# The Cayley isomorphism property for Cayley maps

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**Abstract**

In this paper we study finite groups which have Cayley isomorphism property with respect to Cayley maps, CIM-groups for a brief. We show that the structure of the CIM-groups is very restricted. It is described in Theorem 1.1 where a short list of possible candidates for CIM-groups is given. Theorem 1.2 provides concrete examples of infinite series of CIM-groups.

## 1 Introduction

Let $H$ be a finite group and $S$ a subset of $H \setminus \{1\}$. A *Cayley (di)graph* $\mathsf{Cay}(H,S)$ is defined by having the vertex set $H$ and $g$ is adjacent to $h$ if and only if $g^{-1}h \in S$. The set $S$ is called the *connection set* of the Cayley graph $\mathsf{Cay}(H,S)$. A Cayley graph $\mathsf{Cay}(H,S)$ is *undirected* if and only if $S = S^{-1}$, where $S^{-1} = \left\{ s^{-1} \in H \mid s \in S \right\}$. Every left multiplication via elements of $H$ is an automorphism of $\mathsf{Cay}(H,S)$, so the automorphism group of every Cayley graph over $H$ contains a regular subgroup isomorphic to $H$. Moreover, this property characterises the Cayley graphs of $H$. The group consisting of the elements of the left multiplications will be denoted by $\widehat{H}$ and the left multiplication with $h \in H$ by $\widehat{h}$ (that is $\widehat{h}(x) = hx$). Finally, a *Cayley map* $\mathsf{Cay}(H,S,\rho)$ is an undirected Cayley graph $\mathsf{Cay}(H,S)$ endowed with a cyclic ordering $\rho \in \mathsf{Sym}(S)$ of the connection set.

We say that a map $\mathsf{Cay}(H,S,\rho)$ is *connected* if the underlying Cayley graph is connected, that is $\langle S \rangle = H$.

Using a less combinatorial approach, a Cayley map is a 2-cell embedding of a Cayley graph into oriented surface with the same cyclic rotation around each vertex. For precise definiton of embedding graphs into orientable surfaces, see [10]. Several different subclasses of Cayley maps have been investigated. The notion of a Cayley map first appeared in the paper of Biggs [2] who investigated balanced Cayley maps. A Cayley map $\mathsf{Cay}(H,S,\rho)$ is called *balanced* if $\rho(s^{-1}) = \rho(s)^{-1}$ and it is called *antibalanced* if $\rho(s^{-1}) = \rho^{-1}(s)^{-1}$. Further, a Cayley map $M$ is called *regular* if its automorphism group is transitive on the arcs as well. Following Jajcay and Siran [9], we say that for a group $H$ a permutation

$\phi \in \mathsf{Sym}(H)$ is a *skew-morphism* if there exists a mapping $\pi : H \mapsto \mathbb{N}$ such that $\phi(gh) = \phi(g)\phi^{\pi(g)}(h)$ for every $g, h \in G$.

Given two Cayley maps $M_1 = \mathsf{Cay}(H_1, S_1, \rho_1)$ and $M_2 = \mathsf{Cay}(H_2, S_2, \rho_2)$, a bijection $\phi : H_1 \to H_2$ is a *map isomorphism* from $M_1$ to $M_2$ if $\phi$ is an isomorphism of the underlying Cayley graphs and for all $h \in H_1, s \in S_1$ it holds that $\phi(h)^{-1}\phi(h\rho_1(s)) = \rho_2(\phi(h)^{-1}\phi(hs))$. Denoting by $\Delta_h\phi$ the "differential" map $s \mapsto \phi(h)^{-1}\phi(hs), s \in S_1$ one can rewrite the latter condition as follows $(\Delta_h\phi)\rho_1 = \rho_2(\Delta_{\phi(h)}\phi)$. Notice that since $\phi$ is a graph isomorphism the map $\Delta_h\phi$ is a bijection between $S_1$ and $S_2$ for every $h \in H$.

In what follows we say that $M_1$ and $M_2$ are *Cayley isomorphic* if there exists a group isomorphism $\phi : H_1 \to H_2$ which is simulteneously a map isomorphism, that is $\phi(S_1) = S_2$ and $\phi(\rho_1(s)) = \rho_2(\phi(s))$ holds for each $s \in S_1$.

The *automorphism group* of a Cayley map $M = \mathsf{Cay}(H, S, \rho)$ is the set of all isomorphisms from $M$ to $M$ and it will be denoted by $\mathsf{Aut}(M)$. It is clear that $\widehat{H} \leq \mathsf{Aut}(M)$. Thus $\mathsf{Aut}(\mathsf{Cay}(H, S, \rho))$ contains the regular subgroup $\widehat{H}$. Every group automorphism $\sigma \in \mathsf{Aut}(H)$ induces Cayley isomorphism between the maps $\mathsf{Cay}(H, S, \rho)$ and $\mathsf{Cay}(H, \sigma(S), \sigma'\rho(\sigma')^{-1})$ where $\sigma' = \sigma|_S$ is the restriction of $\sigma$ on $S$. Thus a group automorphism $\sigma$ is an automorphism of a map $\mathsf{Cay}(H, S, \rho)$ if and only if $\sigma(S) = S$ and $\sigma|_S\rho = \rho\sigma|_S$. Since $\rho$ is a full cycle, the latter condition is equivalent to $\sigma|_S = \rho^k$ for some integer $k$.

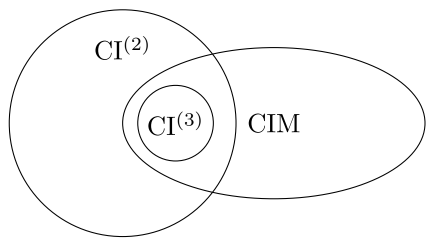
The so-called *CI (Cayley isomorphism)* property of groups is well studied with respect to Cayley graphs. A group $H$ is called a *CI-group* with respect to graphs (CIG-groups, for short) if two Cayley graphs of $H$ are isomorphic if and only if they are isomorphic by a group automorphism as well. A Cayley graph $\Gamma = \mathsf{Cay}(H, S)$ is called a *CI-graph* if every Cayley graph $\mathsf{Cay}(H, T)$ isomorphic $\Gamma$ is Cayley isomorphic to $\Gamma$. For an old but excellent survey about CI-groups, see [11] and further results can be found in [12]. Similarly to the original definition of the CI property we say that a Cayley map $M = \mathsf{Cay}(H, S, \rho)$ is a *CI-map* of $H$ if every Cayley map $M'$ over $H$ isomorphic to $M$ is also Cayley isomorphic to $M$. We call a group $H$ a *CIM-group* if for every Cayley map $\mathsf{Cay}(H, S, \rho)$ is a CI-map.

