymbol{y}\|^q \, P_t(\boldsymbol{X}_0, \mathrm{d}\boldsymbol{y})$ as $n \to \infty$. By $\mathbb{E}(\|\boldsymbol{Y}_t\|^q) \leqslant f_{q,\boldsymbol{y}}(t) < \infty$, we get

$$\mathbb{E}(\|\boldsymbol{X}_t\|^q \,|\, \boldsymbol{X}_0) = \int_{\mathbb{R}_+^d} \|\boldsymbol{y}\|^q \, P_t(\boldsymbol{X}_0, \mathrm{d}\boldsymbol{y}) \leqslant f_{q,\boldsymbol{X}_0}(t) \quad \mathbb{P}\text{-a.s.},$$

hence we conclude (4.21).

The aim of the following discussion is to show that the processes

$$(I_{K,\boldsymbol{w},k}(t))_{t\in\mathbb{R}_+} \quad \text{and} \quad (J_{K,\boldsymbol{w},k,i}(t))_{t\in\mathbb{R}_+}, \quad i \in \{0, 1\},$$

are martingales for all $K \in (1, \infty)$, $\boldsymbol{w} \in \mathbb{R}^d$ and $k \in \mathbb{N}$. These follow similarly to the earlier discussion, since the estimates (4.7) yield

$$\mathbb{E}\left(\int_0^t (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \boldsymbol{Y}_{K,s})^{2k-2} (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \boldsymbol{e}_i)^2 Y_{K,s,i} \, \mathrm{d}s\right) \leqslant \|\boldsymbol{w}\|^{2k} c(t)^{2k} \int_0^t f_{K,2k-1,\boldsymbol{y}}(s) \, \mathrm{d}s < \infty,$$

$$\begin{aligned}
&\mathbb{E}\left(\int_0^t \int_{U_d} \left|(\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \boldsymbol{Y}_{K,s-})^\ell (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \boldsymbol{r})^{k-\ell}\right| \mathrm{d}s \, \nu_K(\mathrm{d}\boldsymbol{r})\right) \\
&\qquad\qquad \leqslant \|\boldsymbol{w}\|^k c(t)^k \int_0^t f_{K,\ell,\boldsymbol{y}}(s) \, \mathrm{d}s \int_{U_d} \|\boldsymbol{r}\|^{k-\ell} \mathbb{1}_{\{\|\boldsymbol{r}\| < K\}} \, \nu(\mathrm{d}\boldsymbol{r}) < \infty,
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\left(\int_0^t \int_{U_d} \int_{U_1} \left|(\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \boldsymbol{Y}_{K,s-})^\ell (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \boldsymbol{z} \mathbb{1}_{\{u \leqslant Y_{K,s-,j}\}})^{k-\ell}\right|^2 \mathbb{1}_{\{\|\boldsymbol{z}\| < 1\}} \, \mathrm{d}s \, \mu_{K,j}(\mathrm{d}\boldsymbol{z}) \, \mathrm{d}u\right) \\
&\qquad\qquad \leqslant \|\boldsymbol{w}\|^{2k} c(t)^{2k} \int_0^t f_{K,2\ell+1,\boldsymbol{y}}(s) \, \mathrm{d}s \int_{U_d} \|\boldsymbol{z}\|^{2(k-\ell)} \mathbb{1}_{\{\|\boldsymbol{z}\| < 1\}} \, \mu_j(\mathrm{d}\boldsymbol{z}) < \infty,
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\left(\int_0^t \int_{U_d} \int_{U_1} \left|(\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \boldsymbol{Y}_{K,s-})^\ell (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \boldsymbol{z} \mathbb{1}_{\{u \leqslant Y_{K,s-,j}\}})^{k-\ell}\right| \mathbb{1}_{\{\|\boldsymbol{z}\| \geqslant 1\}} \, \mathrm{d}s \, \mu_{K,j}(\mathrm{d}\boldsymbol{z}) \, \mathrm{d}u\right) \\
&\qquad\qquad \leqslant \|\boldsymbol{w}\|^k c(t)^k \int_0^t f_{K,\ell+1,\boldsymbol{y}}(s) \, \mathrm{d}s \int_{U_d} \|\boldsymbol{z}\|^{k-\ell} \mathbb{1}_{\{1 \leqslant \|\boldsymbol{z}\| < K\}} \, \mu_j(\mathrm{d}\boldsymbol{z}) < \infty
\end{aligned}$$

for all $\ell \in \{0, 1, \ldots, k-1\}$, where we used (4.12), (4.13) and (4.14). Thus, taking again expectations of (4.10) and putting $\boldsymbol{w} = \mathrm{e}^{t\widetilde{\boldsymbol{B}}_K^\top} \boldsymbol{e}_j$, $j \in \{1, \ldots, d\}$, we conclude

$$\begin{aligned}
\mathbb{E}\left(Y_{K,t,j}^k\right) &= (\boldsymbol{e}_j^\top \mathrm{e}^{t\widetilde{\boldsymbol{B}}_K} \boldsymbol{y})^k + k \int_0^t (\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}_K} \widetilde{\boldsymbol{\beta}}_K) \, \mathbb{E}\big[(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}_K} \boldsymbol{Y}_{K,s})^{k-1}\big] \, \mathrm{d}s \\
&\quad + k(k-1) \sum_{i=1}^d c_i \int_0^t (\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}_K} \boldsymbol{e}_i)^2 \, \mathbb{E}\big[(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}_K} \boldsymbol{Y}_{K,s})^{k-2} Y_{K,s,i}\big] \, \mathrm{d}s \\
&\quad + \sum_{\ell=0}^{k-2} \binom{k}{\ell} \sum_{i=1}^d \int_0^t \int_{U_d} (\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}_K} \boldsymbol{z})^{k-\ell} \, \mathbb{E}\big[(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}_K} \boldsymbol{Y}_{K,s})^\ell Y_{K,s,i}\big] \, \mathrm{d}s \, \mu_{K,i}(\mathrm{d}\boldsymbol{z}) \\
&\quad + \sum_{\ell=0}^{k-2} \binom{k}{\ell} \int_0^t \int_{U_d} (\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}_K} \boldsymbol{z})^{k-\ell} \, \mathbb{E}\big[(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}_K} \boldsymbol{Y}_{K,s})^\ell\big] \, \mathrm{d}s \, \nu_K(\mathrm{d}\boldsymbol{z})
\end{aligned} \tag{4.22}$$

for all $j \in \{1, \ldots, d\}$, $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$ with $k \geqslant 2$.

