PARTITION MODELS, PERMUTATIONS OF INFINITE SETS WITHOUT FIXED POINTS, AND WEAK FORMS OF AC

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Abstract. In set theory without the Axiom of Choice (AC), we observe new relations of the following statements with weak choice forms.

- There does not exist an infinite Hausdorff space X such that every infinite subset of X contains an infinite compact subset.
- If a field has an algebraic closure then it is unique up to isomorphism.
- For every set X there is a set A such that there exists a choice function on the collection $[A]^2$ of two-element subsets of A and satisfying $|X| \leq |2^{[A]^2}|$.
- Van Douwen's Choice Principle (Every family $X = \{(X_i, \leq_i) : i \in I\}$ of linearly ordered sets isomorphic with $(\mathbb{Z}, \leq)$ has a choice function, where $\leq$ is the usual ordering on $\mathbb{Z}$).

We also extend the research works of B.B. Bruce [4]. Moreover, we prove that the principle "Any infinite locally finite connected graph has a spanning m-bush for any even integer $m \geq 4$" is equivalent to Kőnig's Lemma in ZF (i.e., the Zermelo–Fraenkel set theory without AC). We also give a new combinatorial proof to show that any infinite locally finite connected graph has a chromatic number if and only if Kőnig's Lemma holds.

1. Introduction and Abbreviations

1.1. **Forms 269, 233, and 304.** An infinite set X is *amorphous* if there does not exist a partition of X into two infinite sets. As usual, ω denotes the set of natural numbers. For sets X and Y, we write $|X| \leq |Y|$ if there is an injection $f : X \to Y$. We recall the necessary weak choice forms.

- [8, **Form 269**]: For every set X there is a set A such that there exists a choice function on the collection $[A]^2$ of two-element subsets of A and satisfying $|X| \leq |2^{[A]^2}|$.
- [8, **Form 233**]: If a field has an algebraic closure then it is unique up to isomorphism.
- [8, **Form 304**]: There does not exist an infinite Hausdorff space X such that every infinite subset of X contains an infinite compact subset.
- $\mathsf{AC}^{\mathsf{LO}}$ [8, **Form 202**]: Every linearly ordered family of non-empty sets has a choice function.
- $\mathsf{AC}^{-}_{\mathsf{n}}$ for each $n \in \omega\backslash\{0, 1\}$ [8, **Form 342(n)**]: Every infinite family $\mathcal{A}$ of n-element sets has a partial choice function, i.e., $\mathcal{A}$ has an infinite subfamily $\mathcal{B}$ with a choice function.
- The *Chain/Antichain Principle*, CAC [8, **Form 217**]: Every infinite partially ordered set (poset) has an infinite chain or an infinite antichain.

It is known that **Form 269** fails, whereas **Form 304** and **Form 233** hold in the basic Fraenkel model (labeled as Model $\mathcal{N}_1$ in [8]) (cf. [8, **Notes 41, 91, 116**]). Fix any $n \in \omega\backslash\{0, 1\}$. Halbeisen–Tachtsis [11, **Theorem 8**] constructed a permutation model (we denote by $\mathcal{N}^1_{HT}(n)$) where there exists an amorphous set, $\mathsf{AC}^{-}_{\mathsf{n}}$ fails, but CAC holds. In **section 3**, we prove the following:

(1) In ZFA (i.e., ZF with the Axiom of Extensionality weakened to allow the existence of atoms), $\mathsf{AC}^{\mathsf{LO}}$ does not imply **Form 269**.
(2) **Form 269** fails in $\mathcal{N}^1_{HT}(n)$ whereas **Form 233** and **Form 304** hold in $\mathcal{N}^1_{HT}(n)$.

Thus, **Form 233** + **Form 304** implies neither $\mathsf{AC}^{-}_{\mathsf{n}}$ nor "There are no amorphous sets" in ZFA.

1.2. **Partition models.** Bruce [4] introduced the finite partition model (denoted in [4] by $\mathcal{V}_p$), which is a variant of $\mathcal{N}_1$, replacing sets of atoms with finite partitions of the set of atoms as supports. Many, but not all, properties of $\mathcal{N}_1$ transfer to $\mathcal{V}_p$. In particular, Bruce proved that the set of atoms has no amorphous subset in $\mathcal{V}_p$ unlike in $\mathcal{N}_1$, whereas every well-ordered family of well-orderable sets has a choice function in $\mathcal{V}_p$ as in $\mathcal{N}_1$. Let A be an uncountable set of atoms, $\mathcal{G}$ be the group of all permutations of A, and the supports be countable partitions of A. We call the corresponding permutation model $\mathcal{V}^+_p$. At the end of [4], Bruce asked about the status of different weak choice forms in $\mathcal{V}^+_p$. We study the status of the following weak choice forms in $\mathcal{V}_p$ and $\mathcal{V}^+_p$.

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- EPWFP: For any infinite set X, there exists a permutation of X without fixed points (cf. [16, **section 2**]).
- $\mathsf{AC_n}$ for each $n \in \omega, n \geq 2$ [8, **Form 61**]: Every family of n-element sets has a choice function.
- $\mathsf{W}_{\aleph_1}$ (cf. [12, **Chapter 8**]): For every X, either $|X| \leq \aleph_1$ or $|X| \geq \aleph_1$.

Fix any integer $n \geq 2$. In **section 4**, we prove that if $X \in \{\mathsf{EPWFP}, \mathsf{AC_n}\}$, then X fails in $\mathcal{V}_p$ and if $Y \in \{\mathsf{EPWFP}, \mathsf{AC_n}, \mathsf{W}_{\aleph_1}\}$, then Y fails in $\mathcal{V}_p^+$.

1.3. **Van Douwen's Choice Principle in two permutation models.** Howard–Tachtsis [10, **Theorem 3.4**] constructed a permutation model $\mathcal{M}$ where the principle "Every denumerable family of denumerable sets has a multiple choice function ($\mathsf{MC}_{\aleph_0}^{\aleph_0}$)" fails. The authors of [5, proof of **Theorem 3.3**] constructed another permutation model $\mathcal{N}$ where the principle "Every countable union of countable sets is a cuf set ($\mathsf{UT}(\aleph_0, \aleph_0, \mathsf{cuf})$)" holds.[1] Keremedis–Tachtsis–Wajch [14, **Theorem 13**] proved that the principle "Every infinite compact metrizable space is Dedekind-infinite ($\mathsf{M}(\mathsf{IC}, \mathsf{DI})$)" fails in $\mathcal{N}$. In **section 5**, we modify the argument of [9, **p.175**] to prove that the following weak choice form holds in $\mathcal{N}$ and $\mathcal{M}$.

- *Van Douwen's Choice Principle,* vDCP: Every family $X = \{(X_i, \leq_i) : i \in I\}$ of linearly ordered sets isomorphic with $(\mathbb{Z}, \leq)$ ($\leq$ is the usual ordering on $\mathbb{Z}$) has a choice function.

Thus, the statements "$\mathsf{vDCP} \land \mathsf{UT}(\aleph_0, \aleph_0, \mathsf{cuf}) \land \neg \mathsf{M}(\mathsf{IC}, \mathsf{DI})$" and "$\mathsf{vDCP} \land \neg \mathsf{MC}_{\aleph_0}^{\aleph_0}$" have permutation models.

1.4. **Spanning subgraphs and chromatic number.** Fix any $n, k \in \omega \backslash \{0, 1, 2\}$ and any even integer $m \geq 4$ throughout this section. Delhommé–Morillon [6, **Corollary 1, Remark 1**] proved that AC is equivalent to *"Every bipartite connected graph has a spanning subgraph omitting $K_{n,n}$"* as well as *"Every connected graph admits a spanning m-bush"*. Galvin–Komjáth [13] proved that AC is equivalent to *"Every graph has a chromatic number"*. We study the relations between variants of the above statements and the following weak forms of AC.

- $\mathsf{AC}_{\leq \mathsf{n}}^{\omega}$: Every denumerable family of non-empty sets, each with at most n elements, has a choice function.
- $\mathsf{AC}_{\leq \mathsf{n}}^{\mathsf{WO}}$: Every well-orderable family of non-empty sets, each with at most n elements, has a choice function.
- $\mathsf{AC}_{\mathsf{fin}}^{\omega}$ [8, **Form 10**]: Every denumerable family of non-empty finite sets has a choice function.
- WUT [8, **Form 231**]: The union of a well-ordered collection of well-orderable sets is well-orderable.
- $\mathsf{AC}_{\mathsf{WO}}^{\mathsf{WO}}$ [8, **Form 165**]: Every well-orderable family of non-empty well-orderable sets has a choice function.

We introduce the following abbreviations:

- $\mathcal{Q}_{lf,c}^{n}$: Any infinite locally finite connected graph has a spanning subgraph omitting $K_{2,n}$.
- $\mathcal{Q}_{lw,c}^{k,n}$: Any infinite locally well-orderable connected graph has a spanning subgraph omitting $K_{k,n}$.
- $\mathcal{P}_{lf,c}^{m}$: Any infinite locally finite connected graph has a spanning m-bush.
- $\mathcal{C}_{lf,c}$: Any infinite locally finite connected graph has a chromatic number.