A Cayley map $M = \mathsf{Cay}(H, S, \rho)$ can also be considered as a ternary relational structure on the vertices of the underlying graph. Three vertices $(x, y, z)$ are in the relation $\mathcal{R}$ if and only if $x^{-1}y, x^{-1}z \in S$ and $\rho(x^{-1}y) = x^{-1}z$. The automorphism group $\mathsf{Aut}(M)$ consists of all those permutations of the vertices which preserves the relation $\mathcal{R}$. In particular, it is a 3-closed permutation group. This observation allows us to use the technique developed by Babai to solve problems concerning CIM-groups. Moreover a theorem of Pálfy [14] shows that the groups which are CI-groups for every $m$-ary relational structures are the cyclic groups of order $n$, where $(n, \phi(n)) = 1$ and the Klein group. Pálfy also proved that if a group is not a CI-group with respect to some $m$-ary relation, then it is not a CI-group with respect to 4-ary relational structures.

CI-groups with respect to ternary relations ($\mathrm{CI}^{(3)}$-groups, for short) were investigated by Dobson [4],[5] and later by Dobson and Spiga [6]. Although the class of $\mathrm{CI}^{(3)}$-groups is rather narrow, its full classification is not finished yet.

The latest results may be found in [5] and [6]. Since map automorphism group is 3-closed, each $\mathrm{CI}^{(3)}$-group is a CIM-group. The converse is not true. For example, every elementary abelian 2-groups of rank at least 6 is a CIM-group but not a $\mathrm{CI}^{(3)}$-group.

As it was also pointed out by Dobson and Spiga [6] every $\mathrm{CI}^{(3)}$-group is also a $\mathrm{CI}^{(2)}$-group, that is a group which has a the CI-property with respect to binary relational structures. However, we will prove that there are CIM-groups which are not $\mathrm{CI}^{(2)}$-groups. The Venn diagram below reflects the relationships between the three classes of CI-groups.




Our first result formulates necessary conditions for being a CIM-group.

**Theorem 1.1.** *Let $H$ be a CIM-group. Then $H$ is isomorphic to one of the following groups*

*(a)* $\mathbb{Z}_m \times \mathbb{Z}_2^r,\ \mathbb{Z}_m \times \mathbb{Z}_4,\ \mathbb{Z}_m \times \mathbb{Z}_8,\ \mathbb{Z}_m \times Q_8$;

*(b)* $\mathbb{Z}_m \rtimes \mathbb{Z}_{2^e}, e = 1, 2, 3,$

*where $m$ is an odd square-free number.*

The second main result provides several infinite series of CIM-groups.

**Theorem 1.2.** *The following groups are CI-groups with respect to Cayley maps.*

$$\mathbb{Z}_m \times \mathbb{Z}_2^r,\ \mathbb{Z}_m \times \mathbb{Z}_4,\ \mathbb{Z}_m \times Q_8$$

*where $m$ is an odd square-free number.*

As an immediate corollary of the above Theorems we obtain the following criterion.

**Theorem 1.3.** *A group $H$ of odd order is a CIM-group if and only if $H$ is a cyclic group of a square free order.*

Notice that obtained results do not provide a complete classification of cyclic CIM-group. This is because we do not know which of the groups $\mathbb{Z}_m \times \mathbb{Z}_8$, $m$ is odd and square-free, are CIM-groups. Proposition 5.7 shows that $\mathbb{Z}_8$ is a CIM-group. We believe that all groups of the above structure have the CIM-property.

Our paper is organised as follows. In Section 2 we collect a few general results about CI-property which will be used later. In Section 3 we characterize Sylow subgroups of CIM-groups. Section 4 is devoted to the proof of Theorem 1.1. The last section provides proofs of Theorems 1.2 and 1.3.

Most of the group-theoretical notation used in the paper are standard and can be found in [17].

## 2 General observations

The original CI property for graphs is inherited by subgroups which gives us a strong tool to determine the list of possible CI-groups. Similar, but a weaker, property holds for CIM-groups as well. Let us call a group $H$ to be a *connected* CIM-group if it is a CI-group with respect to connected maps.

**Lemma 2.1.** *Every subgroup of a CIM-group is a connected CIM-group.*

*Proof.* Let $G$ be a CIM-group and $H \leq G$. Let us assume that $\mathsf{Cay}(H, S, \rho)$ and $\mathsf{Cay}(H, S', \rho')$ are isomorphic connected Cayley maps of $H$. Let $f$ be a map isomorphism from $\mathsf{Cay}(H, S, \rho)$ to $\mathsf{Cay}(H, S', \rho')$. Then $\widehat{g_2} f \widehat{g_1}^{-1}$ is an isomorphism between the connected component of $\mathsf{Cay}(G, S, \rho)$ on $g_1 H$ and the one $\mathsf{Cay}(G, S', \rho')$ on $g_2 H$. This shows that the connected components of $\mathsf{Cay}(G, S, \rho)$ and $\mathsf{Cay}(G, S', \rho')$ are isomorphic. Therefore $\mathsf{Cay}(G, S, \rho)$ and $\mathsf{Cay}(G', S', \rho')$ are isomorphic Cayley maps. Since $G$ is a CIM-group there exists $\alpha \in \mathsf{Aut}(G)$, which is an isomorphism from $\mathsf{Cay}(G, S, \rho)$ to $\mathsf{Cay}(G, S', \rho')$. Since the Cayley map $\mathsf{Cay}(H, S, \rho)$ is a connected component of $\mathsf{Cay}(G, S, \rho)$, its image $\mathsf{Cay}(\alpha(H), \alpha(S), \alpha|_S \rho (\alpha|_S)^{-1})$ is a connected component of $\mathsf{Cay}(G, S', \rho')$. Therefore $\alpha(H)$ is a left coset of $H$ implying $\alpha(H) = H$. Hence $\alpha|_H$ is a Cayley isomorphism between the above maps. ∎

This result suggests that it is worth investigating $p$-groups which arise as the Sylow $p$-subgroups of finite groups.

Another important observation is that if $\mathsf{Cay}(H, S, \rho)$ is a Cayley map with $|S| \leq 2$, then the Cayley graph $\mathsf{Cay}(H, S,)$ has to be a CI-graph since there exists only one cyclic ordering on one or two elements. This shows that the automorphism group of a CIM-group $H$ has only one orbit on the elements of order 2 and for every $g, h \in H$ with the same order there exists $\alpha \in \mathsf{Aut}(H)$ with $\alpha(g) = h$ or $\alpha(g) = h^{-1}$. Groups having this property were investigated by Li and Praeger [13].

The following lemma is due to Babai [1] and applies for every Cayley relational structures.