Next we show (4.5) with $\boldsymbol{X}_0 = \boldsymbol{y}$ for all $k \in \{1, \ldots, q\}$, $j \in \{1, \ldots, d\}$ and $t \in \mathbb{R}_+$. By monotone convergence theorem, $\widetilde{\boldsymbol{\beta}}_K \to \widetilde{\boldsymbol{\beta}}$ and $\widetilde{\boldsymbol{B}}_K \to \widetilde{\boldsymbol{B}}$ as $K \to \infty$. We will show by the dominated convergence theorem that the integrals in (4.22) tends to those in (4.5) as $K \to \infty$. First, we check that the integrands converge pointwise. For all $t \in \mathbb{R}_+$, $s \in [0, t]$ and $j \in \{1, \ldots, d\}$, we have

$$\mathbb{E}\left[(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}_K} \boldsymbol{Y}_{K,s})^\ell\right] \to \mathbb{E}\left[(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}} \boldsymbol{Y}_s)^\ell\right] = \mathbb{E}\left[(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}} \boldsymbol{X}_s)^\ell\right]$$

as $K \to \infty$ for all $\ell \in \{1, \dots, k-1\}$, and

$$\mathbb{E}\left[(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}_K}\boldsymbol{Y}_{K,s})^\ell Y_{K,s,i}\right] \to \mathbb{E}\left[(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}}\boldsymbol{Y}_s)^\ell Y_{s,i}\right] = \mathbb{E}\left[(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}}\boldsymbol{X}_s)^\ell X_{s,i}\right]$$

as $K \to \infty$ for all $\ell \in \{1, \dots, k-2\}$. Indeed, $\mathbb{E}\big[(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}_K}\boldsymbol{Y}_{K,s})^\ell\big]$ is a linear combination of $\mathbb{E}(Y_{K,s,i_1} \cdots Y_{K,s,i_\ell})$, $i_1, \dots, i_\ell \in \{1, \dots, d\}$. By Theorem 3.3, $\boldsymbol{Y}_{K,t} \uparrow \boldsymbol{Y}_t$ a.s. as $K \to \infty$, hence $Y_{K,s,i_1} \cdots Y_{K,s,i_\ell} \uparrow Y_{s,i_1} \cdots Y_{s,i_\ell}$ a.s. as $K \to \infty$, which yields $\lim_{K\to\infty} \mathbb{E}(Y_{K,s,i_1} \cdots Y_{K,s,i_\ell}) = \mathbb{E}(Y_{s,i_1} \cdots Y_{s,i_\ell}) \in [0, \infty]$ by monotone convergence theorem. Using $\mathbb{E}(\|\boldsymbol{Y}_s\|^q) < \infty$, we have $\mathbb{E}(Y_{s,i_1} \cdots Y_{s,i_\ell}) < \infty$, and we can use again $\widetilde{\boldsymbol{B}}_K \to \widetilde{\boldsymbol{B}}$. The expectation $\mathbb{E}\big[(\boldsymbol{e}_j^\top \mathrm{e}^{(t-a)\widetilde{\boldsymbol{B}}_K}\boldsymbol{Y}_{K,s})^\ell Y_{K,s,i}\big]$ can be handled in the same way (we only note that $\mathbb{E}(Y_{s,i_1} \cdots Y_{s,i_\ell} Y_{s,i}) < \infty$). Next we check that the integrands can be bounded by integrable functions uniformly in $K \in (1, \infty)$. Applying (4.15) and (4.17) with $t = s$ and $\boldsymbol{w} = \mathrm{e}^{t\widetilde{\boldsymbol{B}}_K^\top}\boldsymbol{e}_j$, and using that $\boldsymbol{0} \leqslant \widetilde{\boldsymbol{\beta}}_K \leqslant \widetilde{\boldsymbol{\beta}}$, we obtain

$$\sup_{K\in(1,\infty)} \left|(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}_K}\widetilde{\boldsymbol{\beta}}_K)\, \mathbb{E}\left[(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}_K}\boldsymbol{Y}_{K,s})^{k-1}\right]\right| \leqslant \|\widetilde{\boldsymbol{\beta}}\| c(t)^k \big(\|\boldsymbol{y}\|^{k-1} + g_{k-1,\boldsymbol{y}}(s)\big)$$

for all $t \in \mathbb{R}_+$, $s \in [0, t]$ and $j \in \{1, \dots, d\}$. The integrals in the first sum can be handled in a similar way. Further,

$$\sup_{K\in(1,\infty)} \left|(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}_K}\boldsymbol{z})^{k-\ell}\, \mathbb{E}\left[(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}_K}\boldsymbol{Y}_{K,s})^\ell Y_{K,s,i}\right]\right| \leqslant \|\boldsymbol{z}\|^{k-\ell} c(t)^k \big(\|\boldsymbol{y}\|^{\ell+1} + g_{\ell,\boldsymbol{y}}(s)\big)$$

for all $t \in \mathbb{R}_+$, $s \in [0, t]$, $j \in \{1, \dots, d\}$, $\boldsymbol{z} \in \mathbb{R}_+^d$, $\ell \in \{0, 1, \dots, k-2\}$ and $k \in \{1, \dots, q\}$, where the function $\mathbb{R}_+^d \ni \boldsymbol{z} \mapsto \|\boldsymbol{z}\|^{k-\ell}$ is integrable with respect to the measures μ_i, $i \in \{1, \dots, d\}$, by (4.13) and (4.20). The integrals in the third sum can be handled in a similar way using (4.19). Hence we can apply dominated convergence theorem to obtain (4.5) with $\boldsymbol{X}_0 = \boldsymbol{y}$. By the law of total expectation we obtain (4.5) whenever $\mathbb{E}(\|\boldsymbol{X}_0\|^q) < \infty$.

