ON THE LOCAL AND GLOBAL COMPARISON OF GENERALIZED BAJRAKTAREVIĆ MEANS

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ABSTRACT. Given two continuous functions $f, g : I \to \mathbb{R}$ such that g is positive and f/g is strictly monotone, a measurable space $(T, \mathcal{A})$, a measurable family of d-variable means $m : I^d \times T \to I$, and a probability measure μ on the measurable sets $\mathcal{A}$, the d-variable mean $M_{f,g,m;\mu} : I^d \to I$ is defined by

$$M_{f,g,m;\mu}(\boldsymbol{x}) := \left(\frac{f}{g}\right)^{-1}\left(\frac{\int_T f\big(m(x_1,\dots,x_d,t)\big)\,\mathrm{d}\mu(t)}{\int_T g\big(m(x_1,\dots,x_d,t)\big)\,\mathrm{d}\mu(t)}\right) \qquad (\boldsymbol{x} = (x_1,\dots,x_d) \in I^d).$$

The aim of this paper is to study the local and global comparison problem of these means, i.e., to find conditions for the generating functions (f, g) and (h, k), for the families of means m and n, and for the measures μ, ν such that the comparison inequality

$$M_{f,g,m;\mu}(\boldsymbol{x}) \le M_{h,k,n;\nu}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in I^d)$$

be satisfied.

1. INTRODUCTION

In a recent paper [20], Losonczi and Páles investigated a general class of two-variable means given by the formula

$$M_{f,g;\mu}(x_1, x_2) := \left(\frac{f}{g}\right)^{-1}\left(\frac{\int_0^1 f(tx_1 + (1-t)x_2)\,\mathrm{d}\mu(t)}{\int_0^1 g(tx_1 + (1-t)x_2)\,\mathrm{d}\mu(t)}\right) \qquad ((x_1, x_2) \in I^2),$$

where $f, g : I \to \mathbb{R}$ are continuous functions such that g is positive and f/g is strictly monotone and μ is a probability measure on the Borel measurable subsets of $[0, 1]$. This definition includes many former classical and important settings. In [20] local and global comparison theorems (that provided necessary and in some cases also sufficient conditions) have been established for the comparison of two two-variable means from this general class.

The purpose of this note is to extend the results of [20] in several ways. In our approach we will use Chebyshev systems, measurable families of means and measures for the definition of a general class of d-variable means.

Throughout this paper, the symbols $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_+$ will stand for the sets of natural, real, and positive real numbers, respectively, and I will always denote a nonempty open real interval. The classes of continuous strictly monotone and continuous positive real-valued functions defined on I will be denoted by $\mathcal{CM}(I)$ and $\mathcal{CP}(I)$, respectively.

In the sequel, a function $M : I^d \to I$ is called a *d-variable mean* on I if the following so-called mean value property

$$\min(x_1,\dots,x_d) \le M(\boldsymbol{x}) \le \max(x_1,\dots,x_d) \qquad (\boldsymbol{x} = (x_1,\dots,x_d) \in I^d) \tag{1}$$

holds. Also, if both of the inequalities in (1) are strict for all $x_1,\dots,x_d \in I$ with $x_i \ne x_j$ for some $i \ne j$, then we say that M is a *strict mean* on I. The *arithmetic* and *geometric* means are well known

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instances for strict means on $\mathbb{R}_+$. More generally, if p is a real number, then the d-variable *Hölder mean* $H_p : \mathbb{R}_+^d \to \mathbb{R}$ is defined as

$$H_p(\boldsymbol{x}) := \begin{cases} \left(\dfrac{x_1^p + \cdots + x_d^p}{d}\right)^{\frac{1}{p}} & \text{if } p \neq 0 \\ \sqrt[d]{x_1 \cdots x_d} & \text{if } p = 0 \end{cases} \qquad \big(\boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}_+^d\big).$$

Obviously, H_1 and H_0 equal the arithmetic and geometric mean, respectively. It is easy to see that Hölder means are strict means. The d-variable minimum and maximum functions are instances for non-strict means.

A classical generalization of Hölder means is the notion of *d-variable quasi-arithmetic mean* (cf. [12]), which is introduced as follows: For $f \in \mathcal{CM}(I)$ define

$$A_f(\boldsymbol{x}) := f^{-1}\left(\frac{f(x_1) + \cdots + f(x_d)}{d}\right) \qquad \big(\boldsymbol{x} = (x_1, \dots, x_d) \in I^d\big). \tag{2}$$

More generally, if S_d denotes the $(d-1)$-dimensional simplex given by

$$S_d := \{(t_1, \dots, t_d) \mid t_1, \dots, t_d \geq 0,\, t_1 + \cdots + t_d = 1\}, \tag{3}$$

then we can also define

$$A_f(\boldsymbol{x}, \boldsymbol{t}) := f^{-1}\big(t_1 f(x_1) + \cdots + t_d f(x_d)\big) \qquad \big(\boldsymbol{x} = (x_1, \dots, x_d) \in I^d,\, \boldsymbol{t} = (t_1, \dots, t_d) \in S_d\big), \tag{4}$$

which is called the *weighted d-variable quasi-arithmetic mean on I*.

In this paper, we consider a much more general class of means. For their definition, we recall the notion of Chebyshev system. Let $f, g : I \to \mathbb{R}$ be continuous function. We say that the pair (f, g) forms a *(two-dimensional) Chebyshev system on I* if, for any distinct elements x, y of I, the determinant

$$\mathcal{D}_{f,g}(x, y) := \begin{vmatrix} f(x) & f(y) \\ g(x) & g(y) \end{vmatrix} \qquad (x, y \in I)$$

is different from zero. If, for $x < y$, this determinant is positive, then (f, g) is called a *positive system*, otherwise we call (f, g) a *negative system*. Due to the connectedness of the triangle $\{(x, y) \mid x < y,\, x, y \in I\}$, it follows that every Chebyshev system is either positive or negative. Obviously, if (f, g) is a positive Chebyshev system, then (g, f) is a negative one.

The most standard positive Chebyshev system on $\mathbb{R}$ is given by $f(x) = 1$ and $g(x) = x$. More generally, if $f, g : I \to \mathbb{R}$ are continuous functions with $g \in \mathcal{CP}(I)$, $f/g \in \mathcal{CM}(I)$, then (f, g) is a Chebyshev system. Indeed, we have

$$\mathcal{D}_{f,g}(x, y) := \begin{vmatrix} f(x) & f(y) \\ g(x) & g(y) \end{vmatrix} = g(x)g(y)\left(\frac{f(x)}{g(x)} - \frac{f(y)}{g(y)}\right) \qquad (x, y \in I). \tag{5}$$

From, here it is obvious that $\mathcal{D}_{f,g}(x, y)$ vanishes if and only if $x = y$. Moreover, if f/g is decreasing (resp. increasing), then, for $x < y$, we have that $\mathcal{D}_{f,g}(x, y) > 0$ (resp. $\mathcal{D}_{f,g}(x, y) < 0$), i.e., (f, g) is a positive (resp. negative) Chebyshev system. By symmetry, analogous properties can be established if f is positive and g/f strictly monotone.

For the sake of convenience and brevity, now we make the following hypotheses. We say that $m : I^d \times T \to I$ is a *measurable family of d-variable means on I* if

(H1) I is a nonvoid open real interval,
(H2) $(T, \mathcal{A})$ is a measurable space, where $\mathcal{A}$ is the σ-algebra of measurable sets of T,
(H3) for all $t \in T$, $m(\cdot, t)$ is a d-variable mean on I,
(H4) for all $\boldsymbol{x} \in I^d$, the function $m(\boldsymbol{x}, \cdot)$ is measurable over T.

If, instead of (H2) and H(4), we have that

(H2+) T is a topological space and $\mathcal{A}$ equals the σ-algebra $\mathcal{B}(T)$ of the Borel sets of T,
(H4+) for all $\boldsymbol{x} \in I^d$, the function $m(\boldsymbol{x}, \cdot)$ is continuous over T,

then $m : I^d \times T \to I$ will be called a *continuous family of d-variable means on I.*

For a measurable family of d-variable means $m : I^d \times T \to I$, we introduce the notations:

$$\underline{m}(\boldsymbol{x}) := \inf_{t\in T} m(\boldsymbol{x}, t) \qquad \text{and} \qquad \overline{m}(\boldsymbol{x}) := \sup_{t\in T} m(\boldsymbol{x}, t) \qquad (\boldsymbol{x} \in I^d).$$

Obviously, by property (H3), for all $\boldsymbol{x} \in I^d$, we have that $\min(\boldsymbol{x}) \le \underline{m}(\boldsymbol{x}) \le \overline{m}(\boldsymbol{x}) \le \max(\boldsymbol{x})$. Provided that T is a compact and connected topological space and $m : I^d \times T \to I$ is a continuous family of d-variable means on I, we have that

$$[\underline{m}(\boldsymbol{x}), \overline{m}(\boldsymbol{x})] = \{m(\boldsymbol{x}, t) \mid t \in T\} \qquad (\boldsymbol{x} \in I^d). \tag{6}$$

For the construction of a mean in terms of a Chebyshev system, a measurable family of means, and a probability measure, we need the following basic lemma.