**Lemma 2.2** (Babai)**.** *Let* $\mathsf{Cay}(H, \mathcal{R})$ *be a Cayley relational structure. Then* $\mathsf{Cay}(H, \mathcal{R})$ *has the CI-property if and only if for every regular subgroup* $\mathring{H}$ *of* $\mathsf{Aut}(\mathsf{Cay}(G, \mathcal{R}))$ *there exists* $\alpha \in \mathsf{Aut}(\mathsf{Cay}(G, \mathcal{R}))$ *with* $\alpha(\mathring{H}) = \widehat{H}$.

In what follows we refer to a regular permutation subgroup isomorphic to $H$ as $H$*-regular* subgroup.

The statement below describes the structure of the Cayley map automorphism group. Although it was proven by Jajcay [8] we prefer to provide its proof here to make the paper self-contained.

**Lemma 2.3.** *Let* $M := \mathsf{Cay}(H, S, \rho)$ *be a connected Cayley map and* $G := \mathsf{Aut}(M)$ *its automorphism group. Then* $G_e$ *acts faithfully on* $S$ *and its restriction* $(G_e)|_S$ *is contained in* $\langle \rho \rangle$*. In particular,* $G_e$ *is cyclic.*

*Proof.* Pick an arbitrary $\phi \in G_e$. Then $\Delta_e\phi = \phi|_S$ implying $\rho\phi|_S = \phi|_S\rho$. Since $\rho$ is a full cycle on $S$, any permutation commuting with it belongs to $\langle\rho\rangle$. Therefore $(G_e)|_S \leq \langle\rho\rangle$. This inclusion also implies that for each $s \in S$ the two-point stabilizer $G_{e,s}$ acts trivially on $S$. Therefore $G_{h,hs}$ acts trivially on $hS$ for any $h \in H$ and $s \in S$. Thus if $\phi$ fixes $e$ and $s \in S$, then it fixes pointwise the sets $S, S^2, S^3$ etc. Since $\mathsf{Cay}(H,S)$ is connected, we conclude that $G_{e,s}$ is trivial, i.e. $G_e$ acts faithfully on $S$. ■

The above statement shows the full automorphism group $G$ of a connected $\mathsf{Cay}(H,S,\rho)$ is a product of $\widehat{H}$ with the cyclic group $G_e$. Moreover the restriction of $G_e$ on $S$ is contained in $\langle\rho\rangle$.

# 3 Sylow subgroups of CIM-groups

Similarly to the classical case of CI-groups, it follows from Lemma 2.1 that it is important to investigate $p$-groups. Babai and Frankl proved that if a group $H$ is a $CI^{(2)}$-group of prime power order, then $H$ is either elementary abelian $p$-group, the quaternion group of order 8 or a cyclic group of small order. The statement below describes odd order Sylow subgroups of a $CIM$-group.

**Lemma 3.1.** *A Sylow $p$-subgroup of a CIM-group $H$ corresponding to an odd prime divisor $p$ of $|H|$ has order $p$.*

*Proof.* It follows from Lemma 2.1 that it is sufficient to show that any subgroup of order $p^2$ is not a connected CIM-subgroup.

Let $K$ be a group of order $p^2$. Then either $K \cong \mathbb{Z}_p^2$ or $K \cong \mathbb{Z}_{p^2}$. In both cases there exists an automorphism $\beta \in \mathsf{Aut}(K)$ of order $p$ (the concrete examples of $\beta$ are given below). A direct check shows that the bijection $\alpha \in \mathsf{Sym}(K)$ defined via $\alpha(x) = -\beta(x)$ is an automorphism of $K$ of order $2p$. It follows from $\alpha^p = -1$ that each non-zero $\alpha$-orbits is symmetric, and, therefore, has even cardinality. This implies that at least one orbit of $\alpha$ contains $2p$ element. Let us denote this orbit as $S$. Clearly $\langle S\rangle = K$. Consider a Cayley map $M = \mathsf{Cay}(K,S,\alpha|_K)$. The group $G := \mathsf{Aut}(M)$ contains the semidirect product $\widehat{K} \rtimes \langle\alpha\rangle \leq \mathsf{Sym}(K)$. Combining this with $|\mathsf{Aut}(M)| \leq |K||S| = |K||\langle\alpha\rangle|$ we conclude that $G = \widehat{K} \rtimes \langle\alpha\rangle$ (so, $M$ is a balanced regular map). We claim that $M$ is not a CI-map. According to Lemma 2.2 it is enough to find two $K$-regular subgroups of $G$ which are not conjugate in $G$. Since $\widehat{K}$ is normal in $G$, it is sufficient to find a $K$-regular subgroup of $G$ distinct from $\widehat{K}$. To point out such a subgroup we consider the cases of $K \cong \mathbb{Z}_{p^2}$ and $K \cong \mathbb{Z}_p^2$ separately. In both cases we use the fact that $\beta = \alpha^{p+1} \in G$.

**Case of $K \cong \mathbb{Z}_{p^2}$.**
In this case we chose $\beta \in \mathsf{Aut}(K)$ defined via $\beta(x) = (1+p)x$. The permutation $\gamma(x) := \beta(x) + 1 = (1+p)x + 1$ belongs to the group $G$ because $\gamma = \widehat{1}\beta$. A direct check shows that $\gamma^p(x) = x + p$ implying $o(\gamma^p) = p$, and, consequently, $o(\gamma) = p^2$. Therefore $\langle\gamma\rangle$ is a regular cyclic subgroup of $G$ different from $\widehat{K}$.

**Case of $K \cong \mathbb{Z}_p^2$.**
In this case we chose $\beta \in \mathsf{Aut}(\mathbb{Z}_p^2)$ defined via $\beta((x,y)) = (x+y, y)$. Then the group $G$ contains the subgroup $\widehat{\mathbb{Z}_p^2} \rtimes \langle\beta\rangle$ which consists of all permutations of the form $(x,y) \mapsto (x+ay+u, y+v)$ where $a,u,v \in \mathbb{Z}_p$. A direct check shows that the permutations $\tau_{a,b} : (x,y) \mapsto (x+ay+b, y+a)$, $a,b \in \mathbb{Z}_p$ form a subgroup, say $T$, of $G$ isomorphic to $\mathbb{Z}_p^2$. It is easy to check that $T$ acts regularly on $\mathbb{Z}_p^2$. ■

## 3.1 Sylow 2-subgroups of CIM-groups

**Proposition 3.2.** *For every $n \geq 4$ the cyclic group $\mathbb{Z}_{2^n}$ is not a connected CIM-group.*

*Proof.* The element $a = 1 + 2^{n-1} \in \mathbb{Z}_{2^n}$ has multiplicative order 2. Therefore the automorphism $\alpha \in \mathsf{Aut}(\mathbb{Z}_{2^n})$ defined via $\alpha(x) = ax$ has order two as well. We construct an antibalanced Cayley map the automorphism group of which contains the subgroup $\widehat{\mathbb{Z}}_{2^n} \rtimes \langle\alpha\rangle$. Let $S = \{1, -1, 3, -3a, a, -a, 3a, -3\}$ be a set of 8 different elements, and let $\rho = (1, -1, 3, -3a, a, -a, 3a, -3)$ be an 8-cycle. The permutation $\alpha$ is an automorphism of the map $M := \mathsf{Cay}(\mathbb{Z}_{2^n}, S, \rho)$, because $\alpha(S) = S$ and $\alpha|_S = \rho^4$. Thus the full autmorphism group $G := \mathsf{Aut}(M)$ contains the subgroup $A := \widehat{\mathbb{Z}_{2^n}} \rtimes \langle\alpha\rangle$.