Now we turn to prove (4.6). Again by the law of total probability, it is enough to prove (4.6) for $\boldsymbol{Y}$. Using the recursion (4.5) for $\boldsymbol{Y}$, we obtain the existence of suitable polynomials $Q_{t,k,j}$, $t \in \mathbb{R}_+$, $k \in \{1, \dots, q\}$, $j \in \{1, \dots, d\}$, by induction with respect to k. Indeed, for $k = 1$, we have $\mathbb{E}(Y_{t,j}) = \boldsymbol{e}_j^\top \mathrm{e}^{s\widetilde{\boldsymbol{B}}}\boldsymbol{y} + \int_0^s \boldsymbol{e}_j^\top \mathrm{e}^{v\widetilde{\boldsymbol{B}}}\widetilde{\boldsymbol{\beta}}\,\mathrm{d}v$, $j \in \{1, \dots, d\}$, $t \in \mathbb{R}_+$. Now, suppose that for some $k \in \mathbb{N}$ with $k + 1 \leqslant q$, suitable polynomials $Q_{t,1,j}, \dots, Q_{t,k,j}$ exist for all $t \in \mathbb{R}_+$ and $j \in \{1, \dots, d\}$. We apply the recursion (4.5) for $k + 1$. Then the function $\mathbb{R}_+^d \ni \boldsymbol{y} \mapsto (\boldsymbol{e}_j^\top \mathrm{e}^{t\widetilde{\boldsymbol{B}}}\boldsymbol{y})^{k+1}$ is a polynomial of degree at most $k + 1$. Moreover, for each $\ell \in \{0, 1, \dots, k\}$ and $s, t \in \mathbb{R}_+$ with $s \leqslant t$, the function

$$\mathbb{R}_+^d \ni \boldsymbol{y} \mapsto \mathbb{E}\big[(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}}\boldsymbol{Y}_s)^\ell\big]$$

is a polynomial of degree at most $\ell \leqslant k$. Further, for each $\ell \in \{0, 1, \dots, k-1\}$ and $s, t \in \mathbb{R}_+$ with $s \leqslant t$, the function

$$\mathbb{R}_+^d \ni \boldsymbol{y} \mapsto \mathbb{E}\big[(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}}\boldsymbol{Y}_s)^\ell Y_{s,i}\big]$$

is a polynomial of degree at most $\ell + 1 \leqslant k$. Consequently, by (4.5), $\mathbb{R}_+^d \ni \boldsymbol{y} \mapsto \mathbb{E}\big[(Y_{t,j})^{k+1}\big]$ is a polynomial of degree at most $k+1$, and we conclude the existence of suitable polynomials $Q_{t,k+1,j}$ for all $t \in \mathbb{R}_+$ and $j \in \{1, \dots, d\}$. □

For mixed moments, we have the following corollary.

4.4 Corollary. *Let* $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ *be a CBI process with parameters* $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ *such that* $\mathbb{E}(\|\boldsymbol{X}_0\|^q) < \infty$ *and the moment conditions (4.4) hold with some* $q \in \mathbb{N}$. *Then for all* $t \in \mathbb{R}_+$, $k \in \{1, \ldots, q\}$ *and* $i_1, \ldots, i_k \in \{1, \ldots, d\}$, *there exists a polynomial* $Q_{t,k,i_1,\ldots,i_k} : \mathbb{R}^d \to \mathbb{R}$ *having degree at most* k *such that*

$$\mathbb{E}(X_{t,i_1} \cdots X_{t,i_k}) = \mathbb{E}(Q_{t,k,i_1,\ldots,i_k}(\boldsymbol{X}_0)).$$

The coefficients of the polynomial $Q_{t,k,i_1,\ldots,i_k}$ *depend on* d, $\boldsymbol{c}$, $\boldsymbol{\beta}$, $\boldsymbol{B}$, ν, μ_1, $\ldots$, μ_d.

Proof. By the method of the proof of Theorem 4.3 (formally replacing $\boldsymbol{e}_j$ by $\boldsymbol{w} \in \mathbb{R}^d$ in (4.5)), one can derive

$$\begin{aligned}
\mathbb{E}\left[\langle \boldsymbol{w}, \boldsymbol{X}_t\rangle^k\right] &= \mathbb{E}\left[(\boldsymbol{w}^\top \mathrm{e}^{t\widetilde{\boldsymbol{B}}}\boldsymbol{X}_0)^k\right] + k\int_0^t (\boldsymbol{w}^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}}\widetilde{\boldsymbol{\beta}})\,\mathbb{E}[(\boldsymbol{w}^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}}\boldsymbol{X}_s)^{k-1}]\,\mathrm{d}s \\
&\quad + k(k-1)\sum_{i=1}^d c_i \int_0^t (\boldsymbol{w}^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}}\boldsymbol{e}_i)^2\,\mathbb{E}\big[(\boldsymbol{w}^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}}\boldsymbol{X}_s)^{k-2} X_{s,i}\big]\,\mathrm{d}s \\
&\quad + \sum_{\ell=0}^{k-2}\binom{k}{\ell}\int_0^t\int_{\mathcal{U}_d} (\boldsymbol{w}^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}}\boldsymbol{r})^{k-\ell}\,\mathbb{E}\Big[\big(\boldsymbol{w}^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}}\boldsymbol{X}_s\big)^\ell\Big]\,\mathrm{d}s\,\nu(\mathrm{d}\boldsymbol{r})
\end{aligned}$$

$$+ \sum_{\ell=0}^{k-2}\binom{k}{\ell}\sum_{i=1}^d\int_0^t\int_{\mathcal{U}_d} (\boldsymbol{w}^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}}\boldsymbol{z})^{k-\ell}\,\mathbb{E}\Big[\big(\boldsymbol{w}^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}}\boldsymbol{X}_s\big)^\ell X_{s,i}\Big]\,\mathrm{d}s\,\mu_i(\mathrm{d}\boldsymbol{z})$$

for all $t \in \mathbb{R}_+$, $k \in \{1, \ldots, q\}$ and $\boldsymbol{w} \in \mathbb{R}^d$. Hence, by the proof of Theorem 4.3, for each $t \in \mathbb{R}_+$, $k \in \{1, \ldots, q\}$ and $\boldsymbol{w} \in \mathbb{R}^d$, there exists a polynomial $Q_{t,k,\boldsymbol{w}} : \mathbb{R}^d \to \mathbb{R}$ having degree at most k such that

$$\mathbb{E}\left[\langle \boldsymbol{w}, \boldsymbol{X}_t\rangle^k\right] = \mathbb{E}\big[Q_{t,j,\boldsymbol{w}}(\boldsymbol{X}_0)\big],$$

where the coefficients of the polynomial $Q_{t,k,\boldsymbol{w}}$ depends on d, $\boldsymbol{c}$, $\boldsymbol{\beta}$, $\boldsymbol{B}$, ν, μ_1, $\ldots$, μ_d.