In **section 6**, we prove that $\mathsf{AC}_{\leq \mathsf{n-1}}^{\omega} + \mathcal{Q}_{lf,c}^{n}$ is equivalent to $\mathsf{AC}_{\mathsf{fin}}^{\omega}$, and $\mathcal{P}_{lf,c}^{m}$ is equivalent to $\mathsf{AC}_{\mathsf{fin}}^{\omega}$ in ZF. Moreover, WUT implies $\mathsf{AC}_{\leq \mathsf{n-1}}^{\mathsf{WO}} + \mathcal{Q}_{lw,c}^{k,n}$ and the latter implies $\mathsf{AC}_{\mathsf{WO}}^{\mathsf{WO}}$ in ZF. Recently, Stawiski [15] proved that $\mathcal{C}_{lf,c}$ is equivalent to $\mathsf{AC}_{\mathsf{fin}}^{\omega}$ in ZF. In **Remark 6.4**, we write a new proof to show that $\mathcal{C}_{lf,c}$ is equivalent to $\mathsf{AC}_{\mathsf{fin}}^{\omega}$ in ZF.

1.5. **Diagram.** Fix any $2 < n, k \in \omega$, any $2 \leq q \in \omega$, and any even integer $m \geq 4$. In **Figure 1**, known results are depicted with dashed arrows, new results in ZF are mentioned with simple arrows, and new results in ZFA are mentioned with thick dotted arrows. We denote "known equivalence" by $\sim$.

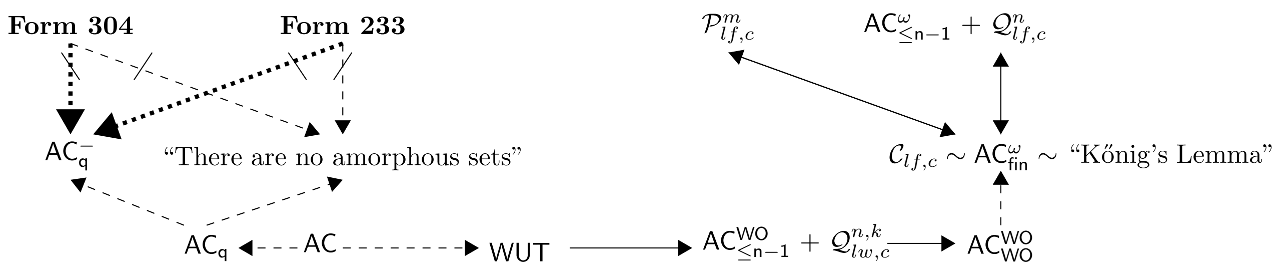

FIGURE 1. *In the above figure, we summarize the results of this note from **sections 3, 6**.*

[1] A set X is called *cuf set* if X is expressible as a countable union of finite sets.

2. Basics

Definition 2.1. Let $\mathbf{X} = (X, \tau)$ be a topological space. We say $\mathbf{X}$ is *compact* if for every $U \subseteq \tau$ such that $\bigcup U = X$ there is a finite subset $V \subseteq U$ such that $\bigcup V = X$. The space $\mathbf{X}$ is called a *Hausdorff (or T_2-space)* if any two distinct points in X can be separated by disjoint open sets. The *degree* of a vertex $v \in V_G$ of a graph $G = (V_G, E_G)$ is the number of edges emerging from v. A graph $G = (V_G, E_G)$ is *locally finite* if every vertex of G has a finite degree. We say that a graph $G = (V_G, E_G)$ is *locally well-orderable* if for every $v \in V_G$, the set of neighbors of v is well-orderable. Given a non-negative integer n, a *path of length* n in the graph $G = (V_G, E_G)$ is a one-to-one finite sequence $\{x_i\}_{0\leq i\leq n}$ of vertices such that for each $i < n$, $\{x_i, x_{i+1}\} \in E_G$; such a path joins x_0 to x_n. The graph G is *connected* if any two vertices are joined by a path of finite length. For each integer $n \geq 3$, an *n-cycle* of G is a path $\{x_i\}_{0\leq i<n}$ such that $\{x_{n-1}, x_0\} \in G$ and an *n-bush* is any connected graph with no n-cycles. We denote by K_n the complete graph on n vertices. We denote by C_n the circuit of length n. A *forest* is a graph with no cycles and a *tree* is a connected forest. A *spanning* subgraph $H = (V_H, E_H)$ of $G = (V_G, E_G)$ is a subgraph that contains all the vertices of G i.e., $V_H = V_G$. A *complete bipartite graph* is a graph $G = (V_G, E_G)$ whose vertex set V_G can be partitioned into two subsets V_1 and V_2 such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is a part of the graph. A complete bipartite graph with partitions of size $|V_1| = m$ and $|V_2| = n$, is denoted by $K_{m,n}$ for any natural number m, n. A *C-coloring* of a graph $G = (V_G, E_G)$ with a color set C is a mapping $f : V_G \to C$ such that for every $\{x, y\} \in E_G$, $f(x) \neq f(y)$. The *chromatic number* of a graph G is the smallest cardinal κ such that G can be colored by κ colors. Let $(P, \leq)$ be a partially ordered set or a poset. A subset $D \subseteq P$ is called a *chain* if $(D, \leq\restriction D)$ is linearly ordered. A subset $A \subseteq P$ is called an *antichain* if no two elements of A are comparable under $\leq$. A subset $C \subseteq P$ is called *cofinal* in P if for every $x \in P$ there is an element $c \in C$ such that $x \leq c$. A subset $A \subseteq P$ is called an *antichain* if no two elements of A are comparable under $\leq$. The size of the largest antichain of the poset $(P, \leq)$ is known as its *width*.

2.1. **Permutation models.** In this subsection, we provide a brief description of the construction of Fraenkel-Mostowski permutation models of ZFA from [12, **Chapter 4**]. Let M be a model of $\mathsf{ZFA} + \mathsf{AC}$ where A is a set of atoms. Let $\mathcal{G}$ be a group of permutations of A. Let $\mathcal{F}$ be a normal filter of subgroups of $\mathcal{G}$. For $x \in M$, we say

$$sym_{\mathcal{G}}(x) = \{g \in \mathcal{G} : g(x) = x\} \text{ and } \text{fix}_{\mathcal{G}}(x) = \{\phi \in \mathcal{G} : \forall y \in x(\phi(y) = y)\}. \quad (1)$$

We say x is *symmetric* if $sym_{\mathcal{G}}(x) \in \mathcal{F}$ and x is *hereditarily symmetric* if x is symmetric and each element of the transitive closure of x is symmetric. We define the permutation model $\mathcal{N}$ with respect to $\mathcal{G}$ and $\mathcal{F}$, to be the class of all hereditarily symmetric sets. It is well-known that $\mathcal{N}$ is a model of ZFA (cf. [12, **Theorem 4.1**]). If $\mathcal{I} \subseteq \mathcal{P}(A)$ is a normal ideal, then the set $\{\text{fix}_{\mathcal{G}}(E) : E \in \mathcal{I}\}$ generates a normal filter (say $\mathcal{F}_{\mathcal{I}}$) over $\mathcal{G}$. Let $\mathcal{N}$ be the permutation model determined by M, $\mathcal{G}$, and $\mathcal{F}_{\mathcal{I}}$. We say $E \in \mathcal{I}$ is a *support* of a set $\sigma \in \mathcal{N}$ if $\text{fix}_{\mathcal{G}}(E) \subseteq sym_{\mathcal{G}}(\sigma)$.

Lemma 2.2. *The following hold:*

(1) *An element x of $\mathcal{N}$ is well-orderable in $\mathcal{N}$ if and only if* $\text{fix}_{\mathcal{G}}(x) \in \mathcal{F}_{\mathcal{I}}$ (cf. [12, **Equation (4.2), p.47**]). *Thus, an element x of $\mathcal{N}$ with support E is well-orderable in $\mathcal{N}$ if* $\text{fix}_{\mathcal{G}}(E) \subseteq \text{fix}_{\mathcal{G}}(x)$.
(2) *For all $\pi \in \mathcal{G}$ and all $x \in \mathcal{N}$ such that E is a support of x, $sym_{\mathcal{G}}(\pi x) = \pi\ sym_{\mathcal{G}}(x)\pi^{-1}$ and* $\text{fix}_{\mathcal{G}}(\pi E) = \pi\ \text{fix}_{\mathcal{G}}(E)\pi^{-1}$ (cf. [12, proof of **Lemma 4.4**]).

A *pure set* in a model M of ZFA is a set with no atoms in its transitive closure. The *kernel* is the class of all pure sets of M. In this paper,

- Fix an integer $n \geq 2$. We denote by $\mathcal{N}^1_{HT}(n)$ the permutation model constructed in [11, **Theorem 8**].
- We denote by $\mathcal{V}_p$ the finite partition model constructed in [4] and by $\mathcal{V}^+_p$ the countable partition model mentioned in [4, **section 5**].
- We denote by $\mathcal{N}_1$ the basic Fraenkel model (cf. [8]).
- We denote by $\mathcal{N}_{12}(\aleph_1)$, the following variant of the basic Fraenkel model: Let A be an uncountable set of atoms, $\mathcal{G}$ be the group of all permutations of A, and the supports are countable subsets of A (cf. [8]).

3. **Form 269**, **Form 233**, and **Form 304**

We recall that $\mathsf{AC}^{\mathsf{LO}}$ implies LW (Every linearly ordered set can be well-ordered) in ZFA and that the latter implication is not reversible in ZFA (cf. [8]).

Theorem 3.1. $\mathsf{AC}^{\mathsf{LO}}$ *does not imply **Form 269** in* ZFA. *So, neither* LW *nor* $\mathsf{AC}^{\mathsf{WO}}$ *implies **Form 269** in* ZFA.

Proof. We present two known models.

