# How many varieties of cylindric algebras

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#### Abstract

Cylindric algebras, or concept algebras in another name, form an interface between algebra, geometry and logic; they were invented by Alfred Tarski around 1947. We prove that there are  $2^{\alpha}$  many varieties of geometric (i.e., representable)  $\alpha$ -dimensional cylindric algebras, this means that  $2^{\alpha}$  properties of definable relations of (possibly infinitary) models of first order logic theories can be expressed by formula schemes using  $\alpha$  variables, where  $\alpha$  is infinite. This solves Problem 4.2 in the 1985 Henkin-Monk-Tarski monograph [19], the problem is restated in [34, 4]. For solving this problem, we had to devise a new kind of construction, which we then use to solve Problem 2.13 of the 1971 Henkin-Monk-Tarski monograph [18] which concerns the structural description of geometric cylindric algebras. There are fewer varieties generated by locally finite dimensional cylindric algebras, and we get a characterization of these among all the  $2^{\alpha}$  varieties. As a by-product, we get a simple, natural recursive enumeration of all the equations true of geometric cylindric algebras, and this can serve as a solution to Problem 4.1 of the 1985 Henkin-Monk-Tarski monograph. All this has logical content and implications concerning ordinary first order logic with a countable number of variables.

## 1 Introduction

Cylindric algebras (or concept algebras in another name) are an algebraic form of first order logic, analogous to Boolean algebras which are an algebraic form of propositional logic. Cylindric algebras were created by Alfred Tarski around 1947, they are Boolean algebras endowed with complemented closure operators—one for each quantifier  $\exists v_i$ —and constants  $v_i = v_j$  for representing equality. Set cylindric algebras analogous to set Boolean algebras are called representable cylindric algebras, these latter are equationally definable (i.e., form a variety), this was proved by Tarski in 1952. However, unlike the propositional case, not all cylindric algebras are representable, and this cannot be repaired easily since J. Donald Monk proved in 1969 that the representable algebras are not finite schema axiomatizable. This gap between "abstract" and representable cylindric algebras is in the center of algebraic logic studies, it is a source of insights and problems.

A set cylindric algebra is the algebra of all definable relations of a model. Representable cylindric algebras are the same, just for classes of models, i.e., for theories, in place of single models. We often call them concept algebras, since they are natural algebras of concepts of a theory. (The name "cylindric algebras" on the other hand refers to the geometrical meaning of these algebras.) Properties of a model, or of a theory, can be read off its concept algebra. E.g., it can be expressed by an equation whether dense linear order can be defined in the model. Equations of concept algebras talk about properties of all definable relations in a model, as opposed to talking about the primitive relations only. An equation in the algebraic language corresponds to a formula schema of first order logic where the formula variables range over formulas with free variables of the language, this way they "talk about" definable relations of models. In this context, the problem asking about the number of varieties of representable cylindric algebras asks how many properties of definable relations we can express by first order logic schemata. In this connection, the present work is not unrelated to Shelah's classification theory. (For some connections, see [40, 7].)

Let  $\alpha$  denote the number of variables we have in the first order language the algebraic versions of which we investigate. In this paper we deal with infinite  $\alpha$  only, let RCA<sub> $\alpha$ </sub> denote the class of representable cylindric algebras of this language. Then we have  $\alpha$  many closure operations called cylindrifications, and  $\alpha \times \alpha$  many constants called diagonal constants apart from the Boolean operations, so our equational language has size  $\alpha$  which means that there are at most  $2^{\alpha}$  many different equational theories in the algebraic language. It was known that RCA<sub> $\alpha$ </sub> has at least continuum many subvarieties ([19, Thm.4.1.24]) and [19, Problem 4.2] asks whether there are  $2^{\alpha}$  many for uncountable  $\alpha$ . There are theorems pointing to the answer being continuum (i.e.,  $2^{\omega}$ , where  $\omega$  is the least infinite ordinal) and there are theorems pointing to the answer being  $2^{\alpha}$ . For example, it is proved in [34] that in concept algebras of finite models we can only express the size of models nothing else, in the algebraic language the theorem says that the concept algebras of models of size n generate a variety which is an atom of the lattice of varieties of  $\mathsf{RCA}_{\alpha}$ . On the other hand, we can say lots of things about definable relations in infinite models. E.g., the equational theory of the concept algebra of a single infinite model in which we have no primitive relations (i.e., in which all the definable relations are the ones we can define by using the equality only) is not recursively enumerable, see [34, Thm.1(i)]. We prove in this paper that there are  $2^{\alpha}$  many subvarieties of  $\mathsf{RCA}_{\alpha}$ , but we also show that this is only part of the answer since there is a large subclass of the distinguished representable cylindric algebras which has only continuum many subvarieties.

From the logical point of view, at first, the subject of the above discussed problem might look "esoteric", since it concerns the difference between using countably or uncountably infinite number of variables in the logical languages having only finite formulas. However, our solution has impact on the countable case, too. We mentioned already that the gap between abstract and representable cylindric algebras is a source of insights and problems. One of the insights has been that this gap is concerned with the number of "free" extra variables  $v_i$  we have for a formula or proof in the logical language. The crucial thing is not that each first order formula uses only finitely many variables, but instead that for each finite set of formulas there is always at least one variable that they do not use. Let us call a cylindric algebra dimension-complemented if to each finite subset there is a cylindrification all elements of this finite subset are fixed-points of; one can prove then that all dimension-complemented cylindric algebras are representable. The strongest and most beautiful form of this insight is Leon Henkin's theorem saying that an abstract cylindric algebra is representable if and only if it can be embedded to one having infinitely many more cylindric operations so that each of the elements of the original algebra are fixed-points of all the new operations. Efforts were made to find structural properties of abstract cylindric algebras, similar to being dimension-complemented mentioned above, which refer to only the original cylindric algebra without comparing it to others. Theorem 2.6.50 in [18] summarizes how far they got in this respect, and [18, Problem 2.13] asks if their last description (which we call here endodimension-complemented) captures being representable or not.

Our full answer to [19, Problem 4.2] shows that when proving that there are many varieties of  $\mathsf{RCA}_{\alpha}$ , we had to use unusual features of representable cylindric algebras, we had to devise a new kind of construction. This contributes to understanding the structural properties of representable cylindric algebras. Namely, our construction shows that "endo-dimension-complemented" is not the final answer in [18, Thm.2.6.50], since our new constructions are representable yet not endo-dimension-complemented, solving Problem 2.13 in the negative. However, here, too, something positive can be added as a result of this solution. A new property extending "endo-dimension-complemented" toward representability emerges by an analysis of our construction. We call an abstract cylindric algebra inductive iff from the fact that an equation holds for all elements which are fixed-points of a given cylindric operation, we can infer that this equation holds for the whole algebra, provided that the given cylindrification does not occur in the equation. In a way, this property also talks about extra free variables. We prove that each endodimension-complemented algebra is inductive and each inductive algebra is representable, but none of the reverse implications holds. This way we get a new representability theorem extending the chain of classes in [18, Thm.2.6.50], and we get a new insight about representable algebras.

A cylindric algebra is called locally finite dimensional if each of its elements is the fixed-point of all but finitely many cylindrifications,  $Lf_{\alpha}$  denotes their class. These algebras correspond to theories of first order logic in which all primitive relations have finite rank. The rest of the representable algebras correspond to theories of first order logic when the primitive relations may have arbitrary ranks, let us call this logic infinitary first order logic to distinguish it from the former which we call finitary or ordinary (following [19]). It was known that the infinitary and finitary first order logics have the same valid formulas, in algebraic form this theorem says that  $Lf_{\alpha}$  generates the variety  $\mathsf{RCA}_{\alpha}$ . However, it was not known whether the two logics have the same theories, i.e., whether all subvarieties of  $\mathsf{RCA}_{\alpha}$  are generated as varieties by their locally finite-dimensional members. In this paper we give a negative answer to this: there are subvarieties of  $\mathsf{RCA}_{\alpha}$  not generated by their locally finite-dimensional members, even in the  $\alpha = \omega$  case, so the theories of finitary and infinitary first order logics are different. The notion of inductivity is the key notion to finding this answer. The property of a cylindric algebra being inductive can easily be converted to a new logical rule, which we call inductive rule. We prove that this rule is admissible for ordinary first order logic but it is not admissible for infinitary first order logic. In particular, we prove that a theory is one of ordinary first order logic iff it is closed under the inductive rule.

The above theorem readily yields a simple, natural recursive enumeration for all equations valid in  $\mathsf{RCA}_{\alpha}$ . This gives a possible solution to [19, Problem 4.1], this problem is restated in numerous other places, e.g., in [4, Problem 19, p.735], or in logical form as [34, Problem 2.9].

The structure of the paper is the following. In the rest of the introduction we recall the definitions, notation, necessary background needed in the paper. Section 2 contains the proof that there are  $2^{\alpha}$  varieties of  $\mathsf{RCA}_{\alpha}$ , section 3 contains the proof of an analogous statement for abstract cylindric algebras. Section 4 concerns with subclasses where there are only continuum many varieties and with the results we can get from the proof contained in section 2. We introduce the notions of symmetric, endo-dimension-complemented (endo-dc in short), and inductive algebras. Section 4.1 investigates the relationship between symmetric and endo-dc algebras leading to the solution of [18, Problem 2.13]. Section 4.2 reveals connections with the polyadic substitution operations. Section 4.3 concerns inductive algebras. It places this notion between endo-dc and symmetric and shows representability of inductive algebras by proving that they are exactly the algebras equationally indistinguishable from a member of  $Lf_{\alpha}$ . As a corollary we get a  $\Delta_2$ -formula separating  $Lf_{\alpha}$  and  $\mathsf{RCA}_{\alpha}$ . Section 4.4 contains a simple recursive enumeration of the equational theory of  $\mathsf{RCA}_{\alpha}$  and the characterization of varieties generated by subclasses of  $Lf_{\alpha}$ . Finally, section 4.5 contains a proof that locally finite dimensional set algebras with infinite bases have indeed continuum many subvarieties.

#### 1.1 Background

Where not specified otherwise, we use the notation of [18, 19], but we try to be self-contained.

Let  $\alpha$  be any set. An algebra  $\langle A, +, -, c_i, d_{ij} : i, j \in \alpha \rangle$  is a *cylindric* algebra of dimension  $\alpha$  if  $\langle A, +, - \rangle$  is a Boolean algebra and for all distinct  $i, j, k \in \alpha$  the following hold. The operations  $c_i$  are unary, they are commuting complemented closure operators, i.e., for all  $x, y \in A$  we have  $c_i c_j x = c_j c_i x, x \leq c_i x = c_i c_i x, c_i (x + y) = c_i x + c_i y, c_i - c_i x = -c_i x$ . The operations  $d_{ij}$  are nullary, i.e., they are constants and they satisfy the following equations:  $d_{ii} = 1, d_{ij} = d_{ji}, c_j (d_{ij} \cdot d_{jk}) = d_{ik}, c_i d_{ij} = 1, d_{ij} \cdot c_i (d_{ij} \cdot x) = d_{ij} \cdot x$ . In the above,  $\cdot$  is the Boolean intersection defined from +, - the usual way. The extra-Boolean operations  $c_i$  and  $d_{ij}$  are called *cyindrifications* and *diagonal constants*, respectively. The schemata of equations  $(C_0) - (C_7)$  in [18, p.161] express the above.  $CA_{\alpha}$  denotes the class of all cylindric algebras.

The cylindric set algebras of dimension  $\alpha$  are Boolean set algebras of  $\alpha$ -dimensional spaces, where the extra-Boolean operations have natural geometric interpretations. Let U be any set, the points of the  $\alpha$ -dimensional space  $^{\alpha}U$  over U are the U-termed  $\alpha$ -sequences, i.e., the set of all functions

from  $\alpha$  to U. We use the terms sequence and function to mean the same thing. If s is any sequence, then  $\mathsf{Dom}(s), \mathsf{Rg}(s)$  denote its domain and range,  $s_i$  denotes s(i), and s(i/u) denotes the sequence we get from s by changing its value at i to be u if  $i \in \mathsf{Dom}(s)$  (and s(i/u) denotes s if  $i \notin \mathsf{Dom}(s)$ ). For  $V \subseteq {}^{\alpha}U$  the full cylindric set algebra with unit V is

$$\langle \mathcal{P}(V), \cup, -, C_i^V, D_{ij}^V : i, j \in \alpha \rangle$$
 where  $\mathcal{P}(V)$  is the powerset of V and  $C_i^V X := \{s(i/u) \in V : s \in X, u \in U\}, \quad D_{ij}^V := \{s \in V : s_i = s_j\}.$ 

Algebras isomorphic to subalgebras of full cylindric set algebras with units as disjoint unions of  $\alpha$ -dimensional spaces are called *geometric*, or *representable*,  $\mathsf{RCA}_{\alpha}$  denotes their class. (For technical reasons, when  $\alpha$  is a one-element set,  $\mathsf{RCA}_{\alpha}$  is defined as the class of subdirect products of these.) Cylindric set algebras with units of form  $^{\alpha}U$  are called simply *cylindric set algebras*, U is called their *base set*, and their class is denoted as  $\mathsf{Cs}_{\alpha}$ .

Geometric cylindric algebras have natural logical meaning, too. Let us have a first order equality language  $\mathcal{L}$  with variables as  $\alpha$  (or  $v_i$  with  $i \in \alpha$ ), and with the logical connectives  $\lor, \neg, \exists v_i, v_i = v_j$  for  $i \in \alpha$ , and let  $\mathfrak{M}$  be a relational structure with universe U and with at most  $\alpha$ -place primitive relations. The points of the  $\alpha$ -dimensional space over U are also evaluations of variables. For any formula  $\varphi$  in this language let  $\varphi^{\mathfrak{M}}$  denote the set of evaluations of variables under which  $\varphi$  is true in  $\mathfrak{M}$ . Then  $(\varphi \lor \psi)^{\mathfrak{M}} =$  $\varphi^{\mathfrak{M}} \cup \psi^{\mathfrak{M}}, (\neg \varphi)^{\mathfrak{M}} = {}^{\alpha}U - \varphi^{\mathfrak{M}}, (\exists v_i \varphi^{\mathfrak{M}}) = C_i \varphi^{\mathfrak{M}}$  and  $(v_i = v_j)^{\mathfrak{M}} = D_{ij}$ (with  $V = {}^{\alpha}U$ ). Thus

$$\mathrm{Ca}^{\mathfrak{M}} := \{\varphi^{\mathfrak{M}} : \varphi \in \mathcal{L}\}$$

is a subuniverse of the full cylindric set algebra with unit  $^{\alpha}U$ . We call the subalgebra with this universe the *concept algebra* of  $\mathfrak{M}$ , its universe is the set of all definable relations over  $\mathfrak{M}$ . Concept algebras for theories can be defined analogously, the unit of such a concept algebra is a disjoint union of sets of form  $^{\alpha}U$ , so it is in  $\mathsf{RCA}_{\alpha}$ . Two theories are definitionally equivalent iff their concept algebras are isomorphic, and homomorphisms from one concept algebra to another correspond exactly to the interpretations between the two theories. Thus, the category  $\mathsf{RCA}_{\alpha}$  with homomorphisms corresponds to the category of all theories and interpretations between them (see [19, sec.4.3]). This feature of concept algebras comes handy in applications of logic, e.g., in physics. For this kind of applications see, e.g., [5, 8, 26, 27, 28, 46]. Equations holding for concept algebras have several logical interpretations, see e.g., [19, sec.4.3]. The most direct of these is where we interpret the algebraic variables occurring in the equation as schemes for formulas in the corresponding logical form. This way we get the logic of formula schemata, for definitions see e.g., [34], [36, sec.3.7]. This schema logic talks directly about the definable relations in a model. Alternatively, we can interpret the algebraic variables as primitive  $\alpha$ -place relation symbols (full restricted first order logic in [19]), or as primitive relation symbols of unspecified but finite arity (type-free logic in [19]). Via these logical interpretations, results about varieties of RCA<sub> $\alpha$ </sub> imply corresponding results about these logics. We deal with the algebraic aspects in this paper, the logical consequences are dealt with in a separate paper.

A class of algebras that can be axiomatized/defined by a set of equations is called a *variety*. Let K be a class of algebras of the same similarity class. Then Eq(K) denotes the set of all equations (using a countable set of prespecified algebraic variables) valid in all algebras in K, and K is a variety iff K consists of all algebras in which Eq(K) is true. The variety generated by K is the least variety containing K, this is the class of all algebras in which Eq(K) holds. Varieties are one of the main subjects of universal algebra, Birkhoff's theorem says, e.g., that all members of the variety generated by K can be obtained from members of K as homomorphic images of subalgebras of products of members of K. The class of algebras isomorphic to members of K is denoted by **IK**, the class of subalgebras of members of K is denoted by **S**K.

The following are the main facts, in connection with the present paper, known about varieties of cylindric algebras, these are contained in [18, 19, 20] if not specified otherwise.  $\mathsf{RCA}_{\alpha} \subseteq \mathsf{CA}_{\alpha}$  is a variety, it is not finitely axiomatizable iff  $|\alpha| > 2$ , where if  $\alpha$  is any set,  $|\alpha|$  denotes its cardinality. J. D. Monk [30, 31] characterized the lattice of all subvarieties of  $\mathsf{RCA}_{\alpha}$  for  $|\alpha| = 1$ , and a similar characterization for  $|\alpha| = 2$  is contained in [9]. Let  $\alpha$ be infinite. Some of the distinguished subvarieties of  $\mathsf{RCA}_{\alpha}$  are  $\mathbf{I}_{\infty}\mathsf{Cs}_{\alpha}$  and  ${}_{n}\mathsf{RCA}_{\alpha}$  for finite n.  $\mathbf{I}_{\infty}\mathsf{Cs}_{\alpha}$  is the class of all algebras isomorphic to a cylindric set algebra with unit of form  ${}^{\alpha}U$  for infinite U, and  ${}_{n}\mathsf{RCA}_{\alpha}$  is the class of all algebras isomorphic to cylindric set algebras with unit as disjoint union of sets of form  ${}^{\alpha}U$  where |U| = n. It is proved in [34] that  ${}_{n}\mathsf{RCA}_{\alpha}$  are atoms in the lattice of subvarieties of  $\mathsf{RCA}_{\alpha}$ , but  $\mathbf{I}_{\infty}\mathsf{Cs}_{\alpha}$  is not an atom. The structure of subvarieties is interesting and is investigated for other kinds of algebras related to logic as well, see e.g., [10, 11, 25, 16, 24, 3].