For all $a_1, \ldots, a_k \in \mathbb{R}$, we have

$$a_1 \cdots a_k = \frac{1}{k!2^k}\sum_{\ell_1=0}^1 \cdots \sum_{\ell_k=0}^1 (-1)^{\ell_1+\cdots+\ell_k}\left[(-1)^{\ell_1}a_1 + \cdots + (-1)^{\ell_k}a_k\right]^k.$$

Indeed, applying the multinomial theorem,

$$\sum_{\ell_1=0}^1 \cdots \sum_{\ell_k=0}^1 (-1)^{\ell_1+\cdots+\ell_k}\left[(-1)^{\ell_1}a_1 + \cdots + (-1)^{\ell_k}a_k\right]^k$$

$$= \sum_{\ell_1=0}^1 \cdots \sum_{\ell_k=0}^1 (-1)^{\ell_1+\cdots+\ell_k} \sum_{\substack{j_1+\cdots+j_k=k,\\ j_1,\ldots,j_k\in\mathbb{Z}_+}} \frac{k!}{j_1!\cdots j_k!}((-1)^{\ell_1}a_1)^{j_1}\cdots((-1)^{\ell_k}a_k)^{j_k} = S_1 + S_2,$$

where

$$S_1 := \sum_{\ell_1=0}^{1} \dots \sum_{\ell_k=0}^{1} (-1)^{\ell_1+\cdots+\ell_k} k! (-1)^{\ell_1} a_1 \cdots (-1)^{\ell_k} a_k,$$

$$S_2 := \sum_{\ell_1=0}^{1} \dots \sum_{\ell_k=0}^{1} (-1)^{\ell_1+\cdots+\ell_k} \sum_{\substack{j_1+\cdots+j_k=k,\, j_1\cdots j_k=0 \\ j_1,\dots,j_k\in\mathbb{Z}_+}} \frac{k!}{j_1!\cdots j_k!} ((-1)^{\ell_1} a_1)^{j_1} \cdots ((-1)^{\ell_k} a_k)^{j_k}.$$

Clearly $S_1 = 2^k k! a_1 \cdots a_k$, and $S_2 = 0$ because of cancellations. Hence

$$\begin{aligned}
&\mathbb{E}\left(X_{t,i_1} \cdots X_{t,i_k}\right) \\
&= \frac{1}{k!2^k} \sum_{\ell_1=0}^{1} \dots \sum_{\ell_k=0}^{1} (-1)^{\ell_1+\cdots+\ell_k} \, \mathbb{E}\left[\langle (-1)^{\ell_1} \boldsymbol{e}_{i_1} + \cdots + (-1)^{\ell_k} \boldsymbol{e}_{i_k}, \boldsymbol{X}_t \rangle^k\right] \\
&= \frac{1}{k!2^k} \sum_{\ell_1=0}^{1} \dots \sum_{\ell_k=0}^{1} (-1)^{\ell_1+\cdots+\ell_k} \, \mathbb{E}\big[Q_{t,k,(-1)^{\ell_1}\boldsymbol{e}_{i_1}+\cdots+(-1)^{\ell_k}\boldsymbol{e}_{i_k}}(\boldsymbol{X}_0)\big] =: \mathbb{E}[Q_{t,k,i_1,\dots,i_k}(\boldsymbol{X}_0)],
\end{aligned}$$

which implies the statement. □

For central moments, we have the following recursion.

4.5 Theorem. *Let* $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ *be a CBI process with parameters* $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ *such that* $\mathbb{E}(\|\boldsymbol{X}_0\|^q) < \infty$ *and the moment conditions* (4.4) *hold with some* $q \in \mathbb{N}$. *Then*

$$\begin{aligned}
&\mathbb{E}\left[(X_{t,j} - \mathbb{E}(X_{t,j}))^k\right] \\
&= k(k-1) \sum_{i=1}^{d} c_i \int_0^t (\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}} \boldsymbol{e}_i)^2 \, \mathbb{E}\big[(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}} (\boldsymbol{X}_s - \mathbb{E}(\boldsymbol{X}_s)))^{k-2} X_{s,i}\big] \mathrm{d}s \\
&\quad + \sum_{\ell=0}^{k-2} \binom{k}{\ell} \sum_{i=1}^{d} \int_0^t \int_{U_d} (\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}} \boldsymbol{z})^{k-\ell} \, \mathbb{E}\Big[\big(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}} (\boldsymbol{X}_s - \mathbb{E}(\boldsymbol{X}_s))\big)^\ell X_{s,i}\Big] \mathrm{d}s \, \mu_i(\mathrm{d}\boldsymbol{z}) \\
&\quad + \sum_{\ell=0}^{k-2} \binom{k}{\ell} \int_0^t \int_{U_d} (\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}} \boldsymbol{z})^{k-\ell} \, \mathbb{E}\Big[\big(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}} (\boldsymbol{X}_s - \mathbb{E}(\boldsymbol{X}_s))\big)^\ell\Big] \mathrm{d}s \, \nu(\mathrm{d}\boldsymbol{z})
\end{aligned} \tag{4.23}$$

for all $k \in \{1, \dots, q\}$, $j \in \{1, \dots, d\}$ *and* $t \in \mathbb{R}_+$. *Moreover, for each* $t \in \mathbb{R}_+$, $k \in \{1, \dots, q\}$ *and* $j \in \{1, \dots, d\}$, *there exists a polynomial* $P_{t,k,j} : \mathbb{R}^d \to \mathbb{R}$ *having degree at most* $\lfloor k/2 \rfloor$ *such that*

$$\mathbb{E}\left[(X_{t,j} - \mathbb{E}(X_{t,j}))^k\right] = \mathbb{E}\left[P_{t,k,j}(\boldsymbol{X}_0)\right], \qquad t \in \mathbb{R}_+. \tag{4.24}$$

The coefficients of the polynomial $P_{t,k,j}$ *depend on* d, $\boldsymbol{c}$, $\boldsymbol{\beta}$, $\boldsymbol{B}$, ν, μ_1, …, μ_d.

4.6 Remark. Note that in case of $\mathbb{E}(\boldsymbol{X}_t) = \boldsymbol{0}$, $t \in \mathbb{R}_+$, formulae (4.5) and (4.23) coincide. Indeed, if $\mathbb{E}(\boldsymbol{X}_t) = \boldsymbol{0}$, $t \in \mathbb{R}_+$, then, by (2.6), we have

$$\mathbb{E}(\boldsymbol{e}_j^\top \mathrm{e}^{t\widetilde{\boldsymbol{B}}} \boldsymbol{X}_0) + \int_0^t \boldsymbol{e}_j^\top \mathrm{e}^{u\widetilde{\boldsymbol{B}}} \widetilde{\boldsymbol{\beta}} \, \mathrm{d}u = 0, \qquad t \in \mathbb{R}_+, \qquad j \in \{1, \ldots, d\}.$$