First model: Fix a successor aleph $\aleph_{\alpha+1}$. We describe the permutation model $\mathcal{V}$ given in the proof of [12, **Theorem 8.9**]. We start with a model M of $\mathsf{ZFA} + \mathsf{AC}$ with a set A of atoms of cardinality $\aleph_{\alpha+1}$. Let $\mathcal{G}$ be the group of all permutations of A and let $\mathcal{F}$ be the normal filter of subgroups of $\mathcal{G}$ generated by $\{\text{fix}_{\mathcal{G}}(E) : E \in [A]^{<\aleph_{\alpha+1}}\}$. Let $\mathcal{V}$ be the permutation model determined by M, $\mathcal{G}$, and $\mathcal{F}$. In the proof of [12, **Theorem 8.9**], Jech proved that $\mathsf{AC^{WO}}$ (Every well-orderable set of non-empty sets has a choice function) holds in $\mathcal{V}$.

We recall a variant of $\mathcal{V}$ from [16, **Theorem 3.5(i)**]. Let M and A as above, and let $\mathcal{N}$ be the permutation model determined by M, $\mathcal{G}'$ and $\mathcal{F}'$, where $\mathcal{G}'$ is the group of permutations of A which move at most $\aleph_\alpha$ atoms, and $\mathcal{F}'$ is the normal filter on $\mathcal{G}'$ generated by $\{\text{fix}_{\mathcal{G}'}(E) : E \in [A]^{<\aleph_{\alpha+1}}\}$. Tachtsis [16, **Theorem 3.5(i)**] proved that $\mathcal{N} = \mathcal{V}$ and $\mathsf{AC^{LO}}$ holds in $\mathcal{N}$. We slightly modify the arguments of [8, **Note 91**] to prove that **Form 269** fails in $\mathcal{N}$. We show that for any set X in $\mathcal{N}$ if the set $[X]^2$ of two element subset of X has a choice function, then X is well-orderable in $\mathcal{N}$. Assume that X is such a set and let E be a support of X and a choice function f on $[X]^2$. In order to show that X is well-orderable in $\mathcal{N}$, it is enough to prove that $\text{fix}_{\mathcal{G}'}(E) \subseteq \text{fix}_{\mathcal{G}'}(X)$ (cf. **Lemma 2.2(1)**). Assume $\text{fix}_{\mathcal{G}'}(E) \nsubseteq \text{fix}_{\mathcal{G}'}(X)$, then there is a $y \in X$ and a $\phi \in \text{fix}_{\mathcal{G}'}(E)$ with $\phi(y) \neq y$. Under such assumptions, Tachtsis constructed a permutation $\psi \in \text{fix}_{\mathcal{G}'}(E)$ such that $\psi(y) \neq y$ but $\psi^2(y) = y$ (cf. the proof of LW in $\mathcal{N}$ from [16, **Theorem 3.5(i)**]). This contradicts our choice of E as a support for a choice function on $[X]^2$ since ψ fixes $\{\psi(y), y\}$ but moves both of its elements. So **Form 269** fails in $\mathcal{N}$.

Second model: We consider the permutation model $\mathcal{N}$ given in the proof of [18, **Theorem 4.7**] where $\mathsf{AC^{LO}}$ and therefore LW hold. Following the above arguments and the arguments in [18, **claim 4.10**], we can see that **Form 269** fails in $\mathcal{N}$. □

We recall a result of Pincus, which we need in order to prove **Theorem 3.3**.

Lemma 3.2. (**Pincus;** [8, **Note 41**]) *If $\mathcal{K}$ is an algebraically closed field, if π is a non-trivial automorphism of $\mathcal{K}$ satisfying $\pi^2 = 1_{\mathcal{K}}$ (the identity on $\mathcal{K}$), and if $i \in \mathcal{K}$ is a square root of -1, then $\pi(i) = -i \neq i$.*

Theorem 3.3. *Fix any $2 \leq n \in \omega$. There is a model $\mathcal{M}$ of* ZFA *where* $\mathsf{AC^-_n}$ *and the statement 'there is no amorphous set' fail. Moreover,* **Form 269** *fails in $\mathcal{M}$ whereas* **Form 233** *and* **Form 304** *hold in $\mathcal{M}$.*

Proof. We consider the permutation model $\mathcal{N}^1_{HT}(n)$ constructed by Halbeisen–Tachtsis [11, **Theorem 8**] where for arbitrary integer $n \geq 2$, $\mathsf{AC^-_n}$ fails. We fix an arbitrary integer $n \geq 2$ and recall the model constructed in the proof of [11, **Theorem 8**]. We start with a model M of $\mathsf{ZFA} + \mathsf{AC}$ where A is a countably infinite set of atoms written as a disjoint union $\bigcup\{A_i : i \in \omega\}$ where for each $i \in \omega$, $A_i = \{a_{i_1}, a_{i_2}, ..., a_{i_n}\}$ and $|A_i| = n$. The group $\mathcal{G}$ is defined in [11] in a way so that if $\eta \in \mathcal{G}$, then η only moves finitely many atoms and for all $i \in \omega$, $\eta(A_i) = A_k$ for some $k \in \omega$. Let $\mathcal{F}$ be the filter of subgroups of $\mathcal{G}$ generated by $\{\text{fix}_{\mathcal{G}}(E) : E \in [A]^{<\omega}\}$. Then $\mathcal{N}^1_{HT}(n)$ is the permutation model determined by M, $\mathcal{G}$, and $\mathcal{F}$. Following **point 1** in the proof of [11, **Theorem 8**], both A and $\mathcal{A} = \{A_i\}_{i\in\omega}$ are amorphous in $\mathcal{N}^1_{HT}(n)$. If X is a set in $\mathcal{N}^1_{HT}(n)$, then without loss of generality we may assume that $E = \bigcup_{i=0}^m A_i$ is a support of X for some $m \in \omega$.

claim 3.4. *Suppose X is not a well-ordered set in $\mathcal{N}^1_{HT}(n)$, and let $E = \bigcup_{i=0}^m A_i$ be a support of X. Then there is a $t \in X$ with support $F \supsetneq E$, a permutation $\psi \in \text{fix}_{\mathcal{G}}E$ and a $y \in X$ such that $t \neq y$, $\psi(t) = y$ and $\psi(y) = t$.*

Proof. Since X is not well-ordered, and E is a support of X, $\text{fix}_{\mathcal{G}}(E) \nsubseteq \text{fix}_{\mathcal{G}}(X)$ by **Lemma 2.2(1)**. So there is a $t \in X$ and a $\psi \in \text{fix}_{\mathcal{G}}(E)$ such that $\psi(t) \neq t$. Let F be a support of t containing E. Without loss of generality, we may assume that F is a union of finitely many A_i's. We slightly modify the arguments of [18, **claim 4.10**]. Let $W = \{a \in A : \psi(a) \neq a\}$. We note that W is finite since if $\eta \in \mathcal{G}$, then η only moves finitely many atoms. Let U be a finite subset of A which is disjoint from $F \cup W$ and such that there exists a bijection $H : tr(U) \to tr((F \cup W)\backslash E)$ (where for a set $x \subseteq A$, $tr(x) = \{i \in \omega : A_i \cap x \neq \emptyset\}$) with the property that if $i \in tr((F \cup W)\backslash E)$ is such that $A_i \subseteq (F \cup W)\backslash E$ then $A_{H^{-1}(i)} \subseteq U$; otherwise if $A_i \nsubseteq (F \cup W)\backslash E$, which means that $A_i \cap F = \emptyset$ and $A_i \nsubseteq W$, then $|W \cap A_i| = |U \cap A_{H^{-1}(i)}|$. Let $f : U \to (F \cup W)\backslash E$ be a bijection such that $\forall i \in tr(U)$, $f \restriction U \cap A_i$ is a one-to-one function from $U \cap A_i$ onto $((F \cup W)\backslash E) \cap A_{H(i)}$. Let $f' : \bigcup_{i\in tr(U)} A_i\backslash(U \cap A_i) \to \bigcup_{i\in tr(U)} A_{H(i)}\backslash(((F \cup W)\backslash E) \cap A_{H(i)})$ be a bijection such that $\forall i \in tr(U)$, $f' \restriction (A_i\backslash(U \cap A_i))$ is a one-to-one function from $A_i\backslash(U \cap A_i)$ onto $A_{H(i)}\backslash(((F \cup W)\backslash E) \cap A_{H(i)})$. Let

$$\delta = \prod_{u\in U}(u, f(u)) \prod_{u\in\bigcup_{i\in tr(U)} A_i\backslash(U\cap A_i)}(u, f'(u))$$

be a product of disjoint transpositions. Then δ only moves finitely many atoms, and for all $i \in \omega$, $\delta(A_i) = A_k$ for some $k \in \omega$. Moreover, $\delta \in \text{fix}_{\mathcal{G}}(E)$, $\delta^2(t) = t$, and $\delta(t) \neq t$ by the arguments in [18, **claim 4.10**]. □

claim 3.5. *In $\mathcal{N}^1_{HT}(n)$, the following hold:*

(1) **Form 269** *fails.*
(2) **Form 304** *holds.*
(3) **Form 233** *holds.*

Proof. (1). Following **claim 3.4** and the arguments in the proof of **Theorem 3.1**, **Form 269** fails in $\mathcal{N}^1_{HT}(n)$.