Lemma 1. *Let $m : I^d \times T \to I$ be a measurable family of d-variable means, let μ be a probability measure on $(T, \mathcal{A})$ and let (f, g) be a Chebyshev system on I. Then, for all $\boldsymbol{x} \in I^d$, there exists a unique element $y \in [\underline{m}(\boldsymbol{x}), \overline{m}(\boldsymbol{x})]$ such that*

$$\int_T \mathcal{D}_{f,g}(m(\boldsymbol{x}, t), y)\,\mathrm{d}\mu(t) = 0. \tag{7}$$

Furthermore, if (f, g) is a positive Chebyshev system, then, for all $u \in I$,

$$\int_T \mathcal{D}_{f,g}(m(\boldsymbol{x}, t), u)\,\mathrm{d}\mu(t) \begin{matrix} < \\ = \\ > \end{matrix} 0 \qquad \text{if and only if} \qquad u \begin{matrix} < \\ = \\ > \end{matrix} y.$$

In addition, if g is positive and f/g is strictly monotone, then

$$y = \left(\frac{f}{g}\right)^{-1} \left(\frac{\int_T f\big(m(\boldsymbol{x}, t)\big)\,\mathrm{d}\mu(t)}{\int_T g\big(m(\boldsymbol{x}, t)\big)\,\mathrm{d}\mu(t)}\right). \tag{8}$$

Proof. Without loosing the generality, we may assume that (f, g) is a positive Chebyshev system throughout this proof.

For fixed $\boldsymbol{x} \in I^d$, consider now the following function

$$h(u) := \int_T \mathcal{D}_{f,g}(m(\boldsymbol{x}, t), u)\,\mathrm{d}\mu(t) = g(u) \int_T f(m(\boldsymbol{x}, t))\,\mathrm{d}\mu(t) - f(u) \int_T g(m(\boldsymbol{x}, t))\,\mathrm{d}\mu(t) \qquad (u \in I).$$

By the continuity of f and g, we have that h is continuous on I. If $\overline{m}(\boldsymbol{x}) < u$, then, for all $t \in T$ we have that $m(\boldsymbol{x}, t) < u$, hence $\mathcal{D}_{f,g}(m(\boldsymbol{x}, t), u) > 0$. This implies that $h(u)$ is positive for all $u \in I$ with $\overline{m}(\boldsymbol{x}) < u$. Similarly, for all $u \in I$ with $u < \underline{m}(\boldsymbol{x})$, we have that $h(u) < 0$. Therefore, by the intermediate value property of continuous functions, h must have a zero between $\underline{m}(\boldsymbol{x})$ and $\overline{m}(\boldsymbol{x})$.

To prove the uniqueness, assume that y and z are distinct zeros of h between $\underline{m}(\boldsymbol{x})$ and $\overline{m}(\boldsymbol{x})$. Then we have that

$$g(y) \int_T f(m(\boldsymbol{x}, t))\,\mathrm{d}\mu(t) - f(y) \int_T g(m(\boldsymbol{x}, t))\,\mathrm{d}\mu(t) = 0,$$
$$g(z) \int_T f(m(\boldsymbol{x}, t))\,\mathrm{d}\mu(t) - f(z) \int_T g(m(\boldsymbol{x}, t))\,\mathrm{d}\mu(t) = 0.$$

This means that the two unknowns $\xi := \int_T f(m(\boldsymbol{x}, t))\,\mathrm{d}\mu(t)$ and $\eta := \int_T g(m(\boldsymbol{x}, t))\,\mathrm{d}\mu(t)$ are solutions of the following system of linear equations:

$$g(y)\xi - f(y)\eta = 0,$$
$$g(z)\xi - f(z)\eta = 0.$$

Because y and z are distinct, the determinant of this system is nonzero, hence $\xi = \eta = 0$. In this case, $h(u) = 0$ for all $u \in I$, which contradicts the property that $h(u) > 0$ for $\overline{m}(\boldsymbol{x}) < u$. The contradiction obtained shows that $y = z$, which proves the uniqueness of the solution y of equation (7). The uniqueness also implies that $h(u) > 0$ for $u > y$ and $h(u) < 0$ for $u < y$.

Finally, assume that g is positive and f/g is strictly monotone (then (f, g) is a Chebyshev system). In this case, the equation $h(y) = 0$ can be rewritten as

$$\frac{f(y)}{g(y)} = \frac{\int_T f(m(\boldsymbol{x}, t))\,\mathrm{d}\mu(t)}{\int_T g(m(\boldsymbol{x}, t))\,\mathrm{d}\mu(t)}.$$

By applying the inverse function of f/g to this equation side by side, we obtain that y is of the form (8). □

The above lemma allows us to define a d-variable mean $M_{f,g,m;\mu} : I^d \to I$. Given $\boldsymbol{x} \in I^d$, let $M_{f,g,m;\mu}(\boldsymbol{x})$ denote the unique solution y of equation (7). In the particular case when g is positive and f/g is strictly monotone, we have that

$$M_{f,g,m;\mu}(\boldsymbol{x}) := \left(\frac{f}{g}\right)^{-1}\left(\frac{\int_T f\big(m(\boldsymbol{x}, t)\big)\,\mathrm{d}\mu(t)}{\int_T g\big(m(\boldsymbol{x}, t)\big)\,\mathrm{d}\mu(t)}\right) \qquad (\boldsymbol{x} \in I^d). \tag{9}$$

This mean will be called a d-variable *generalized Bajraktarević mean* in the sequel. When $g = 1$, then

$$M_{f,1,m;\mu}(\boldsymbol{x}) = (f)^{-1}\left(\int_T f\big(m(\boldsymbol{x}, t)\big)\,\mathrm{d}\mu(t)\right) \qquad (\boldsymbol{x} \in I^d)$$

which will be termed a d-variable *generalized quasi-arithmetic mean*. If $T = \{1, \dots, d\}$, $\mu = \frac{\delta_1 + \cdots + \delta_d}{d}$ (where δ_t denotes the Dirac measure concentrated at t) and $m(\boldsymbol{x}, t) = x_t$, then

$$M_{f,g,m;\mu}(\boldsymbol{x}) = M_{f,g}(\boldsymbol{x}) := \left(\frac{f}{g}\right)^{-1}\left(\frac{f(x_1) + \cdots + f(x_d)}{g(x_1) + \cdots + g(x_d)}\right) \qquad \big(\boldsymbol{x} = (x_1 \dots, x_d) \in I^d\big),$$

which was introduced and studied by Bajraktarević [1], [2]. When $g = 1$, then $M_{f,1,m;\mu}(\boldsymbol{x}) = A_f(\boldsymbol{x})$, which is the d-variable quasi-arithmetic mean introduced in (2).

To define d-variable *generalized Gini means*, let $p, q \in \mathbb{R}$ and assume that $I \subseteq \mathbb{R}_+$. By taking

$$\begin{aligned} &f(x) = x^p, && g(x) = x^q && \text{if} \quad p \neq q,\\ &f(x) = x^p \log(x), && g(x) = x^p && \text{if} \quad p = q, \end{aligned} \tag{10}$$

we can define $G_{p,q,m;\mu}$ in the following manner:

$$G_{p,q,m;\mu}(\boldsymbol{x}) := \begin{cases} \left(\dfrac{\int_T \big(m(\boldsymbol{x}, t)\big)^p\,\mathrm{d}\mu(t)}{\int_T \big(m(\boldsymbol{x}, t)\big)^q\,\mathrm{d}\mu(t)}\right)^{\frac{1}{p-q}} & \text{if } p \neq q,\\ \exp\left(\dfrac{\int_T \big(m(\boldsymbol{x}, t)\big)^p \log\big(m(\boldsymbol{x}, t)\big)\,\mathrm{d}\mu(t)}{\int_T \big(m(\boldsymbol{x}, t)\big)^p\,\mathrm{d}\mu(t)}\right) & \text{if } p = q. \end{cases} \qquad (\boldsymbol{x} \in I^d). \tag{11}$$

In the particular case when $T = \{1, \dots, d\}$, $\mu = \frac{\delta_1 + \cdots + \delta_d}{d}$ and $m(\boldsymbol{x}, t) = x_t$, the above formula reduces to the so-called d-variable *Gini mean* $G_{p,q}$ (cf. [11]):

$$G_{p,q}(\boldsymbol{x}) := \begin{cases} \left(\dfrac{x_1^p + \cdots + x_d^p}{x_1^q + \cdots + x_d^q}\right)^{\frac{1}{p-q}} & \text{if } p \neq q,\\ \exp\left(\dfrac{x_1^p \log(x_1) + \cdots + x_d^p \log(x_d)}{x_1^p + \cdots + x_d^p}\right) & \text{if } p = q, \end{cases} \qquad \big(\boldsymbol{x} = (x_1 \dots, x_d) \in \mathbb{R}_+^d\big).$$

Obviously, $G_{p,0} = H_p$, i.e., Hölder means are particular Gini means.