Since $\boldsymbol{e}_j^\top \mathrm{e}^{t\widetilde{\boldsymbol{B}}} \boldsymbol{X}_0$ is a non-negative random variable and $\mathbb{R}_+ \ni t \mapsto \boldsymbol{e}_j^\top \mathrm{e}^{t\widetilde{\boldsymbol{B}}} \widetilde{\boldsymbol{\beta}}$ is a non-negative continuous function, we obtain $\mathbb{P}(\boldsymbol{e}_j^\top \mathrm{e}^{t\widetilde{\boldsymbol{B}}} \boldsymbol{X}_0 = 0) = 1$ and $\boldsymbol{e}_j^\top \mathrm{e}^{t\widetilde{\boldsymbol{B}}} \widetilde{\boldsymbol{\beta}} = 0$ for all $t \in \mathbb{R}_+$. Consequently,

$$\mathbb{E}\left[(\boldsymbol{e}_j^\top \mathrm{e}^{t\widetilde{\boldsymbol{B}}} \boldsymbol{X}_0)^k\right] + k \int_0^t (\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}} \widetilde{\boldsymbol{\beta}}) \, \mathbb{E}\left[(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}} \boldsymbol{X}_s)^{k-1}\right] \mathrm{d}s = 0$$

for all $t \in \mathbb{R}_+$, $j \in \{1, \ldots, d\}$ and $k \in \mathbb{N}$, which yields that formulae (4.5) and (4.23) coincide. □

Proof of Theorem 4.5. Consider objects (E1)–(E4) with initial value $\boldsymbol{\xi} = \boldsymbol{y} = (y_1, \ldots, y_d)^\top \in \mathbb{R}_+^d$. For each $K \in \mathbb{N}$, let $(\boldsymbol{Y}_{K,t})_{t \in \mathbb{R}_+}$ be a pathwise unique $\mathbb{R}_+^d$-valued strong solution to the SDE (3.1) with initial value $\boldsymbol{y}$. Using (4.9), we obtain

$$\begin{aligned}
\boldsymbol{w}^\top \mathrm{e}^{-t\widetilde{\boldsymbol{B}}_K} (\boldsymbol{Y}_{K,t} - \mathbb{E}(\boldsymbol{Y}_{K,t})) &= \sum_{i=1}^d \int_0^t \boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} \boldsymbol{e}_i \sqrt{2c_i Y_{K,s,i}} \, \mathrm{d}W_{s,i} \\
&\quad + \int_0^t \int_V \boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}_K} h(\boldsymbol{Y}_{K,s-}, \boldsymbol{r}) \, \widetilde{N}_K(\mathrm{d}s, \mathrm{d}\boldsymbol{r})
\end{aligned}$$

for all $\boldsymbol{w} \in \mathbb{R}^d$ and $t \in \mathbb{R}_+$. By the method of the proof of Theorem 4.3, for a CBI process $(\boldsymbol{Y}_t)_{t \in \mathbb{R}_+}$ having parameters $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ with initial value $\boldsymbol{y}$, one can derive

$$\begin{aligned}
&\mathbb{E}\left[(\boldsymbol{w}^\top \mathrm{e}^{-t\widetilde{\boldsymbol{B}}} (\boldsymbol{Y}_t - \mathbb{E}(\boldsymbol{Y}_t))^k\right] \\
&= k(k-1) \sum_{i=1}^d c_i \, \mathbb{E}\left(\int_0^t (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}} \boldsymbol{e}_i)^2 (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}} (\boldsymbol{Y}_s - \mathbb{E}(\boldsymbol{Y}_s))^{k-2} \, Y_{s,i} \, \mathrm{d}s\right) \\
&\quad + \sum_{\ell=0}^{k-2} \binom{k}{\ell} \mathbb{E}\left(\int_0^t \int_V (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}} (\boldsymbol{Y}_s - \mathbb{E}(\boldsymbol{Y}_s))^\ell \big(\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}} h(\boldsymbol{Y}_s, \boldsymbol{r})\big)^{k-\ell} \, \mathrm{d}s \, m(\mathrm{d}\boldsymbol{r})\right)
\end{aligned} \tag{4.25}$$

for all $k \in \{2, \ldots, q\}$, where

$$\begin{aligned}
&\mathbb{E}\left(\int_0^t \int_V (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}} (\boldsymbol{Y}_s - \mathbb{E}(\boldsymbol{Y}_s))^\ell \big(\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}} h(\boldsymbol{Y}_s, \boldsymbol{r})\big)^{k-\ell} \, \mathrm{d}s \, m(\mathrm{d}\boldsymbol{r})\right) \\
&= \mathbb{E}\left(\int_0^t \int_{U_d} (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}} (\boldsymbol{Y}_s - \mathbb{E}(\boldsymbol{Y}_s))^\ell \big(\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}} \boldsymbol{r}\big)^{k-\ell} \, \mathrm{d}s \, \nu(\mathrm{d}\boldsymbol{r})\right) \\
&\quad + \sum_{i=1}^d \mathbb{E}\left(\int_0^t \int_{U_d} \int_{U_1} (\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}} (\boldsymbol{Y}_s - \mathbb{E}(\boldsymbol{Y}_s))^\ell \big(\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}} \boldsymbol{z} \mathbb{1}_{\{s \leqslant Y_{s,i}\}}\big)^{k-\ell} \, \mathrm{d}s \, \mu_i(\mathrm{d}\boldsymbol{z}) \, \mathrm{d}u\right)
\end{aligned}$$

$$= \int_0^t \int_{U_d} \big(\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}} \boldsymbol{r}\big)^{k-\ell} \, \mathbb{E}\big[(\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}} (\boldsymbol{Y}_s - \mathbb{E}(\boldsymbol{Y}_s))^\ell\big] \, \mathrm{d}s \, \nu(\mathrm{d}\boldsymbol{r})$$

$$+ \sum_{i=1}^d \int_0^t \int_{U_d} \big(\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}} \boldsymbol{z}\big)^{k-\ell} \, \mathbb{E}\big[(\boldsymbol{w}^\top \mathrm{e}^{-s\widetilde{\boldsymbol{B}}} (\boldsymbol{Y}_s - \mathbb{E}(\boldsymbol{Y}_s))^\ell Y_{s,i}\big] \, \mathrm{d}s \, \mu_i(\mathrm{d}\boldsymbol{z}).$$

As in the proof of Theorem 4.3, this yields that the recursion (4.23) holds for $\boldsymbol{Y}$, and, by the law of total probability, we obtain (4.23) for $\boldsymbol{X}$ as well.