Straightforward calculation shows that $(\widehat{1}\alpha)^2(x) = x + a + 1 = x + 2^{n-1} + 2$ implying that $(\widehat{1}\alpha)^2$ has order $2^{n-1}$. Hence the order of $\widehat{1}\alpha$ is $2^n$. Therefore the subgroup $A$ of $G$ contains at least two regular subgroups isomorphic to $\mathbb{Z}_{2^n}$, both of index two. These subgroups are not conjugate in $A$, since they are normal in $A$. Thus it is enough to prove that $A = G$. The latter is equivalent to showing that the point stabilizer of $G_0$ has order two. Assume, towards a contradiction, that $|G_0| > 2$. The group $G_0$ is cyclic and acts on $S$ faithfully and semi-regularly. Therefore there exists an element $\sigma \in G_0$ such that $\sigma^2 = \alpha$. In particular, $\sigma$ has order 4. Since $\sigma|_S$ commutes with $\rho$, we conclude that $\sigma|_S = \rho^2 = (1, 3, a, 3a)(-1, -3a, -a, -3)$.

Consider the subset $T := \{x \in \mathbb{Z}_{2^n} \,|\, |S \cap (S+x)| = 6\}$[1]. Since $\sigma$ is an automorphism of $\mathsf{Cay}(\mathbb{Z}_{2^n}, S)$ stabilizing 0, it satisfies the equation $\sigma(x+S) = \sigma(x) + S$ for every $x \in \mathbb{Z}_{2^n}$. Thus $T$ is $\sigma$-invariant. A direct calculation yields us $T = \{2, -2, 2+2^{n-1}, -2+2^{n-1}\}$.

Consider the set $\sigma(S \setminus (S+2)) = \sigma(\{-3, -3+2^{n-1}\})$. Since $\sigma$ is an automorphism of the graph $\mathsf{Cay}(\mathbb{Z}_{2^n}, S)$, we can write $\sigma(S+2) = S + \sigma(2)$. Therefore

$$\sigma(S \setminus (S+2)) = \sigma(\{-3, -3+2^{n-1}\}) \implies S \setminus (S + \sigma(2))) = \{-1, -1+2^{n-1}\}$$

Since $T$ is $\sigma$-invariant $\sigma(2) \in T$, none of the elements $t \in T$ satisfies $S \setminus (S+t)) = \{-1, -1+2^{n-1}\}$, a contradiction.

■

**Proposition 3.3.** *Let $P \leq H$ be a Sylow 2-subgroup of an CIM-group $H$. Then $P$ is either elementary abelian or cyclic $C_{2^n}, n \leq 3$ or $Q_8$.*

[1] These are the elements at distance two from 0 in $\mathsf{Cay}(\mathbb{Z}_{2^n}, S)$, each of them is connected to 0 by 6 paths of length two.

*Proof.* Assume that $\exp(P) > 2$. Then $P$ contains a cyclic subgroup $C_4 = \langle c \rangle$ of order 4. We claim that $P$ doesn't contain the Klein subgroup $K_4 \cong \mathbb{Z}_2^2$, Indeed, if $K_4 = \{1, u, v, w\} < P$ is the Klein subgroup, then the Cayley map $M(K_4, \{u, v\}, (u, v))$ is isomorphic, as a map, to the Cayley map $\mathsf{Cay}(C_4, \{c, c^{-1}\}, (c, c^{-1}))$. Hence there should exists an automorphism $\alpha \in \mathsf{Aut}(H)$ which maps the first map onto the second one. Since both maps are connected, this would imply $\alpha(C_4) = K_4$, a contradiction.

Thus $P$ does not contain $K_4$. By Burnside's Theorem [3], $P$ is either cyclic or generalized quaternion. If $P$ is cyclic, then by Proposition 3.2 its order is bounded by 8.

Assume now that $P$ is a generalized quaternion group distinct from $Q_8$. Then $P$ contains a characteristic cyclic subgroup $C = \langle c \rangle$ of index 2. Then it follows from Lemma 2.1 and Proposition 3.2 that $|C| \leq 8$. Together with $P \not\cong Q_8$ we obtain that $|C| = 8$, and, consequently $|P| = 16$.

Let $a \in P$ denote an element of order 4 outside of $C$. Then $\langle a, c^2 \rangle \cong Q_8$. Let $\alpha$ be an automorphism of $\langle a, c^2 \rangle$ whose action is described by the formulas $\alpha(a) = c^2$ and $\alpha(c^2) = a^{-1}$. Its orbit $\{a, c^2, a^{-1}, c^{-2}\}$ is symmetric and generates $\langle a, c^2 \rangle$. Therefore $M = \mathsf{Cay}(\langle a, c^2 \rangle, \{a, c^2, a^{-1}, c^{-2}\}, \alpha)$ is a regular balanced Cayley map with $\mathsf{Aut}(M) = \langle a, c^2 \rangle \rtimes \alpha$. The element $\widehat{a}\alpha \in \langle a, c^2 \rangle \rtimes \alpha$ has order 8 and acts regularly on the point set $\langle a, c^2 \rangle$ of the map $M$. Therefore there exists a regular Cayley map $M'$ over the cyclic group of order 8 isomorphic to $M$. Thus $M \cong M' = \mathsf{Cay}(C, S, \rho)$ for some $S \subseteq C$ and an appropriate rotation $\rho$.