In what follows, we describe further interesting particular cases of formula (9). If $T = \{0, 1, -1\}$, $I = \mathbb{R}_+$, $\mu = \mu_{-1}\delta_{-1} + \mu_0\delta_0 + \mu_1\delta_1$ (where $\mu_{-1}, \mu_0, \mu_1 \in \mathbb{R}_+$ with $\mu_{-1} + \mu_0 + \mu_1 = 1$), and $m(\boldsymbol{x}, t) = H_t(\boldsymbol{x})$ (where H_t stands for the t-th Hölder mean), then

$$M_{f,g,m;\mu}(\boldsymbol{x}) = \left(\frac{f}{g}\right)^{-1}\left(\frac{\mu_{-1}f(H_{-1}(\boldsymbol{x})) + \mu_0 f(H_0(\boldsymbol{x})) + \mu_1 f(H_1(\boldsymbol{x}))}{\mu_{-1}g(H_{-1}(\boldsymbol{x})) + \mu_0 g(H_0(\boldsymbol{x})) + \mu_1 g(H_1(\boldsymbol{x}))}\right) \qquad (\boldsymbol{x} \in \mathbb{R}_+^d).$$

In the next example we use the notations introduced in (3) and (4). If $T = S_d$, λ is the $(d-1)$-dimensional Lebesgue measure on S_d, and $m(\boldsymbol{x}, \boldsymbol{t}) = A_\varphi(\boldsymbol{x}, \boldsymbol{t})$, where $\varphi \in \mathcal{CM}(I)$, then

$$M_{f,g,m;\mu}(\boldsymbol{x}) = M_{f,g,A_\varphi;\lambda}(\boldsymbol{x}) = \left(\frac{f}{g}\right)^{-1}\left(\frac{\int_{S_d} f\big(A_\varphi(\boldsymbol{x}, \boldsymbol{t})\big)\,\mathrm{d}\lambda(\boldsymbol{t})}{\int_{S_d} g\big(A_\varphi(\boldsymbol{x}, \boldsymbol{t})\big)\,\mathrm{d}\lambda(\boldsymbol{t})}\right) \qquad (\boldsymbol{x} = (x_1, \dots, x_d) \in I^d).$$

If μ is the Lebesgue measure on $[0, 1]$ and $f, g : I \to \mathbb{R}$ are continuously differentiable functions such that $g' > 0$ and f'/g' is strictly monotone, and $m(\boldsymbol{x}, t) = tx_1 + (1-t)x_2$, then, by the Fundamental Theorem of Calculus, one can easily see that

$$M_{f',g',m;\mu}(\boldsymbol{x}) = C_{f,g}(\boldsymbol{x}) = \begin{cases} \left(\dfrac{f'}{g'}\right)^{-1}\left(\dfrac{f(x_2) - f(x_1)}{g(x_2) - g(x_1)}\right) & \text{if } x_1 \neq x_2 \\ x_1 & \text{if } x_1 = x_2 \end{cases} \qquad (\boldsymbol{x} = (x_1, x_2) \in I^2),$$

which is called a *Cauchy* or *difference mean* in the literature. Their equality problem was solved by Losonczi [17].

By taking f and g given in (10), the mean so obtained is the so-called *Stolarsky mean*, which was introduced in the papers [26] and [13]. Their comparison problem was solved by Leach and Sholander [14] on unbounded intervals and by Páles [22], [25] and by Czinder and Páles [9] on bounded intervals.

The aim of this paper is to study the *global comparison problem*

$$M_{f,g,m;\mu}(\boldsymbol{x}) \le M_{h,k,n;\nu}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in I^d) \tag{12}$$

and also its *local* analogue. In terms of the Chebyshev systems (f, g) and (h, k), the measurable families of d-variable means $m : I^d \times T \to I$ and $n : I^d \times S \to I$, and the measures μ, ν, we give necessary conditions (which, in general, are not sufficient) and also sufficient conditions (that are also necessary in a certain sense) for (12) to hold. Our main results generalize that of the paper by Losonczi and Páles [20] and also many former results obtained in various particular cases of this problem, cf. [7], [8], [10], [18], [19], [21], [23].

2. Invariants with respect to equality of means

In order to describe the regularity conditions related to the two generating functions f, g of the mean $M_{f,g,m;\mu}$, we introduce some regularity classes. The class $\mathcal{C}_0(I)$ consists of all those pairs of continuous functions $f, g : I \to \mathbb{R}$ that form a Chebyshev system over I.

If $k \ge 1$, then we say that the pair (f, g) is in the class $\mathcal{C}_k(I)$ if f, g are k-times continuously differentiable functions such that $(f, g) \in \mathcal{C}_0(I)$ and the *Wronski determinant*

$$\begin{vmatrix} f'(x) & f(x) \\ g'(x) & g(x) \end{vmatrix} = \partial_1 \mathcal{D}_{f,g}(x, x) \qquad (x \in I) \tag{13}$$

does not vanish on I. Provided that g is positive, then we have that

$$\left(\frac{f}{g}\right)'(x) = \frac{\partial_1 \mathcal{D}_{f,g}(x, x)}{g^2(x)} \tag{14}$$

hence condition $\partial_1 \mathcal{D}_{f,g}(x,x) \neq 0$ implies that f/g is strictly monotone, whence it follows that $(f,g) \in \mathcal{C}_0(I)$. Obviously, $\mathcal{C}_0(I) \supseteq \mathcal{C}_1(I) \supseteq \mathcal{C}_2(I) \supseteq \cdots$.

It is easy to see that if $(f,g), (f^*,g^*) \in \mathcal{C}_0(I)$ and

$$\begin{aligned} f &= \alpha f^* + \beta g^*, \\ g &= \gamma f^* + \delta g^*, \end{aligned} \tag{15}$$

where the constants $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ satisfy $\alpha\delta - \beta\gamma \neq 0$, then

$$\mathcal{D}_{f,g}(x,y) = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \cdot \mathcal{D}_{f^*,g^*}(x,y) \qquad (x,y \in I). \tag{16}$$

This implies that the identity

$$M_{f,g,m;\mu} = M_{f^*,g^*,m;\mu} \tag{17}$$

also holds for any measurable family $m : I^d \times T \to I$ and probability measure μ.

If (15) holds for some constants $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, then we say that the pairs (f,g) and (f^*,g^*) are *equivalent*. It is obvious that any necessary and/or sufficient condition for (12) has to be invariant with respect to the equivalence of the generating functions.

The following result, which is based on [5, Theorem 3], allows us to assume more regularity on Chebyshev systems.

Lemma 2. *Let $k \in \mathbb{N} \cup \{0\}$ and $(f,g) \in \mathcal{C}_k(I)$. Then there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $\alpha\delta - \beta\gamma \neq 0$ and $(f^*,g^*) \in \mathcal{C}_k(I)$ such that* (15) *holds and g^* is positive and f^*/g^* is strictly monotone. Furthermore, if $k \geq 1$, then the derivative of f^*/g^* does not vanish on I.*

Proof. In the case $k = 0$, the statement is a direct consequence of the result [5, Theorem 3] obtained by Bessenyei and Páles.

Assume now that $k \geq 1$. Then, $(f,g) \in \mathcal{C}_k(I)$ means that f and g are k-times continuously differentiable and $\partial_1 \mathcal{D}_{f,g}(x,x)$ does not vanish for $x \in I$. Using what we have established in the case $k = 0$, there exist constants $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $\alpha\delta - \beta\gamma \neq 0$ and $(f^*,g^*) \in \mathcal{C}_0(I)$ such that (15) holds and g^* is positive and f^*/g^* is strictly monotone. The condition $\alpha\delta - \beta\gamma \neq 0$ and (15) imply that there exist $a, b, c, d \in \mathbb{R}$ with $ad - bc \neq 0$ such that

$$\begin{aligned} f^* &= af + bg, \\ g^* &= cf + dg. \end{aligned}$$

Hence f^* and g^* are also k-times continuously differentiable. This immediately implies that

$$\partial_1 \mathcal{D}_{f^*,g^*}(x,x) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \cdot \partial_1 \mathcal{D}_{f,g}(x,x) \qquad (x \in I),$$

which shows that $\partial_1 \mathcal{D}_{f^*,g^*}(x,x)$ does not vanish for $x \in I$. Therefore, $(f^*,g^*) \in \mathcal{C}_k(I)$ holds, too.