Monographs and books on these algebras and their logical applications

include [17, 18, 12, 20, 19, 4, 29, 36, 23, 14, 32, 2, 15].

## 2 Number of subvarieties of $\mathsf{RCA}_{\alpha}$

Let  $\alpha$  be any infinite ordinal, throughout the rest of the paper. (We assume that  $\alpha$  is an ordinal, and not just any set, for convenience, this way  $\alpha$  is ordered by the elementhood relation.) It is proved in [19, 4.1.24] that there are at least  $2^{\omega}$  many subvarieties of  $\mathsf{RCA}_{\alpha}$ . Since in the language of  $\mathsf{CA}_{\alpha}$ there are  $|\alpha|$  many equations, there can be at most  $2^{|\alpha|}$  many subvarieties of any  $\mathsf{K} \subseteq \mathsf{CA}_{\alpha}$ . Problem 4.2 in [19] asks if there are  $2^{|\alpha|}$  many subvarieties of  $\mathsf{RCA}_{\alpha}$  and of  $\mathsf{ICs}_{\alpha} \subseteq \mathsf{RCA}_{\alpha}$ , if  $\alpha$  is uncountable. The problem is restated in [4, Problem 41, p.738]. In this section, we prove that indeed there are maximum number of subvarieties of  $\mathsf{RCA}_{\alpha}$  as well as of  $\mathbf{I}_{\infty}\mathsf{Cs}_{\alpha} \subseteq \mathsf{ICs}_{\alpha}$ . Note that both  $\mathsf{RCA}_{\alpha}$  and  $\mathbf{I}_{\infty}\mathsf{Cs}_{\alpha}$  are varieties but  $\mathsf{ICs}_{\alpha}$  is not.

**Theorem 2.1** (Solution of [19, Problem 4.2]) Let  $\alpha$  be infinite. There are  $2^{|\alpha|}$  many subvarieties of RCA<sub> $\alpha$ </sub> as well as of  $\mathbf{I}_{\infty} \mathbf{Cs}_{\alpha}$ .

The proof of Theorem 2.1 is contained in subsections 2.1-2.4. The idea of the proof is the following. We exhibit a set of equations E of cardinality  $|\alpha|$ which is independent in the sense that no element of E follows from the rest of the equations in E. All these equations will be variants of a single equation e such that we rename the indices of the operations occurring in e. We will show independence of E by constructing one algebra  $\mathfrak{A} \in \mathbf{I}_{\infty} Cs_{\alpha}$  in which e fails but the rest of the equations in E hold. Then the algebras in which we rename the operations  $c_i$  and  $d_{ij}$  according to appropriate permutations of  $\alpha$  will show independence of the whole set E. This will show that all the subvarieties of  $\mathbf{I}_{\infty} Cs_{\alpha}$  specified by the  $2^{|\alpha|}$  many subsets of E are distinct. We begin with constructing the "witness" algebra  $\mathfrak{A}$ , because it will give intuition for writing up the "master" equation e.

#### 2.1 Construction of the witness algebra $\mathfrak{A}$

Let  $\langle V_i : i \in \alpha \rangle$  be a system of sets such that

 $V_0 = V_1 = V_2$  is the set of rational numbers,

and all the other  $V_i$ 's are pairwise disjoint two-element sets disjoint from  $V_0$ , too (i.e.,  $V_i \cap V_j = \emptyset$  for  $2 \le i < j < \alpha$ ). Let U be the union of these sets, i.e.,

$$U := \bigcup \{ V_i : i \in \alpha \}$$

let p be an  $\alpha$ -sequence such that  $p_i \in V_i$  for all  $i \in \alpha$  and let V be the set of U-termed  $\alpha$ -sequences that deviate from p only at finitely many places, i.e.,

$$V := {}^{\alpha}U^{(p)} = \{ s \in {}^{\alpha}U : |\{i \in \alpha : s_i \neq p_i\}| < \omega \}.$$

This V will be the unit of our algebra. Since  $V_0$  is the set of rational numbers, we will use the usual operations and ordering < between rational numbers. Our algebra is generated by a single element, namely by

 $g := \{ s \in V : s_0 < s_1 < s_2 \text{ and } s_i \in V_i \text{ for all } i \in \alpha \text{ and} \\ s_1 = (s_0 + s_2)/2 \quad \text{if } |\{s_i : s_i \neq p_i, i > 2\}| \text{ is even}, \\ s_1 \neq (s_0 + s_2)/2 \quad \text{if } |\{s_i : s_i \neq p_i, i > 2\}| \text{ is odd } \}.$ 

Let  $\mathfrak{A}$  denote the subalgebra of the full set algebra with unit V that is generated by g.

Sets of form  ${}^{\alpha}U^{(p)}$  for some set U and  $p \in {}^{\alpha}U$  are called *weak spaces* and algebras with unit a weak space are called *weak set algebras*, their class is denoted by  $Ws_{\alpha}$ . It is proved in [19, 3.1.102] that  $Ws_{\alpha} \subseteq ICs_{\alpha}$ , thus our above constructed algebra  $\mathfrak{A}$  is in  $I_{\infty}Cs_{\alpha}$  since the fact that U is infinite is reflected by an equation holding in  $\mathfrak{A}$  (see, [18, 2.4.61]). (We note that we could have used for our witness algebra the subalgebra of the full cylindric set algebra with unit  ${}^{\alpha}U$  generated by g, the proofs would be only slightly more complicated.) The set U is called the *base set* of  $\mathfrak{A}$ .

## 2.2 Describing the elements of $\mathfrak{A}$

The idea behind the construction of the witness algebra  $\mathfrak{A}$ , defined in the previous section, is that we put some information in g at the indices 0, 1, 2 which information cannot be transferred by the cylindric operations to higher indices  $i, j, k \in \alpha$ . Then we express lack of this information by an equation e. If we succeed with realizing this idea, then e would fail in  $\mathfrak{A}$  at 0, 1, 2 while e would be valid in  $\mathfrak{A}$  at higher indices. We now turn to elaborating this idea.

Let  $R \subseteq {}^{n}U$  be an *n*-place relation on U, where *n* is any ordinal. We say that X is a *sensitive cut* of R if  $c_i X = c_i (R - X)$  for all i < n. (Here,  $c_i$ 

denotes cylindrification of the full  $Cs_n$  with base set U, i.e.,  $c_i X = \{s(i/u) \in {}^{n}U : s \in X\}$ .) Thus, as soon as we apply a cylindrification to X, the information on how X cuts R into two parts is lost. This technique is widely applicable, cf., e.g., [1, 41, 42, 39]. We are going to show that our generator g is a sensitive cut of

$$T := \{ s \in V : s_0 < s_1 < s_2 \text{ and } s_i \in V_i \text{ for all } i \in \alpha \}.$$

Intuitively, the proof will be a kind of "flip-flop" play between the two independent conditions  $s_1 = (s_0 + s_2)/2$  and  $|\{s_i : s_i \neq p_i, i > 2\}|$  being even in the definition of g. Indeed, let  $i \in \alpha$  and  $s \in T$ . We show that  $s(i/u) \in g$  while  $s(i/v) \in T - g$  for some  $u, v \in U$ . This will show that  $c_ig = c_i(T - g)$ . By  $s \in T$  we have  $s_0 < s_1 < s_2$ . Let  $\Sigma = |\{s_i : s_i \neq p_i, i > 2\}|, u = (s_0 + s_2)/2$ , and let v be such that  $v \neq u$ ,  $s_0 < v < s_2$ . Assume first that i = 1. Now, if  $\Sigma$  is even then  $s(1/u) \in g$ ,  $s(1/v) \in T - g$  and if  $\Sigma$  is odd then  $s(1/u) \in T - g$ ,  $s(1/v) \in g$ . For i = 0 choose  $u = 2s_1 - s_2$  and v < u, for i = 2 choose  $u = 2s_1 - s_0$  and v > u, for these choices the same is true as in the case of i = 1. Assume i > 2 and let v be the element of  $V_i$  distinct from  $s_i$ . Assume  $s_1 = (s_0 + s_2)/2$ . Then  $s \in g$ ,  $s(i/v) \in T - g$  if  $\Sigma$  is even, and otherwise  $s \in T - g$ ,  $s(i/v) \in g$ . Assume  $s_1 \neq (s_0 + s_2)/2$ . Then just the other way round, namely  $s \in T - g$ ,  $s(i/v) \in g$  if  $\Sigma$  is even, and otherwise  $s \in g$ ,  $s(i/v) \in T - g$ . We have shown that

(1) 
$$c_i g = c_i (T - g) = c_i T$$
 for all  $i \in \alpha$ .

Next we show that this last property implies that the elements of A are those that are generated by T and perhaps one of g, T - g added:

**Lemma 2.1** Let  $\mathfrak{B}$  be the weak set algebra of dimension  $\alpha$  with unit V and generated by T. Then

$$A = \{x + h : x \in B \text{ and } h \in \{0, g, T - g\}\}.$$

**Proof.** Each element of the form x + h is generated by g, since  $T = c_0 g \cdot c_2 g$ . To finish the proof, we are going to show that A is closed under the cylindric operations  $c_i$  and  $d_{ij}$  as well as under the Boolean operations +, -. Let  $i, j \in \alpha$ . Now, A is closed under  $c_i$  by (1) since  $c_i(x + h) = c_i x + c_i h \in B$  by  $x \in B$  and  $c_i h \in \{0, c_i T\} \subseteq B$ . Also,  $d_{ij} \in A$  by  $d_{ij} \in B$ . The set A is closed under Boolean addition + by its definition and by  $g + (T - g) = T \in B$ .

To see that A is closed under Boolean complementation, first we show that T is an atom in B. We will use the following property of < later on, too: (\*) Assume that  $a_1 < a_2 < \cdots < a_n$  and  $b_1 < b_2 < \cdots < b_n$ . There is an automorphism  $\pi$  of  $\langle V_0, < \rangle$  mapping  $a_1, \ldots, a_n$  to  $b_1, \ldots, b_n$ , respectively. If  $a_1 = b_1$  and  $a_n = b_n$  then  $\pi$  can be chosen such that it is the identity on elements smaller than  $a_1$  or bigger than  $a_n$ .

To prove (\*), for a < b let  $[a, b] = \{x \in V_0 : a \le x \le b\}$  denote the closed interval between a and b. Let  $u, v \in V_0$  be such that  $u < a_0, u < b_0$  and  $v > a_n, v > b_n$  and define  $a_0 := b_0 := u$  and  $a_{n+1} := b_{n+1} := v$ . For  $k \le n$ and  $x \in [a_k, a_{k+1}]$  let  $\pi_k(x) := (x - a_k) \cdot (b_{k+1} - b_k)/(a_{k+1} - a_k) + b_k$ . Then  $\pi_k$ is an isomorphism between  $\langle [a_k, a_{k+1}], < \rangle$  and  $\langle [b_k, b_{k+1}], < \rangle$ , thus their union  $\sigma := \pi_0 \cup \cdots \cup \pi_n$  is an automorphism of  $\langle [a_0, a_{n+1}], < \rangle$  taking  $a_0, \ldots, a_{n+1}$ to  $b_0, \ldots, b_{n+1}$ , respectively. We now can choose  $\pi$  to be the identity outside  $[a_0, a_{n+1}]$  and  $\sigma$  on the interval. This proves (\*).

Returning to showing that T is an atom, let  $s, z \in T$  be arbitrary. Then  $s_0 < s_1 < s_2$  and  $z_0 < z_1 < z_2$  and  $s_i, z_i \in V_i$  for all  $i \in \alpha$  by the definition of T. Let  $\pi$  be a permutation of U which takes  $s_i$  to  $z_i$  for all  $i \in \alpha$ , is a permutation of  $V_i$  for all  $i \in \alpha$ , and is an automorphism of  $\langle V_0, < \rangle$ . By (\*), there is such a  $\pi$ . Then clearly,  $\pi$  takes s to z in the sense that  $z = \pi(s) := \langle \pi(s_k) : k \in \alpha \rangle$  while leaving T as well as V fixed, i.e.,  $T = \pi(T) := \{\pi(q) : q \in T\}$  and  $V = \pi(V) := \{\pi(q) : q \in V\}$ . This implies that  $\pi(x) = x$  for all  $x \in B$  (see, e.g., [19, 3.1.36]), so  $s \in x$  implies  $z \in x$  for all  $x \in B$ . Since  $s, z \in T$  were chosen arbitrarily, this shows that T is an atom in B. We are ready to show that A is closed under complementation. Let  $x \in B$  and  $h \in \{0, g, T - g\}$ . If T is disjoint from x then  $V - (x + h) = (V - x) + (T - h) \in A$ , and if  $T \leq x$  then x = x + h and we are done with proving Lemma 2.1.

### **2.3** The set *E* of independent equations

The "master equation" e will express about an element that it is not similar to our generator g. Namely, it will say about an element x that either it is not a sensitive cut of its closure  $c_0x \cdot c_2x$  (this is T in the case of g), or else this closure is not like T in the sense that the first two coordinates of  $c_2x$  form a strict linear order  $<_x$  and the ternary beginning of the closure is  $\{\langle u, v, w \rangle : u <_x v <_x w\}$ . An equation can talk about finitely many indices only, our equation will concern the first three indices 0, 1, 2.

We begin writing up the equation e. First we write up a term we will use in checking that x is not a sensitive cut of its closure  $z := c_0 x \cdot c_2 x$  (precise statements about the meanings of the terms below can be found in the proof of Lemma 2.2 that is to be stated soon).

$$\beta(x) := c_0 x \oplus c_0(z - x) + c_1 x \oplus c_1(z - x) + c_2 x \oplus c_2(z - x),$$

where  $\oplus$  denotes Boolean symmetric difference, i.e.,  $x \oplus y := (x \cdot y) + (-x \cdot -y)$ . In writing up the rest of the terms, it will be convenient to use the following notation. It concerns rearranging sequences in set algebras, see (4)-(7) somewhat later.

$$s_{j}^{i}x := c_{i}(d_{ij} \cdot x) \text{ for } i \neq j,$$
  

$$s_{12}^{01}x := s_{1}^{0}s_{2}^{1}x,$$
  

$$s_{10}^{01}x := {}_{2}s(0, 1)c_{2}x = s_{0}^{2}s_{1}^{0}s_{2}^{1}c_{2}x,$$
  

$$s_{01}^{12}x := s_{1}^{2}s_{0}^{1}x.$$

Next we write up the terms we use in expressing that it is not the case that the binary relations at places 01 and 12 of x coincide:

$$\gamma(x) := c_2 x \oplus s_{01}^{12} c_0 x.$$

Finally, the terms for expressing that  $c_2 x$  is not a strict linear order:

$\iota(x) := c_2 x \cdot d_{01}$	not irreflexive,
$\sigma(x) := c_2 x \cdot s_{10}^{01} c_2 x$	not antisymmetric,
$\tau(x) := c_2 x \cdot s_{12}^{01} c_2 x - s_2^1 c_2 x$	not transitive,
$\lambda(x) := c_1 c_2 x \cdot c_0 c_2 x \cdot -c_2 x \cdot -s_{10}^{01} c_2 x$	not linear,
$\phi(x) := \iota(x) + \sigma(x) + \tau(x) + \lambda(x).$	

Let

(2) 
$$e(x) := x \le c_{(3)}(\beta(x) + \gamma(x) + \phi(x)),$$

where  $c_{(3)}y := c_0c_1c_2y$ .

We now turn to stating precisely what the equation e(x) expresses in a setalgebra about an element x. We will use the following notation extensively. Let U be a set,  $s \in {}^{\alpha}U$ , let  $n \in \omega$ , let  $H \in {}^{n}\alpha$  be repetition-free, and let  $q \in {}^{n}U$ . Then s(H/q) denotes the sequence we get from s by changing  $s(H_k)$  to  $q_k$ , simultaneously, for all k < n. We will write finite sequences  $\langle i_0, i_1, ..., i_n \rangle$  in the simplified form  $i_0i_1...i_n$  when this is not likely to lead to confusion. E.g., s(12/uv) = s(1/u)(2/v). Further, if  $x \subseteq {}^{\alpha}U$ , then x[s, H] denotes the *n*-place relation defined as

$$x[s,H] := \{q \in {}^{n}U : s(H/q) \in x\}.$$

For example,  $x[s, 01] = \{uv : s(01/uv) \in x\}$ , and  $x[s, 0] = \{u : s(0/u) \in x\}$ .

**Lemma 2.2** Let  $\mathfrak{C}$  be any  $\alpha$ -dimensional set algebra with unit a disjoint union of weak spaces. Then the equation e as defined in (2) is true in  $\mathfrak{C}$  at  $x \in C$  iff for all  $s \in x$  it is true that either x[s,012] is not a sensitive cut of  $Z := (c_0x \cdot c_2x)[s,012]$ , or  $<_x:= c_2x[s,01]$  is not a strict linear order on  $W := c_1c_2x[s,0]$ , or else  $Z \neq \{uvw \in {}^{3}W : u <_x v <_x w\}$ .

**Proof.** Let  $z := c_0 x \cdot c_2 x$  and  $s \in x$ .

(3) 
$$s \notin c_{(3)}\beta(x)$$
 iff  $x[s, 012]$  is a sensitive cut of  $z[s, 012]$ .

Indeed,

$$s \in -c_{(3)}(c_0 x \oplus c_0(z-x))$$
 iff

$$s(012/uvw) \notin c_0 x \oplus c_0(z-x)$$
 for all  $u, v, w$  iff

$$s(012/uvw) \in (c_0x \cdot c_0(z-x) + (-c_0x \cdot -c_0(z-x)))$$
 for all  $u, v, w$  iff

$$(c_0 x)[s, 012] = (c_0(z-x))[s, 012]$$
 iff

$$c_0(x[s, 012]) = c_0(z[s, 012] - x[s, 012]).$$

In the last step we used  $(c_0x)[s,012] = c_0(x[s,012])$  and (z-x)[s,012] = z[s,012] - x[s,012] which statements are easy to verify by using the definitions. All the above hold also for  $c_1$  and  $c_2$  in place of  $c_0$ , so we get (3).