The generalized quaternion group of order 16 contains both $Q_8$ and $\mathbb{Z}_8$, therefore if $H = Q_{16}$ is a CIM-group, there exists $\beta \in \mathsf{Aut}(H)$ which maps $M$ on $M'$. But in this case $\langle a, c^2 \rangle \cong C$, a contradiction. ■

# 4 Proof of Theorem 1.1

We start with the following

**Lemma 4.1.** *Let $H$ be a group which admits a decomposition $H = CK$ such that $K \cap C = \{1\}$ and $K \triangleleft H$ and $C = \langle c \rangle$ is cyclic of odd order $m$. Assume that a there exists a faithful $C$-orbit $O = \{k, k^c, ..., k^{c^{m-1}}\}$ such that $\langle OO^{(-1)} \rangle = K$. Then $H$ is not a connected CIM-group.*

*Proof.* It is sufficient to provide an example of a connected non-CI map over $H$. Take $S := cO = \{ck_0, ck_1, ..., ck_{m-1}\}$ where $k_i := k^{c^i}, i = 0, ..., m-1$. Then $S^{(-1)} \cap S = \emptyset$ because the images of $S$ and $S^{(-1)}$ in $H/K \cong C$ are $c$ and $c^{-1}$, respectively.

Take a Cayley map $M = \mathsf{Cay}(H, S \cup S^{(-1)}, \rho)$ where

$$\rho = (ck_0, (ck_\ell)^{-1}, ck_1, (ck_{\ell+1})^{-1}, ...)$$

and $\ell = \frac{m+1}{2}$. Notice that the condition $\langle OO^{(-1)} \rangle = K$ implies that the map is connected.

It follows from the construction that $\rho^2 = \sigma|_{S\cup S^{(-1)}}$, where $\sigma$ is the inner automorphism of $H$ mapping $x$ to $x^c$. Therefore $\sigma \in \mathsf{Aut}(M)$ and $G := \widehat{H} \rtimes \langle\sigma\rangle \leq \mathsf{Aut}(M)$.

In order to build a regular subgroup of $\widehat{H} \rtimes \langle\sigma\rangle$ different from $\widehat{H}$ we notice, first, that this group is isomorphic to a direct product $H \times C$ where the isomorphism is defined via $\psi : \widehat{h}\sigma^i \mapsto (hc^i, c^{-i})$. Under this isomorphism the point stabilizer $G_1 = \langle\sigma\rangle$ is mapped onto the subgroup $\psi(G_1) = \{(d^{-1}, d) \,|\, d \in C\}$.

Let $\pi : H \to C$ be a projection homomorphism defined via $\pi(xk) := x$ for $x \in C$ and $k \in K$. Then $F := \{(h, \pi(h)) \,|\, h \in H\}$ is a subgroup of $H \times C$ which intersects $\psi(G_1)$ trivially. Indeed,

$$(h, \pi(h)) \in \psi(G_1) \iff \pi(h) = h^{-1} \implies h \in C \implies \pi(h) = h \implies h = h^{-1}.$$

By assumption $C$ has odd order. Therefore $h = 1$.

It follows from $F \cap \psi(G_1) = 1$ that $\psi^{-1}(F)$ is a regular subgroup of $G$. Thus $G$ contains two regular subgroups isomorphic to $H$, which are $\widehat{H}$ and $\psi^{-1}(F)$. Since $\widehat{H} \lhd G$, it is not conjugate to $\psi^{-1}(F)$ inside $G$.

Since $G_1$ has two orbits on the connection set $S \cup S^{-1}$, either $\mathsf{Aut}(M) = G$ or $[\mathsf{Aut}(M) : G] = 2$. In the first case we already have two $H$-regular subgroups of $G$ which are non-conjugate in $G$. In the second case it follows from $\rho(x^{-1}) = \rho(x)^{-1}$ that $M$ is a regular balanced map over $H$. It was proved in [15] that $\widehat{H} \trianglelefteq \mathsf{Aut}(M)$. Since $G$ contains a $H$-regular subgroup distinct from $\widehat{H}$, it is not conjugate to $\widehat{H}$ inside $\mathsf{Aut}(M)$. ■

**Remark.** The condition $\langle OO^{(-1)}\rangle = K$ is always fulfilled if $K$ does not contain a proper non-trivial $C$-normalized subgroups. For example, if $K$ is of prime order, then $\langle OO^{(-1)}\rangle = K$ holds for any non-trivial orbit $O$.

Now we are ready to prove Theorem 1.1.

*Proof.* Let $T$ denote a Sylow 2-subgroup of $H$. Our proof is divided into few steps.

**Step 1.** Any normal subgroup $N$ of $H$ of odd order is cyclic.
Since all Sylow subgroups of $N$ have prime order, it is sufficient to prove that any Sylow subgroup of $N$ is normal in $H$. This would follow if we prove that each Sylow subgroup of $N$ has a normal complement. To show that let us fix a Sylow subgroup $P$ of order $p$, where $p$ is prime. By Burnside Theorem the existence of a normal complement follows from $\mathbf{N}_N(P) = \mathbf{C}_N(P)$. Assume towards a contradiction that there exists $g \in \mathbf{N}_N(P)$ which does not centralize $P$. We may assume that $o(g)$ is a prime power. By Lemma 3.1 any Sylow subgroup of $N$ has a prime order. Therefore $o(g)$ is prime distinct from $p$. In this case the group $\langle g\rangle P$ satisfies the assumptions of Lemma 4.1 and therefore, is not a connected CIM group. A contradiction.

**Step 2.** $T$ has a normal complement.
By Proposition 3.3, $T$ is isomorphic to one of the groups $\mathbb{Z}_2^r, \mathbb{Z}_4, \mathbb{Z}_8$ or $Q_8$. If $T$ is cyclic, then the result follows from the Cayley normal 2-complement Theorem.

Assume now that $T$ is not cyclic, i.e. $T \cong \mathbb{Z}_2^r$ or $T \cong Q_8$. By Frobenius normal $p$-complement Theorem it is sufficient to show that $\mathbf{N}_H(T)/\mathbf{C}_H(T)$ is a 2-group. Notice that $\mathbf{N}_H(T)/\mathbf{C}_H(T)$ is embedded into $\mathsf{Aut}(T)$.

If $T \not\cong Q_8$, then $T \cong \mathbb{Z}_2^e$ for some $e \geq 1$. Assume, towards a contradiction, that $\mathbf{N}_H(T)/\mathbf{C}_H(T)$ is not a 2-group. Then there exists an element $g \in \mathbf{N}_H(T)$ of odd order which acts on $T$ nontrivially. Without loss of generality, we may assume that $o(g)$ is a $p$-power for some odd prime divisor $p$ of $|H|$. Since $|H|_p = p$, we conclude $o(g) = p$. Since $T$ is elementary abelian 2-group, it contains a minimal $g$-invariant subgroup $T_1$ on which $g$ acts non-trivially. The group $\langle g \rangle T_1$ satisfies the assumptions of Proposition 4.1. Therefore $\langle g \rangle T_1$ is not a connected CIM-group. A contradiction.