Applying the identity (14) for f^* and g^* instead of f and g, we can also see that the derivative of f^*/g^* does not vanish on I. □

For the computation of the first- and second-order partial derivatives of the mean $M_{f,g,m;\mu}$ at the diagonal of I^d, we will establish a result below. For brevity, we introduce the following notation: If $\boldsymbol{p} \in I^d$ and $\delta > 0$ then let $B(\boldsymbol{p}, \delta)$ stand for the ball $\{\boldsymbol{x} \in I^d \colon |\boldsymbol{x} - \boldsymbol{p}| \leq \delta\}$. Furthermore, if μ is a probability measure on the measurable space $(T, \mathcal{A})$ and $q \geq 1$, then the space of measurable functions $\varphi : T \to \mathbb{R}$ such that $|\varphi|^q$ is μ-integrable will be denoted by $L^q(T, \mathcal{A}, \mu)$ or shortly by L^q.

If $\varphi, \psi : T \to \mathbb{R}$ are measurable functions such that $\varphi\psi$ is μ-integrable (for instance, $\varphi, \psi \in L^2$), then we set

$$\langle \varphi, \psi \rangle_\mu := \int_T \varphi(t)\psi(t) \, \mathrm{d}\mu(t).$$

More generally, if $\varphi, \psi : I^d \times T \to \mathbb{R}$, and for some $\boldsymbol{x} \in I^d$, the map $t \mapsto \varphi(\boldsymbol{x}, t)\psi(\boldsymbol{x}, t)$ is μ-integrable, then we write

$$\langle \varphi, \psi \rangle_\mu(\boldsymbol{x}) := \int_T \varphi(\boldsymbol{x}, t)\psi(\boldsymbol{x}, t) \,\mathrm{d}\mu(t).$$

Given a number $q \geq 1$, a function $\varphi : I^d \times T \to \mathbb{R}$ is said to be of L^q-type at $\boldsymbol{p} \in I^d$, if $\varphi(\boldsymbol{p}, \cdot)$ is measurable, furthermore, there exist $\delta > 0$ and a function $a \in L^q$ such that

$$|\varphi(\boldsymbol{x}, t)| \leq a(t) \qquad (t \in T,\, \boldsymbol{x} \in B(\boldsymbol{p}, \delta)).$$

Let $\mathcal{C}_1(I^d \times T)$ denote the class of measurable families of d-variable means $m : I^d \times T \to I$ with the following two additional properties:

(H5) For every $t \in T$, the function $m(\cdot, t)$ is continuously partially differentiable over I^d such that, for all $\boldsymbol{p} \in I^d$, $i \in \{1, \dots, d\}$, the function $\partial_i m$ is of L^1-type at $\boldsymbol{p}$.

Analogously, we define $\mathcal{C}_2(I^d \times T)$ to be the following subclass of $\mathcal{C}_1(I^d \times T)$:

(H6) For every $t \in T$, the function $m(\cdot, t)$ is twice continuously partially differentiable over I^d such that, for all $\boldsymbol{p} \in I^d$ and $i, j \in \{1, \dots, d\}$, the function $\partial_i m$ is of L^2-type and $\partial_i\partial_j m$ is of L^1-type at $\boldsymbol{p}$.

Lemma 3. *Let $k \in \{1, 2\}$ and let $\varphi : I \to \mathbb{R}$ be a k-times continuously differentiable function and $m \in \mathcal{C}_k(I^d \times T)$. Then the function $\Phi : I^d \to \mathbb{R}$ defined by*

$$\Phi(\boldsymbol{x}) := \int_T \varphi(m(\boldsymbol{x}, t)) \,\mathrm{d}\mu(t) \tag{18}$$

is k-times continuously differentiable on I^d. Furthermore, for $i \in \{1, \dots, d\}$,

$$\partial_i\Phi(\boldsymbol{p}) = \int_T \varphi'(m(\boldsymbol{p}, t))\, \partial_i m(\boldsymbol{p}, t) \,\mathrm{d}\mu(t) \qquad (\boldsymbol{p} \in I^d) \tag{19}$$

and, for $i, j \in \{1, \dots, d\}$,

$$\partial_i\partial_j\Phi(\boldsymbol{p}) = \int_T \Big[\varphi''(m(\boldsymbol{p}, t))\, \partial_i m(\boldsymbol{p}, t)\, \partial_j m(\boldsymbol{p}, t) + \varphi'(m(\boldsymbol{p}, t))\, \partial_i\partial_j m(\boldsymbol{p}, t)\Big] \,\mathrm{d}\mu(t) \qquad (\boldsymbol{p} \in I^d) \tag{20}$$

provided that $k = 2$.

Proof. If m is measurable family of d-variable means, then, by the continuity of φ and the mean value property of m, it easily follows that Φ is well-defined for all $\boldsymbol{x} \in I^d$. Due to the continuity of φ' and the assumption $m \in \mathcal{C}_1(I^d \times T)$, it easily follows that the integral on the right hand side of (19) is well-defined. Furthermore, if φ'' is continuous and $m \in \mathcal{C}_2(I^d \times T)$, then also the right hand side of (20) exists.

First we elaborate the case $k = 1$. We need to show that, for every $\boldsymbol{p} \in I^d$, the function Φ is partially differentiable at $\boldsymbol{p}$ with formula (19) and that the partial derivatives are continuous at $\boldsymbol{p}$.

Before proceeding to the proof, we shall establish, for every $\boldsymbol{p} \in I^d$, the following equality

$$\lim_{\delta \to 0} \int_T \sup_{\boldsymbol{x} \in B(\boldsymbol{p}, \delta)} \big|\varphi'(m(\boldsymbol{x}, t))\, \partial_i m(\boldsymbol{x}, t) - \varphi'(m(\boldsymbol{p}, t))\, \partial_i m(\boldsymbol{p}, t)\big| \,\mathrm{d}\mu(t) = 0. \tag{21}$$

First choose $\delta_0 > 0$ so that $\alpha := \min\{p_1, \dots, p_d\} - \delta_0$ and $\beta := \max\{p_1, \dots, p_d\} + \delta_0$ be elements of I. Let K be the supremum of $|\varphi'|$ over the compact interval $[\alpha, \beta]$. The continuity of φ' implies that K is finite. By the mean value property of m, for every $t \in T$ and $\boldsymbol{x} \in B(\boldsymbol{p}, \delta_0)$, we have that $\alpha \leq m(\boldsymbol{x}, t) \leq \beta$. Hence, for every $t \in T$ and $\boldsymbol{x} \in B(\boldsymbol{p}, \delta_0)$, the inequality $|\varphi'(m(\boldsymbol{x}, t))| \leq K$ holds.

Using the assumption that $\partial_i m$ is of L^1-type at $\boldsymbol{p}$, we can find $0<\delta_1\leq\delta_0$ and a function $a\in L^1$ such that

$$|\partial_i m(\boldsymbol{x},t)|\leq a(t)\qquad (t\in T,\,\boldsymbol{x}\in B(\boldsymbol{p},\delta_1)).$$

Let $\delta_n>0$ be an arbitrary sequence converging to 0 with $\delta_n\leq\delta_1$ for all $n\in\mathbb{N}$. By the Lagrange Mean Value Theorem, for every $\boldsymbol{x}\in B(\boldsymbol{p},\delta_n)$ and for every $t\in T$, there exists $\lambda\in[0,1]$ such that

$$|m(\boldsymbol{x},t)-m(\boldsymbol{p},t)|\leq\sum_{i=1}^{d}|\partial_i m(\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{p},t)||x_i-p_i|\leq d\delta_n a(t).$$

Using the continuity of φ' at $m(\boldsymbol{p},t)$, it immediately follows that the sequence of measurable functions $\psi_n:T\to\mathbb{R}$ defined by

$$\psi_n(t):=\sup_{\boldsymbol{x}\in B(\boldsymbol{p},\delta_n)}|\varphi'(m(\boldsymbol{x},t))-\varphi'(m(\boldsymbol{p},t))|\leq 2K$$

converges to zero for every $t\in T$. By the continuity of the partial derivative $\partial_i m(\cdot,t)$ at $\boldsymbol{p}$, we also have that the sequence of measurable functions $\chi_n:T\to\mathbb{R}$ defined by

$$\chi_n(t):=\sup_{\boldsymbol{x}\in B(\boldsymbol{p},\delta_n)}|\partial_i m(\boldsymbol{x},t)-\partial_i m(\boldsymbol{p},t)|\leq 2a(t)$$

converges to zero for every $t\in T$. Using the above estimations, we can now obtain that

$$\begin{aligned}
&\sup_{\boldsymbol{x}\in B(\boldsymbol{p},\delta_n)}\big|\varphi'(m(\boldsymbol{x},t))\,\partial_i m(\boldsymbol{x},t)-\varphi'(m(\boldsymbol{p},t))\,\partial_i m(\boldsymbol{p},t)\big|\\
&\leq\sup_{\boldsymbol{x}\in B(\boldsymbol{p},\delta_n)}|\varphi'(m(\boldsymbol{x},t))||\partial_i m(\boldsymbol{x},t)-\partial_i m(\boldsymbol{p},t)|+\sup_{\boldsymbol{x}\in B(\boldsymbol{p},\delta_n)}|\varphi'(m(\boldsymbol{x},t))-\varphi'(m(\boldsymbol{p},t))||\partial_i m(\boldsymbol{p},t)|\\
&\leq K\chi_n(t)+\psi_n(t)a(t).
\end{aligned}$$