For dealing with the rest of the terms, we make some preparations. We will check the following: Assume  $i \neq j$ .

(4)  $s \in \mathbf{s}_j^i x$  iff  $s(i/s_j) \in x$ ,

(5) 
$$s \in \mathbf{s}_{01}^{12} x$$
 iff  $s(12/s_0 s_1) \in x$ ,

(6) 
$$s \in \mathbf{s}_{10}^{01} c_2 x$$
 iff  $s(01/s_1 s_0) \in c_2 x$ ,

(7)  $s \in \mathbf{s}_{12}^{01} x$  iff  $s(01/s_1s_2) \in x$ .

Indeed, (4) is true because

$$s \in \mathbf{s}_{j}^{i} x = c_{i}(d_{ij} \cdot x)$$
 iff

$$s(i/u) \in d_{ij} \cdot x$$
 for some  $u$  iff  
 $u = s_j$  and  $s(i/u) \in x$  for some  $u$  iff  
 $s(i/s_j) \in x$ .

(5) is true because

$$s \in s_{01}^{12} x = s_1^2 s_0^1 x$$
 iff, by (4)  

$$s(2/s_1) \in s_0^1 x$$
 iff, by (4)  

$$(s(2/s_1))(1/s_0) \in x$$
 iff  

$$s(12/s_0 s_1) \in x.$$

Checking (6):

$$s \in \mathbf{s}_{10}^{01}c_{2}x = \mathbf{s}_{0}^{2}\mathbf{s}_{1}^{0}\mathbf{s}_{2}^{1}c_{2}x \qquad \text{iff, by (4)} \\ s(2/s_{0}) \in \mathbf{s}_{1}^{0}\mathbf{s}_{2}^{1}c_{2}x \qquad \text{iff, by (4)} \\ (s(2/s_{0}))(0/s_{1}) \in \mathbf{s}_{2}^{1}c_{2}x \qquad \text{iff, by (4)} \\ ((s(2/s_{0}))(0/s_{1}))(1/s_{0}) \in c_{2}x \qquad \text{iff} \\ s(012/s_{1}s_{0}s_{0}) \in c_{2}x \qquad \text{iff} \\ s(01/s_{1}s_{0}) \in c_{2}x. \end{aligned}$$

Checking (7):

$$s \in \mathbf{s}_{12}^{01} x = \mathbf{s}_1^0 \mathbf{s}_2^1 x \qquad \text{iff, by (4)} \\ s(0/s_1) \in \mathbf{s}_2^1 x \qquad \text{iff, by (4)} \\ (s(0/s_1))(1/s_2) \in x \qquad \text{iff} \\ s(01/s_1s_2) \in x.$$

We are ready to continue with the terms occurring in e.

(8)  $s \notin c_{(3)}\gamma(x)$  iff  $c_2x[s,01] = c_0x[s,12].$ 

Indeed,

$$s \in -c_{(3)}\gamma(x) = -c_{(3)}(c_2x \oplus \mathsf{s}_{01}^{12}c_0x) \qquad \text{iff} \\ s(012/uvw) \notin c_2x \oplus \mathsf{s}_{01}^{12}c_0x \quad \text{for all } u, v, w \qquad \text{iff} \\ s(012/uvw) \in (c_2x \cdot \mathsf{s}_{01}^{12}c_0x + (-c_2x \cdot -\mathsf{s}_{01}^{12}c_0x) \quad \text{for all } u, v, w \qquad \text{iff} \\ s(012/uvw) \in c_2x \quad \text{iff } s(012/uvw) \in \mathsf{s}_{01}^{12}c_0x \quad \text{for all } u, v, w \qquad \text{iff, by (5)} \\ uv \in c_2x[s, 01] \quad \text{iff } s(012/wuv) \in c_0x \quad \text{for all } u, v, w \qquad \text{iff} \\ uv \in c_2x[s, 01] \quad \text{iff } uv \in c_0x[s, 12], \quad \text{for all } u, v \qquad \text{iff} \\ c_2x[s, 01] = c_0x[s, 12]. \end{cases}$$

By this, (8) has been proved. We note that  $c_2x[s,01] = c_0x[s,12]$  implies that  $\mathsf{Rg}(c_2x[s,01]) = \mathsf{Dom}(c_2x[s,01])$ . Indeed,

$u \in Rg(c_2 x[s, 01])$		iff
$vu \in c_2 x[s, 01]$	for some $v$	iff
$s(012/vuw) \in x$	for some $v, w$	iff
$s(012/vuw) \in c_0 x$	for some $v, w$	iff
$uw \in c_0 x[s, 12]$	for some $w$	iff
$u \in Dom(c_0 x[s, 12])$		iff, by $c_0 x[s, 12] = c_2 x[s, 01]$
$u \in Dom(c_2 x[s, 01]).$		

We say that a binary relation R is *linear on* W iff W is both the domain and range of R and  $\langle u, v \rangle \in R$  or  $\langle v, u \rangle \in R$  for all  $u, v \in W$ . We have seen that  $s \in -c_{(3)}\gamma(x)$  implies that the domain of  $c_2x[s,01]$  coincides with its range. Assume for (9) below that the domain and range of  $c_2x[s,01]$  coincide. We can do this because we will use (9) only when  $s \notin c_{(3)}\gamma(x)$ .

(9)  $s \notin c_{(3)} \phi(x)$  iff  $c_2 x[s, 01]$  is a strict linear order on  $c_1 c_2 x[s, 0]$ .

Indeed,

$s \in -c_{(3)}\iota(x) = -c_{(3)}(c_2x \cdot d_{01})$	iff
$s(012/uvw) \notin (c_2 x \cdot d_{01})$ for all $u, v, w$	iff
$s(012/uvw) \in c_2 x$ implies $u \neq v$ for all $u, v, w$	iff
$u \neq v$ for all $uvw \in c_2 x[s, 012]$	iff
$u \neq v$ for all $uv \in c_2 x[s, 01]$	iff
$c_2 x[s, 01]$ is irreflexive.	

$$s \in -c_{(3)}\sigma(x) = -c_{(3)}(c_2x \cdot \mathbf{s}_{10}^{01}c_2x)$$
 iff

$$s(012/uvw) \notin (c_2x \cdot \mathbf{s}_{10}^{01}c_2x) \text{ for all } u, v, w \qquad \text{iff} \\ s(012/uvw) \in c_2x \text{ implies } s(012/uvw) \notin \mathbf{s}_{10}^{01}c_2x \qquad \text{iff, by (6)} \\ uv \in c_2x[s, 01] \text{ implies } vu \notin c_2x[s, 01] \qquad \text{iff} \\ c_2x[s, 01] \text{ is antisymmetric.}$$

$$s \in -c_{(3)}\tau(x) = -c_{(3)}(c_2x \cdot \mathbf{s}_{12}^{01}c_2x - \mathbf{s}_2^{1}c_2x) \qquad \text{iff} \\ s(012/uvw) \notin (c_2x \cdot \mathbf{s}_{12}^{01}c_2x - \mathbf{s}_2^{1}c_2x) \quad \text{for all } u, v, w \qquad \text{iff} \\ s(012/uvw) \in c_2x \cdot \mathbf{s}_{12}^{01}c_2x \text{ implies } s(012/uvw) \in \mathbf{s}_2^{1}c_2x \qquad \text{iff, by } (7), (4) \\ uv, vw \in c_2x[s, 01] \text{ implies } uw \in c_2x[s, 01] \qquad \text{iff} \\ c_2x[s, 01] \text{ is transitive.} \end{cases}$$

$$s \in -c_{(3)}\lambda(x) = -c_{(3)}(c_1c_2x \cdot c_0c_2x \cdot -c_2x \cdot -\mathbf{s}_{10}^{01}c_2x)$$
 iff  

$$s(012/uvw) \notin (c_1c_2x \cdot c_0c_2x \cdot -c_2x \cdot -\mathbf{s}_{10}^{01}c_2x) \text{ for all } u, v, w$$
 iff  

$$s(012/uvw) \in c_1c_2x \cdot c_0c_2x \Rightarrow s(012/uvw) \in c_2x + \mathbf{s}_{10}^{01}c_2x$$
 iff, by (6)  

$$u \in \mathsf{Dom}(c_2x[s, 01]), v \in \mathsf{Rg}(c_2x[s, 01]) \Rightarrow uv \in c_2x[s, 01] \text{ or } vu \in c_2x[s, 01]$$
 iff  

$$c_2x[s, 01] \text{ is linear on its field if } \mathsf{Dom}(c_2x[s, 01]) = \mathsf{Rg}(c_2x[s, 01]) .$$

By the above, (9) has been proved. Let  $x \in C$ . By (2), then  $\mathfrak{C} \models e(x)$ iff for all  $s \in x$  we have  $s \in c_{(3)}(\beta(x) + \gamma(x) + \phi(x))$ . Now,  $s \in c_{(3)}(\beta(x) + \gamma(x) + \phi(x))$  iff  $s \in c_{(3)}\beta(x)$  or  $s \in c_{(3)}\gamma(x)$  or we have  $s \in c_{(3)}\phi(x)$  when  $s \notin c_{(3)}\gamma(x)$ . By (9), (8), (3), then  $s \in c_{(3)}(\beta(x) + \gamma(x) + \phi(x))$  iff it is not the case that  $\langle x = c_2x[s, 01]$  is a strict linear order,  $Z = c_0x \cdot c_2x[s, 012] = \{\langle u, v, w \rangle : u <_x v <_x w\}$  and x[s, 012] is a sensitive cut of Z[s, 012].  $\Box$ 

#### 2.4 Checking the equations in the witness algebra

**Lemma 2.3** The equation e fails in  $\mathfrak{A}$ .

**Proof.** We show that e fails in  $\mathfrak{A}$  at g. Let  $s \in g$  be arbitrary such that s agrees with p on all indices i > 2. There is such a sequence. Then,  $\langle g = c_2g[s, 01] = c_2T[s, 01] = \{\langle u, v \rangle : u, v \in V_0, u < v\}$  is a strict linear order on  $c_1c_2g[s, 0] = c_1c_2T[s, 0] = V_0$ . Also,  $Z = (c_0g \cdot c_2g)[s, 012] = T[s, 012] = \{\langle u, v, w \rangle \in {}^{3}V_0 : u < v < w\}$ . Finally,  $g[s, 012] = \{\langle u, (u + v)/2, v \rangle : u < v, u, v \in V_0\}$  is a sensitive cut of Z. Then Lemma 2.2 implies that e fails in  $\mathfrak{A}$  at g.

Let  $i, j, k \in \alpha - \{0, 1, 2\}$  be distinct and let  $e_{ijk}$  denote the equation we get from e by replacing the indices 0, 1, 2 everywhere with i, j, k respectively. We are going to show that  $e_{ijk}$  holds in  $\mathfrak{A}$ . Lemma 2.2 is true with systematically replacing the indices 0, 1, 2 by i, j, k. Thus, to show that  $e_{ijk}$  holds in  $\mathfrak{A}$ , we have to show that x[s, ijk] is not a sensitive cut of a ternary relation built up from a linear order  $<_x$  in the way T is built up from <, for all  $x \in A$  and  $s \in x$ . In proving this, we will use the following lemma, which says that certain permutations of U leave the relations x[s, ijk], that determine the validity of  $e_{ijk}$ , fixed. We agree on some terminology first.

Let  $\pi$  be a permutation of U, and let n be an ordinal. Then  $\pi(s) := \langle \pi(s_i) : i < n \rangle$  if  $s \in {}^nU$ , and  $\pi(R) := \{\pi(s) : s \in R\}$  for  $R \subseteq {}^nU$ . We say that  $\pi$  leaves R fixed iff  $\pi(R) = R$ . In the present section we shall often use a certain property of permutations of U, so we give it a temporary name.

**Definition 2.1** We say that  $\pi$  is good iff it satisfies (i)-(iii) below.

- (i)  $\pi$  leaves < fixed, i.e., u < v iff  $\pi(u) < \pi(v)$  for all  $u, v \in V_0$ ,
- (ii)  $\pi$  leaves all the  $V_ms$  fixed, i.e.,  $\pi(V_m) = V_m$  for all  $m \in \omega$ ,
- (iii)  $\pi$  is the identity on all but a finite number of  $V_m s$ , i.e.,  $\{m \in \alpha : \forall u \in V_m(\pi(u) = u)\}$  is a co-finite subset of  $\alpha$ .

**Lemma 2.4** For any  $x \in A$  and  $s \in V$  there is a finite  $S \subseteq U$  such that any good permutation of U which is identity on S leaves x[s, ijk] fixed.

**Proof.** Let  $x \in A$ . Then x = y + h for some y generated by T and for some  $h \in \{0, g, T - g\}$ , by Lemma 2.1. Assume that  $y = \xi(T)$  for a term  $\xi$ . Let  $\Delta \subseteq \alpha$  be finite such that it contains all the indices occurring in  $\xi$  as well as 0, 1, 2. We show that for all s, s'

(10) 
$$s \in T \leftrightarrow s' \in T, \ s \upharpoonright \Delta = s' \upharpoonright \Delta \quad \text{imply} \quad s \in y \leftrightarrow s' \in y.$$

We prove (10) by induction on elements z generated from T by the use of indices from  $\Delta$ . Clearly, (10) holds for T and  $d_{mn}$  for  $m, n \in \Delta$ . Assume that (10) holds for z, z'. Then clearly it holds for -z and  $z \cdot z'$ . Let  $m \in \Delta$  and assume that s, s' satisfy the conditions. Now,

 $\begin{array}{ll} s \in c_m z & \text{iff, by the definition of } c_m \\ s(m/u) \in z \text{ for some } u & \text{iff, by the induction hyp., see details below} \\ s'(m/u) \in z \text{ for the same } u & \text{iff, by the definition of } c_m \\ s' \in c_m z. \end{array}$ 

Above, in the step from the second to third line we used that s'(m/u) agrees with s(m/u) on  $\Delta$  and  $s(m/u) \in T$  iff  $s'(m/u) \in T$  (by  $s \in T$  iff  $s' \in T$ , the definition of T, and  $0, 1, 2 \in \Delta$ ). By this, (10) has been proved.

Let now  $s \in V$  and

$$S := \{s_m : m \in \Delta\} \cup V_i \cup V_j \cup V_k.$$

Then  $S \subseteq U$  is finite. Let  $\pi$  be a good permutation of U which is the identity on S, we want to show that  $\pi$  leaves x[s, ijk] fixed. Recall that x = y + h where  $h \in \{0, g, T - g\}$ . Then x[s, ijk] = y[s, ijk] + h[s, ijk]. Thus  $\pi(x[s, ijk]) = \pi(y[s, ijk]) + \pi(h[s, ijk])$ , so it is enough to show that  $\pi$  leaves both y[s, ijk] and h[s, ijk] fixed. We begin with the second. Indeed,  $h[s, ijk] \subseteq V_i \times V_j \times V_k$  when  $h \in \{0, g, T - g\}$ , thus  $\pi(uvw) = uvw$  for all  $uvw \in h[s, ijk]$  by  $V_i \cup V_j \cup V_k \subseteq S$  and  $\pi$  being the identity on S.

We turn to y[s, ijk]. First we note that  $\pi$  being good implies that  $\pi(V) = V$  by Def.2.1(ii), and then  $\pi(T) = T$  by Def.2.1(i),(ii). Since y is generated by T we then have (by, e.g., [19, 3.1.36])

(11) 
$$\pi(y) = y$$

We want to show that

(12) 
$$uvw \in y[s, ijk]$$
 iff  $\pi(uvw) \in y[s, ijk].$ 

Indeed,

$$\begin{array}{ll} uvw \in y[s,ijk] & \text{iff, by the definition of } y[s,ijk] \\ s(ijk/uvw) \in y & \text{iff, by (11)} \\ \pi(s(ijk/uvw)) \in y & \text{iff, by (10) and see below} \\ s(ijk/\pi(uvw)) \in y & \text{iff, by the definition of } y[s,ijk] \\ \pi(uvw) \in y[s,ijk]. \end{array}$$

In the argument from the third to fourth line we used that  $\pi(s(ijk/uvw))$  and  $s(ijk/\pi(uvw))$  agree on i, j, k by their definitions, they agree on  $\Delta - \{i, j, k\}$  by  $\pi$  being the identity on  $S \supseteq \{s_m : m \in \Delta\}$ ; further, one of them is in T iff the other is:

$$\pi(s(ijk/uvw)) \in T \qquad \text{iff, by } \pi(T) = T$$
  

$$s(ijk/uvw) \in T \qquad \text{iff, by Def.2.1(ii)}$$
  

$$s(ijk/\pi(uvw)) \in T.$$

This proves (12), and Lemma 2.4 has been proved.

**Lemma 2.5** The equation  $e_{ijk}$  is valid in  $\mathfrak{A}$  when  $i, j, k \in \alpha - \{0, 1, 2\}$  are distinct.

**Proof.** For checking validity of  $e_{ijk}$ , we will use Lemma 2.2 (with 0, 1, 2 systematically replaced by i, j, k). Let  $x \in A$ ,  $s \in x$  and R := x[s, ijk],  $W := c_j c_k x[s, i]$ ,  $Z := (c_i x \cdot c_k x)[s, ijk]$ . Assume that  $\langle x := c_k x[s, ij]$  is a strict linear order on W and  $Z = \{\langle u, v, w \rangle \in {}^{3}W : u <_{x} v <_{x} w\}$ . We have to show that R is not a sensitive cut of Z. Let  $S \subseteq U$  be such that

(13)  $\pi(R) = R$  for all good permutations  $\pi$  of U that are identity on S.