Now we turn to prove (4.24). As it was explained before, by the law of total probability, it is enough to prove (4.24) for $\boldsymbol{Y}$. Using the recursion (4.23), we obtain the existence of suitable polynomials $P_{t,k,j}$, $t \in \mathbb{R}_+$ $k \in \{1, \ldots, q\}$, $j \in \{1, \ldots, d\}$, by induction with respect to k. Indeed, for $k = 1$, we have $\mathbb{E}[Y_{t,j} - \mathbb{E}(Y_{t,j})] = 0$, $j \in \{1, \ldots, d\}$, $t \in \mathbb{R}_+$. For $k = 2$, by (4.23), we have

$$\mathbb{E}\left[(Y_{t,j} - \mathbb{E}(Y_{t,j}))^2\right] = 2 \sum_{i=1}^d c_i \int_0^t (\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}} \boldsymbol{e}_i)^2 \, \mathbb{E}(Y_{s,i}) \, \mathrm{d}s$$

(4.26)

$$+ \sum_{i=1}^d \int_0^t \int_{U_d} (\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}} \boldsymbol{z})^2 \, \mathbb{E}(Y_{s,i}) \, \mathrm{d}s \, \mu_i(\mathrm{d}\boldsymbol{z}) + \int_0^t \int_{U_d} (\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}} \boldsymbol{z})^2 \mathrm{d}s \, \nu(\mathrm{d}\boldsymbol{z})$$

for all $j \in \{1, \ldots, d\}$ and $t \in \mathbb{R}_+$. Thus $\mathbb{E}\left[(Y_{t,j} - \mathbb{E}(Y_{t,j}))^2\right] = P_{t,2,j}(\boldsymbol{y})$, where $P_{t,2,j} : \mathbb{R}^d \to \mathbb{R}$ is a polynomial of degree at most 1, since $\mathbb{E}(Y_{s,i}) = \boldsymbol{e}_i^\top \mathrm{e}^{s\widetilde{\boldsymbol{B}}} \boldsymbol{y} + \int_0^s \boldsymbol{e}_i^\top \mathrm{e}^{u\widetilde{\boldsymbol{B}}} \widetilde{\boldsymbol{\beta}} \, \mathrm{d}u$, $s \in \mathbb{R}_+$, from (2.6).

Now, suppose that for some $k' \in \mathbb{N}$ with $2k' + 1 \leqslant q$, suitable polynomials $P_{t,1,j}, \ldots, P_{t,2k',j}$ exist for all $t \in \mathbb{R}_+$ and $j \in \{1, \ldots, d\}$. We apply the recursion (4.23) for $k = 2k'+1$. Then for each $\ell \in \{0, 1, \ldots, 2k' - 1\}$ and $s, t \in \mathbb{R}_+$ with $s \leqslant t$, the function

$$\mathbb{R}_+^d \ni \boldsymbol{y} \mapsto \mathbb{E}\Big[\big(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}} (\boldsymbol{Y}_s - \mathbb{E}(\boldsymbol{Y}_s))\big)^\ell\Big]$$

is a polynomial of degree at most $\lfloor \ell/2 \rfloor \leqslant \lfloor (2k'-1)/2 \rfloor = k' - 1$. Moreover, for each $\ell \in \{0, 1, \ldots, 2k' - 1\}$ and $s, t \in \mathbb{R}_+$ with $s \leqslant t$, the function

$$\mathbb{R}_+^d \ni \boldsymbol{y} \mapsto \mathbb{E}\Big[\big(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}} (\boldsymbol{Y}_s - \mathbb{E}(\boldsymbol{Y}_s))\big)^\ell Y_{s,i}\Big]$$

is a polynomial of degree at most $\max\{\lfloor \ell/2 \rfloor + 1, \lfloor (\ell+1)/2 \rfloor\} \leqslant \max\{k', \lfloor (2k')/2 \rfloor\} = k'$, since, by (2.6),

$$Y_{s,j} = \mathbb{E}(Y_{s,j}) + (Y_{s,j} - \mathbb{E}(Y_{s,j})) = \boldsymbol{e}_j^\top \mathrm{e}^{s\widetilde{\boldsymbol{B}}} \boldsymbol{y} + \int_0^s \boldsymbol{e}_j^\top \mathrm{e}^{v\widetilde{\boldsymbol{B}}} \widetilde{\boldsymbol{\beta}} \, \mathrm{d}v + (Y_{s,j} - \mathbb{E}(Y_{s,j})).$$

Consequently, by (4.23), $\mathbb{R}_+^d \ni \boldsymbol{y} \mapsto \mathbb{E}\left[(Y_{t,j} - \mathbb{E}(Y_{t,j}))^{2k'+1}\right]$ is a polynomial of degree at most $k' = \lfloor (2k'+1)/2 \rfloor$, and we conclude the existence of suitable polynomials $P_{t,2k'+1,j}$ for all $t \in \mathbb{R}_+$ and $j \in \{1, \ldots, d\}$.

In a similar way, if for some $k' \in \mathbb{N}$ with $2k' + 2 \leqslant q$, suitable polynomials $P_{t,1,j}, \ldots, P_{t,2k'+1,j}$ exist for all $t \in \mathbb{R}_+$ and $j \in \{1, \ldots, d\}$, then we apply the recursion (4.23) for

$k = 2k'+2$. Then for each $\ell \in \{0,1,\ldots,2k'\}$, the function $\mathbb{R}_+^d \ni \boldsymbol{y} \mapsto \mathbb{E}\Big[\big(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}}(\boldsymbol{Y}_s - \mathbb{E}(\boldsymbol{Y}_s))\big)^\ell\Big]$ is a polynomial of degree at most $\lfloor \ell/2 \rfloor \leqslant \lfloor (2k')/2 \rfloor = k'$. Further, for each $\ell \in \{0,1,\ldots,2k'\}$, the function $\mathbb{R}_+^d \ni \boldsymbol{y} \mapsto \mathbb{E}\Big[\big(\boldsymbol{e}_j^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}}(\boldsymbol{Y}_s - \mathbb{E}(\boldsymbol{Y}_s))\big)^\ell Y_{s,i}\Big]$ is a polynomial of degree at most $\max\{\lfloor \ell/2\rfloor+1, \lfloor(\ell+1)/2\rfloor\} \leqslant \max\{k'+1, \lfloor(2k'+1)/2\rfloor\} = k'+1$. Consequently, by (4.23), $\mathbb{R}_+^d \ni \boldsymbol{y} \mapsto \mathbb{E}\left[(Y_{t,j} - \mathbb{E}(Y_{t,j}))^{2k'+2}\right]$ is a polynomial of degree at most $k'+1 = \lfloor (2k'+2)/2 \rfloor$, and we conclude the existence of suitable polynomials $P_{t,2k'+2,j}$ for all $t \in \mathbb{R}_+$ and $j \in \{1,\ldots,d\}$. $\square$

For mixed central moments, we have the following corollary.