(2). We modify the arguments of [8, **Note 116**] to prove that **Form 304** holds in $\mathcal{N}^1_{HT}(n)$. Let X be an infinite Hausdorff space in $\mathcal{N}^1_{HT}(n)$, and $E = \bigcup_{i\in K} A_i$ be a support of X and its topology where $K \in [\omega]^{<\omega}$. We show there is an infinite $Y \subseteq X$ in $\mathcal{N}^1_{HT}(n)$ such that Y has no infinite compact subsets in $\mathcal{N}^1_{HT}(n)$. If X is well-orderable, then we can use transfinite induction without using any form of choice to finish the proof. Suppose X is not well-orderable in $\mathcal{N}^1_{HT}(n)$. By **Lemma 2.2(1)**, there is an $x \in X$ and a $\phi \in \text{fix}_{\mathcal{G}}(E)$ such that $\phi(x) \neq x$. Let $F = \bigcup_{i\in K'} A_i$ be a support of x where $K' \in [\omega]^{<\omega}$. Since E is not a support of x, $F\backslash E \neq \emptyset$. Without loss of generality assume that $E \subsetneq F$. We also assume that $\{A_i : i \in K'\}$ has the fewest possible copies A_j outside $\{A_i : i \in K\}$. Let $i_0 \in K'$ such that $A_{i_0} \cap E = \emptyset$. We define the following set.

$$f = \{(\psi(x), \psi(A_{i_0})) : \psi \in \text{fix}_{\mathcal{G}}(F\backslash A_{i_0})\}$$

Tacthsis proved that f is a function with $dom(f) \subseteq X$ and $ran(f) = \mathcal{A}\backslash\{A_i : i \in K', i \neq i_0\}$, where $\mathcal{A} = \{A_i : i \in \omega\}$ and $Y = dom(f)$ is an amorphous subset of X (cf. proof of [20, **Lemma 2**]).[2] Since $\phi(x) \neq x$ and X is an infinite Hausdorff space, we can choose open sets C and D so that $x \in C$, $\phi(x) \in D$ and $C \cap D = \emptyset$. Since Y is amorphous in $\mathcal{N}^1_{HT}(n)$, every subset of Y in the model must be finite or cofinite. Thus at least one of $Y \cap C$ or $Y \cap D$ is finite. We may assume that $Y \cap C$ is finite. Then we can conclude that $\mathcal{C} = \{\psi(C) \cap Y : \psi \in \text{fix}_{\mathcal{G}}(F\backslash A_{i_0})\}$ is an open cover for Y and each element of $\mathcal{C}$ is finite. So there is an infinite $Y \subseteq X$ in $\mathcal{N}^1_{HT}(n)$ such that for any infinite subset Z of Y, $\mathcal{C}$ is an open cover for Z without a finite subcover.

(3). We follow the arguments due to Pincus from [8, **Note 41**] and use **claim 3.4** to prove that **Form 233** holds in $\mathcal{N}^1_{HT}(n)$. For the reader's convenience, we write down the proof. Let $(\mathcal{K}, +, \cdot, 0, 1)$ be a field in $\mathcal{N}^1_{HT}(n)$ with finite support $E \subset A$ and assume that $\mathcal{K}$ is algebraically closed. Without loss of generality assume that $E = \bigcup_{i=0}^{m} A_i$. We show that every element of $\mathcal{K}$ has support E which implies that $\mathcal{K}$ is well-orderable in $\mathcal{N}^1_{HT}(n)$ and therefore the standard proof of the uniqueness of algebraic closures (using AC) is valid in $\mathcal{N}^1_{HT}(n)$. For the sake of contradiction, assume that $x \in \mathcal{K}$ does not have support E. Let $F = \bigcup_{i=0}^{m+k} A_i$ be a support of x containing E. By **claim 3.4**, there is a permutation ψ in $\text{fix}_{\mathcal{G}}E$ such that $\psi(x) \neq x$ and ψ^2 is the identity. The permutation ψ induces an automorphism of $(\mathcal{K}, +, \cdot, 0, 1)$ and we can therefore apply **Lemma 3.2** to conclude that $\psi(i) = -i \neq i$ for some square root i of -1 in $\mathcal{K}$. We can follow the arguments from [8, **Note 41**] to see that for every permutation π of A that fixes E pointwise, $\pi(i) = i$ for every square root i of -1 in $\mathcal{K}$. Hence we arrive at a contradiction. □

□

Remark 3.6. For each regular $\aleph_\alpha$,[3] we denote by $\mathsf{CAC}_1^{\aleph_\alpha}$ the statement *"If in a poset all antichains are finite and all chains have size at most $\aleph_\alpha$ and there exists at least one chain with size $\aleph_\alpha$ then the poset has size $\aleph_\alpha$"*. In [1, **Theorem 4.3, Remark 4.4**] we proved that the statement "For every regular $\aleph_\alpha$, $\mathsf{CAC}_1^{\aleph_\alpha}$" holds in $\mathcal{N}^1_{HT}(n)$ and $\mathcal{N}_1$. We present different proofs. First, we recall the following results communicated to us by Tachtsis (cf. [1, **Lemma 4.1, Corollary 4.2**]):

(1) The statement *"If $(P, \leq)$ is a poset such that P is well-ordered, and if all antichains in P are finite and all chains in P are countable, then P is countable"* holds in any Fraenkel-Mostowski model.
(2) $\mathsf{UT}(\aleph_\alpha, \aleph_\alpha, \aleph_\alpha)$ implies the statement *"If $(P, \leq)$ is a poset such that P is well-ordered, and if all antichains in P are finite and all chains in P have size $\aleph_\alpha$, then P has size $\aleph_\alpha$"* for any regular $\aleph_\alpha$ in ZF. [4]

Fix $\mathcal{N} \in \{\mathcal{N}^1_{HT}(n), \mathcal{N}_1\}$. Let $(P, \leq)$ be a poset in $\mathcal{N}$ such that all antichains in P are finite and all chains in P have size $\aleph_\alpha$. Let $E \in [A]^{<\omega}$ be a support of $(P, \leq)$.

Case (i): Let $\mathcal{N} = \mathcal{N}^1_{HT}(n)$. Then for each element $p \in P$, either $\text{Orb}_{\text{fix}_{\mathcal{G}}E}(p) = \{\phi(p) : \phi \in \text{fix}_{\mathcal{G}}(E)\} = \{p\}$ or $\text{Orb}_{\text{fix}_{\mathcal{G}}E}(p)$ is infinite (cf. [20, **Remark 2.2**]). Following the arguments of [17, **claims 3.5, 3.6**], $Orb_{\text{fix}_{\mathcal{G}}E}(p)$ is an antichain in P for each $p \in P$ and $\mathcal{O} = \{Orb_{\text{fix}_{\mathcal{G}}E}(p) : p \in P\}$ is a well-ordered partition of P. So by assumption, $\text{Orb}_{\text{fix}_{\mathcal{G}}E}(p) = \{p\}$. Thus P is well-orderable. The rest follows from (2), since WUT holds in $\mathcal{N}$ and WUT implies $\mathsf{UT}(\aleph_\alpha, \aleph_\alpha, \aleph_\alpha)$ in any permutation model (cf. [8, **p. 176**]).

Case (ii): Let $\mathcal{N} = \mathcal{N}_1$. If P is well-orderable, then we are done. Suppose P is not well-orderable. By **Lemma 2.4(1)**, there is a $t \in P$ and a $\pi \in \text{fix}_{\mathcal{G}}(E)$ such that $\pi(t) \neq t$. Let $F \cup \{a\}$ be a support of t where $a \in A\backslash(E \cup F)$. Under such assumptions, Blass [3, **p.389**] proved that

$$f = \{(\pi(a), \pi(t)) : \pi \in \text{fix}_{\mathcal{G}}(E \cup F)\}$$

[2] If $f : X \to Y$ is a function, then we denote "the range of f" by $ran(f)$ and "the domain of f" by $dom(f)$.

[3] We assume that α is an ordinal and that $\aleph_\alpha$ is the α^{th} infinite initial ordinal (where an ordinal β is "initial" if β is not equipotent with an ordinal $\gamma \in \alpha$).

[4] $\mathsf{UT}(\aleph_\alpha, \aleph_\alpha, \aleph_\alpha)$ is the statement "If A and every member of A has cardinality $\aleph_\alpha$, then $|\bigcup A| = \aleph_\alpha$" which is [8, **Form 23**].

is a bijection from $A-(E \cup F)$ onto $ran(f) \subset P$. Now, $ran(f)=\mathrm{Orb}_{\mathrm{fix}_{\mathcal{G}}(E\cup F)}(t)=\{\pi(t):\pi\in \mathrm{fix}_{\mathcal{G}}(E\cup F)\}$ is an infinite antichain of P (cf. the proof [12, **Lemma 9.3**]) in $\mathcal{N}$, which contradicts our assumption.

Fix $k,n \in \omega\backslash\{0,1\}$. Tachtsis [17, **Theorem 3.7, Remark 3.8**] proved that the statement "If P is a poset with width k while at least one k-element subset of P is an antichain, then P can be partitioned into k chains", abbreviated as DT, holds in $\mathcal{N}_1$ and $\mathcal{N}^1_{HT}(2)$. Using the above arguments we can give a different proof of DT in $\mathcal{N}^1_{HT}(n)$ and $\mathcal{N}_1$ since DT for well-ordered infinite posets with finite width is provable in ZF [17, **Theorem 3.1(i)**].

4. Permutations of infinite sets in Partition models

For a set A, $\mathrm{Sym}(A)$ and $\mathrm{FSym}(A)$ denote the set of all permutations of A and the set of all $\phi\in\mathrm{Sym}(A)$ such that $\{x\in A:\phi(x)\neq x\}$ is finite. For a set A of size at least $\aleph_\alpha$, $\aleph_\alpha\mathrm{Sym}(A)$ denotes the set of all $\phi\in\mathrm{Sym}(A)$ such that $\{x\in A:\phi(x)\neq x\}$ has cardinality at most $\aleph_\alpha$ (cf. [16, **section 2**]). We recall a fact.