There is such an S by Lemma 2.4. We note that

(14) 
$$\pi(R) = R$$
 implies  $\pi(<_x) = <_x$  and  $\pi(W) = W$ 

This is true because  $\langle x = c_2 R$  by their definitions:  $\langle x = c_k x[s, ij] = \{uv : s(ij/uv) \in c_k x\} = c_2 \{uvw : s(ijk/uvw) \in x\} = c_2 R$ . Below, we will use (\*) from the proof of Lemma 2.1 several times.

We turn to showing that R is not a sensitive cut of Z. By  $s \in x$  we have that  $s_i s_j \in c_k x[s, ij]$  so  $<_x$  is nonzero. Since by our assumption  $<_x$  is a strict linear order on W, it does not have a maximal element (by  $W = \text{Dom}(<_x) =$  $\text{Rg}(<_x)$ ), so W is infinite by  $W \neq \emptyset$ . Assume  $m \ge 3$  is such that S is disjoint from  $V_m$ . We show that  $V_m$  is disjoint from W. Assume  $W \cap V_m \neq \emptyset$ . Let  $\pi$ be the permutation of U that interchanges the elements of  $V_m$  and it leaves all the other elements of U fixed. (Recall that the  $V_m$ 's for  $m \ge 3$  have two elements.) Then  $\pi$  is good and it is identity on S, so it leaves W as well as  $<_x$  fixed, by (13),(14). This implies that  $V_m \subseteq W$  by  $V_m \cap W \neq \emptyset$ , so by  $<_x$ being linear on W, we have  $a <_x b$  for some distinct  $a, b \in V_m$ . By  $\pi$  leaving  $<_x$  fixed, then we have  $b <_x a$  (by  $\pi(a) = b, \pi(b) = a$ ). This contradicts  $<_x$ being antisymmetric.

Thus W intersects only finitely many of the  $V_m$ s. Then  $W \cap V_0$  is infinite because all the  $V_m$ s disjoint from  $V_0$  are finite. Let

$$K := W \cap V_0 \cap S$$
 and  $W' := (W \cap V_0) - S$ .

Thus K is finite and W' is infinite. Therefore, there are distinct  $u, v \in W'$  such that no element of K lies in between u, v according to <. (Indeed, let

 $K = \{k_1, \ldots, k_n\}$  with  $k_1 < \cdots < k_n$ . Then there are at least two elements of W' that lie in the same interval determined by the  $k_m$ s, and they will do.) We may assume u < v. Since  $<_x$  is linear on W and  $u, v \in W$ , we have either  $u <_x v$  or  $v <_x u$ . We assume  $u <_x v$ , the case  $v <_x u$  will be completely analogous, see (16). So assume

$$[u, v] \cap K = \emptyset, \quad u < v, \quad \text{and} \quad u <_x v$$

and we are going to show that for all  $w \in U$  we have

(15) 
$$u <_x w <_x v \quad \text{iff} \quad u < w < v.$$

Indeed, to prove (15), let first u < w < v be arbitrary, we want to show  $u <_x w <_x v$ . Let u' < u be such that there is no element of K between u'and u, there is such an u' because < is dense and K is finite. Then there is no element of K between u' and v. Take a  $\pi$  as in (\*) for u' < u < v and u' < w < v and extend it to U by being the identity on  $U - V_0$ . Then this  $\pi$ is identity on S and it is good. So it leaves  $<_x$  fixed, by (13),(14). By  $u <_x v$ then we have  $w = \pi(u) <_x \pi(v) = v$ . By a similar argument we get  $u <_x w$ . (Indeed, choose v' > v such that there is no element of K between v and v' and apply (\*) with u < v < v' and u < w < v'.) We have seen  $u <_x w <_x v$ . To prove the other direction, assume that  $w \in U$  and it is not the case that u < w < v. Thus either w < u < v or u < v < w. In either cases, there is a good permutation  $\pi$  of U which is identity on S, leaves w fixed and takes u to v, and there is also a good permutation  $\pi$  of U which is identity on S, leaves w fixed and takes v to u. (Indeed, take u' < u and v < v' such that no element of  $K \cup \{w\}$  lies between u' and v', and then apply (\*).) Hence  $w <_x u$  iff  $w <_x v$  and  $u <_x w$  iff  $v <_x w$ . Hence, it is not the case that  $u <_x w <_x v$ , as it was desired. The equation (15) has been proved.

Assume now the other case, i.e., that

$$[u, v] \cap K = \emptyset, \quad u < v, \quad \text{and} \quad v <_x u.$$

We are going to show that for all  $w \in U$  we have

(16) 
$$v <_x w <_x u \quad \text{iff} \quad u < w < v.$$

To prove (16), let first u < w < v be arbitrary, we want to show  $u <_x w <_x v$ . Let u' < u be such that there is no element of K between u' and u. Take a  $\pi$  as in (\*) for u' < u < v and u' < w < v and extend it to U by being the identity on  $U - V_0$ . Then this  $\pi$  is identity on S and it is good. So it leaves  $<_x$  fixed, by (13),(14). By  $v <_x u$  then we have  $v = \pi(v) <_x \pi(u) = w$ . We get  $w <_x u$  by choosing v' > v such that there is no element of K between v and v' and applying (\*) with u < v < v' and u < w < v'. We have seen  $v <_x w <_x u$ . The proof of the other direction is the same as in the proof for (15). The equation (16) has been proved.

We are ready to prove that R is not a sensitive cut of Z. Assume that  $Z \subseteq c_1R$ , we will show that  $Z \not\subseteq c_1(Z - R)$ . By (15),(16) and < being dense we have that  $uwv \in Z$  for some w (and the u, v chosen as before), so  $uw'v \in R$  for some w' by  $Z \subseteq c_1R$ . Let u < w'' < v be arbitrary and take a good permutation  $\pi$  of U that takes w' to w'' and leaves everything outside the open interval (u, v) fixed. There is such a  $\pi$  by (\*). This  $\pi$  leaves S fixed since no element of S lies between u and v (according to <). Then it leaves R fixed by (13). So  $uw''v \in R$  by  $uw'v \in R$  and  $\pi(uv'v) = uv''v$ . By (15),(16) this means that R(uw'v) for all w' such that  $uw'v \in Z$ . Hence  $uwv \in Z$  is such that  $uwv \notin c_1(Z - R)$ , and we are done with proving Lemma 2.5.

We now round up the proof of Theorem 2.1. Let  $I \subseteq \alpha \times \alpha \times \alpha$  be such that  $|I| = |\alpha|$  and for all distinct  $ijk, lmn \in I$  we have  $\{i, j, k\} \cap \{l, m, n\} = \emptyset$ ,  $|\{i, j, k\}| = 3$  and  $012 \in I$ . There is such an I since  $\alpha$  is infinite. Let

$$E := \{e_{ijk} : ijk \in I\}.$$

Then  $|E| = |\alpha|$ . For all  $ijk \in I$  let  $\mathfrak{A}_{ijk}$  denote the algebra we get from  $\mathfrak{A}$  by interchanging (renaming) the operations  $c_m, d_{mn}$  for  $m, n \in \{0, 1, 2\}$  with those for  $m, n \in \{i, j, k\}$ , respectively. This algebra is denoted by  $\mathfrak{RO}^{\rho}\mathfrak{A}$  where  $\rho$  is the permutation of  $\alpha$  which interchanges i, j, k with 0, 1, 2 respectively, see [18, Def.2.6.1]. By Lemmas 2.3, 2.5 then we have

(17) 
$$\mathfrak{A}_{ijk} \not\models e_{ijk}$$
 while  $\mathfrak{A}_{ijk} \models e_{klm}$  for all  $klm \in I, klm \neq ijk$ .

Also,  $\mathfrak{A}_{ijk} \in \mathbf{IWs}_{\alpha}$  by [19, 3.1.119] and, so  $\mathfrak{A}_{ijk} \in \mathbf{I}_{\infty}\mathsf{Cs}_{\alpha}$  by [19, 3.1.102] and because the fact that  $\mathfrak{A}$  has an infinite base is reflected on its equational theory. (We note that  $\mathfrak{A}_{ijk}$  is isomorphic to the algebra we get from  $\mathfrak{A}$  by replacing in its construction 0, 1, 2 with i, j, k systematically, and changing nothing else. An isomorphism showing this takes  $x \in \mathfrak{A}_{ijk}$  to  $\{\rho(s) : s \in x\}$ .)

For  $G \subseteq E$  define

$$\mathsf{V}_G := \{\mathfrak{B} \in \mathbf{I}_{\infty}\mathsf{Cs}_{\alpha} : \mathfrak{B} \models G\}.$$

Then  $V_G$  is a subvariety of  $\mathbf{I}_{\infty} \mathsf{Cs}_{\alpha}$ . Assume  $G, H \subseteq E$  are distinct. Then there is  $ijk \in I$  distinguishing them, we may assume  $ijk \in G$  and  $ijk \notin H$ . By (17) we have that  $\mathfrak{A}_{ijk} \notin V_G$  but  $\mathfrak{A}_{ijk} \in V_H$ , so  $V_G \neq V_H$ . Therefore, there are  $2^{|\alpha|}$  distinct subvarieties of  $\mathbf{I}_{\infty} \mathsf{Cs}_{\alpha}$ . The same is true for  $\mathsf{RCA}_{\alpha}$  by  $\mathbf{I}_{\infty} \mathsf{Cs}_{\alpha} \subseteq \mathsf{RCA}_{\alpha}$ . Theorem 2.1 has been proved.

We close this section with discussing some properties necessary for our construction  $\mathfrak{A}$  to work.

**Remark 2.1** (i) It is necessary that the base of  $\mathfrak{A}$  (i.e., the set U) be at least of cardinality  $|\alpha|$ . This is true because algebras of smaller base are diagonal (roughly: each of their elements intersects many diagonal elements, for precise definition see [18, p.416]), and we will prove that all diagonal algebras are symmetric, see Theorem 4.1. Clearly,  $\mathfrak{A}$  has to be non-symmetric to play its role in the proof.

(ii) In the equation e it was necessary to code a property that can occur on an infinite set only, this is the role of using the ordering on rational numbers  $V_0 = V_1 = V_2$  in the definition of the generator element g. In more detail: an equation e(x) using indices from  $\{i, j, k\}$  can talk in a set algebra about the ternary relation x[s, ijk] only. However, all ternary relations on a finite set occur as x[s, ijk] in a set algebra when the base set is infinite. (This is the main idea used in [34].) Since we want  $e_{012}(x)$  to hold and  $e_{ijk}(x)$  to fail in our witness algebra, e(x) has to code a property of x[s, ijk] which can be realized only on infinite sets.

(iii) The equation e fails in  $\mathfrak{A}$  at g, but e is true in  $\mathfrak{A}$  for all elements that are closed to at least one cylindrification  $c_i$ . Indeed, we can see that e holds for  $c_i a \in A$  as follows. Lemma 2.1 and (1) together with  $T \in B$ imply that  $c_i a \in B$ . The proof of Lemma 2.4 works for  $x \in B$  and ijk =012, and then the proof of Lemma 2.5 works to show that e holds for  $c_i a$ . Thus, in  $\mathfrak{A}$  cylindrification-closed elements satisfy more equations than all the elements. This behavior of our witness algebra  $\mathfrak{A}$  is necessary, because each algebra in which no such behavior occurs is symmetric (see Theorem 4.5(i) in subsection 4.3).

## 3 Subvarieties of $CA_{\alpha}$ containing $RCA_{\alpha}$

This section contains an unpublished theorem from [33]. The proof is analogous to the proof of Theorem 2.1. **Theorem 3.1** There are  $2^{|\alpha|}$  distinct subvarieties of  $\mathsf{CA}_{\alpha}$  all containing  $\mathsf{RCA}_{\alpha}$ .

**Proof.** We are going to exhibit an equation e valid in  $\mathsf{RCA}_{\alpha}$  and an algebra  $\mathfrak{A} \in \mathsf{CA}_{\alpha}$  such that  $\mathfrak{A} \not\models e$  while  $\mathfrak{A} \models e_{i1}$  for appropriate versions  $e_{i1}$  of e. This e is Henkin's equation  $e_{ij}(x, y)$  with ij taken as 01:

(18) 
$$e_{ij}(x,y) := c_j(x \cdot y \cdot c_i(x-y)) \le c_i(c_j x - d_{ij}),$$

see [19, 3.2.65]. For a simplified version of this equation see [44, chap.3.5], and for a drawing see [35, p.551]. Henkin's equation expresses that if the domains of R and S coincide and this common domain is a singleton, then R and S are disjoint iff their ranges are disjoint. Now,  $\mathsf{RCA}_{\alpha} \models e_{ij}(x, y)$ , by e.g. [19, 3.2.65].

We now turn to constructing our "witness" algebra  $\mathfrak{A}$ . It is obtained from a representable algebra  $\mathfrak{B}$  in which we split an atom whose domain is a singleton into two parts both having the same domain and range as the original atom. Henkin's equation then will fail for the split elements. In some sense this will be a "nonrepresentable counterpart" of the construction we used in the proof of Theorem 2.1.

Let  $\langle V_i : i \in \alpha \rangle$  be a system of disjoint sets such that  $V_0$  is a singleton, and  $V_i$  for  $i \ge 1$  have more than one elements. Let U be the union of these sets, let  $V := {}^{\alpha}U$  and let g be the direct product of the  $V_i$ , i.e.,

$$g := \prod \langle V_i : i \in \alpha \rangle := \{ s \in V : s_i \in V_i \text{ for all } i \in \alpha \}.$$

Let  $\mathfrak{B}$  denote the cylindric set algebra with base set U and generated by g. In  $\mathfrak{B}$ , the element g is an atom, this can be seen by using permutations of U exactly as in the proof of Lemma 2.1. Now, g is below all the diversity elements  $-d_{ij}$ ,  $i < j < \alpha$ , so we can split it into two disjoint parts g', g'' obtaining the algebra  $\mathfrak{A} \in \mathsf{CA}_{\alpha}$  defined as follows. Let  $\langle A, +, - \rangle$  be the Boolean algebra which contains  $\langle B, +, - \rangle$  as a subalgebra, in which g' and g'' are disjoint nonzero elements such that g = g' + g'' and which is generated by  $B \cup \{g'\}$ . Then the elements of A are

$$A = \{b + h : b \in B \text{ and } h \in \{0, g', g''\}\}.$$

The cylindric operations are defined in  $\mathfrak{A}$  so that  $\mathfrak{B}$  is a subalgebra of  $\mathfrak{A}$  and

$$c_i(b+g') := c_i(b+g'') := c_i(b+g) \quad \text{for all } b \in B.$$

Now,  $\mathfrak{A} \in \mathsf{CA}_{\alpha}$  can be checked directly by checking that the cylindric equations  $(C_0) - (C_7)$  of [18, 1.1.1] hold in  $\mathfrak{A}$ , or by checking that  $\mathfrak{A}$  is the algebra we get from  $\mathfrak{B}$  by splitting g in it by the method in [18, 2.6.12].

We show that  $\mathfrak{A} \not\models e_{01}(g,g')$ . Indeed,  $c_1(g \cdot g' \cdot c_0(g-g')) = c_1(g' \cdot c_0(g'')) = c_1(g' \cdot c_0g) = c_1g$  while  $c_0(c_1g - d_{01}) = c_0(V_0 \times (U - V_0) \times V_2 \times \dots)$  which does not contain  $c_1g = V_0 \times U \times V_2 \times \dots$ 

Next we show that  $\mathfrak{A} \models e_{ij}(x, y)$  when  $i \neq 0$ , i.e., we show

$$\mathfrak{A}\models c_i(x\cdot y\cdot c_j(x-y))\leq c_i(c_jx-d_{ij}).$$

Let  $x, y \in B$  be arbitrary. Then  $x \cdot y, x - y$  are of form a + h, b + k with  $a, b \in B, h, k \in \{0, g', g''\}$  and a, b, g pairwise disjoint as well as h, k disjoint, by our construction of  $\mathfrak{A}$ . Since negation – occurs in the equation only in form of  $-d_{ij}$ , the terms at the two sides of the equation are additive, and since  $a, b \in B$  and  $\mathfrak{B} \subseteq \mathfrak{A}$  is representable, we have that the equation is true for a, b. So, if both h and k are 0, then we are done. Assume therefore that  $h + k \neq 0$ . Then we get a bigger term on the lhs of the inequality if we replace h, k with g, g respectively. We get then  $c_i(x \cdot y \cdot c_j(x - y)) = c_i((a + g) \cdot c_j(b + g)) = c_i(a \cdot c_j(b + g)) + c_i(g \cdot c_j(b + g))$ . On the other side of the inequality we have  $c_i(c_jx - d_{ij}) = c_i(c_j(a + b + h + k) - d_{ij}) = c_i(c_j(a + b + g) - d_{ij})$ . (We used  $c_j(h + k) = c_jg$  in the last step.) This is now an equation concerning the representable algebra  $\mathfrak{B}$  since all the elements occurring are in  $\mathfrak{B}$ . Now,  $c_i(a \cdot c_j(b + g)) \leq c_i(c_j(a + b + g) - d_{ij})$  since this is an instance of Henkin's equation by a and b + g being disjoint. We only have to show  $c_i(g \cdot c_j(b + g)) \leq c_i(c_j(a + b + g) - d_{ij}$ .

(19) 
$$c_i(g) \le c_i(c_jg - d_{ij})$$

holds because  $V_i$  has at least two elements:  $c_i(c_jg - d_{ij}) = c_i\{s \in V : (\forall k \neq j)s_k \in V_k \text{ and } s_j \neq s_i\} = c_ig$ . Then

$c_i(g \cdot c_j(b+g))$	=	by $g$ being an atom
$c_i(g)$	$\leq$	by (19)
$c_i(c_jg - d_{ij})$	$\leq$	by monotony of the terms involved
$c_i(c_j(a+b+g)-d_{ij}).$		

To finish the proof of Theorem 3.1, let  $E := \{e_{i1} : i \in \alpha, i \neq 1\}$  and for all  $H \subseteq E$  let  $V_H$  be the subvariety of  $CA_{\alpha}$  axiomatized by H. Then  $V_H \supseteq RCA_{\alpha}$  for all  $H \subseteq E$ , by  $RCA_{\alpha} \models E$ . Also,  $V_H \neq V_G$  for distinct  $H, G \subseteq E$  since if,

say,  $e_{i1} \in G - H$  then  $\mathfrak{Ro}^{\rho}\mathfrak{A} \in (\mathsf{V}_H - \mathsf{V}_G)$  whenever  $\rho$  is a permutation of  $\alpha$  with  $\rho(i) = 0$ .