The expression on the right hand side of this inequality converges to zero for each $t\in T$, and these functions are dominated by the integrable function $4Ka$. Hence, by Lebesgue's Dominated Convergence Theorem, it follows that

$$\lim_{n\to\infty}\int\limits_T\sup_{\boldsymbol{x}\in B(\boldsymbol{p},\delta_n)}\big|\varphi'(m(\boldsymbol{x},t))\,\partial_i m(\boldsymbol{x},t)-\varphi'(m(\boldsymbol{p},t))\,\partial_i m(\boldsymbol{p},t)\big|=0.$$

Because the sequence (δ_n) converging to 0 was arbitrary, it follows that (21) holds.

Let $\boldsymbol{p}\in I^d$ be fixed and let $\boldsymbol{e}_i$ denote the ith vector of the standard basis on $\mathbb{R}^n$. For the proof that the ith partial derivative of Φ at $\boldsymbol{p}$ is given by (19), consider the following estimation for $s\in(I-p_i)$, $s\neq 0$:

$$\begin{aligned}
\Delta_i(s):&=\bigg|\frac{\Phi(\boldsymbol{p}+s\boldsymbol{e}_i)-\Phi(\boldsymbol{p})}{s}-\int_T\varphi'(m(\boldsymbol{p},t))\,\partial_i m(\boldsymbol{p},t)\,\mathrm{d}\mu(t)\bigg|\\
&=\bigg|\frac{1}{s}\bigg(\int_T\varphi(m(\boldsymbol{p}+s\boldsymbol{e}_i,t))\,\mathrm{d}\mu(t)-\int_T\varphi(m(\boldsymbol{p},t))\,\mathrm{d}\mu(t)\bigg)-\int_T\varphi'(m(\boldsymbol{p},t))\,\partial_i m(\boldsymbol{p},t)\,\mathrm{d}\mu(t)\bigg|\\
&\leq\frac{1}{|s|}\int_T\big|\varphi(m(\boldsymbol{p}+s\boldsymbol{e}_i,t))-\varphi(m(\boldsymbol{p},t))-s\varphi'(m(\boldsymbol{p},t))\,\partial_i m(\boldsymbol{p},t)\big|\,\mathrm{d}\mu(t).
\end{aligned}\tag{22}$$

Applying the Lagrange Mean Value Theorem for the function

$$\kappa_t(s):=\varphi(m(\boldsymbol{p}+s\boldsymbol{e}_i,t))-s\varphi'(m(\boldsymbol{p},t))\,\partial_i m(\boldsymbol{p},t),$$

for every $t\in T$, we can find an element σ_t between 0 and s such that $\kappa_t(s)-\kappa_t(0)=\kappa_t'(\sigma_t)s$, that is

$$\begin{aligned}
&\varphi(m(\boldsymbol{p}+s\boldsymbol{e}_i,t))-\varphi(m(\boldsymbol{p},t))-s\varphi'(m(\boldsymbol{p},t))\,\partial_i m(\boldsymbol{p},t)\\
&\qquad=\big(\varphi'(m(\boldsymbol{p}+\sigma_t\boldsymbol{e}_i,t))\,\partial_i m(\boldsymbol{p}+\sigma_t\boldsymbol{e}_i,t)-\varphi'(m(\boldsymbol{p},t))\,\partial_i m(\boldsymbol{p},t)\big)s.
\end{aligned}$$

Using this formula, inequality (22) and the equality (21), for $s \in I - p_i$, it follows that

$$\limsup_{s\to 0} \Delta_i(s) \le \limsup_{s\to 0} \int\limits_T \big|\varphi'(m(\boldsymbol{p}+\sigma_t\boldsymbol{e}_i,t))\,\partial_i m(\boldsymbol{p}+\sigma_t\boldsymbol{e}_i,t) - \varphi'(m(\boldsymbol{p},t))\,\partial_i m(\boldsymbol{p},t)\big|\,\mathrm{d}\mu(t)$$

$$\le \lim_{s\to 0} \int\limits_T \sup_{\boldsymbol{x}\in B(\boldsymbol{p},|s|)} \big|\varphi'(m(\boldsymbol{x},t))\,\partial_i m(\boldsymbol{x},t) - \varphi'(m(\boldsymbol{p},t))\,\partial_i m(\boldsymbol{p},t)\big|\,\mathrm{d}\mu(t) = 0.$$

Thus, we have proved that $\Delta_i(s)$ tends to zero as $s \to 0$. This completes the proof of the partial differentiability of Φ with respect to the ith variable at $\boldsymbol{p}$ and also the validity of formula (19).

Finally, we show that the function $\partial_i\Phi$ is continuous at every $\boldsymbol{p} \in I^d$. Let $(\boldsymbol{x}_n)$ be an arbitrary sequence in $B(\boldsymbol{p},\delta_0)$ converging to $\boldsymbol{p}$ and denote $\delta_n := |\boldsymbol{x}_n - \boldsymbol{p}|$. Then (δ_n) is a null sequence and we have that

$$|\partial_i\Phi(\boldsymbol{x}_n) - \partial_i\Phi(\boldsymbol{p})| \le \int\limits_T \big|\varphi'(m(\boldsymbol{x}_n,t))\,\partial_i m(\boldsymbol{x}_n,t) - \varphi'(m(\boldsymbol{p},t))\,\partial_i m(\boldsymbol{p},t)\big|\,\mathrm{d}\mu(t)$$

$$\le \int\limits_T \sup_{\boldsymbol{u}\in B(\boldsymbol{p},\delta_n)} \big|\varphi'(m(\boldsymbol{u},t))\,\partial_i m(\boldsymbol{u},t) - \varphi'(m(\boldsymbol{p},t))\,\partial_i m(\boldsymbol{p},t)\big|\,\mathrm{d}\mu(t).$$

Due to the equality (21), the right hand side in the above inequality tends to zero as $n \to \infty$, whence it follows that $(\partial_i\Phi(\boldsymbol{x}_n))$ converges to $\partial_i\Phi(\boldsymbol{p})$, which proves the continuity of $\partial_i\Phi$ at $\boldsymbol{p}$.

Analogously, using a similar argument as in the proof of (21), for the case $k = 2$, the reader can show that the following two equalities hold:

$$\lim_{\delta\to 0} \int\limits_T \sup_{\boldsymbol{x}\in B(\boldsymbol{p},\delta)} \big|\varphi''(m(\boldsymbol{x},t))\,\partial_i m(\boldsymbol{x},t)\,\partial_j m(\boldsymbol{x},t) - \varphi''(m(\boldsymbol{p},t))\,\partial_i m(\boldsymbol{p},t)\,\partial_j m(\boldsymbol{p},t)\big|\,\mathrm{d}\mu(t) = 0, \tag{23}$$

$$\lim_{\delta\to 0} \int\limits_T \sup_{\boldsymbol{x}\in B(\boldsymbol{p},\delta)} \big|\varphi'(m(\boldsymbol{x},t))\,\partial_i\partial_j m(\boldsymbol{x},t) - \varphi'(m(\boldsymbol{p},t))\,\partial_i\partial_j m(\boldsymbol{p},t)\big|\,\mathrm{d}\mu(t) = 0. \tag{24}$$