Lemma 4.1. (**Tachtsis;** [16, **Theorem 3.1**]) EPWFP *implies "For every infinite set X, $Sym(X)\neq FSym(X)$".*

4.1. **Finite partition model.** We recall the finite partition model $\mathcal{V}_p$ from [4]. Let A be a countably infinite set of atoms. Let $\mathcal{G}$ be the group of all permutations of A, S be the set of all finite partitions of A, and $\mathcal{F}=\{H :$ H is a subgroup of $\mathcal{G}$, $H\supseteq \mathrm{fix}_{\mathcal{G}}(P)$ for some $P\in S\}$ be the normal filter of subgroups of $\mathcal{G}$. The model $\mathcal{V}_p$ is the permutation model determined by M, $\mathcal{G}$, and $\mathcal{F}$. In $\mathcal{V}_p$, WUT holds, A has no infinite amorphous subset and $\mathcal{P}(A)$ is Dedekind-finite (cf. [4, **Proposition 4.3**]). We note that $\mathsf{MC}^{\aleph_0}$ (Every denumerable family $\mathcal{A}$ of non-empty sets has a multiple choice function) fails in $\mathcal{V}_p$. Suppose $\mathsf{MC}^{\aleph_0}$ is true and f is a multiple choice function for the denumerable set $\{[A]^n : 2\leq n\in\omega\}$. Then $\{\bigcup f[A]^n : 2\leq n\in\omega\}$ is a denumerable subset of $\mathcal{P}(A)$, contradicting the fact that $\mathcal{P}(A)$ is Dedekind-finite in $\mathcal{V}_p$.

Theorem 4.2. *The following fail in $\mathcal{V}_p$:*

(1) EPWFP.
(2) $\mathsf{AC_n}$ *for any integer $n\geq 2$.*

Proof. (1). By **Lemma 4.1**, it is enough to show that $(\mathrm{Sym}(A))^{\mathcal{V}_p}=\mathrm{FSym}(A)$. For the sake of contradiction, assume that f is a permutation of A in $\mathcal{V}_p$, which moves infinitely many atoms. Let $P=\{P_j : j\leq k\}$ be a support of f for some $k\in\omega$. Without loss of generality, assume that $P_0,...,P_n$ are the singleton and tuple blocks for some $n<k$. Then there exists $n<i\leq k$ where $a\in P_i$ and $b\in\bigcup P\backslash(P_0\cup...\cup P_n\cup\{a\})$ such that $b=f(a)$.

Case (i): Let $b\in P_i$. Consider $\phi\in\mathrm{fix}_{\mathcal{G}}(P)$ such that ϕ fixes all the atoms in all the blocks other than P_i and ϕ moves every atom in P_i except b. Thus, $\phi(b)=b$, $\phi(a)\neq a$, and $\phi(f)=f$ since P is the support of f. Thus $(a,b)\in f \implies (\phi(a),\phi(b))\in\phi(f)\implies(\phi(a),b)\in f$. So f is not injective; a contradiction.

Case (ii): Let $b\notin P_i$. Consider $\phi\in\mathrm{fix}_{\mathcal{G}}(P)$ such that ϕ fixes all the atoms in all the blocks other than P_i and ϕ moves every atom in P_i. Then again we can obtain a contradiction as in Case (i).

(2). Fix any integer $n\geq 2$. The set $S=\{x : x\in[A]^n\}$ has no choice function in $\mathcal{V}_p$. Assume that f is a choice function of S and let P be a support of f. Since A is countably infinite and P is a finite partition of A, there is a $p\in P$ such that $|p|$ is infinite. Let $a_1,...,a_n\in p$ and $\pi\in\mathrm{fix}_{\mathcal{G}}(P)$ be such that $\pi a_1=a_2$, $\pi a_2=a_3$,..., $\pi a_{n-1}=a_n$, $\pi a_n=a_1$. Without loss of generality, assume that $f(a_1,...,a_n)=a_1$. Thus, $\pi f(a_1,...,a_n)=\pi a_1 \implies f(\pi(a_1),...,\pi(a_n))=a_2\implies f(a_2,a_3,...,a_n,a_1)=a_2$. Thus f is not a function; a contradiction. □

4.2. **Countable partition model.** Let A be an uncountable set of atoms, $\mathcal{G}$ be the group of all permutations of A, and S be the set of all countable partitions of A.

Lemma 4.3. $\mathcal{F}=\{H : H$ *is a subgroup of* $\mathcal{G}$, $H\supseteq\mathrm{fix}_{\mathcal{G}}(P)$ *for some* $P\in S\}$ *is a normal filter of subgroups of* $\mathcal{G}$.

Proof. We modify the arguments of [4, **Lemma 4.1**] and verify the clauses of a normal filter from [12, **p.46**].

(1). We can see that $\mathcal{G}\in\mathcal{F}$.

(2). Let $H\in\mathcal{F}$ and K be a subgroup of $\mathcal{G}$ such that $H\subseteq K$. Then there exist $P\in S$ such that $\mathrm{fix}_{\mathcal{G}}(P)\subseteq H$. So, $\mathrm{fix}_{\mathcal{G}}(P)\subseteq K$ and $K\in\mathcal{F}$.

(3). Let $K_1,K_2\in\mathcal{F}$. Then there exist $P_1,P_2\in S$ such that $\mathrm{fix}_{\mathcal{G}}(P_1)\subseteq K_1$ and $\mathrm{fix}_{\mathcal{G}}(P_2)\subseteq K_2$. Let $P_1\wedge P_2=\{p\cap q : p\in P_1, q\in P_2, p\cap q\neq\emptyset\}$. Clearly, $\mathrm{fix}_{\mathcal{G}}(P_1\wedge P_2)\subseteq\mathrm{fix}_{\mathcal{G}}(P_1)\cap\mathrm{fix}_{\mathcal{G}}(P_2)\subseteq K_1\cap K_2$. Since the product of two countable sets is countable, $P_1\wedge P_2\in S$. Thus $K_1\cap K_2\in\mathcal{F}$.

(4). Let $\pi\in\mathcal{G}$ and $H\in\mathcal{F}$. Then there exists $P\in S$ such that $\mathrm{fix}_{\mathcal{G}}(P)\subseteq H$. Since $\mathrm{fix}_{\mathcal{G}}(\pi P)=\pi\ \mathrm{fix}_{\mathcal{G}}(P)\pi^{-1}\subseteq\pi H\pi^{-1}$ by **Lemma 2.2(2)**, it is enough to show $\pi P\in S$. Clearly, πP is countable, since P is countable. Following the arguments of [4, **Lemma 4.1(iv)**], we can see that πP is a partition of A.

(5). Fix any $a \in A$. Consider any countable partition P of A where $\{a\}$ is a singleton block of P. We can see that $\text{fix}_{\mathcal{G}}P \subseteq \{\pi \in \mathcal{G} : \pi(a) = a\}$. Thus, $\{\pi \in \mathcal{G} : \pi(a) = a\} \in \mathcal{F}$. □

We call the permutation model (denoted by $\mathcal{V}_p^+$) determined by M, $\mathcal{G}$, and $\mathcal{F}$, the countable partition model. Tachtsis [16, **Theorem 3.1(2)**] proved that $\mathsf{DF} = \mathsf{F}$ (Every Dedekind-finite set is finite) implies "For every infinite set X, Sym(X) $\neq$ FSym(X)" in ZF. Inspired by that idea we first prove the following.

Proposition 4.4. (ZF) *The following hold:*

(1) $\mathsf{W}_{\aleph_{\alpha+1}}$ *implies "for any set X of size $\aleph_{\alpha+1}$, Sym(X) $\neq \aleph_\alpha$Sym(X)".*
(2) EPWFP *implies "for any set X of size $\aleph_{\alpha+1}$, Sym(X) $\neq \aleph_\alpha$Sym(X)".*

Proof. (1). Let X be a set of size $\aleph_{\alpha+1}$ and let us assume Sym(X) $= \aleph_\alpha$Sym(X). We prove that there is no injection f from $\aleph_{\alpha+1}$ into X. Assume there exists such an f. Let $\{y_n\}_{n\in\aleph_{\alpha+1}}$ be an enumeration of the elements of $Y = f(\aleph_{\alpha+1})$. We can use transfinite recursion, without using any form of choice, to construct a bijection $f : Y \to Y$ such that $f(x) \neq x$ for any $x \in Y$. Define $g : X \to X$ as follows: $g(x) = f(x)$ if $x \in Y$, and $g(x) = x$ if $x \in X \backslash Y$. Clearly $g \in$ Sym(X)$\backslash$ $\aleph_\alpha$Sym(X), and hence Sym(X) $\neq \aleph_\alpha$Sym(X), a contradiction.

(2). This is straightforward. □

Theorem 4.5. *The following hold:*

(1) $\mathcal{N}_{12}(\aleph_1) \subset \mathcal{V}_p^+$.
(2) EPWFP *fails in* $\mathcal{V}_p^+$.
(3) $\mathsf{AC_n}$ *fails in* $\mathcal{V}_p^+$ *for any integer* $n \geq 2$.
(4) A *cannot be linearly ordered in* $\mathcal{V}_p^+$.
(5) $\mathsf{W}_{\aleph_1}$ *fails in* $\mathcal{V}_p^+$.

Proof. (1). Let $x \in \mathcal{N}_{12}(\aleph_1)$ with support E. So $\text{fix}_{\mathcal{G}}(E) \subseteq sym_{\mathcal{G}}(x)$. Then $P = \{\{a\}\}_{a\in E} \cup \{A\backslash E\}$ is a countable partition of A, and $\text{fix}_{\mathcal{G}}(P) = \text{fix}_{\mathcal{G}}(E)$. Thus $\text{fix}_{\mathcal{G}}(P) \subseteq sym_{\mathcal{G}}(x)$ and so $x \in \mathcal{V}_p^+$ with support P.