4.7 Corollary. *Let $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ be a CBI process with parameters $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ such that $\mathbb{E}(\|\boldsymbol{X}_0\|^q) < \infty$ and the moment conditions (4.4) hold with some $q \in \mathbb{N}$. Then for all $t \in \mathbb{R}_+$, $k \in \{1,\ldots,q\}$ and $i_1,\ldots,i_k \in \{1,\ldots,d\}$, there exists a polynomial $P_{t,k,i_1,\ldots,i_k} : \mathbb{R}^d \to \mathbb{R}$ having degree at most $\lfloor k/2 \rfloor$ such that*

$$\mathbb{E}\left[(X_{t,i_1} - \mathbb{E}(X_{t,i_1}))\cdots(X_{t,i_k} - \mathbb{E}(X_{t,i_k}))\right] = \mathbb{E}(P_{t,k,i_1,\ldots,i_k}(\boldsymbol{X}_0)). \tag{4.27}$$

The coefficients of the polynomial $P_{t,k,i_1,\ldots,i_k}$ depend on d, $\boldsymbol{c}$, $\boldsymbol{\beta}$, $\boldsymbol{B}$, ν, μ_1, …, μ_d.

Proof. Replacing $\boldsymbol{w}$ by $\mathrm{e}^{t\widetilde{\boldsymbol{B}}^\top}\boldsymbol{w}$ in (4.25), and then using the law of total probability, one obtains

$$\begin{aligned}
&\mathbb{E}\left[\langle \boldsymbol{w}, \boldsymbol{X}_t - \mathbb{E}(\boldsymbol{X}_t)\rangle^k\right] \\
&= k(k-1)\sum_{i=1}^d c_i \int_0^t (\boldsymbol{w}^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}}\boldsymbol{e}_i)^2\, \mathbb{E}\big[(\boldsymbol{w}^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}}(\boldsymbol{X}_s - \mathbb{E}(\boldsymbol{X}_s)))^{k-2} X_{s,i}\big]\,\mathrm{d}s \\
&\quad + \sum_{\ell=0}^{k-2}\binom{k}{\ell}\sum_{i=1}^d \int_0^t \int_{\mathcal{U}_d} (\boldsymbol{w}^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}}\boldsymbol{z})^{k-\ell}\, \mathbb{E}\Big[\big(\boldsymbol{w}^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}}(\boldsymbol{X}_s - \mathbb{E}(\boldsymbol{X}_s))\big)^\ell X_{s,i}\Big]\,\mathrm{d}s\,\mu_i(\mathrm{d}\boldsymbol{z}) \\
&\quad + \sum_{\ell=0}^{k-2}\binom{k}{\ell}\int_0^t \int_{\mathcal{U}_d} (\boldsymbol{w}^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}}\boldsymbol{z})^{k-\ell}\, \mathbb{E}\Big[\big(\boldsymbol{w}^\top \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}}(\boldsymbol{X}_s - \mathbb{E}(\boldsymbol{X}_s))\big)^\ell\Big]\,\mathrm{d}s\,\nu(\mathrm{d}\boldsymbol{z})
\end{aligned}$$

for all $t \in \mathbb{R}_+$, $k \in \{1,\ldots,q\}$ and $\boldsymbol{w} \in \mathbb{R}^d$, and hence, by the proof of Theorem 4.5, for each $t \in \mathbb{R}_+$, $k \in \{1,\ldots,q\}$ and $\boldsymbol{w} \in \mathbb{R}^d$, there exists a polynomial $P_{t,k,\boldsymbol{w}} : \mathbb{R}^d \to \mathbb{R}$ having degree at most $\lfloor k/2 \rfloor$, such that

$$\mathbb{E}\left[\langle \boldsymbol{w}, \boldsymbol{X}_t - \mathbb{E}(\boldsymbol{X}_t)\rangle^k\right] = \mathbb{E}\big[P_{t,k,\boldsymbol{w}}(\boldsymbol{X}_0)\big],$$

where the coefficients of the polynomial $P_{t,k,\boldsymbol{w}}$ depend on d, $\boldsymbol{c}$, $\boldsymbol{\beta}$, $\boldsymbol{B}$, ν, μ_1, …, μ_d. The proof can be finished as the proof of Corollary 4.4. $\square$

4.8 Proposition. *Let $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ be a CBI process with parameters $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ such that $\mathbb{E}(\|\boldsymbol{X}_0\|^2) < \infty$ and the moment conditions (4.4) hold with $q = 2$. Then for all $t \in \mathbb{R}_+$,* *we have*

$$\operatorname{Var}(\boldsymbol{X}_t) = \sum_{\ell=1}^{d} \int_0^t (\boldsymbol{e}_\ell^\top \mathrm{e}^{(t-u)\widetilde{\boldsymbol{B}}}\, \mathbb{E}(\boldsymbol{X}_0)) \mathrm{e}^{u\widetilde{\boldsymbol{B}}} \boldsymbol{C}_\ell \mathrm{e}^{u\widetilde{\boldsymbol{B}}^\top} \mathrm{d}u + \int_0^t \mathrm{e}^{u\widetilde{\boldsymbol{B}}} \left(\int_{U_d} \boldsymbol{z}\boldsymbol{z}^\top \nu(\mathrm{d}\boldsymbol{z}) \right) \mathrm{e}^{u\widetilde{\boldsymbol{B}}^\top} \mathrm{d}u$$

$$+ \sum_{\ell=1}^{d} \int_0^t \left(\int_0^{t-u} \boldsymbol{e}_\ell^\top \mathrm{e}^{v\widetilde{\boldsymbol{B}}} \widetilde{\boldsymbol{\beta}}\, \mathrm{d}v \right) \mathrm{e}^{u\widetilde{\boldsymbol{B}}} \boldsymbol{C}_\ell \mathrm{e}^{u\widetilde{\boldsymbol{B}}^\top} \mathrm{d}u,$$

where

$$\boldsymbol{C}_\ell := 2c_\ell \boldsymbol{e}_\ell \boldsymbol{e}_\ell^\top + \int_{U_d} \boldsymbol{z}\boldsymbol{z}^\top \mu_\ell(\mathrm{d}\boldsymbol{z}) \in \mathbb{R}_+^{d\times d}, \qquad \ell \in \{1, \ldots, d\}.$$