If $T \cong Q_8$ and $\mathbf{N}_H(T)/\mathbf{C}_H(T)$ is not a 2-group, then this group contains an element of order 3. Hence $\mathbf{N}_H(T)$ contains an element $g$ of order 3 which acts on $T$ non-trivially. Applying Lemma 4.1 once more we get a contradiction.

**Step 3.** If $T$ is non-cyclic, then $H \cong N \times T$.
As it was mentioned before, a CIM-group has the property that any two elements of the same order are either conjugate or inverse conjugate by an automorphism of $H$. In particular, this implies that all involtiuons of $H$ are $\mathsf{Aut}(H)$-conjugate.

If $T$ is non-cyclic, then either it is elementary abelian or $Q_8$. Let us assume first that $T$ is an elementary abelian 2-group of order at least 4. Then all non-trivial elements of $T$ are $\mathsf{Aut}(H)$-conjugate. Since $N$ is characteristic in $H$, the subgroups $\mathbf{C}_N(s)$ and $\mathbf{C}_N(t)$ are $\mathsf{Aut}(H)$-conjugate for any $s \neq t \in T \setminus \{1\}$. Since any subgroup of $N$ is characteristic in $H$, we conclude that $K := \mathbf{C}_N(s) = \mathbf{C}_N(t) = \mathbf{C}_N(ts)$. Let $L \leq N$ be a unique subgroup complementary to $K$ in $N$. Then both $t$ and $s$ invert the elements of $L$, Therefore $st$ acts trivially on $L$ implying $L \leq K$, and consequently $L = 1$. Thus any element of $T$ centralizes $N$. Therefore $H \cong N \times T$.

It remains to settle the case when $T \cong Q_8$. In this case all cyclic subgroups of order 4 are $\mathsf{Aut}(H)$-conjugate. Since $\mathsf{Aut}(N)$ is abelian the commutator subgroup $Z$ of $T$ acts trivially on $N$. The quotient group $\bar{H} = H/Z$ is isomorphic to $N \rtimes \mathbb{Z}_2^2$. Moreover all involutions of $\mathbb{Z}_2^2$ are $\mathsf{Aut}(\bar{H})$-conjugate. From the previous paragraph we obtain that $\mathbb{Z}_2^2$ acts trivially on $N$. So, the semi-direct product $N \rtimes \mathbb{Z}_2^2$ is, in fact, the direct one. Therefore $H \cong N \times T$. ■

# 5 Proof of Theorem 1.2

We start with introducing the notation $\mathscr{M}$ for the set of groups $\mathbb{Z}_n \times \mathbb{Z}_2^r, \mathbb{Z}_n \times \mathbb{Z}_4, \mathbb{Z}_n \times Q_8$ where $n$ is a square-free odd number. The statement below collects the properties of these groups. We omit the proof because it is straightforward.

**Proposition 5.1.** *The following properties hold*

- *(a) The subgroups and factor groups of $H \in \mathscr{M}$ belong to $\mathscr{M}$;*
- *(b) Any two subgroups $A, B \leq H \in \mathscr{M}$ of the same order are conjugate by an automorphism of $H$;*

*(c) Any subgroup automorphism $\beta \in \mathsf{Aut}(A), A \leq H$ may be extended to an automorphsim of $H$;*

*(d) The groups in $\mathscr{M}$ are Hamiltonian.*

Our first step provides a reduction of Theorem 1.2 to the connected case.

**Proposition 5.2.** *If the groups of $\mathscr{M}$ are connected CIM-group, then they are CIM groups.*

*Proof.* Let $M = \mathsf{Cay}(H, S, \rho)$ and $M' = \mathsf{Cay}(H, S', \rho')$ be two isomorphic map over a group $H \in \mathscr{M}$. Then $|\langle S\rangle| = |\langle S'\rangle|$, and by Proposition 5.1 there exists an automorphism $\alpha \in \mathsf{Aut}(H)$ such that $\alpha(\langle S'\rangle) = \langle S\rangle$. Thus replacing $M'$ by $\alpha(M')$ we may assume that $\langle S\rangle = \langle S'\rangle$. Since $M$ and $M'$ are isomorphic, their connected components $M_1 := \mathsf{Cay}(\langle S\rangle, S, \rho)$ and $M_1' := \mathsf{Cay}(\langle S\rangle, S', \rho')$ are isomorphic too. Both $M_1$ and $M_1'$ are connected maps over the group $\langle S\rangle \in \mathscr{M}$. Therefore there exists $\beta \in \mathsf{Aut}(\langle S\rangle)$ such that $\beta(M_1) = M_1'$. By Proposition 5.1 $\beta$ can be extended up to an automorphism of $H$, $\alpha$ say. Then $\alpha(M) = M'$ hereby proving the claim. ■

To prove Theorem 1.2 for connected maps we provide a little bit more general result.

**Theorem 5.3.** *Let $G \leq \mathsf{Sym}(\Omega)$ be a transitive permutation group with cyclic point stabilizer which contains a regular subgroup $H \in \mathscr{M}$. Then any $H$-regular subgroup of $G$ is conjugate to $H$ in $G$.*

If $H$ is abelian then by Ito's theorem [7] the group $G$ is metabelian and therefore $G$ is solvable. If $H$ is non-abelian, then it is nilpotent and $G$ is solvable by Kegel-Wielandt theorem.

We will prove Theorem 5.3 by induction on $|G|$ and assume that $G$ is a counterexample of a minimal order. In particular, this implies that the theorem is correct for any proper subgroup $X$ where $H \leq X < G$. Since $G$ is a counerexample, there exists an $H$-regular subgroup $F$ of $G$ which is not conjugate to $\widehat{H}$ inside $G$. We fix $F$ till the end of the proof. By the minimality of $G$, we may assume that $\langle \widehat{H}, F^g\rangle = G$ for each $g \in G$. We write the order of $H$ by $2^r n$. Recall that $n$ is an odd square-free number.

Below the following notiation is used. If $G$ is a group acting on a set $X$, then $G_X$ denote the kernel of this action and $G^X$ denote the image of $G$ in $\mathsf{Sym}(X)$.