(2). Similarly to the proof of $\neg\mathsf{EPWFP}$ in $\mathcal{V}_p$ (cf. the proof of **Theorem 4.2(1)**), one may verify that if f is a permutation of A in $\mathcal{V}_p^+$, then the set $\{x \in A : f(x) \neq x\}$ has cardinality at most $\aleph_0$. Since A is uncountable, it follows that "for any uncountable X, Sym(X) $\neq \aleph_0$Sym(X)" fails in $\mathcal{V}_p^+$.

(3, 4). Fix any integer $n \geq 2$. Similarly to the proof of **Theorem 4.2(2)**, one may verify that the set $S = \{x : x \in [A]^n\}$ has no choice function in $\mathcal{V}_p^+$. Consequently, $\mathsf{AC_n}$ fails in $\mathcal{V}_p^+$ and A cannot be linearly ordered.

(5). We can use the arguments in (2) and **Proposition 4.4(1)**. However, we write a different argument. We prove that there is no injection $f : \aleph_1 \to A$. Assume there exists such an f with support P, and let $\pi \in \text{fix}_{\mathcal{G}}(P)$ be such that π moves every atom in each non-singleton block of P. Since P contains only countably many singletons, π fixes only countably many atoms. Fix $n \in \aleph_1$. Since n is in the kernel, $\pi(n) = n$. Thus $\pi(f(n)) = f(\pi(n)) = f(n)$. But f is one-to-one, and thus, π fixes $\aleph_1$ many values of f in A, a contradiction. □

5. Van Douwen's Choice Principle in two permutation models

We recall two known permutation models, say $\mathcal{N}$ and $\mathcal{M}$, from [5, proof of **Theorem 3.3**] and [10, proof of **Theorem 3.4**] respectively.

Model $\mathcal{N}$: We start with a model M of $\mathsf{ZFA} + \mathsf{AC}$ with a set A of atoms such that A has a denumerable partition $\{A_i : i \in \omega\}$ into denumerable sets, and for each $i \in \omega$, A_i has a denumerable partition $P_i = \{A_{i,j} : j \in \mathbb{N}\}$ into finite sets such that, for every $j \in \mathbb{N}$, $|A_{i,j}| = j$. Let $\mathcal{G} = \{\phi \in Sym(A) : (\forall i \in \omega)(\phi(A_i) = A_i)$ and $|\{x \in A : \phi(x) \neq x\}| < \aleph_0\}$, where $Sym(A)$ is the group of all permutations of A. Let $\mathbf{P}_i = \{\phi(P_i) : \phi \in \mathcal{G}\}$ for each $i \in \omega$ and let $\mathbf{P} = \bigcup\{\mathbf{P}_i : i \in \omega\}$. Let $\mathcal{F}$ be the normal filter of subgroups of $\mathcal{G}$ generated by the filter base $\{\text{fix}_{\mathcal{G}}(E) : E \in [\mathbf{P}]^{<\omega}\}$. Then $\mathcal{N}$ is the permutation model determined by M, $\mathcal{G}$ and $\mathcal{F}$.

Model $\mathcal{M}$: Let A be a denumerable set of atoms which is written as a disjoint union $\bigcup\{A_n : n \in \omega\}$, where $|A_n| = \aleph_0$ for all $n \in \omega$. For each $n \in \omega$, let $\mathcal{G}_n$ be the group of all permutations of A_n which move only finitely many elements of A_n. Let $\mathcal{G}$ be the weak direct product of the $\mathcal{G}_n$'s for $n \in \omega$. Consequently, *every permutation of A in $\mathcal{G}$ moves only finitely many atoms.* Let $\mathcal{I}$ be the normal ideal of subsets of A which is generated by finite unions of A_n's. Let $\mathcal{F}$ be the normal filter on $\mathcal{G}$ generated by the subgroups $\text{fix}_G(E)$, $E \in \mathcal{I}$, and $\mathcal{M}$ be the Fraenkel–Mostowski model determined by M, $\mathcal{G}$, and $\mathcal{F}$.

Proposition 5.1. vDCP *holds in* $\mathcal{N}$ *and* $\mathcal{M}$.

Proof. We modify the arguments from the 1^{st}-paragraph of [9, **p.175**] to prove that vDCP holds in $\mathcal{N}$. Let $\mathcal{A} = \{(A_i, \leq_i) : i \in I\}$ be a family as in vDCP. Without loss of generality, we assume that $\mathcal{A}$ is pairwise disjoint. Let $R = \bigcup \mathcal{A}$. We partially order R by requiring $x \prec y$ if and only if there exists an index $i \in I$ such that $x, y \in A_i$ and $x \leq_i y$. Let $E \in [\mathbf{P}]^{<\omega}$ be a support of $(R, \prec)$. Following the arguments of [17, **claims 3.5, 3.6**], $Orb_{\text{fix}_{\mathcal{G}}E}(p) = \{\phi(p) : \phi \in \text{fix}_{\mathcal{G}}E\}$ is an antichain in R for each $p \in R$ and $\mathcal{O} = \{Orb_{\text{fix}_{\mathcal{G}}E}(p) : p \in R\}$ is a well-ordered partition of R. Thus, R can be written as a well-ordered disjoint union $\bigcup\{W_\alpha : \alpha < \kappa\}$ of antichains. For each $i \in I$, let $\alpha_i = min\{\alpha \in \kappa : A_i \cap W_\alpha \neq \emptyset\}$. Since for all $i \in I$, A_i is linearly ordered, it follows that $A_i \cap W_{\alpha_i}$ is a singleton for each $i \in I$. Consequently, $f = \{(i, \bigcup(A_i \cap W_{\alpha_i})) : i \in I\}$ is a choice function of $\mathcal{A}$.

Similarly, vDCP holds in $\mathcal{M}$. □

Remark 5.2. In every permutation model, CS (Every poset without a maximal element has two disjoint cofinal subsets) implies vDCP (cf. [9, **Theorem 3.15(3)**]). We can see that in the above-mentioned permutation models (i.e., $\mathcal{N}$ and $\mathcal{M}$) CS and CWF (Every poset has a cofinal well-founded subset) hold following the arguments of **Proposition 5.1**, [9, proof of **Theorem 3.26**], and [19, proof of **Theorem 10 (ii)**].

6. Spanning subgraphs, chromatic number, and weak choice forms

We recall the following result to prove **Proposition 6.2**.

Lemma 6.1. (ZF; **Delhomme–Morillon**; [6, **Lemma 1**]) *Given a set X and a set A which is the range of no mapping with domain X, consider a mapping $f : A \to \mathcal{P}(X)\backslash\{\emptyset\}$. Then*

1. *There are distinct a and b in A such that $f(a) \cap f(b) \neq \emptyset$.*
2. *If the set A is infinite and well-orderable, then for every positive integer p, there is an $F \in [A]^p$ such that $\bigcap f[F] := \bigcap_{a\in F} f(a)$ is non-empty.*

Proposition 6.2. (ZF) *The following hold:*

1. $\mathsf{AC}^{\omega}_{\leq \mathsf{n}-1} + \mathcal{Q}^{n}_{lf,c}$ *is equivalent to* $\mathsf{AC}^{\omega}_{\mathsf{fin}}$ *for any* $2 < n \in \omega$.
2. WUT *implies* $\mathsf{AC}^{\mathsf{WO}}_{\leq \mathsf{n}-1} + \mathcal{Q}^{n,k}_{lw,c}$ *and the latter implies* $\mathsf{AC}^{\mathsf{WO}}_{\mathsf{WO}}$ *for any* $2 < n, k \in \omega$.
3. $\mathcal{P}^{m}_{lf,c}$ *is equivalent to* $\mathsf{AC}^{\omega}_{\mathsf{fin}}$ *for any even integer* $m \geq 4$.

Proof. (1). (⇐) We assume $\mathsf{AC}^{\omega}_{\mathsf{fin}}$. Fix any $2 < n \in \omega$. We know that $\mathsf{AC}^{\omega}_{\mathsf{fin}}$ implies $\mathsf{AC}^{\omega}_{\leq \mathsf{n}-1}$ in ZF. Moreover, $\mathsf{AC}^{\omega}_{\mathsf{fin}}$ implies $\mathcal{Q}^{n}_{lf,c}$ in ZF (cf. [6, **Theorem 2**]).

(⇒) Fix any $2 < n \in \omega$. We show that $\mathsf{AC}^{\omega}_{\leq \mathsf{n}-1} + \mathcal{Q}^{n}_{lf,c}$ implies $\mathsf{AC}^{\omega}_{\mathsf{fin}}$ in ZF. Let $\mathcal{A} = \{A_i : i \in \omega\}$ be a countably infinite set of non-empty finite sets. Without loss of generality, we assume that $\mathcal{A}$ is disjoint. Let $A = \bigcup_{i\in\omega} A_i$. Consider a countably infinite family $(B_i, <_i)_{i\in\omega}$ of well-ordered sets such that $|B_i| = |A_i| + k$ for some fixed $1 \leq k \in \omega$, for each $i \in \omega$, B_i is disjoint from A and the other B_j's, and there is no mapping with domain A_i and range B_i (cf. the proof of [6, **Theorem 1**, **Remark 6**]). Let $B = \bigcup_{i\in\omega} B_i$. Consider another countably infinite sequence $T = \{t_i : i \in \omega\}$ disjoint from A and B. We construct a graph $G_1 = (V_{G_1}, E_{G_1})$ (cf. **Figure 2**).