## 4 Counterpoint: classes with only continuum many varieties

Let us call a cylindric algebra  $\mathfrak{A}$  symmetric iff  $\mathfrak{A} \models e$  implies  $\mathfrak{A} \models \rho(e)$  for all permutations  $\rho$  of  $\alpha$ , where  $\rho(e)$  denotes the equation we get from e by systematically replacing each index  $i \in \alpha$  in it with  $\rho(i) \in \alpha$ . The proofs of the previous theorems were based on the existence of non-symmetric algebras. We will show that, surprisingly, many  $\mathsf{CA}_{\alpha}$ s, almost all in some sense, are symmetric. In particular, all dimension-complemented, all diagonal cylindric algebras, and more generally, all the algebras occurring in [18, Thm.2.6.50(i)-(iii)] are symmetric. Clearly, symmetric algebras can generate at most continuum many varieties since their equational theories are determined by equations written in the first  $\omega$  indices. In section 4.5 we show that this maximal possible number  $2^{\omega}$  is indeed achieved using only a small subclass of symmetric  $\mathsf{CA}_{\alpha}$ s: locally finite dimensional regular cylindric set algebras with infinite bases generate indeed continuum many varieties, for all infinite  $\alpha$ .

Thus,  $\mathsf{RCA}_{\alpha}$  has  $2^{\alpha}$  subvarieties, but locally finite dimensional ones generate only  $2^{\omega}$  many. What is the property that the  $\mathsf{Lf}_{\alpha}$ -generated varieties have but not all of the subvarieties have? Clearly, being symmetric is such a distinguishing property. (We call a variety symmetric iff it is generated by symmetric algebras.) However, being symmetric does not characterize the  $\mathsf{Lf}_{\alpha}$ -generated subvarieties: we will show that there is a symmetric subvariety of  $\mathsf{RCA}_{\alpha}$  that is not generated by a subclass of  $\mathsf{Lf}_{\alpha}$ . In section 4.3 we introduce the notion of *inductive* algebras and inductive varieties and we prove that this property characterizes the subvarieties generated by  $\mathsf{Lf}_{\alpha}$ s, the property of being inductive singles out the  $2^{\omega}$  many  $\mathsf{Lf}_{\alpha}$ -generated subvarieties among all the  $2^{\alpha}$  many subvarieties of  $\mathsf{RCA}_{\alpha}$ . By this we also get a simple characterization, and recursive enumeration, of the equational theory of  $\mathsf{RCA}_{\alpha}$ , much simpler than either one of the three enumerations presented in [19, pp.112-119]. These results contribute to solving [19, Problem 4.1] which is asking for a simple equational basis for  $\mathsf{RCA}_{\alpha}$ .

Being inductive is a nice property: inductive algebras are all representable,

they are symmetric, and their equational theories coincide with the one of an  $Lf_{\alpha}$ . We show that there are more inductive algebras than the widest class dealt with in [18, 2.6.50]. This provides us with a new representation theorem for  $CA_{\alpha}s$ . All this strengthen, extend and improve [18, 2.6.50], whose significance was discussed in the introduction to the present paper. The notion of being inductive can be described by a set of  $\Delta_2$  first order logic formulas. Since inductive algebras are all symmetric and we have constructed a nonsymmetric representable algebra in section 2.1 here, we get a  $\Delta_2$ -formula distinguishing  $Lf_{\alpha}$  and  $RCA_{\alpha}$ .

# 4.1 Endo-dimension-complemented algebras are symmetric

Let  $Lf_{\alpha}$ ,  $Dc_{\alpha}$ , and  $Di_{\alpha}$  denote the classes of all locally finite dimensional, dimension-complemented, and diagonal CA's, respectively. Let us call the elements of the wider class introduced in (iii) of [18, 2.6.50] endo-dimensioncomplemented (endo-dc in short): an algebra  $\mathfrak{A} \in \mathsf{CA}_{\alpha}$  is called endo-dc if for each finite  $\Gamma \subseteq \alpha$  and each nonzero  $x \in A$  there are a  $\kappa \in \alpha - \Gamma$  and an endomorphism h of the  $\Gamma$ -reduct  $\mathfrak{Rd}_{\Gamma}\mathfrak{A} := \langle A, +, -, c_i, d_{ij} \rangle_{i,j \in \Gamma}$  of  $\mathfrak{A}$  such that  $h(x) \neq 0$  and each element of the range of h is  $\kappa$ -closed, i.e.,  $c_{\kappa}h(a) = h(a)$ for all  $a \in A$ . Let  $\mathsf{Edc}_{\alpha}$  denote the class of all endo-dimension-complemented  $\mathsf{CA}_{\alpha}$ s. It is proved in [18, 2.6.50] that  $\mathsf{Lf}_{\alpha} \subset \mathsf{Dc}_{\alpha} \subset \mathsf{Di}_{\alpha} \subset \mathsf{Edc}_{\alpha} \subseteq \mathsf{RCA}_{\alpha}$  and it is asked as [18, Problem 2.13] whether the last inclusion is proper or not. We are going to show that this inclusion is proper: the algebra constructed in section 2.1 here is representable but not endo-dc. More specifically, we will show that each endo-dc algebra is symmetric, which implies  $\mathsf{Edc}_{\alpha} \neq \mathsf{RCA}_{\alpha}$ since our witness algebra  $\mathfrak{A}$  in the proof of Theorem 2.1 was designed to be non-symmetric but it is representable. We also show that  $\mathsf{RCA}_{\alpha}$  is close to  $\mathsf{Edc}_{\alpha}$  in the sense that  $\mathsf{RCA}_{\alpha}$  is the closure of  $\mathsf{Edc}_{\alpha}$  under taking subalgebras. On the other hand, to indicate the distance between  $\mathsf{Edc}_{\alpha}$  and  $\mathsf{RCA}_{\alpha}$  we show that the class  $Sy_{\alpha} \cap RCA_{\alpha}$  of symmetric representable algebras lies strictly in between  $\mathsf{Edc}_{\alpha}$  and  $\mathsf{RCA}_{a}$ , i.e.,  $\mathsf{Edc}_{\alpha} \subset \mathsf{Sy}_{\alpha} \cap \mathsf{RCA}_{\alpha} \subset \mathsf{RCA}_{\alpha}$ .

#### Theorem 4.1 Each endo-dc algebra is symmetric.

**Proof.** Because the notion of a symmetric algebra involves renaming indices of operations, in this and the coming proofs we will often deal with renaming operations in equations and algebras. Therefore we begin the proof with

introducing notation for these. We will use these notation, except for ind(e), only in proofs.

If  $\tau$  is a term in the language of  $\mathsf{CA}_{\alpha}$  and  $\rho : \alpha \to \alpha$  then  $\rho(\tau)$ , the term we get from  $\tau$  by renaming the indices occurring in it according to  $\rho$ , is defined by induction as  $\rho(d_{ij}) := d_{\rho(i)\rho(j)}, \ \rho(c_i\sigma) := c_{\rho(i)}\rho(\sigma)$  and  $\rho(x) := x$  if x is a variable,  $\rho(\sigma + \delta) := \rho(\sigma) + \rho(\delta), \ \rho(-\sigma) := -\rho(\sigma)$ . If e is an equation of form  $\tau = \sigma$  then  $\rho(e)$  is  $\rho(\tau) = \rho(\sigma)$ .

 $\operatorname{ind}(\tau)$  denotes the set of indices occurring in  $\tau$ , this is defined by induction as follows.  $\operatorname{ind}(d_{ij}) := \{i, j\}, \operatorname{ind}(c_i \tau) := \{i\} \cup \operatorname{ind}(\tau), \text{ and } \operatorname{ind}(x) := \emptyset,$  $\operatorname{ind}(\tau + \sigma) := \operatorname{ind}(\tau) \cup \operatorname{ind}(\sigma), \operatorname{ind}(-\tau) := \operatorname{ind}(\tau).$  If e is an equation of form  $\tau = \sigma$  then  $\operatorname{ind}(e) := \operatorname{ind}(\tau) \cup \operatorname{ind}(\sigma).$ 

Assume  $\mathfrak{A} \in \mathsf{CA}_{\alpha}$ ,  $\Gamma$  is any set and  $\rho : \Gamma \to \alpha$  is a one-to-one function. Then  $\mathfrak{Rd}^{\rho}\mathfrak{A}$  denotes an algebra whose signature is that of  $\mathsf{CA}_{\Gamma}$ , whose Boolean reduct  $\langle A, +, - \rangle$  is the same as that of  $\mathfrak{A}$ , whose operation denoted by  $c_i$  for  $i \in \Gamma$  is the operation of  $\mathfrak{A}$  denoted by  $c_{\rho(i)}$ , and similarly for the diagonals,  $d_{ij}$  of  $\mathfrak{Rd}^{\rho}\mathfrak{A}$  is the same as  $d_{\rho(i)\rho(j)}$  of  $\mathfrak{A}$ . In symbols,

$$\mathfrak{Rd}^{\rho}\mathfrak{A} := \langle A, +, -, c_{\rho(i)}, d_{\rho(i)\rho(j)} : i, j \in \Gamma \rangle.$$

It is not difficult to check that  $\mathfrak{Rd}^{\rho}\mathfrak{A} \in \mathsf{CA}_{\Gamma}$  and  $\mathfrak{Rd}^{\rho}\mathfrak{A} \models e$  iff  $\mathfrak{A} \models \rho(e)$ , for any equation e. This algebra is called a generalized reduct of  $\mathfrak{A}$  and it is introduced in [18, 2.6.1].

We begin the proof of Theorem 4.1. Assume  $\mathfrak{A} \in \mathsf{CA}_{\alpha}$  is endo-dc, we want to show that it is symmetric. This means showing that  $\mathfrak{A} \models e$  implies  $\mathfrak{A} \models \rho(e)$  for all equations e and permutations  $\rho$  of  $\alpha$ . For this, it is enough to prove

(20) 
$$\mathfrak{A} \not\models e \text{ implies } \mathfrak{A} \not\models \rho(e), \text{ for all } e \text{ and } \rho,$$

since each equation e is of form  $\rho(e')$  and  $\rho^{-1}\rho(e') = e'$ . Assume  $\mathfrak{A} \not\models e$ . We may assume that e is of form  $\tau = 0$  for some  $\tau$ . Let  $\Gamma := \operatorname{ind}(\tau)$  and let  $\rho : \Gamma \to \Delta$  be a bijection. We have  $\tau(a) \neq 0$  for some  $a \in A$  by  $\mathfrak{A} \not\models e$ , and we want to show that  $\rho(\tau)(b) \neq 0$  for some  $b \in A$ . (In fact,  $\tau$  may have more than one variable, so we should use a sequence  $\overline{a}$  in place of  $a \in A$ . For simplicity, we write out the present proof for the case when  $\tau$  contains one variable.)

We aim for getting a homomorphism  $\mathfrak{Rd}_{\Gamma}\mathfrak{A} \to \mathfrak{Rd}^{\rho}\mathfrak{A}$  which takes  $\tau(a)$  to a nonzero element. The idea of the proof is as follows. Assume  $\Delta =$ 

 $\{k_1, \ldots, k_n\}$  is disjoint from  $\Gamma = \{i_1, \ldots, i_n\}$ . Then the substitution operation  $x \mapsto \mathbf{s}_{k_1}^{i_1} \ldots \mathbf{s}_{k_n}^{i_n}(x)$  is such a homomorphism, but only on the  $\Delta$ -closed elements x, i.e., when  $x = c_{(\Delta)}x := c_{k_1} \ldots c_{k_n}x$ . There are two obstacles to deal with:  $\Delta$  may not be disjoint from  $\Gamma$ , and  $\tau(a)$  may not be  $\Delta$ -closed. We deal with the first obstacle by finding J which is disjoint both from  $\Gamma$  and  $\Delta$ , and finding desired homomorphisms from  $\Gamma$  to J and then from J to  $\Delta$ . We deal with the second obstacle by using the condition  $\mathfrak{A} \in \mathsf{Edc}_{\alpha}$  for finding a homomorphism from  $\Gamma$  to  $\Gamma$  which takes  $\tau(a)$  to a J-closed non-zero element. We begin now to elaborate the just outlined idea.

By [18, (2), p.416],  $\mathfrak{A} \in \mathsf{Edc}_{\alpha}$  implies that there is  $J \subseteq \alpha - (\Gamma \cup \Delta)$ with  $|J| = |\Gamma|$  and there is a homomorphism  $h : \mathfrak{Ro}_{\Gamma}\mathfrak{A} \to \mathfrak{Ro}_{\Gamma}\mathfrak{A}$  such that  $h(\tau(a)) \neq 0$  and  $h(x) = c_{(J)}h(x)$  for all  $x \in A$ . By h being a homomorphism on  $\mathfrak{Ro}_{\Gamma}\mathfrak{A}$  and  $\mathsf{ind}(\tau) \subseteq \Gamma$  we have  $\tau(h(a)) = h(\tau(a)) \neq 0$ .

Now that h provided us with J-closed elements, we can use the usual substitution operations  $\mathbf{s}_{j}^{i}$  to get the homomorphism we seek for, as follows. Let  $c_{j}^{*}\mathfrak{A}$  denote the algebra whose elements are the  $c_{j}$ -closed elements of  $\mathfrak{A}$ and whose operations are those of  $\mathfrak{A}$  except  $c_{j}, d_{jk}, d_{kj}$  for  $k \in \alpha$ . This is indeed an algebra, it is  $\mathfrak{Nr}_{(\alpha-\{j\})}\mathfrak{A}$  in the terminology of [18], but we will use the shorter notation  $c_{j}^{*}\mathfrak{A}$  in the present proof. We will use  $c_{(J)}^{*}\mathfrak{A}$  for the analogous algebra (where  $J \subset \alpha$ ). Let [i/j] denote the function that takes ito j and takes k to k for all  $k \in (\alpha - \{i, j\})$ . Then  $\mathfrak{RO}^{[i/j]}c_{i}^{*}\mathfrak{A}$  is the algebra  $c_{i}^{*}\mathfrak{A}$  except that we rename the operations  $c_{j}, d_{jk}, d_{kj}$  (of  $c_{i}^{*}\mathfrak{A}$ ) as  $c_{i}, d_{ik}, d_{ki}$ , respectively. Thus the similarity types of  $c_{j}^{*}\mathfrak{A}$  and  $\mathfrak{RO}^{[i/j]}c_{i}^{*}\mathfrak{A}$  are equal. We are going to show, by using [18, sec. 1.5], that

(21) 
$$\mathbf{s}_{j}^{i}: c_{j}^{*}\mathfrak{A} \to \mathfrak{R}\mathfrak{d}^{[i/j]}c_{i}^{*}\mathfrak{A}$$
 is an isomorphism

Indeed,  $\mathbf{s}_j^i$  is a Boolean homomorphism by 1.5.3, it is a homomorphism for  $c_k, d_{km}$  for  $k, m \in \alpha - \{i, j\}$  by 1.5.8(ii), 1.5.4(ii), and it takes  $d_{ik}, d_{ki}$  to  $d_{jk}, d_{kj}$  by 1.5.4(i). For the next two steps we need to use that we are mapping  $c_j$ -closed elements.  $\mathbf{s}_i^j$  is the inverse of  $\mathbf{s}_j^i$  on  $c_j$ -closed elements because  $\mathbf{s}_i^j \mathbf{s}_j^i c_j x = c_j x$  by 1.5.10(i), 1.5.8(i).  $\mathbf{s}_j^i$  takes the operation  $c_i$  on  $c_j$ -closed elements to  $c_j$  because  $c_j \mathbf{s}_j^i a = c_i \mathbf{s}_i^j a = c_i \mathbf{s}_i^j c_j a = c_i c_j a = c_i a = \mathbf{s}_j^i c_i a$ , by 1.5.8(i), 1.5.9(i). We are done with proving (21).

Recall that  $J \subseteq \alpha - (\Gamma \cup \Delta)$  and  $|J| = |\Gamma|$ . Let  $i_1, \ldots, i_n$  and  $j_1, \ldots, j_n$ be repetition-free enumerations of  $\Gamma$  and J, respectively. Let  $\eta : \Gamma \to J$  be such that  $\eta(i_1) = j_1, \ldots, \eta(i_n) = j_n$ . Define

$$\mathsf{s}(\eta) := \mathsf{s}_{j_1}^{i_1} \dots \mathsf{s}_{j_n}^{i_n}.$$

By using (21) successively, we get

(22)  $\mathbf{s}(\eta) : \mathfrak{Rd}_{\Gamma}c^*_{(J)}\mathfrak{A} \to \mathfrak{Rd}^{\eta}c^*_{(\Gamma)}\mathfrak{A}$  is an isomorphism.

By letting  $k_{\ell} := \rho(i_{\ell})$  and  $\xi(j_{\ell}) := k_{\ell}$  for  $1 \leq \ell \leq n$  we get that  $k_1, \ldots, k_n$  is a repetition-free enumeration of  $\Delta$ , and  $\rho = \xi \circ \eta$ . By repeating the process leading to (22) we get

(23)  $\mathbf{s}(\xi) : \mathfrak{Rd}^{\eta} c^*_{(\Gamma)} \mathfrak{A} \to \mathfrak{Rd}^{\rho} c^*_{(\Delta)} \mathfrak{A}$  is an isomorphism.

Putting these two isomorphisms together we get

(24)  $\mathbf{s}(\rho): \mathfrak{Rd}_{\Gamma}c^*_{(J)}\mathfrak{A} \to \mathfrak{Rd}^{\rho}c^*_{(\Delta)}\mathfrak{A}$  is an isomorphism.

Let  $g := \mathsf{s}(\rho) \circ h$ , then  $g(\tau(a)) = \mathsf{s}(\rho)h(a) \neq 0$  by  $h(a) \neq 0$ , so

(25)  $g: \mathfrak{Rd}_{\Gamma}\mathfrak{A} \to \mathfrak{Rd}^{\rho}c^*_{(\Delta)}\mathfrak{A}$  is a homomorphism with  $g(\tau(a)) \neq 0$ .