**Proposition 5.4.** *Let $G$ be a minimal counterexample to Theorem 5.3 and $\mathcal{D}$ be a proper non-trivial imprimitivity system of $G$. Then $G^{\mathcal{D}} = H^{\mathcal{D}}$, or equivalently, $G = HG_{\mathcal{D}} \iff G_\omega \leq G_{\mathcal{D}}$.*

*Proof.* Notice that $\mathcal{D}$ is an imprimitivity system of $H$ too. Since $H$ is regular, the setwise stabilizer $H_{\{D\}}$ of a block $D \in \mathcal{D}$ acts regularly on $D$. Since the block stabilizers are conjugate in $H$ and $H$ is a Hamiltonian group, the subgroup $H_{\{D\}}$ does not depend on a choice of $D \in \mathcal{D}$. Therefore the subgroup $H_{\{D\}}, D \in \mathcal{D}$ coincides with $H_{\mathcal{D}}$ implying that $D$ is an orbit of $H_{\mathcal{D}}$. It follows from $H_{\mathcal{D}} \leq G_{\mathcal{D}}$

that $G_{\mathcal{D}}$ acts transitively on each block of $\mathcal{D}$. The group $H^{\mathcal{D}}$ is a regular subgroup of $G^{\mathcal{D}}$. Also $H^{\mathcal{D}} \cong H/H_{\mathcal{D}} \in \mathscr{M}$. The point stabilizer of $G^{\mathcal{D}}$ is isomorphic to $G_\omega G_{\mathcal{D}}/G_{\mathcal{D}} \cong G_\omega/(G_\omega \cap G_{\mathcal{D}})$, and, therefore, is cyclic. Thus $G^{\mathcal{D}} \leq \mathsf{Sym}(\mathcal{D})$ satisfies the assumptions of Theorem 5.3. Since $|G^{\mathcal{D}}| = |G|/|G_{\mathcal{D}}| < |G|$, we may apply the induction hypothesis to $G^{\mathcal{D}}$. It yields us that $F^{\mathcal{D}}$ and $H^{\mathcal{D}}$ are conjugate in $G^{\mathcal{D}}$. Therefore there exists $g \in G$ such that $(F^g)^{\mathcal{D}} = H^{\mathcal{D}}$ implying $G^{\mathcal{D}} = \langle F^g, H\rangle^{\mathcal{D}} = \langle (F^g)^{\mathcal{D}}, H^{\mathcal{D}}\rangle = H^{\mathcal{D}}$. ■

**Proposition 5.5.** *Let $G$ be a minimal counterexample to Theorem 5.3. Then $G$ admits at most one minimal imprimitivity system.*

*Proof.* Assume, towards a contradiction, that $G$ admits two minimal imprimitivity systems, say $\mathcal{D}$ and $\mathcal{E}$. By Proposition 5.4 $G_\omega \leq G_{\mathcal{D}}$ and $G_\omega \leq G_{\mathcal{E}}$. If follows from minimality of $\mathcal{E}$ and $\mathcal{D}$ that $G_{\mathcal{D}} \cap G_{\mathcal{E}} = \{1\}$. Therefore $G_\omega = \{1\}$ implying $G = H$ contrary to $G$ being a counterexample. ■

For a set of elements $S$ of a group acting on a set $X$, we denote by $\mathsf{Fix}(S)$, the elements of $X$ fixed by every $s \in S$.

**Proposition 5.6.** *Let $G \leq \mathsf{Sym}(\Omega)$ be a transitive permutation group with cyclic point stabilizer. Then for each $S \leq G_\omega$ the set $\mathsf{Fix}(S)$ is a block of $G$.*

*Proof.* Assume that $\mathsf{Fix}(S) \cap \mathsf{Fix}(S^g)$ is non-empty, and pick an arbitrary $\delta \in \mathsf{Fix}(S) \cap \mathsf{Fix}(S^g)$. Then $S, S^g \leq G_\delta$. Since $G_\delta$ is cyclic, any two subgroups of $G_\delta$ of the same order coincide. Therefore $S = S^g$ implying that $\mathsf{Fix}(S^g) = \mathsf{Fix}(S)$. ■

**Proof of Theorem 5.3**

Let $\mathcal{P}$ be a minimal imprimitivity system of $G$. Pick an arbitrary block $\Pi \in \mathcal{P}$. Then $G_{\{\Pi\}}^{\Pi}$ is a solvable primitive permutation subgroup of $\mathsf{Sym}(\Pi)$. Therefore $|\Pi|$ is a power of a prime divisor $p$ of $|H|$.

We split the proof into few steps.

**Step 1.** $|\mathcal{P}| > 1$.

Assume the contrary, that is $|\mathcal{P}| = 1$, or, equivalently, $\Pi = \Omega$. In this case $H$ is a $p$-group. By Proposition 5.6 the set $\mathsf{Fix}(G_{\alpha,\beta})$ is a block of $G$ for any pair of points $\alpha, \beta \in \Omega$. Together with primitivity of $G$ this implies that $G_{\alpha,\beta} = 1$ whenever $\alpha \neq \beta$. Therefore $G$ is a Frobenius group the kernel of which, $K$ say, has order $|H|$. Since $K$ is a unique Sylow $p$-subgroup of $G$, we conclude $H = K = F$, a contradiction.

**Step 2.** $G_{\mathcal{P}}$ is a $p$-group.

Assume that there exists a prime divisor $q \neq p$ of $G_{\mathcal{P}}$. Since $G_{\mathcal{P}}$ acts transitively on each block $\Pi \in \mathcal{P}$ and $G_\omega \leq G_{\mathcal{P}}$, we conclude that $|G_{\mathcal{P}}| = |G_\omega| \cdot |\Pi|$. This implies that $q$ divides $|G_\omega|$. Thus $G_\omega$ contains a subgroup $Q$ of order $q$. By Proposition 5.6 the set $\mathsf{Fix}(Q)$ is block of $G$. It follows from Proposition 5.4 that $Q \leq G_\omega \leq G_{\mathcal{P}}$ that $Q$ fixes each block of $\mathcal{P}$ setwise. Since blocks of $\mathcal{P}$ have a $p$-power size, the set $\mathsf{Fix}(Q)$ intersects each block of $\mathcal{P}$ non-trivially. By Proposition 5.5 $\mathcal{P}$ is a unique minimal imprimitivity system of $G$. Therefore each block of $G$ is a union of some blocks of $\mathcal{P}$. Thus $\mathsf{Fix}(Q) = \Omega$ implying that $Q = \{1\}$. A contradiction.

**Step 3.** $G_{p'} = H_{p'} = \mathbf{O}_{p'}(G) \neq \{1\}$.
By Step 2 $G_{\mathcal{P}}$ is a $p$-group. Therefore $|H|_{p'} = |H^{\mathcal{P}}|_{p'}$ and $|G|_{p'} = |G^{\mathcal{P}}|_{p'}$. By Proposition 5.4 $H^{\mathcal{P}} = G^{\mathcal{P}} \cong G/G_{\mathcal{P}}$. Therefore $|G_{p'}| = |H_{p'}|$ implying $G_{p'} = H_{p'}$. By Hall's Theorem there exists $g \in G$ such that $(F^g)_{p'} = (F_{p'})^g = H_{p'}$ implying that $F^g$ normalizes $H_{p'}$. Combining this with $G = \langle H, F^g\rangle$ we conclude that $H_{p'} \trianglelefteq G$. Together with $H_{p'} = G_{p'}$ we obtain that $H_{p'} = G_{p'} = \mathbf{O}_{p'}(G)$.