Let $\boldsymbol{p} \in I^d$ be fixed. To prove equality (20) which establishes the formula for the jth partial derivative of $\partial_i\Phi$ at $\boldsymbol{p}$, consider the following estimation for $r \in (I - p_j)$, $r \ne 0$:

$$\begin{aligned}
\Delta_{ij}(r) &\\
:=&\bigg|\frac{\partial_i\Phi(\boldsymbol{p}+r\boldsymbol{e}_j) - \partial_i\Phi(\boldsymbol{p})}{r} - \int_T \varphi''(m(\boldsymbol{p},t))\,\partial_i m(\boldsymbol{p},t)\,\partial_j m(\boldsymbol{p},t) + \varphi'(m(\boldsymbol{p},t))\,\partial_i\partial_j m(\boldsymbol{p},t)\,\mathrm{d}\mu(t)\bigg|\\
=&\bigg|\frac{1}{r}\bigg(\int_T \varphi'(m(\boldsymbol{p}+r\boldsymbol{e}_j,t))\,\partial_i m(\boldsymbol{p}+r\boldsymbol{e}_j,t)\,\mathrm{d}\mu(t) - \int_T \varphi'(m(\boldsymbol{p},t))\,\partial_i m(\boldsymbol{p},t)\,\mathrm{d}\mu(t)\bigg)\\
&- \int_T \varphi''(m(\boldsymbol{p},t))\,\partial_i m(\boldsymbol{p},t)\,\partial_j m(\boldsymbol{p},t) + \varphi'(m(\boldsymbol{p},t))\,\partial_i\partial_j m(\boldsymbol{p},t)\,\mathrm{d}\mu(t)\bigg|\\
\le&\frac{1}{|r|}\int_T \big|\varphi'(m(\boldsymbol{p}+r\boldsymbol{e}_j,t))\,\partial_i m(\boldsymbol{p}+r\boldsymbol{e}_j,t) - \varphi'(m(\boldsymbol{p},t))\,\partial_i m(\boldsymbol{p},t)\\
&- r\varphi''(m(\boldsymbol{p},t))\,\partial_i m(\boldsymbol{p},t)\,\partial_j m(\boldsymbol{p},t) - r\varphi'(m(\boldsymbol{p},t))\,\partial_i\partial_j m(\boldsymbol{p},t)\big|\,\mathrm{d}\mu(t).
\end{aligned} \tag{25}$$

Applying, for every $t \in T$, the Lagrange Mean Value Theorem for the function

$$\theta_t(r) := \varphi'(m(\boldsymbol{p}+r\boldsymbol{e}_j,t))\partial_i m(\boldsymbol{p}+r\boldsymbol{e}_j,t) - r\big(\varphi''(m(\boldsymbol{p},t))\partial_i m(\boldsymbol{p},t)\partial_j m(\boldsymbol{p},t) + \varphi'(m(\boldsymbol{p},t))\partial_i\partial_j m(\boldsymbol{p},t)\big),$$

we can find an element ρ_t between 0 and r such that

$$\theta_t(r) - \theta_t(0) = \theta_t'(\rho_t)r. \tag{26}$$

Now, by using equality (26), inequality (25) and equalities (23) and (24), respectively, with an analogous argument that we applied in the case $k = 1$, we get that $\Delta_{ij}(r)$ tends to zero as $r \to 0$, proving the partial differentiability of $\partial_i\Phi$ at $\boldsymbol{p}$ with respect to the jth variable and formula (20). On the other hand, again by a similar train of thoughts, it easily follows from (23) and (24) that the function $\partial_i\partial_j\Phi$ is continuous on I^d. This completes the proof of the lemma. □

Theorem 4. *Let $(f,g) \in \mathcal{C}_1(I)$, let $m \in \mathcal{C}_1(I^d \times T)$ be a measurable family of means, and let μ be a probability measure on the measurable space $(T,\mathcal{A})$. Then $M_{f,g,m;\mu}$ is continuously differentiable on I^d and, for all $i \in \{1,\dots,d\}$ and $x \in I$,*

$$\partial_i M_{f,g,m;\mu}(x,\dots,x) = \langle \partial_i m, 1\rangle_\mu(x,\dots,x). \tag{27}$$

If, in addition, $(f,g) \in \mathcal{C}_2(I)$, let $m \in \mathcal{C}_2(I^d \times T)$, then $M_{f,g,m;\mu}$ is twice continuously differentiable on I^d and, for all $i,j \in \{1,\dots,d\}$ and $x \in I$,

$$\begin{aligned}\partial_i\partial_j M_{f,g,m;\mu}(x,\dots,x)\\ = \big(\langle \partial_i m, \partial_j m\rangle_\mu - \langle \partial_i m, 1\rangle_\mu \langle \partial_j m, 1\rangle_\mu\big)(x,\dots,x)\frac{\partial_1^2 \mathcal{D}_{f,g}(x,x)}{\partial_1 \mathcal{D}_{f,g}(x,x)} + \langle \partial_i\partial_j m, 1\rangle_\mu(x,\dots,x).\end{aligned} \tag{28}$$

Proof. Let $k \in \{1,2\}$ and assume that $(f,g) \in \mathcal{C}_k(I)$, $m \in \mathcal{C}_k(I^d \times T)$. In view of Lemma 2, we may assume that g is positive, f/g is strictly monotone with a non-vanishing first-order derivative. Then f, g and the inverse of f/g are k-times continuously differentiable and, by Lemma 3, we also have that the mappings

$$\boldsymbol{x} \mapsto \int_T f\big(m(\boldsymbol{x},t)\big)\,\mathrm{d}\mu(t) \qquad \text{and} \qquad \boldsymbol{x} \mapsto \int_T g\big(m(\boldsymbol{x},t)\big)\,\mathrm{d}\mu(t)$$

are k-times continuously differentiable on I^d. On the other hand, we now also have formula (9) for the d-variable mean $M_{f,g,m;\mu}$. Thus, using the standard calculus rules, it follows that $M_{f,g,m;\mu}$ is k-times continuously differentiable on I^d

To prove the first formula stated in (27), let us consider the case $k = 1$. Differentiating the identity (7) with respect to the variable x_i once, we get

$$\int_T \big[\partial_1\mathcal{D}_{f,g}(m(\boldsymbol{x},t), M_{f,g,m;\mu}(\boldsymbol{x}))\partial_i m(\boldsymbol{x},t) + \partial_2\mathcal{D}_{f,g}(m(\boldsymbol{x},t), M_{f,g,m;\mu}(\boldsymbol{x}))\partial_i M_{f,g,m;\mu}(\boldsymbol{x})\big]\,\mathrm{d}\mu(t) = 0.$$

Now taking $x \in I$ and substituting $\boldsymbol{x} = (x,\dots,x)$, the above equation simplifies to

$$\partial_1\mathcal{D}_{f,g}(x,x)\int_T \partial_i m(x,\dots,x,t)\,\mathrm{d}\mu(t) + \partial_2\mathcal{D}_{f,g}(x,x)\partial_i M_{f,g,m;\mu}(x,\dots,x) = 0.$$

Observe that $\partial_1\mathcal{D}_{f,g}(x,x) = -\partial_2\mathcal{D}_{f,g}(x,x) \neq 0$ for all $x \in I$, hence the former equation yields the desired equality (27).

Now consider the case $k = 2$. Differentiating the identity that we obtained in the first lines of the proof with respect to the variable x_j, we obtain

$$\begin{aligned}&\int_T \big[\partial_1^2\mathcal{D}_{f,g}(m(\boldsymbol{x},t), M_{f,g,m;\mu}(\boldsymbol{x}))\partial_i m(\boldsymbol{x},t)\partial_j m(\boldsymbol{x},t)\\ &+ \partial_1\partial_2\mathcal{D}_{f,g}(m(\boldsymbol{x},t), M_{f,g,m;\mu}(\boldsymbol{x}))\big(\partial_j M_{f,g,m;\mu}(\boldsymbol{x})\partial_i m(\boldsymbol{x},t) + \partial_i M_{f,g,m;\mu}(\boldsymbol{x})\partial_j m(\boldsymbol{x},t)\big)\\ &+ \partial_2^2\mathcal{D}_{f,g}(m(\boldsymbol{x},t), M_{f,g,m;\mu}(\boldsymbol{x}))\partial_j M_{f,g,m;\mu}(\boldsymbol{x})\partial_i M_{f,g,m;\mu}(\boldsymbol{x})\\ &+ \partial_1\mathcal{D}_{f,g}(m(\boldsymbol{x},t), M_{f,g,m;\mu}(\boldsymbol{x}))\partial_j\partial_i m(\boldsymbol{x},t) + \partial_2\mathcal{D}_{f,g}(m(\boldsymbol{x},t), M_{f,g,m;\mu}(\boldsymbol{x}))\partial_j\partial_i M_{f,g,m;\mu}(\boldsymbol{x})\big]\,\mathrm{d}\mu(t) = 0,\end{aligned}$$

respectively. Using the identities $\partial_2\mathcal{D}_{f,g}(x,x) = -\partial_1\mathcal{D}_{f,g}(x,x)$, $\partial_2^2\mathcal{D}_{f,g}(x,x) = -\partial_1^2\mathcal{D}_{f,g}(x,x)$, and $\partial_1\partial_2\mathcal{D}_{f,g}(x,x) = 0$ (that are consequences of the asymmetry property $\mathcal{D}_{f,g}(x,y) = -\mathcal{D}_{f,g}(y,x)$), and substituting $\boldsymbol{x} = (x, \dots, x)$, we get that

$$\partial_1^2 \mathcal{D}_{f,g}(x,x) \int_T \partial_i m(x,\dots,x,t)\partial_j m(x,\dots,x,t)\,\mathrm{d}\mu(t) + \partial_1 \mathcal{D}_{f,g}(x,x) \int_T \partial_j\partial_i m(x,\dots,x,t)\,\mathrm{d}\mu(t)$$
$$- \partial_1^2 \mathcal{D}_{f,g}(x,x)\partial_j M_{f,g,m;\mu}(x,\dots,x)\partial_i M_{f,g,m;\mu}(x,\dots,x) - \partial_1 \mathcal{D}_{f,g}(x,x)\partial_j\partial_i M_{f,g,m;\mu}(x,\dots,x) = 0.$$

Dividing both sides of this equation by $\partial_1 \mathcal{D}_{f,g}(x,x) \neq 0$ and using (27), for the second-order partial derivative $\partial_j\partial_i M_{f,g,m;\mu}(x,\dots,x)$, we obtain the formula stated in (28). □

One of the most important particular case of the above theorem is when the d-variable family of means is a family of weighted d-variable arithmetic means.