Now,  $\rho(\tau)$  in  $\mathfrak{A}$  is the same as  $\tau$  in  $\mathfrak{Rd}^{\rho}\mathfrak{A}$ , by definition. Therefore,  $\rho(\tau)(ga)$  in  $\mathfrak{A}$  is the same as  $\tau(ga)$  in  $\mathfrak{Rd}^{\rho}\mathfrak{A}$ , which is the same as  $g(\tau(a))$  which is nonzero by (25) and  $\tau(a) \neq 0$ . We are done with showing that  $\mathfrak{A}$  is symmetric.  $\Box$ 

**Lemma 4.1** Each full cylindric set algebra with unit a disjoint union of weak spaces is endo-dc.

**Proof.** The proof in [18, 2.6.51, p.417] for showing "(iii) does not imply (ii)" in fact proves the present Lemma 4.1.

**Theorem 4.2** (Solution of [18, Problem 2.13]) There is an  $\mathsf{RCA}_{\alpha}$  which is not endo-dc, but each  $\mathsf{RCA}_{\alpha}$  can be embedded into an endo-dc algebra. In symbols:  $\mathsf{Edc}_{\alpha} \subset \mathsf{SEdc}_{\alpha} = \mathsf{RCA}_{\alpha}$ .

**Proof.** The algebra we based the proof of Theorem 2.1 on is not symmetric, hence not endo-dc by Theorem 4.1. Clearly, it is representable. This shows  $\mathsf{Edc}_{\alpha} \neq \mathsf{RCA}_{\alpha}$ .  $\mathsf{Edc}_{\alpha} \subseteq \mathsf{RCA}_{\alpha}$  is proved as (iii) $\Rightarrow$ (iv) in [18, Thm.2.6.50].  $\mathsf{RCA}_{\alpha} = \mathsf{SEdc}_{\alpha}$  follows from Lemma 4.1 immediately, since each representable algebra is embeddable into a full one.

**Remark 4.1** In the proof above, we used Theorem 4.1 to show that the algebra  $\mathfrak{A}$  we used in the proof of Theorem 2.1 is not endo-dc. A concrete  $\Gamma \subseteq \alpha$  and nonzero  $a \in A$  for which there are no  $\kappa \in \alpha$  and endomorphism h with the required properties are  $\{0, 1, 2\}$  and g. Indeed, let  $\tau := x - c_{(3)}(\beta + \gamma + \phi)$ , see (2). Then e(x) fails iff  $\tau(x) \neq 0$ , by (2). Hence,  $\tau(g) \neq 0$  but  $\tau(c_{\kappa}x) = 0$  for all  $\kappa$  by Remark 2.1(iii), and this implies that there is no endomorphism h of  $\mathfrak{Ro}_{\{0,1,2\}}\mathfrak{A}$  with range inside  $c_{\kappa}^*A$  and  $h(g) \neq 0$ .

**Theorem 4.3** Not all symmetric algebras are representable, and not all representable algebras are symmetric. In symbols,

$$\mathsf{Sy}_{\alpha} \cap \mathsf{RCA}_{\alpha} \subset \mathsf{Sy}_{\alpha}$$
 and  $\mathsf{Sy}_{\alpha} \cap \mathsf{RCA}_{\alpha} \subset \mathsf{RCA}_{\alpha}$ .

**Proof.** To exhibit a symmetric algebra that is nonrepresentable, take any nonrepresentable  $\mathfrak{A} \in CA_{\alpha}$ , we "turn" it symmetric. Indeed, let

$$\mathfrak{B} := \prod \langle \mathfrak{R}\mathfrak{d}^{\rho}\mathfrak{A} : \rho \text{ is a permutation of } \alpha \rangle.$$

That  $\mathfrak{B}$  is symmetric can be seen by

$\mathfrak{B}\models e$	iff	by the definition of $\mathfrak{B}$
$\mathfrak{Rd}^{\rho}\mathfrak{A}\models e \text{ for all }\rho$	iff	by the definition of $\Re \mathfrak{d}^\rho$
$\mathfrak{A}\models\rho(e)\text{ for all }\rho$	iff	by the nature of permutations
$\mathfrak{A}\models\rho(\eta(e))\text{ for all }\rho,\eta$	iff	by previous step
$\mathfrak{Rd}^{\rho}\mathfrak{A}\models\eta(e)\text{ for all }\rho$	iff	by first step
$\mathfrak{B} \models \eta(e).$		

That  $\mathfrak{B}$  is not representable follows from the facts that  $\mathfrak{A} \notin \mathsf{RCA}_{\alpha}$  is a homomorphic image of  $\mathfrak{B}$  (as  $\mathfrak{A} = \mathfrak{RO}^{\rho}\mathfrak{A}$  with  $\rho$  being the identity permutation of  $\alpha$ ) and  $\mathsf{RCA}_{\alpha}$ , being a variety ([19, 3.1.103]), is closed under homomorphic images.

The algebra used in the proof of Theorem 2.1 is representable and nonsymmetric, this proves the second part of the theorem, i.e.,  $\mathsf{RCA}_{\alpha} \cap \mathsf{Sy}_{\alpha} \subset \mathsf{RCA}_{\alpha}$ .

### 4.2 Polyadic algebras are symmetric

We have seen in the proof of Theorem 4.1 that substitution operations are useful in proving an algebra be symmetric. In fact, the proof of Theorem 2.1 hinges over the fact that the polyadic substitution operations  $\mathbf{p}_{ij}$ are not expressible in the witness algebra  $\mathfrak{A}$ . In this section we very briefly talk about Halmos' polyadic algebras. We show that  $\alpha$ -dimensional quasipolyadic equality algebras indeed have only  $2^{\omega}$  many subvarieties, since all their members are symmetric (in an appropriate sense). We then state some of the corollaries of our construction that concern polyadic algebras.

Polyadic equality algebras ( $\mathsf{PEA}_{\alpha}$ s) were introduced by Paul Halmos [17], they are basically cylindric algebras endowed with unary substitution operations  $\mathbf{s}_{\rho}$  for  $\rho : \alpha \to \alpha$ . In the set algebras with unit  $^{\alpha}U$  these are interpreted as

$$S_{\rho}(X) := \{ s \in {}^{\alpha}U : \rho \circ s \in X \}.$$

Quasi-polyadic equality algebras were also defined by Halmos in [17], they retain only those substitutions where  $\rho$  is finite. Let  $QPEA_{\alpha}$  denote their class, for precise definition see, e.g., [19, p.266, item 9] or [39].

**Theorem 4.4** QPEA<sub> $\alpha$ </sub> has exactly  $2^{\omega}$  many subvarieties.

**Proof.** The idea of the proof is to show that each  $QPEA_{\alpha}$  is symmetric in the sense analogous to the notion used in  $CA_{\alpha}$ . However, the indices of the  $QPEA_{\alpha}$ -operations have some structure, it is not so clear how we are to change the indices in an equation systematically/uniformly. (For more on this see the introduction of [39].) Therefore, we will use the more indexfriendly version  $FPEA_{\alpha}$  of  $QPEA_{\alpha}$  defined in [39]. Since the two varieties are term-definitionally equivalent, proved as [39, Thm.1(ii)], it is enough to show that  $FPEA_{\alpha}$  has only  $2^{\omega}$  subvarieties.

We are going to show that each element of  $\mathsf{FPEA}_{\alpha}$  is symmetric in the very analogous sense to  $\mathsf{CA}_{\alpha}$ , this will prove that  $\mathsf{QPEA}_{\alpha}$  has at most continuum many subvarieties (since each equation is equivalent to one which uses indices from  $\omega$  only). That  $\mathsf{QPEA}_{\alpha}$  has indeed continuum many varieties can be seen by repeating the proof of [19, Thm.4.1.24] for  $\mathsf{QPEA}_{\alpha}$ .

The extra-cylindric operations in an  $\mathsf{FPEA}_{\alpha}$  are denoted as  $\mathsf{p}_{ij}$  for  $i, j \in \alpha$ . The operation  $\mathsf{p}_{ij}$  stands for  $\mathsf{s}_{\rho}$  where  $\rho$  is [i, j], the latter being the permutation of  $\alpha$  that interchanges i and j and leaves all the other elements fixed. Now, the definitions of  $\rho(\tau)$  and  $\mathsf{ind}(\tau)$  for  $\mathsf{FPEA}_{\alpha}$ -terms  $\tau$  can easily be extended from  $CA_{\alpha}$ . We will show that each  $\mathfrak{A} \in \mathsf{FPEA}_{\alpha}$  is symmetric in the sense that

$$\mathfrak{A} \models e$$
 iff  $\mathfrak{A} \models \rho(e)$ , for all permutations  $\rho$  of  $\alpha$ .

Indeed, let  $\mathfrak{A} \in \mathsf{FPEA}_{\alpha}$  and let  $\tau$  be a term in the language of  $\mathsf{FPEA}_{\alpha}$ , let  $\rho$  be a permutation of  $\alpha$ . Then  $\mathsf{ind}(\tau)$  is finite, so we may assume that  $\rho$  is finite, too. Each finite permutation is a composition of transpositions [i, j], so we may assume that  $\rho$  is indeed a transposition [i, j] with  $i \neq j$ . In the sequel we will write  $\tau(\bar{x})$  and  $\tau(\mathsf{p}_{ij}\bar{x})$  for  $\tau(x_1, \ldots, x_n)$  and  $\tau(\mathsf{p}_{ij}x_1, \ldots, \mathsf{p}_{ij}x_n)$ . The following can be proved by induction on  $\tau$ :

(26) 
$$\mathsf{FPEA}_{\alpha} \models \mathsf{p}_{ij}\tau(\bar{x}) = \rho(\tau(\mathsf{p}_{ij}\bar{x}))$$

with the use of the following equations that can be proved to hold in  $\mathsf{FPEA}_{\alpha}$ :

$$\begin{aligned} \mathbf{p}_{ij}(x+y) &= \mathbf{p}_{ij}x + \mathbf{p}_{ij}y, \quad \mathbf{p}_{ij}(-x) = -\mathbf{p}_{ij}x, \\ \mathbf{p}_{ij}\mathbf{p}_{ij}x &= x, \quad \mathbf{p}_{ij}x = \mathbf{p}_{ji}x, \\ \mathbf{p}_{ij}c_kx &= c_{k'}\mathbf{p}_{ij}x, \quad \mathbf{p}_{ij}d_{kl} = d_{k'l'}, \quad \mathbf{p}_{ij}\mathbf{p}_{kl}x = \mathbf{p}_{k'l'}\mathbf{p}_{ij}x, \end{aligned}$$

where  $k' = \rho(k)$  and  $l' = \rho(l)$ . Now, let *e* be any equation, we may assume that it is of form  $\tau(\bar{x}) = 1$ .

$$\begin{split} \mathfrak{A} &\models e & \text{iff} & \text{by } e \text{ being } \tau = 1 \\ \mathfrak{A} &\models \tau = 1 & \text{implies} & \text{by } \mathsf{p}_{ij} 1 = 1 \\ \mathfrak{A} &\models \mathsf{p}_{ij} \tau = 1 & \text{iff} & \text{by } (26) \\ \mathfrak{A} &\models \rho(\tau(\mathsf{p}_{ij}\bar{x})) = 1 & \text{implies} & \text{by } \mathsf{p}_{ij} \mathsf{p}_{ij} x = x \\ \mathfrak{A} &\models \rho(\tau) = 1 & \text{iff} & \text{by } e \text{ being } \tau = 1 \\ \mathfrak{A} &\models \rho(e). \end{split}$$

It is proved in [19, 5.4.18] that the cylindric reducts of  $\mathsf{PEA}_{\alpha}$ s are all representable, in symbols  $\mathsf{Rd}_{ca}\mathsf{PEA}_{\alpha} \subseteq \mathsf{RCA}_{\alpha}$ . Our results imply that this inclusion is a strict one. Further, the cylindric reducts of (quasi-)polyadic (equality)-algebras are not closed under subalgebras.

**Corollary 4.1** Not every representable cylindric algebra is the cylindric reduct of a polyadic equality algebra, hence the class of the latter is not closed under subalgebras. Formally:

$$\mathsf{Rd}_{ca}\mathsf{PEA}_{\alpha} \subset \mathsf{RCA}_{\alpha} = \mathbf{SRd}_{ca}\mathsf{PEA}_{\alpha}.$$

*Further*,  $\mathsf{Rd}_{ca}\mathsf{QPEA}_{\alpha} \subset \mathsf{SRd}_{ca}\mathsf{QPEA}_{\alpha}$ .

**Proof.** It follows from the proof of Theorem 4.4 that the cylindric reduct of any quasi-polyadic equality algebra is symmetric. We have seen in Theorem 4.3 that not all representable algebras are symmetric. Take a non-symmetric  $\mathsf{RCA}_{\alpha}$ , it is not in  $\mathsf{Rd}_{ca}\mathsf{QPEA}_{\alpha}$ , hence it is not in  $\mathsf{Rd}_{ca}\mathsf{PEA}_{\alpha}$ , either. Since all full cylindric set algebras are reducts of  $\mathsf{PEA}_{\alpha}$ , our non-symmetric  $\mathsf{RCA}_{\alpha}$  is in  $\mathsf{SRd}_{ca}\mathsf{PEA}_{\alpha}$ .

#### 4.3 Inductive algebras

Let us call a cylindric algebra  $\mathfrak{A}$  inductive iff  $\mathfrak{A} \models e(c_i x_1, \ldots, c_i x_n)$  implies  $\mathfrak{A} \models e(x_1,\ldots,x_n)$  whenever e is an equation and i does not occur as an index of an operation in e. Let  $\mathsf{Ind}_{\alpha}$  denote the class of all inductive  $\mathsf{CA}_{\alpha}$ s. While  $\mathfrak{A} \models e$  implies  $\mathfrak{A} \models e(c_i x)$  always holds, the converse of this would be thought to hold only in rather special cases, if at all. We are going to show that, on the contrary, there is a great variety of inductive algebras: each endo-dc algebra is inductive and we have already seen that there is a great variety of endo-dc algebras. There are even more inductive algebras than endo-dc algebras:  $\mathsf{Edc}_{\alpha} \subset \mathsf{Ind}_{\alpha}$ . We then prove that each inductive algebra is representable and symmetric (but the converse does not hold). Thus, we refine the chain  $Lf_{\alpha} \subset Dc_{\alpha} \subset Di_{\alpha} \subset Edc_{\alpha} \subset Sy \cap RCA_{\alpha} \subset RCA_{\alpha}$ with inserting a new class into it:  $\mathsf{Edc}_{\alpha} \subset \mathsf{Ind}_{\alpha} \subset \mathsf{Sy}_{\alpha} \cap \mathsf{RCA}_{\alpha}$ . This is also a new representation theorem, a sharpening of [18, 2.6.50], since in the chain presented in [18, 2.6.50] the widest representable class was  $\mathsf{Edc}_{\alpha}$ . The new class  $\mathsf{Ind}_{\alpha}$  has an additional significance, namely an algebra is inductive iff it is equationally indistinguishable from an  $Lf_{\alpha}$ . So, inductive algebras are in intimate connection with  $\mathsf{Lf}_{\alpha}$ . This will give us a specific  $\Delta_2$  formula distinguishing  $Lf_{\alpha}$  and  $RCA_{\alpha}$ .

#### Theorem 4.5

(i) Each endo-dc algebra is inductive, and each inductive algebra is symmetric and representable but the converses of these statements do not hold, i.e.,

$$\mathsf{Edc}_{\alpha} \subset \mathsf{Ind}_{\alpha} \subset \mathsf{Sy}_{\alpha} \cap \mathsf{RCA}_{\alpha}.$$

(ii) An algebra is inductive iff there is an  $Lf_{\alpha}$  with the same equational theory, i.e.,

$$\mathfrak{A}$$
 is inductive iff  $\mathsf{Eq}(\mathfrak{A}) = \mathsf{Eq}(\mathfrak{B})$  for some  $\mathfrak{B} \in \mathsf{Lf}_{\alpha}$ .

**Proof.** First we prove part of (i), namely we prove  $\mathsf{Edc}_{\alpha} \subseteq \mathsf{Ind}_{\alpha}$ . This follows almost directly from the definitions and from Theorem 4.1. Let  $\mathfrak{A} \in \mathsf{Edc}_{\alpha}$ and let  $e(x_1, ..., x_n)$  be an equation,  $i \in \alpha$  such that i does not occur in e. In the sequel we will write  $e(\bar{x})$  and  $e(c_i\bar{x})$  in place of  $e(x_1, ..., x_n)$  and  $e(c_ix_1, ..., c_ix_n)$ , respectively. We want to show  $\mathfrak{A} \models e(c_i\bar{x})$  implies  $\mathfrak{A} \models$  $e(\bar{x})$ . To this end, we assume  $\mathfrak{A} \not\models e(\bar{x})$  and we show that  $\mathfrak{A} \not\models e(c_i\bar{x})$ . Let  $a_1, ..., a_n \in A$  be such that  $\mathfrak{A} \not\models e(\bar{a})$ . We may assume that e is of form  $\tau = 0$ , so we have  $\tau(\bar{a}) \neq 0$  in  $\mathfrak{A}$ . Let  $\Gamma := \mathsf{ind}(\tau)$ . By  $\mathfrak{A}$  being endo-dc, there are a homomorphism  $h : \mathfrak{Rd}_{\Gamma}\mathfrak{A} \to \mathfrak{Rd}_{\Gamma}\mathfrak{A}$  and a  $\kappa \in \alpha - \Gamma$  such that  $h(\tau(\bar{a})) \neq 0$  and  $h(b) = c_{\kappa}h(b)$  for all  $b \in A$ . Now,  $h(\tau(\bar{a})) = \tau(h(\bar{a})) = \tau(h(a_1), \ldots, h(a_n))$ by h being a homomorphism wrt. the operations occurring in  $\tau$ . By  $h(a_1) =$  $c_{\kappa}h(a_1), \ldots, h(a_n) = c_{\kappa}h(a_n)$  we then have  $\tau(c_{\kappa}h(\bar{a})) \neq 0$  in  $\mathfrak{A}$ . This means that  $\mathfrak{A} \not\models e(c_i\bar{x})$  since  $\mathfrak{A}$  is symmetric by Theorem 4.1 and  $\kappa, i \notin \mathsf{ind}(e)$ , we get that  $\mathfrak{A} \not\models e(c_i\bar{x})$  as was desired.