If $G_{p'}$ is trivial, then $G$ and $H$ are $p$-groups. Since $\mathcal{P}$ is non-trivial (by minimality) and $|\mathcal{P}| > 1$, we conclude that $|H| = |\Omega| \geq p^2$. Together with $H \in \mathscr{M}$ this implies that $p = 2$ and $H$ is one of the groups: $\mathbb{Z}_2^r, \mathbb{Z}_4, Q_8$. Since $G^{\Pi}_{\{\Pi\}}$ is a primitive 2-group, we conclude that $|\Pi| = 2$. Therefore $G_{\mathcal{P}}$ is an elementary abelian 2-group. By Proposition 5.4 $G_\omega \leq G_{\mathcal{P}}$. Therefore $|G_\omega| = 2$ and both $H$ and $F$ are index two subgroups of $G$. So both of them are normal in $G$ and $G \leq \mathbf{N}_{\mathsf{Sym}(\Omega)}(H)$. If $H$ is isomorphic to one of $\mathbb{Z}_2, \mathbb{Z}_4, Q_8$, then $H$ is a unique $H$-regular subgroup of $\mathbf{N}_{\mathsf{Sym}(\Omega)}(H)$, contrary to $F \leq G \leq \mathbf{N}_{\mathsf{Sym}(\Omega)}(H)$. Therefore $H \cong \mathbb{Z}_2^r, r \geq 2$.

It follows from $H \neq F$ that $G = HF$ and $H \cap F \leq \mathbf{Z}(G)$. It follows from $G = FH$ that a unique involution $s \in G_\omega$ has a presentation $s = h_0 f_0$ with $h_0 \in H$ and $f_0 \in F$. Notice that $h_0 \notin H \cap F$ and $f_0 \notin F \cap H$ (otherwise we would have $s \in (H \cup F) \setminus \{1\}$ which cannot happen because $(H \cup F) \setminus \{1\}$ contains only fixed-point-free permutations). Thus $G = HF = \langle f_0\rangle\langle h_0\rangle(H \cap F)$. It follows from $s^2 = 1$ that $[f_0, h_0] = 1$. Together with $H \cap F \leq \mathbf{Z}(G)$ we conclude that $G$ is an abelian group. Thus $G$ should be regular contrary to $|G_\omega| = 2$.

**Step 4. Getting the final contradiction.** It follows from Step 3 that $\mathbf{O}_{p'}(G)$ is nontrivial. Therefore the orbits of $\mathbf{O}_{p'}(G)$ form a non-trivial imprimitivity system of $G$ with block size coprime to $p$. Since $\mathcal{P}$ is a unique minimal imprimitivity system (Proposition 5.5), the orbits of $\mathbf{O}_{p'}(G)$ are unions of blocks of $\mathcal{P}$. But the this is impossible, since the cardinality of blocks of $\mathcal{P}$ is a $p$-power. ■

We finish this section by resolving the status of the cyclic group of order 8.

**Proposition 5.7.** *A cyclic group $\mathbb{Z}_8$ is a CIM-group.*

*Proof.* Assume towards a contradiction that $M := \mathsf{Cay}(\mathbb{Z}_8, S, \rho)$ is a non-CI map over $\mathbb{Z}_8$. Let $P$ be a Sylow 2-subgroup of $G := \mathsf{Aut}(M)$ which contains $\widehat{\mathbb{Z}_8}$. Then $P$ contains a regular cyclic subgroup which is not conjugate to $\widehat{\mathbb{Z}_8}$ inside $P$. In particular, $|P| \geq 16$. Therefore $|\mathbf{N}_P(\widehat{\mathbb{Z}_8})| \geq 16$. The point stabilizer $\mathbf{N}_P(\widehat{\mathbb{Z}_8})_0$ is cyclic and is contained in $\mathsf{Aut}(\mathbb{Z}_8)$. Therefore $|\mathbf{N}_P(\widehat{\mathbb{Z}_8})_0| = 2$, or, equivalently, $\mathbf{N}_P(\widehat{\mathbb{Z}_8})_0 = \langle\alpha\rangle$ for some $\alpha \in \mathsf{Aut}(\mathbb{Z}_8)$.

If $\widehat{\mathbb{Z}_8}$ is a unique regular cyclic subgroup of $\mathbf{N}_P(\widehat{\mathbb{Z}_8})$, then $\mathbf{N}_P(\mathbf{N}_P(\widehat{\mathbb{Z}_8}))$ normalizes $\widehat{Z_8}$. So, in this case $\mathbf{N}_P(\mathbf{N}_P(\widehat{\mathbb{Z}_8})) = \mathbf{N}_P(\widehat{\mathbb{Z}_8})$ implying $P = \mathbf{N}_P(\widehat{\mathbb{Z}_8})$, because in a $p$-group the normalizer of a proper subgroup is strictly bigger than the subgroup. The latter equality contradicts our assumption that $P$ contains non-conjugate regular cyclic subgroups. Thus $\mathbf{N}_P(\widehat{\mathbb{Z}_8}) = \widehat{\mathbb{Z}_8} \rtimes \langle\alpha\rangle$ contains non-conjugate regular cyclic subgroups. This yields a unique choice for $\alpha \in \mathsf{Aut}(H)$, namely: $\alpha(x) = 5x, x \in \mathbb{Z}_8$. Notice that $\widehat{\mathbb{Z}_8} \rtimes \langle\alpha\rangle$ contains exactly two regular cyclic subgroups $\widehat{\mathbb{Z}_8}$ and $\langle\widehat{1}\alpha\rangle$. Each of these subgroups is normal in $\widehat{\mathbb{Z}_8} \rtimes \langle\alpha\rangle$.

Since $\alpha \in G_0$, it acts semiregularly on $S$. Combining this with $\langle S\rangle = \mathbb{Z}_8$ and $S = -S$ we obtain that the only possibility for $S$ is $\{1, 5, 3, 7\}$. It follows from $\rho^2 = \alpha|_S$ that either $\rho = (1, 3, 5, 7)$ or $\rho = (1, 7, 5, 3)$. In both cases $M$ is an antibalanced map the full automorphism group of which has order 32 and has a decomposition $G = \widehat{\mathbb{Z}_8}\langle\rho\rangle$ where $\rho$ acts trivially on the subgroup $2\mathbb{Z}_8$. In both cases all regular cyclic subgroups are conjugate in $G$. ■

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