Corollary 5. *Let $(f,g) \in \mathcal{C}_1(I)$, let μ be a probability measure on the measurable space $(T,\mathcal{A})$ and let $m \in \mathcal{C}_1(I^d \times T)$ be a measurable family of d-variable means given by*

$$m(\boldsymbol{x},t) := \lambda_1(t)x_1 + \cdots + \lambda_d(t)x_d \qquad (\boldsymbol{x} = (x_1,\dots,x_d) \in I^d,\ t \in T),$$

where $\lambda_1,\dots,\lambda_d : T \to [0,1]$ are measurable functions with $\lambda_1 + \cdots + \lambda_d = 1$. Then $M_{f,g,m;\mu}$ is continuously differentiable on I^d and, for all $i \in \{1,\dots,d\}$ and $x \in I$,

$$\partial_i M_{f,g,m;\mu}(x,\dots,x) = \langle \lambda_i, 1\rangle_\mu. \tag{29}$$

If, in addition, $(f,g) \in \mathcal{C}_2(I)$, let $m \in \mathcal{C}_2(I^d \times T)$, then $M_{f,g,m;\mu}$ is twice continuously differentiable on I^d and, for all $i,j \in \{1,\dots,d\}$ and $x \in I$,

$$\partial_i\partial_j M_{f,g,m;\mu}(x,\dots,x) = \big(\langle \lambda_i,\lambda_j\rangle_\mu - \langle \lambda_i,1\rangle_\mu\langle \lambda_j,1\rangle_\mu\big)\frac{\partial_1^2\mathcal{D}_{f,g}(x,x)}{\partial_1\mathcal{D}_{f,g}(x,x)}. \tag{30}$$

Proof. Observe that we have $\partial_i m(\boldsymbol{x},t) = \lambda_i(t)$ and $\partial_i\partial_j m(\boldsymbol{x},t) = 0$. By the boundedness of the measurable function, it follows that $\partial_i m$ is L^1- and L^2-type, and $\partial_i\partial_j m$ is L^1-type at every point of I^d. Therefore, Theorem 4 applies, and formulas (27) and (28) reduce to (29) and (30), respectively. □

The particular case when $T = [0,1]$, $d = 2$, $\lambda_1(t) = t$ and $\lambda_2 = 1 - t$ was considered by Losonczi and Páles in the paper [20], where also the related local and global comparison problems were investigated. The above Theorem 4 and Corollary 5 generalize the result of [20, Lemma 4].

The following lemma, which is an extension of [20, Lemma 3], will play an important role in establishing the necessary conditions for the (global) comparison of means. We recall that a sequence (ν_k) of probability measures on T is said to *converge weakly to a measure* ν if, for all bounded Borel measurable functions $\varphi : T \to \mathbb{R}$, we have

$$\lim_{k\to\infty} \int_T \varphi(t)\,\mathrm{d}\nu_k(t) = \int_T \varphi(t)\,\mathrm{d}\nu(t).$$

Lemma 6. *Let $(f,g) \in \mathcal{C}_1(I)$ and let (ν_k) be a sequence of probability measures on T weakly converging to a measure ν, let (γ_k) be a null sequence of positive numbers in $[0,1]$ and let $t_0 \in T$. Set $\mu_k := (1-\gamma_k)\delta_{t_0} + \gamma_k\nu_k$ for $k \in \mathbb{N}$. Then*

$$\lim_{k\to\infty} \frac{1}{\gamma_k}\big[M_{f,g,m;\mu_k}(\boldsymbol{x}) - m(\boldsymbol{x},t_0)\big] = \frac{\int_T \mathcal{D}_{f,g}(m(\boldsymbol{x},t),m(\boldsymbol{x},t_0))\,\mathrm{d}\nu(t)}{\partial_1\mathcal{D}_{f,g}(m(\boldsymbol{x},t_0),m(\boldsymbol{x},t_0))} \qquad (\boldsymbol{x} \in I^d). \tag{31}$$

Proof. Let $\boldsymbol{x} \in I^d$ be a fixed vector. By the assumptions of the lemma, we have that μ_k converges to δ_{t_0} weakly. More generally, for an arbitrary bounded sequence of Borel measurable functions $\varphi_k : T \to \mathbb{R}$, which converges uniformly to $\varphi_0 : T \to \mathbb{R}$ as $k \to \infty$, we get

$$\begin{aligned}\int_T \varphi_k \,\mathrm{d}\mu_k(t) &= \int_T \varphi_k d((1-\gamma_k)\delta_{t_0} + \gamma_k \nu_k)(t) \\ &= (1-\gamma_k)\int_T \varphi_k d\delta_{t_0}(t) + \gamma_k \int_T \varphi_0 \,\mathrm{d}\nu_k(t) + \gamma_k \int_T (\varphi_k - \varphi_0)\,\mathrm{d}\nu_k(t) \\ &\longrightarrow \int_T \varphi_0 d\delta_{t_0}(t) = \varphi_0(t_0).\end{aligned} \tag{32}$$

First, we are going to show that the sequence $u_k := M_{f,g,m;\mu_k}(\boldsymbol{x})$ converges to $m(\boldsymbol{x}, t_0)$. We have that $\min(\boldsymbol{x}) \le u_k \le \max(\boldsymbol{x})$ for all $k \in \mathbb{N}$. Hence it is sufficient to prove that every convergent subsequence of (u_k) converges to the same limit point. To show this, let (u_{k_j}) be any convergent subsequence of (u_k) such that $u_{k_j} \to u_0$ as $j \to \infty$. Then, the sequence of Borel measurable functions $\varphi_j(t) := D_{f,g}(m(\boldsymbol{x}, t), u_{k_j})$ tends uniformly to the limit function $\varphi_0(t) := D_{f,g}(m(\boldsymbol{x}, t), u_0)$. Thus, in view of formula (32), we get

$$\lim_{j\to\infty} \int_T D_{f,g}(m(\boldsymbol{x}, t), u_{k_j})\,\mathrm{d}\mu_{k_j}(t) = D_{f,g}(m(\boldsymbol{x}, t_0), u_0).$$

On the other hand, for all j, we have that

$$\int_T D_{f,g}(m(\boldsymbol{x}, t), u_{k_j})\,\mathrm{d}\mu_{k_j}(t) = 0,$$

which implies that $D_{f,g}(m(\boldsymbol{x}, t_0), u_0)$ is zero, i.e., $u_0 = m(\boldsymbol{x}, t_0)$. Hence $u_k \to m(\boldsymbol{x}, t_0)$ as $k \to \infty$.