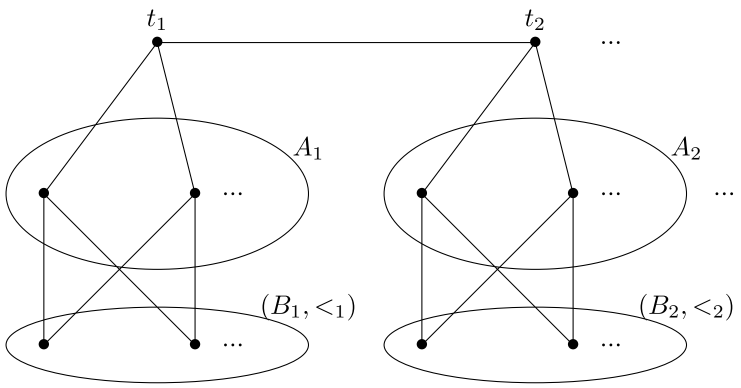

Figure 2. *The graph G_1*

Let $V_{G_1} = A \cup B \cup T$, and

$$E_{G_1} = \{\{t_i, t_{i+1}\} : i \in \omega\} \bigcup \{\{t_i, x\} : i \in \omega, x \in A_i\} \bigcup \{\{x, y\} : i \in \omega, x \in A_i, y \in B_i\}.$$

The graph G_1 is connected and locally finite. By $\mathcal{Q}^n_{lf,c}$, G_1 has a spanning subgraph G'_1 omitting $K_{2,n}$. For each $i \in \omega$, let $f_i : B_i \to \mathcal{P}(A_i)\backslash\emptyset$ map each element of B_i to its neighborhood in G'_1. We can see that for any two distinct ϵ_1 and ϵ_2 in B_i, $f_i(\epsilon_1) \cap f_i(\epsilon_2)$ has at most $n-1$ elements, since G'_1 has no $K_{2,n}$. By **Lemma 6.1(1)**, there are tuples $(\epsilon'_1, \epsilon'_2) \in B_i \times B_i$ s.t. $f_i(\epsilon'_1) \cap f_i(\epsilon'_2) \neq \emptyset$. Consider the first such tuple $(\epsilon''_1, \epsilon''_2)$ w.r.t. the lexicographical ordering on $B_i \times B_i$. Let $A'_i = f_i(\epsilon''_1) \cap f_i(\epsilon''_2)$. By $\mathsf{AC}^{\omega}_{\leq n-1}$, we can obtain a choice function of $\mathcal{A}' = \{A'_i : i \in \omega\}$, which is a choice function of $\mathcal{A}$.

(2). For the first implication, we know that WUT implies $\mathsf{AC}^{\mathsf{WO}}_{\leq n-1}$ as well as the statement *"Every locally well-orderable connected graph is well-orderable"* in ZF. The rest follows from the fact that every well-ordered graph has a spanning tree in ZF.

Similar to the arguments of (1), we show that $\mathsf{AC}^{\mathsf{WO}}_{\leq n-1} + \mathcal{Q}^{n,k}_{lw,c}$ implies $\mathsf{AC}^{\mathsf{WO}}_{\mathsf{WO}}$ by applying **Lemma 6.1(2)**. Let $\mathcal{A} = \{A_n : n \in \kappa\}$ be a disjoint well-orderable collection of non-empty well-orderable sets. Let $A = \bigcup_{i\in\kappa} A_i$. Consider an infinite well-orderable family $(B_i, <_i)_{i\in\kappa}$ of well-orderable sets such that for each $i \in \kappa$, B_i is disjoint from A and the other B_j's, and there is no mapping with domain A_i and range B_i (cf. the proof of [6, **Theorem 1**, **Remark 6**]). Let $B = \bigcup_{i\in\kappa} B_i$. Consider another κ-sequence $T = \{t_n : n \in \kappa\}$ disjoint from A and B. Similar to G_1, we construct a connected and locally well-orderable graph $G_2 = (V_{G_2}, E_{G_2})$ as follows:

$$\text{Let } V_{G_2} = A \cup B \cup T, \text{ and}$$
$$E_{G_2} = \{\{t_i, t_{i+1}\} : i \in \kappa\} \bigcup \{\{t_i, x\} : i \in \kappa, x \in A_i\} \bigcup \{\{x, y\} : i \in \kappa, x \in A_i, y \in B_i\}.$$

By assumption, G_2 has a spanning subgraph G'_2 omitting $K_{k,n}$. For each $i \in \kappa$, let $f_i : B_i \to \mathcal{P}(A_i)\backslash\emptyset$ map each element of B_i to its neighborhood in G'_2 and for any $H_i \in [B_i]^k$, $|\bigcap_{\epsilon\in H_i} f_i(\epsilon)| \leq n-1$ since G'_2 has no $K_{k,n}$. Since each B_i is infinite and well-orderable, by **Lemma 6.1(2)**, there are tuples $(\epsilon_1, \epsilon_2, ...\epsilon_k) \in B_i^k$ s.t. $\bigcap_{i<k} f_i(\epsilon_i) \neq \emptyset$. Consider the first such tuple $(\epsilon_1, \epsilon_2, ...\epsilon_k)$ w.r.t. the well-ordering on B_i^k. Let $A'_i = \bigcap_{i<k} f_i(\epsilon_i)$. By $\mathsf{AC}^{\mathsf{WO}}_{\leq n-1}$, there exists a choice function of $\mathcal{A}' = \{A'_n : n \in \kappa\}$, which is a choice function of $\mathcal{A}$.

(3). ($\Rightarrow$) Fix any even integer $m = 2(k+1) \geq 4$. We prove that $\mathcal{P}^m_{lf,c}$ implies $\mathsf{AC}^{\omega}_{\mathsf{fin}}$. Let $\mathcal{A} = \{A_i : i \in \omega\}$ be a countably infinite set of non-empty finite sets and $A = \bigcup_{i\in\omega} A_i$. Let $T = \{t_i : i \in \omega\}$ be a sequence such that t_i's are pair-wise distinct and belong to no $A_j \times \{1, ..., k\}$, and $R = \{r_i : i \in \omega\}$ be a sequence such that r_i's are pair-wise distinct and belong to no $(A_j \times \{1, ..., k\}) \cup \{t_j\}$ for any $i, j \in \omega$. We construct a locally finite and connected graph $G_3 = (V_{G_3}, E_{G_3})$ (cf. **Figure 3**).

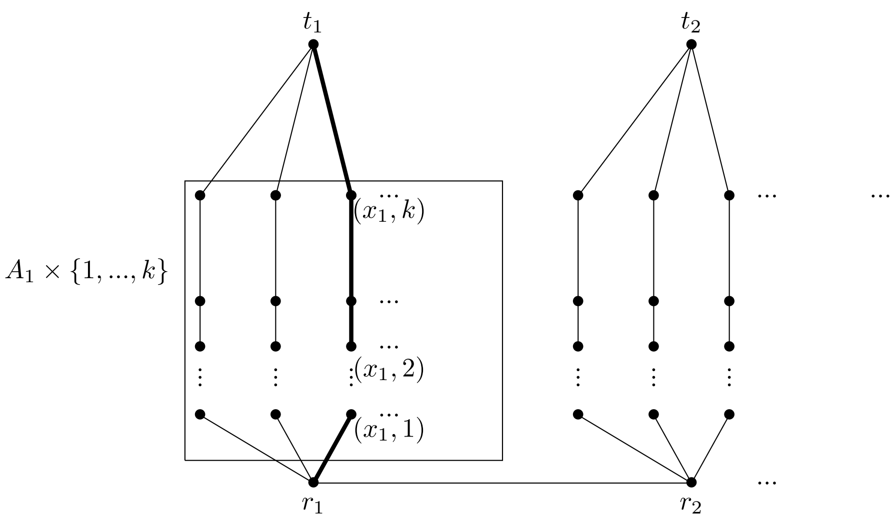

FIGURE 3. *The graph G_3, where $\{t_1, (x_1, k), ...(x_1, 1), r_1\}$ is a path in ζ.*

$$\text{Let } V_{G_3} = (\textstyle\bigcup_{i\in\omega}(A_i \times \{1, ..., k\})) \cup R \cup T, \text{ and}$$
$$E_{G_3} = (\textstyle\bigcup_{i\in\omega, x\in A_i}\{\{r_i, (x,1)\}, \{(x,1), (x,2)\}, ..., \{(x,k), t_i\}\}) \bigcup \{\{r_i, r_{i+1}\} : i \in \omega\}.$$

By assumption, G_3 has a spanning m-bush ζ. We can see that ζ generates a choice function of $\mathcal{A}$: for each $i \in I$, there is a unique $x \in A_i$, say x_i, such that $\{t_i, (x_i, k), ...(x_i, 1), r_i\}$ is a path in ζ.

($\Leftarrow$) Fix any even integer $m \geq 4$. We prove that $\mathsf{AC}^{\omega}_{\mathsf{fin}}$ implies $\mathcal{P}^m_{lf,c}$. We know that $\mathsf{AC}^{\omega}_{\mathsf{fin}}$ implies the statement *"Every infinite locally finite connected graph is countably infinite"* in ZF. The rest follows from the fact that every well-ordered graph has a spanning tree in ZF and any spanning tree is a spanning m-bush. $\square$

Remark 6.3. Let G be a graph. We denote by P_G, the class of infinite graphs whose only components are isomorphic with G. For any graph $G_1 = (V_{G_1}, E_{G_1}) \in P_G$, we construct a graph $G_2 = (V_{G_2}, E_{G_2})$ as follows: Pick a $t \notin V_{G_1}$. Let $V_{G_2} = \{t\} \bigcup V_{G_1}$, $E_{G_1} \subseteq E_{G_2}$ and for each $x \in V_{G_1}$, let $\{t, x\} \in E_{G_2}$. We denote by P'_{G_1}, the class of graphs of the form G_2. Fix any $2 < k \in \omega$ and any $2 \leq p, q < \omega$. We observe the following in ZF:

(1) $\mathsf{AC}_{\mathsf{k}^{\mathsf{k}-2}}$ implies *"Every graph from the class P'_{K_k} has a spanning tree"*.
(2) AC_{k} implies *"Every graph from the class P'_{C_k} has a spanning tree"*.
(3) $(\mathsf{AC}_{\mathsf{p}^{\mathsf{q}-1}\mathsf{q}^{\mathsf{p}-1}} + \mathsf{AC}_{\mathsf{p}+\mathsf{q}})$ implies *"Every graph from the class $P'_{K_{p,q}}$ has a spanning tree"*.