Proof. By (4.26), we have

$$\boldsymbol{e}_j^\top\, \mathbb{E}\left[(\boldsymbol{X}_t - \mathbb{E}(\boldsymbol{X}_t))(\boldsymbol{X}_t - \mathbb{E}(\boldsymbol{X}_t))^\top \right] \boldsymbol{e}_j = \boldsymbol{e}_j^\top \operatorname{Var}(\boldsymbol{X}_t) \boldsymbol{e}_j = \mathbb{E}\left[(X_{t,j} - \mathbb{E}(X_{t,j}))^2 \right]$$

$$= \sum_{\ell=1}^{d} \int_0^t \boldsymbol{e}_j^\top \mathrm{e}^{(t-u)\widetilde{\boldsymbol{B}}} \boldsymbol{C}_\ell\, \mathrm{e}^{(t-u)\widetilde{\boldsymbol{B}}^\top} \boldsymbol{e}_j\, \mathbb{E}(X_{u,\ell})\, \mathrm{d}u + \int_0^t \boldsymbol{e}_j^\top \mathrm{e}^{(t-u)\widetilde{\boldsymbol{B}}} \left(\int_{U_d} \boldsymbol{z}\boldsymbol{z}^\top \nu(\mathrm{d}\boldsymbol{z}) \right) \mathrm{e}^{(t-u)\widetilde{\boldsymbol{B}}^\top} \boldsymbol{e}_j\, \mathrm{d}u,$$

which is finite by (4.4) with $q = 2$ and part (v) of Definition 2.2. Using the identities

$$\boldsymbol{e}_i^\top \operatorname{Var}(\boldsymbol{X}_t) \boldsymbol{e}_j = \frac{1}{4} \left[(\boldsymbol{e}_i + \boldsymbol{e}_j)^\top \operatorname{Var}(\boldsymbol{X}_t)(\boldsymbol{e}_i + \boldsymbol{e}_j) - (\boldsymbol{e}_i - \boldsymbol{e}_j)^\top \operatorname{Var}(\boldsymbol{X}_t)(\boldsymbol{e}_i - \boldsymbol{e}_j) \right]$$

for $i, j \in \{1, \ldots, d\}$, and $\operatorname{Var}(\boldsymbol{X}_t) = \sum_{i=1}^d \sum_{j=1}^d \boldsymbol{e}_i (\boldsymbol{e}_i^\top \operatorname{Var}(\boldsymbol{X}_t) \boldsymbol{e}_j) \boldsymbol{e}_j^\top$, we obtain

$$\operatorname{Var}(\boldsymbol{X}_t) = \sum_{\ell=1}^{d} \int_0^t \mathrm{e}^{(t-u)\widetilde{\boldsymbol{B}}} \boldsymbol{C}_\ell\, \mathrm{e}^{(t-u)\widetilde{\boldsymbol{B}}^\top}\, \mathbb{E}(X_{u,\ell})\, \mathrm{d}u + \int_0^t \mathrm{e}^{(t-u)\widetilde{\boldsymbol{B}}} \left(\int_{U_d} \boldsymbol{z}\boldsymbol{z}^\top \nu(\mathrm{d}\boldsymbol{z}) \right) \mathrm{e}^{(t-u)\widetilde{\boldsymbol{B}}^\top} \mathrm{d}u.$$

By (2.6), we have $\mathbb{E}(X_{u,\ell}) = \boldsymbol{e}_\ell^\top \mathrm{e}^{u\widetilde{\boldsymbol{B}}}\, \mathbb{E}(\boldsymbol{X}_0) + \int_0^u \boldsymbol{e}_\ell^\top \mathrm{e}^{v\widetilde{\boldsymbol{B}}} \widetilde{\boldsymbol{\beta}}\, \mathrm{d}v$, thus

$$\operatorname{Var}(\boldsymbol{X}_t) = \sum_{\ell=1}^{d} \int_0^t (\boldsymbol{e}_\ell^\top \mathrm{e}^{u\widetilde{\boldsymbol{B}}}\, \mathbb{E}(\boldsymbol{X}_0))\, \mathrm{e}^{(t-u)\widetilde{\boldsymbol{B}}} \boldsymbol{C}_\ell \mathrm{e}^{(t-u)\widetilde{\boldsymbol{B}}^\top} \mathrm{d}u + \int_0^t \mathrm{e}^{u\widetilde{\boldsymbol{B}}} \left(\int_{U_d} \boldsymbol{z}\boldsymbol{z}^\top \nu(\mathrm{d}\boldsymbol{z}) \right) \mathrm{e}^{u\widetilde{\boldsymbol{B}}^\top} \mathrm{d}u$$

$$+ \sum_{\ell=1}^{d} \int_0^t \left(\int_0^u \boldsymbol{e}_\ell^\top \mathrm{e}^{v\widetilde{\boldsymbol{B}}} \widetilde{\boldsymbol{\beta}}\, \mathrm{d}v \mathrm{e}^{(t-u)\widetilde{\boldsymbol{B}}} \boldsymbol{C}_\ell \mathrm{e}^{(t-u)\widetilde{\boldsymbol{B}}^\top} \right) \mathrm{d}u$$

$$= \sum_{\ell=1}^{d} \int_0^t (\boldsymbol{e}_\ell^\top \mathrm{e}^{(t-v)\widetilde{\boldsymbol{B}}}\, \mathbb{E}(\boldsymbol{X}_0))\, \mathrm{e}^{v\widetilde{\boldsymbol{B}}} \boldsymbol{C}_\ell \mathrm{e}^{v\widetilde{\boldsymbol{B}}^\top} \mathrm{d}v + \int_0^t \mathrm{e}^{u\widetilde{\boldsymbol{B}}} \left(\int_{U_d} \boldsymbol{z}\boldsymbol{z}^\top \nu(\mathrm{d}\boldsymbol{z}) \right) \mathrm{e}^{u\widetilde{\boldsymbol{B}}^\top} \mathrm{d}u$$

$$+ \sum_{\ell=1}^{d} \int_0^t \left(\int_0^{t-u} \boldsymbol{e}_\ell^\top \mathrm{e}^{v\widetilde{\boldsymbol{B}}} \widetilde{\boldsymbol{\beta}}\, \mathrm{d}v \right) \mathrm{e}^{u\widetilde{\boldsymbol{B}}} \boldsymbol{C}_\ell \mathrm{e}^{u\widetilde{\boldsymbol{B}}^\top} \mathrm{d}u,$$

and hence we obtain the statement. □

Acknowledgements

We would like to thank the referee for his/her comments that helped us to improve the presentation of the paper.

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