Next we prove (ii). For proving the "only-if" part, let  $\mathfrak{A}$  be inductive, we will show that it is equationally indistinguishable from an  $Lf_{\alpha}$ . Let  $\mathfrak{C}$  be an elementary  $\alpha$ -saturated extension of  $\mathfrak{A}$ , and let  $\mathfrak{B}$  be the greatest locally finite dimensional subalgebra of  $\mathfrak{C}$ . (This exists by [18, 2.1.5(ii)].) We are going to show that  $\mathfrak{A}$  and  $\mathfrak{B}$  are equationally indistinguishable. Let e be an equation. If  $\mathfrak{A} \models e$  then  $\mathfrak{C} \models e$  because  $\mathfrak{C}$  is an elementary extension of  $\mathfrak{A}$ , and thus  $\mathfrak{B} \models e$  because  $\mathfrak{B}$  is a subalgebra of  $\mathfrak{C}$ . Assume now  $\mathfrak{A} \not\models e$ . Let  $\Delta := \{i_1, ..., i_n\}$  be disjoint from  $\operatorname{ind}(e)$  with  $i_1, ..., i_n$  being all distinct. Then  $\mathfrak{A} \not\models e(c_{i_1}\bar{x})$  since  $\mathfrak{A}$  is inductive and  $\mathfrak{A} \not\models e$ . But then  $\mathfrak{A} \not\models e(c_{i_1}c_{i_2}\bar{x})$  because  $i_2 \notin \operatorname{ind}(e(c_{i_1}\bar{x}))$ , and so on, showing that  $\mathfrak{A} \not\models e(c_{(\Delta)}\bar{x})$ . Let

$$\Sigma(\bar{x}) := \{\neg e(\bar{x}), c_i \bar{x} = \bar{x} : i \in \alpha - \mathsf{ind}(e)\}.$$

Each finite subset of  $\Sigma$  is satisfiable in  $\mathfrak{C}$  by  $\mathfrak{A} \not\models e(c_{(\Delta)}\bar{x})$  for all finite  $\Delta \subseteq \alpha - \operatorname{ind}(e)$ . By  $\mathfrak{C}$  being  $\alpha$ -saturated, this implies that there are  $b_1, \ldots, b_n \in C$  for which  $\Sigma(\bar{b})$  holds in  $\mathfrak{C}$ . These  $b_j$ s are finite dimensional (by  $c_i(b_j) = b_j$  for all  $i \in \alpha - \operatorname{ind}(e)$ ), and  $\mathfrak{C} \not\models e(\bar{b})$  (by  $\neg e(\bar{x}) \in \Sigma(\bar{x})$ ). Hence  $b_1, \ldots, b_n \in B$  and  $\mathfrak{C} \not\models e(\bar{b})$ , hence  $\mathfrak{B} \not\models e(\bar{b})$ , i.e.,  $\mathfrak{B} \not\models e$ . This finishes the "only-if" part of the proof of (ii). For the "if" part, we have to show that each  $\mathfrak{B} \in \mathsf{Lf}_\alpha$  is inductive. Indeed,  $\mathsf{Lf}_\alpha \subseteq \mathsf{Edc}_\alpha$  by [18, 2.6.50], and  $\mathsf{Edc}_\alpha \subseteq \mathsf{Ind}_\alpha$  by that part of (i) that we have already proved.

It remains to prove the rest of (i). We have already shown  $\mathsf{Edc}_{\alpha} \subseteq \mathsf{Ind}_{\alpha}$ . To show  $\mathsf{Ind}_{\alpha} \subseteq \mathsf{Sy}_{\alpha} \cap \mathsf{RCA}_{\alpha}$  we use (ii), [18, 2.6.50] and Theorem 4.1, as follows. Let  $\mathfrak{A} \in \mathsf{Ind}_{\alpha}$ . Then  $\mathsf{Eq}(\mathfrak{A}) = \mathsf{Eq}(\mathfrak{B})$  for some  $\mathfrak{B} \in \mathsf{Lf}_{\alpha}$ , by (ii). Now,  $\mathsf{Lf}_{\alpha} \subseteq \mathsf{Edc}_{\alpha} \subseteq \mathsf{Sy}_{\alpha}$  by [18, 2.6.50] and Theorem 4.1,  $\mathsf{Lf}_{\alpha} \subseteq \mathsf{RCA}_{\alpha}$  by [19]. Thus,  $\mathfrak{A} \in \mathsf{Sy}_{\alpha} \cap \mathsf{RCA}_{\alpha}$ . We turn to proving that the stated inclusions are proper.

First we want to exhibit an inductive algebra that is not endo-dc. The difference between the two notions, and this will be reflected in the algebra  $\mathfrak{D}$  we exhibit, is that the notion of being inductive talks about the equational theory of the algebra, while the notion endo-dc talks about the inner structure of the algebra. The algebra  $\mathfrak{D}$  is a direct product of the  $\omega$ -generated free RCA<sub> $\alpha$ </sub>-algebra  $\mathfrak{F}$  and another representable algebra  $\mathfrak{A}$ . By this, it is already ensured that  $\mathfrak{D}$  is inductive, as follows.

$\mathfrak{D} \models e(c_i \bar{x})$	implies	by $\mathfrak F$ being a homomorphic image of $\mathfrak D$
$\mathfrak{F} \models e(c_i \bar{x})$	implies	by $\mathfrak{F}$ being a free algebra of $RCA_{\alpha}$
$RCA_{\alpha} \models e(c_i \bar{x})$	implies	by $Lf_{\alpha} \subseteq RCA_{\alpha}$
$Lf_{\alpha} \models e(c_i \bar{x})$	implies	by $Lf_{\alpha} \subseteq Ind_{\alpha}$
$Lf_{\alpha} \models e(\bar{x})$	implies	by $RCA_{\alpha} = \mathbf{Var}(Lf_{\alpha})$
$RCA_{\alpha} \models e(\bar{x})$	implies	by $\mathfrak{D} \in RCA_{\alpha}$
$\mathfrak{D} \models e(\bar{x}).$		

The role of the algebra  $\mathfrak{A}$  in the direct product is to destroy the property endo-dc. The idea is that we split an  $\alpha$ -dimensional atom in  $\mathfrak{A}$  into more parts than there are *i*-closed elements (for some  $i \in \alpha$ ) in  $\mathfrak{D}$ , so each required endomorphism will have to collapse all of the split parts to 0. We begin to elaborate this idea. Let W be a set of cardinality bigger than  $|\alpha|$  and let  $\langle W, +, z \rangle$  be any commutative group on W where z is the zero-element of +(i.e., w + z = w for all  $w \in W$ ). Let  $U_i := W \times \{i\}$ , let  $p := \langle (z, i) : i \in \alpha \rangle$ , let  $U := \bigcup \{U_i : i \in \alpha\}$  and  $T := \{s \in {}^{\alpha}U : s_i \in U_i \text{ for all } i \in \alpha \text{ and } | \{i \in \alpha : s_i \neq p_i\} | < \omega\}$ . Let  $\mathfrak{B}$  be the weak cylindric set algebra of dimension  $\alpha$  with unit element  ${}^{\alpha}U^{(p)}$  and generated by T. Then  $|B| = |\alpha|$  and T is an atom in  $\mathfrak{B}$ . We now split T in  $\mathfrak{B}$  into |W| many parts. For all  $g \in W$  let

$$T_g := \{ s \in T : \sum \{ w : s_i = (w, i) \text{ for some } i \in \alpha \} = g \}.$$

Then the  $T_q$ s  $(g \in W)$  form a disjoint union of T such that

(27) 
$$c_i T_g = c_i T$$
 for all  $i \in \alpha$  and  $g \in W$ 

Let  $\mathfrak{A}$  be the weak set algebra with unit element  ${}^{\alpha}U^{(p)}$  and generated by T together with  $T_g, g \in W$ . It is not hard to check, by using (27), that each element of A is of form

$$b + \sum \{T_g : g \in X\}$$

where  $b \in B$  and X is a finite or co-finite subset of W. Thus, all elements of A-B are  $\alpha$ -dimensional. We now show that  $\mathfrak{D} = \mathfrak{F} \times \mathfrak{A}$  is not endo-dc. Let's fix a  $g \in W$ , let  $a := \langle 0, T_g \rangle \in D$ , let  $\Gamma := \{0\}$ , we want to show that there are no endomorphism h of  $\mathfrak{Rd}_{\Gamma}\mathfrak{D}$  and  $\kappa \in \alpha$  such that h takes a to a nonzero element and each element of the range of h is  $\kappa$ -closed. For, assume the contrary, that h and  $\kappa$  are as described above, we will derive a contradiction. Since there are only  $\alpha$  many  $\kappa$ -closed elements of  $\mathfrak{A}$ , hence of  $\mathfrak{D}$  by  $|F| = |\alpha|$ , and there are more than  $\alpha$  many split parts of T, the endomorphism h has to take two of the elements  $\langle 0, T_w \rangle \in D$  to the same element. But these are all disjoint from each other, so  $h(\langle 0, T_w \rangle) = 0$  for some  $w \in W$ . But then  $h(c_0\langle 0, T_w \rangle) = c_0h(\langle 0, T_w \rangle) = 0$  since h is a homomorphism wrt.  $c_0$ . However,  $c_0\langle 0, T_w \rangle = \langle 0, c_0T_w \rangle = \langle 0, c_0T \rangle \geq \langle 0, T \rangle \geq \langle 0, T_g \rangle$ , showing that  $h(\langle 0, T_g \rangle) = h(a) = 0$ , and this contradicts our assumption  $h(a) \neq 0$ . Thus the algebra  $\mathfrak{D}$  is inductive but not endo-dc.

Finally, we exhibit a symmetric representable algebra which is not inductive. Here, both notions refer to the equational theory of the algebra, but they make different restrictions on it. Symmetry requires that if an equation holds then its versions where we rename the indices hold also, and inductivity requires that the same equations hold for the some-cylindrification-closed elements than for the whole algebra. Our algebra  $\mathfrak{A}$  that we used in the proof of Theorem 2.1 is not symmetric, hence it is not inductive, either, by the already proved part of (i) of the present theorem. We will modify the algebra  $\mathfrak{A}$  so that it becomes symmetric, but the above mentioned difference between the some-cylindrification-closed and  $\alpha$ -dimensional elements remains intact. Let R denote the set of all permutations of  $\alpha$ . Define  $\mathfrak{B}$  as the direct product of all the  $\rho$ -reducts of  $\mathfrak{A}$  for  $\rho \in R$ , i.e.,

$$\mathfrak{B} := \prod \langle \mathfrak{Rd}^{\rho} \mathfrak{A} : \rho \in R \rangle.$$

Clearly,  $\mathfrak{B}$  is symmetric and representable. We show that it is not inductive. Take the equation e used in the proof of Theorem 2.1. We have seen in Remark 2.1 that  $\rho(e(c_0x))$  is valid in  $\mathfrak{A}$  for all  $\rho \in \mathbb{R}$ . Hence,  $e(c_0x)$  is valid in all  $\mathfrak{RO}^{\rho}\mathfrak{A}$ , hence in  $\mathfrak{B}$  by its construction. However, e(x) is not valid in  $\mathfrak{B}$ since it is not valid in  $\mathfrak{A}$ . This shows that  $\mathfrak{B}$  is not inductive.

It is known that the same universal formulas are valid in  $Lf_{\alpha}$  as in  $RCA_{\alpha}$ , see [19, 4.1.29]. There is no existential formula distinguishing  $Lf_{\alpha}$  and  $RCA_{\alpha}$ , either because each  $RCA_{\alpha}$  has a subalgebra in  $Lf_{\alpha}$ . The next complexity class is  $\Delta_2$ -formulas, and our theorems so far imply that  $Lf_{\alpha}$  indeed can be distinguished from  $RCA_{\alpha}$  by a  $\Delta_2$ -formula. We note that it was known that there is a  $\Pi_2$ -formula distinguishing  $Lf_{\alpha}$  and  $RCA_{\alpha}$  (see [18, 2.6.53]).

**Corollary 4.2** There is a  $\Delta_2$ -formula which is valid in  $Lf_{\alpha}$  but is not valid in  $RCA_{\alpha}$ .

**Proof.** The property of being inductive is defined by a set D of formulas of form  $\forall \bar{x}e_1(\bar{x}) \rightarrow \forall \bar{x}e_2(\bar{x})$  where  $e_1, e_2$  are equations using variables occurring in  $\bar{x}$ . All such formulas are known to be  $\Delta_2$ . Indeed, let  $\varphi$  denote the previous formula. Then  $\varphi$  is equivalent both to the  $\Pi_2$ -formula  $\forall \bar{x} \exists \bar{y}(\neg e_2(\bar{x}) \rightarrow \neg e_1(\bar{y}))$ , and to the  $\Sigma_2$  formula  $\exists \bar{x} \forall \bar{y}(\neg e_1(\bar{x}) \lor e_2(\bar{y}))$ . There is a representable algebra  $\mathfrak{A}$  which is not inductive, by Theorem 4.5(i). Since  $\mathfrak{A}$  is not inductive, there is a  $\Delta_2$ -formula  $\varphi \in D$  which is not valid in  $\mathfrak{A}$ . Then  $\mathsf{RCA}_{\alpha} \not\models \varphi$  by  $\mathfrak{A} \in \mathsf{RCA}_{\alpha}$ . However,  $\mathsf{Lf}_{\alpha} \models \varphi$  since  $\mathsf{Lf}_{\alpha} \subseteq \mathsf{Ind}_{\alpha}$  by Theorem 4.5(ii).

**Remark 4.2** (i) We can get a concrete  $\Delta_2$  formula separating  $Lf_{\alpha}$  and  $RCA_{\alpha}$  by using Remark 2.1(ii).

(ii) From the fact that there are more subvarieties of  $\mathsf{RCA}_{\alpha}$  than generated by  $\mathsf{Lf}_{\alpha} \subseteq \mathsf{RCA}_{\alpha}$  we can immediately get that there is a subvariety V of  $\mathsf{RCA}_{\alpha}$ which is not generated by its  $\mathsf{Lf}_{\alpha}$ -members, i.e., V is not generated by  $\mathsf{V} \cap \mathsf{Lf}_{\alpha}$ . (iii) Using (ii) above, from the fact that there are more subvarieties of  $\mathsf{RCA}_{\alpha}$  than generated by  $\mathsf{Lf}_{\alpha} \subseteq \mathsf{RCA}_{\alpha}$  we can immediately get that there is a  $\Delta_2$ -formula distinguishing  $\mathsf{Lf}_{\alpha}$  and  $\mathsf{RCA}_{\alpha}$ , because the structure of subvarieties of a variety V is determined by its  $\Delta_2$ -theory. Indeed, assume that K and L have the same  $\Delta_2$ -theories. Then the same varieties are generated by subclasses of K and L, since all of the formulas of form  $\forall \bar{x}e_1(\bar{x}) \land \cdots \land \forall \bar{x}e_n(\bar{x}) \rightarrow \forall \bar{x}e_0(\bar{x})$  are  $\Delta_2$ . Indeed, let  $\mathsf{K}_0 \subseteq \mathsf{K}$ , let  $E_0 = \mathsf{Eq}(\mathsf{K}_0)$  and let  $\mathsf{L}_0 = \{\mathfrak{A} \in \mathsf{L} : \mathfrak{A} \models E_0\}$ , then  $\mathsf{Eq}(\mathsf{L}_0) = E_0$ , since  $E_0 \subseteq \mathsf{Eq}(\mathsf{L}_0)$  by the definition of  $\mathsf{L}_0$  and for all  $e \notin E_0$  we have  $\mathsf{K} \not\models \Sigma \rightarrow e$  for all finite  $\Sigma \subseteq E$ , so the same is true for L.

(iv) From what we said so far, it follows that for any  $Lf_{\alpha} \subseteq K \subseteq Sy \cap RCA_{\alpha}$  we have that the  $\Delta_2$ -theories of K and  $RCA_{\alpha}$  are different but the corresponding universal and existential theories coincide.

## 4.4 Characterization of the equational theory of $RCA_{\alpha}$

In this section we concentrate on sets of equations, rather than on algebras. Assume E is a set of equations in the language of  $\mathsf{CA}_{\alpha}$ , it contains the cylindric axioms  $(C_0) - (C_7)$  axiomatizing  $\mathsf{CA}_{\alpha}$  and it is semantically closed (i.e.,  $e \in E$  iff  $E \models e$ ). We call E inductive iff  $e(c_i x_1, \ldots, c_i x_n) \in E$  implies  $e(x_1, \ldots, x_n) \in E$  whenever i does not occur as an index of an operation in e. Thus, an algebra is inductive iff its equational theory is such. However, we will see that not all models of an inductive set of equations are inductive. In the next theorem we characterize the inductive sets of equations. We obtain that they coincide with the equational theories of subclasses of  $\mathsf{Lf}_{\alpha}$ . Equational theories of subclasses of  $\mathsf{Lf}_{\alpha}$  are important, because  $\mathsf{Lf}_{\alpha}$ s correspond to ordinary first order logic theories ([19, 4.3.28(iii)]).