We prove (1). Pick some $G_2 = (V_{G_2}, E_{G_2}) \in P'_{K_k}$. Then there is a $G_1 \in P_{K_k}$ such that $V_{G_2} = V_{G_1} \cup \{t\}$ for some $t \notin V_{G_1}$. Let $\{A_i : i \in I\}$ be the components of G_1. By AC_{k} (which follows from $\mathsf{AC}_{\mathsf{k}^{\mathsf{k}-2}}$, since AC_m implies AC_n if m is a multiple of n), we choose a sequence $\{a_i : i \in I\}$ such that $a_i \in A_i$ for all $i \in I$. The number of spanning trees in $A_i = K_k$ is k^{k-2} for any $i \in I$.[5] By $\mathsf{AC}_{\mathsf{k}^{\mathsf{k}-2}}$, we choose a sequence $\{s_i : i \in I\}$ such that $s_i = (V_{s_i}, E_{s_i})$ is a spanning tree of A_i for all $i \in I$. Then the graph $S_{G_2} = (V_{S_{G_2}}, E_{S_{G_2}})$ where $V_{S_{G_2}} = \{t\} \cup \bigcup_{i \in I} V_{s_i}$ and $E_{S_{G_2}} = \{\{t, a_i\} : i \in I\} \cup \bigcup_{i \in I} E_{s_i}$ is a spanning tree of G_2. Similarly, we can prove (2) and (3) since the number of spanning trees in C_k is k and the number of spanning trees in $K_{p,q}$ is $p^{q-1}q^{p-1}$.[6]

Remark 6.4. Galvin–Komjáth [13] proved that any graph based on a well-ordered set of vertices has a chromatic number in ZF. Consequently, $\mathsf{AC}^{\omega}_{\mathsf{fin}}$ implies $\mathcal{C}_{lf,c}$ in ZF. Stawiski proved the other direction in [15]. We give a different argument to prove that $\mathcal{C}_{lf,c}$ implies $\mathsf{AC}^{\omega}_{\mathsf{fin}}$ in ZF. Since $\mathsf{AC}^{\omega}_{\mathsf{fin}}$ is equivalent to its partial version $\mathsf{PAC}^{\omega}_{\mathsf{fin}}$ (Every denumerable family of finite sets has an infinite subfamily with a choice function), it suffices to show that $\mathcal{C}_{lf,c}$ implies $\mathsf{PAC}^{\omega}_{\mathsf{fin}}$ (cf. [8]). Let $\mathcal{A} = \{A_n : n \in \omega\}$ be a denumerable set of non-empty finite sets without a partial choice function. Without loss of generality, we assume that $\mathcal{A}$ is disjoint. Consider a denumerable sequence $T = \{t_n : n \in \omega\}$ disjoint from $\mathcal{A}$. We construct an infinite locally finite connected graph $G = (V_G, E_G)$ as follows:

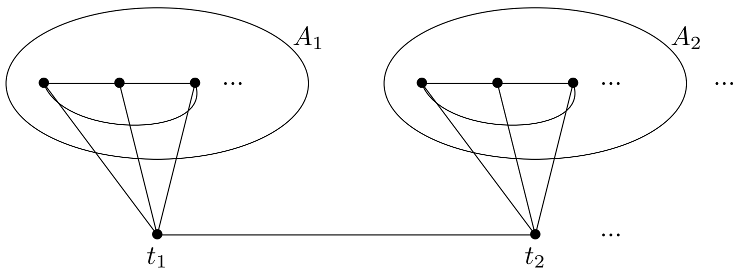

FIGURE 4. *The graph G.*

Let $V_G = (\bigcup_{n \in \omega} A_n) \cup T$. For each $n \in \omega$, let $\{t_n, t_{n+1}\} \in E_G$ and $\{t_n, x\} \in E_G$ for every element $x \in A_n$. Also for each $n \in \omega$, and any two $x, y \in A_n$ such that $x \neq y$, let $\{x, y\} \in E_G$.

Let $f : V_G \to C$ be a C-coloring of G, i.e., a map such that $\{x, y\} \in E_G$ implies $f(x) \neq f(y)$. Then for each $c \in C$, the set $M_c = \{v \in f^{-1}(c) : v \in A_i \text{ for some } i \in \omega\}$ must be finite, otherwise M_c will generate a partial choice function for $\mathcal{A}$. Also the set $f[\bigcup_{n \in \omega} A_n]$ must be infinite. Fix some $c_0 \in f[\bigcup_{n \in \omega} A_n]$. Then $Index(M_{c_0}) = \{n \in \omega : M_{c_0} \cap A_n \neq \emptyset\}$ is finite, and thus there exists some $b_0 \in (f[\bigcup_{n \in \omega} A_n] \backslash \bigcup_{m \in Index(M_{c_0})} f[A_n])$ since finite union of finite sets is finite. Define a coloring $g : \bigcup_{n \in \omega} A_n \to (f[\bigcup_{n \in \omega} A_n] \backslash c_0)$ as follows:

$$g(x) = f(x) \text{ if } f(x) \neq c_0, \text{ and } g(x) = b_0 \text{ otherwise.}$$

Similarly, we can define a coloring $h : \bigcup_{n \in \omega} A_n \to (f[\bigcup_{n \in \omega} A_n] \backslash \{c_0, c_1, c_2\})$ for some $c_0, c_1, c_2 \in f[\bigcup_{n \in \omega} A_n]$. Let $h(t_{2n}) = c_0$ and $h(t_{2n+1}) = c_1$ for all $n \in \omega$. Thus, $h : V_G \to (f[\bigcup_{n \in \omega} A_n] \backslash \{c_2\})$ is a $h[V_G] \backslash \{c_2\}$-coloring of G. Since $|h[V_G] \backslash \{c_2\}| < |h[V_G]| \leq |C|$, the graph G has no chromatic number.

7. Questions, and further studies

Question 7.1. Does CAC, the infinite Ramsey's Theorem (RT) [8, **Form 17**], and **Form 233** hold in $\mathcal{V}_p$?

Question 7.2. (asked in [4]) Does $\mathcal{V}_p$ have any amorphous sets?

Question 7.3. Does WUT hold in $\mathcal{V}^+_p$?

[5]This is Cayley's formula for finite graphs, which works in ZF.
[6]This is Scoin's formula for finite graphs, which works without invoking any form of choice.

We recall that in Feferman–Lévy's model (model $\mathcal{M}_9$ in [8]) the statement "$\aleph_1$ is regular" [8, **Form 34**] fails. We recall a brief description of $\mathcal{M}_9$ from Dimitriou's Ph.D. thesis (cf. [7, **Chapter 1**, §2]).

Forcing notion $\mathbb{P}_1$: Let $\mathbb{P}_1 = \{p : \omega \times \omega \rightharpoonup \aleph_\omega; |p| < \omega \text{ and } \forall(n,i) \in dom(p), p(n,i) < \omega_n\}$ be a forcing notion ordered by reverse inclusion, i.e., $p \leq q$ iff $p \supseteq q$ (We denote by $p : A \rightharpoonup B$ a partial function from A to B).

Group of permutations $\mathcal{G}_1$ of $\mathbb{P}_1$: Let $\mathcal{G}_1$ be the full permutation group of ω. Extend $\mathcal{G}_1$ to an automorphism group of $\mathbb{P}_1$ by letting an $a \in \mathcal{G}_1$ act on a $p \in \mathbb{P}_1$ by $a^*(p) = \{(n, a(i), \beta); (n, i, \beta) \in p\}$. We identify a^* with $a \in \mathcal{G}_1$. We can see that this is an automorphism group of $\mathbb{P}_1$.

Normal filter $\mathcal{F}_1$ of subgroups over $\mathcal{G}_1$: For every $n \in \omega$ define the following sets.

$$E_n = \{p \cap (n \times \omega \times \omega_n); p \in \mathbb{P}_1\}, \text{fix}_{\mathcal{G}_1} E_n = \{a \in \mathcal{G}_1; \forall p \in E_n (a(p) = p)\}. \tag{2}$$

We can see that $\mathcal{F}_1 = \{X \subseteq \mathcal{G}_1; \exists n \in \omega, \text{fix}_{\mathcal{G}_1} E_n \subseteq X\}$ is a normal filter of subgroups over $\mathcal{G}_1$.

$\mathcal{M}_9$ is the symmetric extension (symmetric submodel of a forcing extension where AC can consistently fail) obtained by $\langle \mathbb{P}_1, \mathcal{G}_1, \mathcal{F}_1 \rangle$. For details concerning symmetric extensions, the reader is referred to [12, **Chapter 5**]. It is known that the following statements follow from "$\aleph_1$ is regular" in ZF (cf. [1, 2]).

(*): If P is a poset such that the underlying set has a well-ordering and if all antichains in P are finite and all chains in P are countable, then P is countable.

(**): If P is a poset such that the underlying set has a well-ordering and if all antichains in P are countable and all chains in P are finite, then P is countable.

Question 7.4. Does any of (**) and (*) is true in $\mathcal{M}_9$?

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