## Theorem 4.6

$$E \text{ is inductive} \quad iff \quad E = \mathsf{Eq}(\mathsf{K}) \text{ for some } \mathsf{K} \subseteq \mathsf{Lf}_{\alpha}.$$

**Proof.** Assume that E is inductive. Let  $\mathfrak{F}$  be the E-free  $\omega$ -generated algebra. Then  $E = \mathsf{Eq}(\mathfrak{F})$  and  $\mathfrak{F}$  is inductive, by E being inductive. So, there is  $\mathfrak{B} \in \mathsf{Lf}_{\alpha}$  with  $\mathsf{Eq}(\mathfrak{F}) = \mathsf{Eq}(\mathfrak{B})$ , by Theorem 4.5(ii). This shows that  $E = \mathsf{Eq}(\mathsf{K})$  for  $\mathsf{K} = \{\mathfrak{B}\} \subseteq \mathsf{Lf}_{\alpha}$ . Assume now  $\mathsf{K} \subseteq \mathsf{Lf}_{\alpha}$  and let  $E = \mathsf{Eq}(\mathsf{K})$ . Then E contains the cylindric axioms  $(C_0) - (C_7)$  and is semantically closed. Also, E is inductive by Theorem 4.5(ii). Let us call *inductive rule* the rule according to which from  $e(c_i x_1, \ldots, c_i x_n)$ we can infer  $e(x_1, \ldots, x_n)$  provided that  $i \notin ind(e)$ . Note that this is a decidable rule, because given any equation we can decide whether it is of form  $e(c_i x_1, \ldots, c_i x_n)$  for an equation e such that  $i \notin ind(e)$ .

**Corollary 4.3** The equational theory of  $\mathsf{RCA}_{\alpha}$  is the least set of equations which

contains the equations  $(C_0) - (C_7)$  which define  $CA_{\alpha}$ , is closed under the 5 rules of equational logic, and is closed under the inductive rule defined above.

**Proof.** By definition, a set E of equations contains  $(C_0) - (C_7)$ , is closed under the 5 rules of equational logic, and is closed under the inductive rule iff E is inductive. This is so because equational logic is complete for its five rules. By Theorem 4.6, the least such set axiomatizes the variety generated by the largest subclass of  $Lf_{\alpha}$ , which subclass is  $Lf_{\alpha}$  itself. Now, the variety generated by  $Lf_{\alpha}$  is RCA<sub> $\alpha$ </sub>, e.g., by [19, 4.1.29].

Corollary 4.3 above gives a simple, natural enumeration for the equational theory of  $\mathsf{RCA}_{\alpha}$ . It can be considered as a solution to [19, Problem 4.1] which asks for a simple equational base for  $Eq(RCA_{\alpha})$ . Certainly, the enumeration based on the above Corollary 4.3 is much simpler than any of the three such enumerations given in  $[19, \sec 4.1]$ . It has some resemblance to the second and third enumerations given in [19]. An advantage of the present enumeration is that it stays strictly in the equational language of  $CA_{\alpha}$  while the second method given in [19] uses all first order logic formulas in the language of  $\mathsf{CA}_{\alpha}$ , and the third method even uses symbols outside the language of  $\mathsf{CA}_{\alpha}$ . A drawback of the present enumeration is that it works only for infinite  $\alpha$ , while the three methods given in [19] work for finite  $\alpha$  also. We note that possible solutions for Problems 4.1 and the related Problem 4.16 were also given in Simon [43] and Venema [45]. The root of [19, Problem 4.1] is Monk's theorem saying that  $\mathsf{RCA}_{\alpha}$  is not finite schema axiomatisable, exposing a gap between abstract and representable cylindric algebras. As we mentioned in the Introduction, this gap is addressed many ways in algebraic logic, some works in this direction are [6, 21, 22, 38, 37, 13].

**Remark 4.3** (i) Not all models of an inductive set of equations are inductive. An example is  $Eq(RCA_{\alpha})$ . It is inductive because  $RCA_{\alpha}$  is generated by  $Lf_{\alpha}$  and it has a noninductive algebra by Theorem 4.5(i). Exceptions are the equational theories of the minimal cylindric algebras in the sense that all members of these varieties are inductive. We wonder whether these are the only such exceptions or not.

(ii) Any variety of cylindric algebras generated by a class of locally finite dimensional algebras is also generated by a single  $Lf_{\alpha}$ . This was known, but this also follows from Theorems 4.5, 4.6 as follows. Let V be generated by  $K \subseteq Lf_{\alpha}$ . Then Eq(V) is inductive by Theorem 4.6, so the free algebra  $\mathfrak{F}$  of V is inductive, then it is equationally indistinguishable from a  $\mathfrak{B} \in Lf_{\alpha}$  by Theorem 4.5, and then  $Eq(V) = Eq(\mathfrak{B})$ .

(iii) A set E is inductive iff there is an ordinary first order logic theory Th such that E is the equational theory of all the concept algebras of models of Th. We briefly sketch a proof for this, we deal with the logical connections in detail in another paper. Let E be any inductive set. Then, by (ii) above, it is the equational theory of a single  $\mathfrak{B} \in Lf_{\alpha}$ . Then  $\mathfrak{B}$  is the Lindenbaum-Tarski algebra of an ordinary theory Th, by [19, 4.3.28(ii)]. It is not difficult to see that the Lindenbaum-Tarski algebra is a subdirect product of  $\{Ca^{\mathfrak{M}} : \mathfrak{M} \models Th\}$ , which finishes the proof.

## 4.5 Continuum many inductive varieties

We close the paper with showing that subclasses of concept algebras of ordinary first order logic with infinite universes generate continuum many subvarieties. The proof of this theorem will be analogous to, but simpler than, the proof of Theorem 2.1. Concept algebras of ordinary first order logic with finite universes also generate continuum many varieties, a slightly modified version of the proof of [19, 4.1.24] shows this. This is why we deal with concept algebras of models with infinite universes below.

Let  $a_m := c_{(m)} \prod \{-d_{ij} : i < j < m\}$ , for  $m \in \omega$ , cf. [18, 2.4.61]. We call a cylindric algebra of infinite base iff  $\{e_m : m \in \omega\}$  is valid in it, and  ${}_{\infty}\mathsf{Lf}_{\alpha}$ denotes the class of  $\mathsf{Lf}_{\alpha}$ s of infinite bases. An *inductive variety of infinite* base is a variety whose equational theory is inductive and which contains the equations  $\{a_m = 1 : m \in \omega\}$ . The inductive varieties of infinite base are exactly the varieties generated by subclasses of  ${}_{\infty}\mathsf{Lf}_{\alpha}$ , by Theorem 4.5. Also, they are exactly the varieties generated by concept algebras of ordinary first order logic with infinite bases, by Remark 4.3(iii).

The following is a counterpoint to Theorem 2.1. We know that there can be only continuum many inductive varieties for all  $\alpha$  because inductive

varieties are also symmetric. The following theorem says that there are indeed continuum many inductive varieties for all  $\alpha$ , even if we require the bases to be infinite.

**Theorem 4.7** Subclasses of  ${}_{\infty}\mathsf{Lf}_a$  generate continuum many subvarieties, for all infinite  $\alpha$ . In other words, there are continuum many inductive varieties of infinite base.

**Proof.** As in the proof of Theorem 2.1, we will use a set of independent equations, in this case we will use a countable set of independent equations. The *n*-th equation  $e_n$  will express that there is no partition of the universe (in the form of an equivalence relation as element of the algebra) all of whose blocks have size *n*. Then, for each  $n \in \omega$  we will exhibit an algebra  $\mathfrak{A}_n \in {}_{\infty}\mathsf{Lf}_{\alpha}$  in which  $e_n$  fails, but  $e_k$  holds for all  $k \in \omega - 2$ ,  $k \neq n$ .

We begin to write up the term expressing that "x is not an equivalence relation on the whole base set with each equivalence block having size n". The following terms express the parts of this statement (in the final equation we will replace x with  $c_2 \ldots c_n x$ ). Let  $n \ge 2$ .

The domain of x is not the base set:

$$\delta(x) := c_0 - c_1 x.$$

x is not symmetric:

$$\sigma(x) := c_0 c_1({}_2\mathsf{s}(0,1)x \oplus x)$$

x is not transitive:

$$\tau(x) := c_0 c_1 c_2 (x \cdot \mathbf{s}_{12}^{01} x - \mathbf{s}_{02}^{01} x).$$

x is not reflexive:

$$\rho(x) := c_0 c_1 (d_{01} - x).$$

There is a block in x with size < n:

$$\mu_{<}(x) := c_0 - c_1 \cdots - c_{n-1} (\prod \{-d_{ij} : i < j < n\} \cdot \prod \{\mathbf{s}_{ij}^{01} x : i < j < n\}).$$

There is a block in x with size > n:

$$\mu_{>}(x) := c_{(n+1)} (\prod \{ -d_{ij} : i < j \le n \} \cdot \prod \{ \mathbf{s}_{ij}^{01} x : i < j \le n \}).$$

The sum of all these is

$$\eta(x) := \delta(x) + \sigma(x) + \tau(x) + \rho(x) + \mu_{<}(x) + \mu_{>}(x).$$

The equation  $e_n$  is defined as

$$e_n(x) \quad := \quad \eta(c_2 \dots c_n x) = 1.$$

**Lemma 4.2** Let  $\mathfrak{A} \in \mathsf{Cs}_{\alpha}$ , let  $n \geq 2$  and let  $a = c_2 \dots c_n a \in A$ . Then  $\mathfrak{A} \models e_n(a)$  iff for all  $s \in a$  it is true that a[s, 01] is not an equivalence relation on the base set with each block of size n.

**Proof of Lemma 4.2.** Assume the conditions of the lemma, then  $\mathfrak{A} \models e_n(a)$  iff for all  $s \in {}^{\alpha}U$ , where U is the base set of  $\mathfrak{A}$ , we have  $s \in \eta(a) = \delta(a) + \cdots + \mu_{>}(a)$ . Let  $R := a[s, 01] = \{(u, v) : s(01/uv) \in a\} \subseteq U \times U$ . We have

(28) 
$$s \notin \delta(a)$$
 iff the domain of R is U.

Indeed,  $s \notin \delta(a)$  iff  $s \in -\delta(a) = -c_0 - c_1 a$  iff for all  $u \in U$  there is  $v \in U$  with  $s(01/uv) \in a$ , which means  $(u, v) \in a[s, 01]$ .

(29) 
$$s \notin \sigma(a)$$
 iff  $R$  is symmetric.

Indeed,  $s \notin \sigma(a) = c_0 c_1({}_2\mathfrak{s}(0, 1)a \oplus a)$  iff for all  $u, v \in U$  we have  $s(01/uv) \notin ({}_2\mathfrak{s}(0, 1)a \oplus a)$ , this last thing holds iff  $s(01/vu) \in a \Leftrightarrow s(01/uv) \in a$ , which means that R is symmetric.

(30) 
$$s \notin \tau(a)$$
 iff  $R$  is transitive.

Indeed,  $s \notin \tau(a) = c_0 c_1 c_2 (a \cdot \mathbf{s}_{12}^{01} a - \mathbf{s}_{02}^{01} a)$  iff for all  $u, v, w \in U$  whenever  $s(012/uvw) \in a \cdot \mathbf{s}_{12}^{01} a$  we have  $s(012/uvw) \in \mathbf{s}_{02}^{01} a$ . Now,  $s(012/uvw) \in a \cdot \mathbf{s}_{12}^{01} a$  means that  $(u, v) \in R$  and  $(v, w) \in R$  (we used  $c_2 a = a$ ). Similarly,  $s(012/uvw) \in \mathbf{s}_{02}^{01} a$  means that  $(u, w) \in R$ . Putting these together, we get that R is transitive.

(31) 
$$s \notin \rho(a)$$
 iff  $R$  is reflexive.

Indeed,  $s \notin \rho(a) = c_0 c_1 (d_{01} - a)$  iff for all  $u, v \in U$  we have u = v implies  $(u, v) \in R$ , i.e., R is reflexive.

Assume now that  $s \notin (\delta(a) + \sigma(a) + \tau(a) + \rho(a))$ . Then, by the above, we have that R is an equivalence relation on U.

(32)  $s \in \mu_{\leq}(a)$  iff there is a block in R with size < n.

Indeed,  $s \in \mu_{<}(a) = c_0 - c_1 \cdots - c_{n-1} (\prod \{-d_{ij} : i < j < n\} \cdot \prod \{\mathbf{s}_{ij}^{01}a : i < j < n\})$  iff there is  $u_0 \in U$  such that there are no  $u_1, \ldots, u_{n-1} \in U$  such that  $u_0, \ldots, u_{n-1}$  are all distinct and  $(u_i, u_j) \in R$  for all i < j < n. This means that there is a block in R with size < n.

(33)  $s \in \mu_{>}(a)$  iff there is a block in R with size > n.

Indeed,  $s \in \mu_{>}(a) = c_{(n+1)}(\prod \{-d_{ij} : i < j \leq n\} \cdot \prod \{\mathbf{s}_{ij}^{01}a : i < j \leq n\})$  iff there are  $u_0, \ldots, u_n \in U$  such that they are all distinct and  $(u_i, u_j) \in R$  for all  $i < j \leq n$ , and this means that there is a block in R with size > n.

By the above we have that  $s \in \eta(a)$  iff whenever R = s[a, 01] is an equivalence relation on U, there is either a block with size < n or else there is a block with size > n. This proves Lemma 4.2.

Let  $n \in \omega, n \geq 2$ , let U be an infinite set, and let R be an equivalence relation on U with each block of size n. Let  $\mathfrak{A}_n$  be the  $\mathsf{Cs}_{\alpha}$  with base U and generated by  $g := \{s \in U^{\alpha} : (s_0, s_1) \in R\}.$ 

**Lemma 4.3**  $\mathfrak{A}_n \not\models e_n$  but  $\mathfrak{A}_n \models e_k$  for all  $k \neq n, k \in \omega - 2$ .

**Proof of Lemma 4.3.**  $\mathfrak{A}_n \not\models e_n$  by Lemma 4.2 and  $g \in A_n$ , since clearly s[g,01] = R for all  $s \in g$  and R is an equivalence relation of the kind  $e_n$  prohibits. Let  $k \in \omega - 2$ ,  $k \neq n$ , we want to show that  $\mathfrak{A}_n \models e_k$ . By Lemma 4.2, it is enough to show that a[s,01] is not an equivalence relation on U with all blocks of size k, whenever  $s \in a = c_2 \dots c_n a$ . We begin doing this.

We call an  $X \subseteq {}^{\alpha}U$  regular if, intuitively, X is determined by its restriction to its dimension set  $\Delta(X)$ , formally

 $s \in X$  iff  $z \in X$ , whenever  $s, z \in {}^{\alpha}U$  and s, z agree on  $\Delta(X)$ 

where  $\Delta(X) := \{i \in \alpha : c_i X \neq X\}$ . Since  $\mathfrak{A}_n$  is generated by g which is a locally finite regular element, we have that a is also a regular locally finite element, by [19, 3.1.64]. Let  $S' := \operatorname{Rg}(s \upharpoonright \Delta(a) \cup \{0\})$  and let  $S := \{u \in U : (\exists v \in S')(u, v) \in R\}$ . Then S is finite since  $\Delta(a)$  is finite and each block of

R is finite. Assume that E := a[s, 01] is an equivalence relation on U with all blocks finite, and  $\geq 2$ . If E is not such then we are done by Lemma 4.2 and  $k \geq 2$ .

Let  $u, v \in U - S$  such that  $(u, v) \in E - R$ , we will derive a contradiction. Let  $w \in U - S - u/R$  be arbitrary. There are infinitely many such w. We want to show that  $w \in u/E$ , contradicting our assumption that u/E is finite. Let  $\pi : U \to U$  be a permutation of U which leaves R fixed, is identity on  $S \cup \{u\}$  and takes v to w. There is such a permutation since  $v/R \cup w/R$  is disjoint from  $S \cup \{u\}$  by our assumptions. Since  $\pi$  leaves R fixed and  $\mathfrak{A}_n$  is generated by g, we have that a is closed under  $\pi$ , i.e.,  $z \in a$  iff  $\pi \circ z \in a$  for all z. Now,  $(u, v) \in E = a[s, 01]$  means that  $s(01/uv) \in a$ . Therefore  $z := \pi \circ (s(01/uv)) \in a$ . Now,  $z_0 = \pi(u) = u$ ,  $z_1 = \pi(v) = w$ , and z agrees with s(01/uw) on  $\Delta(a)$  by  $\pi$  being the identity on S. Hence  $s(01/uw) \in a$  by  $z \in a$  and a being regular. This means  $(u, w) \in E$ , i.e.,  $w \in u/R$  as was to be shown.

Assume now that  $u, v \in U - S$  such that  $(u, v) \in R - E$ , we will derive a contradiction. Let  $(w, v) \in E$ ,  $w \neq v$ . There is such by our assumption that each block of E has at least two elements. Let  $\pi : U \to U$  be a permutation of U which is identity on  $U - \{u, v\}$  and interchanges u and v. This  $\pi$  is identity on S by  $u, v \notin S$  and it leaves R fixed by  $(u, v) \in R$ . Thus, ais closed under this  $\pi$ , too. As before,  $(w, v) \in E = a[s, 01]$  means that  $s(01/wv) \in a$ , therefore  $z := \pi \circ (s(01/wv)) \in a$ . Then  $z_0 = \pi(w) = w$ ,  $z_1 = \pi(v) = u$ , and z agrees with s(01/wu) on  $\Delta(a)$  by  $\pi$  being the identity on S. Hence  $s(01/wu) \in a$  by  $z \in a$  and a being regular. This means  $(w, u) \in E$ , contradicting  $(u, v) \notin E$  and  $(v, w) \in E$ .

We have seen that R and E agree on the infinite set U - S. Since each block of R has n elements, this means that E has at least one block with exactly n elements. So,  $e_k(a)$  holds in  $\mathfrak{A}$  by  $k \neq n$  and Lemma 4.2. By this, Lemma 4.3 has been proved.

We are ready for completing the proof of Theorem 4.7. For each  $H \subseteq \omega - 2$ let  $V_H$  be the variety generated by  $K_H := \{\mathfrak{A}_n : n \in H\} \subseteq {}_{\infty}\mathsf{Lf}_{\alpha}$ . Assume  $G, H \subseteq \omega - 2$  are distinct, say  $n \in H - G$ . Then  $\mathfrak{A}_n \in \mathsf{V}_G - \mathsf{V}_H$  by Lemma 4.3, so  $\mathsf{V}_H$  and  $\mathsf{V}_G$  are distinct. This shows that there are at least continuum many varieties generated by subclasses of  ${}_{\infty}\mathsf{Lf}_{\alpha}$ .

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