Moreover, as $k \to \infty$, we similarly obtain

$$\begin{aligned}&\frac{1}{\gamma_k}\int_T D_{f,g}(m(\boldsymbol{x}, t), m(\boldsymbol{x}, t_0))\,\mathrm{d}\mu_k(t) \\ &\quad= \frac{1}{\gamma_k}\int_T D_{f,g}(m(\boldsymbol{x}, t), m(\boldsymbol{x}, t_0))\,\mathrm{d}((1-\gamma_k)\delta_{t_0} + \gamma_k\nu_k)(t) \\ &\quad= \int_T D_{f,g}(m(\boldsymbol{x}, t), m(\boldsymbol{x}, t_0))\,\mathrm{d}\nu_k(t) \longrightarrow \int_T D_{f,g}(m(\boldsymbol{x}, t), m(\boldsymbol{x}, t_0))\,\mathrm{d}\nu(t).\end{aligned} \tag{33}$$

Taking $\Phi_k(u) := \int_T D_{f,g}(m(\boldsymbol{x}, t), u)\,\mathrm{d}\mu_k(t)$ and applying the Lagrange mean value theorem for the differentiable function Φ_k, for every $k \in \mathbb{N}$, we can find a number η_k between u_k and $m(\boldsymbol{x}, t_0)$ such that

$$\Phi_k(m(\boldsymbol{x}, t_0)) - \Phi_k(u_k) = \Phi_k'(\eta_k)(m(\boldsymbol{x}, t_0) - u_k) \tag{34}$$

Since $\Phi_k(u_k) = 0$, therefore it follows that

$$\frac{1}{\gamma_k}\Phi_k(m(\boldsymbol{x}, t_0)) = \Phi_k'(\eta_k)\cdot\frac{1}{\gamma_k}(m(\boldsymbol{x}, t_0) - u_k). \tag{35}$$

Thus,

$$\begin{aligned}&\frac{1}{\gamma_k}\int_T D_{f,g}(m(\boldsymbol{x}, t), m(\boldsymbol{x}, t_0))\,\mathrm{d}\mu_k(t) \\ &\qquad= \int_T \begin{vmatrix} f(m(\boldsymbol{x}, t)) & f'(\eta_k) \\ g(m(\boldsymbol{x}, t)) & g'(\eta_k)\end{vmatrix}\,\mathrm{d}\mu_k(t)\cdot\frac{1}{\gamma_k}\big[m(\boldsymbol{x}, t_0) - M_{f,g,m;\mu_k}(\boldsymbol{x})\big].\end{aligned} \tag{36}$$

Then, obviously, η_k converges to $m(\boldsymbol{x}, t_0)$. By taking the limit of both sides of (36) as $k \to \infty$ and using (33) and (32), we get

$$\begin{aligned}&\int_T \mathcal{D}_{f,g}(m(\boldsymbol{x}, t), m(\boldsymbol{x}, t_0))\,\mathrm{d}\nu(t) \\ &\qquad= \partial_1\mathcal{D}_{f,g}(m(\boldsymbol{x}, t_0), m(\boldsymbol{x}, t_0))\cdot\lim_{k\to\infty}\frac{1}{\gamma_k}\big[M_{f,g,m;\mu_k}(\boldsymbol{x}) - m(\boldsymbol{x}, t_0)\big]\end{aligned} \tag{37}$$

By dividing both sides by $\partial_1\mathcal{D}_{f,g}(m(\boldsymbol{x},t_0),m(\boldsymbol{x},t_0))$, we get

$$\lim_{k\to\infty}\frac{1}{\gamma_k}\left[M_{f,g,m;\mu_k}(\boldsymbol{x})-m(\boldsymbol{x},t_0)\right]=\frac{\int_T\mathcal{D}_{f,g}(m(\boldsymbol{x},t),m(\boldsymbol{x},t_0))\,\mathrm{d}\nu(t)}{\partial_1\mathcal{D}_{f,g}(m(\boldsymbol{x},t_0),m(\boldsymbol{x},t_0))}. \tag{38}$$

This completes the proof of the lemma. □

3. Necessary conditions, sufficient conditions for local comparison of means

Our first result offers a necessary as well as a sufficient condition for the local comparison of means. Given two d-variable means $M,N:I^d\to I$, we say that M *is locally smaller than* N *at* $x_0\in I$ if there exists a neighborhood $U\subseteq I$ of x_0 such that

$$M(\boldsymbol{x})\le N(\boldsymbol{x}) \tag{39}$$

holds for all $\boldsymbol{x}\in U^d$. The case $d=1$ being trivial, we always assume that $d\ge 2$ holds in the subsequent considerations.

Theorem 7. *Let $M,N:I^d\to I$ be d-variable means such that M is locally smaller than N at a point $x_0\in I$. Assume that M and N are partially differentiable at the diagonal point $(x_0,\dots,x_0)\in I^d$. Then, for $x=x_0$ and for all $i\in\{1,\dots,d\}$,*

$$\partial_i M(x,\dots,x)=\partial_i N(x,\dots,x). \tag{40}$$

If, in addition, M and N are twice differentiable at $(x_0,\dots,x_0)\in I^d$, then the symmetric $(d-1)\times(d-1)$-matrix

$$\big(\partial_i\partial_j N(x_0,\dots,x_0)-\partial_i\partial_j M(x_0,\dots,x_0)\big)_{i,j=1}^{d-1} \tag{41}$$

is positive semidefinite.

On the other hand, if, for some $x_0\in I$, the equality (40) *holds for all $i\in\{1,\dots,d\}$ and for all x in a neighborhood of x_0, furthermore, M and N are twice continuously differentiable at $(x_0,\dots,x_0)$ and the symmetric $(d-1)\times(d-1)$-matrix given by* (41) *is positive definite, then M is locally smaller than N at x_0.*

Proof. Assume that M is locally smaller than N at $x_0\in I$, i.e., (39) holds for all $\boldsymbol{x}\in U^d$ in a neighborhood $U\subseteq I$ of x_0. Assume that M and N are partially differentiable at the diagonal point $(x_0,\dots,x_0)\in I^d$. Define the function $D:U^d\to\mathbb{R}$ by

$$D(\boldsymbol{x})=N(\boldsymbol{x})-M(\boldsymbol{x})\qquad(\boldsymbol{x}\in U^d).$$

Then D is nonnegative by inequality (39) and attains its minimum (which equals zero) at $\boldsymbol{x}=(x_0,\dots,x_0)$. Therefore $\partial_i D(x_0,\dots,x_0)=0$ for all $i\in\{1,\dots,d\}$, which yields (40).

If, in addition, M and N are twice differentiable at $(x_0,\dots,x_0)\in I^d$. Then $D''(x_0,\dots,x_0)=\big(\partial_i\partial_j D(x_0,\dots,x_0)\big)_{i,j=1}^{d}$ is a positive semidefinite symmetric $d\times d$-matrix. By the well-known necessary and sufficient conditions of positive semidefiniteness (cf. [6]), this implies that the symmetric $(d-1)\times(d-1)$-matrix $\big(\partial_i\partial_j D(x_0,\dots,x_0)\big)_{i,j=1}^{d-1}$ is also positive semidefinite.

Now let $x_0\in I$ and assume that there exists a neighborhood $U\subseteq I$ of x_0 such that the means M and N are twice differentiable on U^d, their second-order partial derivatives are continuous at $\boldsymbol{x}_0=(x_0,\dots,x_0)$, the equality (40) holds for all $x\in U$ and for all $i\in\{1,\dots,d\}$, and the symmetric $(d-1)\times(d-1)$-matrix-valued function $A(\boldsymbol{x}):=\big(\partial_i\partial_j D(\boldsymbol{x})\big)_{i,j=1}^{d-1}$ is positive definite at $\boldsymbol{x}_0$.

By Sylvester's criterion, $A(\boldsymbol{x}_0)$ is positive definite if and only if all of its leading principal minors are positive. By the continuity of the second-order partial derivatives, A is continuous at $\boldsymbol{x}_0$, hence its leading principal minors are also continuous at $\boldsymbol{x}_0$. Therefore, there is a neighborhood $V\subseteq I^d$ of $\boldsymbol{x}_0$ where these leading principal minors are positive and hence, at the points of V, A is also positive definite. By shrinking the neighborhood U of x_0 if necessary, we may assume that U is an interval and $U^d \subseteq V$. Hence $A(\boldsymbol{x})$ is positive definite for all $\boldsymbol{x} \in U^d$.

In order to show that the inequality (39) holds for all $\boldsymbol{x} \in U^d$, let $\boldsymbol{x} = (x_1, \dots, x_d) \in U^d$ be fixed and apply the Taylor Mean Value Theorem to the function

$$(u_1, \dots, u_{d-1}) \mapsto D(u_1, \dots, u_{d-1}, x_d) \qquad \big((u_1, \dots, u_{d-1}) \in U^{d-1}\big)$$

at the base point $(x_d, \dots, x_d) \in U^{d-1}$. In view of this theorem, there exists $\theta \in [0, 1]$, such that

$$\begin{aligned} D(x_1, \dots, &x_{d-1}, x_d) \\ =& D(x_d, \dots, x_d, x_d) + \sum_{i=1}^{d-1} \partial_i D(x_d, \dots, x_d, x_d)(x_i - x_d) \\ &+ \frac{1}{2} \sum_{i=1}^{d-1} \sum_{j=1}^{d-1} \partial_i \partial_j D(\theta x_1 + (1-\theta)x_d, \dots, \theta x_{d-1} + (1-\theta)x_d, x_d)(x_i - x_d)(x_j - x_d). \end{aligned} \tag{42}$$

We have that $D(x_d, \dots, x_d, x_d) = M(x_d, \dots, x_d, x_d) - N(x_d, \dots, x_d, x_d) = x_d - x_d = 0$, equation (40) applied for $x = x_d$ implies that $\partial_i D(x_d, \dots, x_d, x_d) = 0$ for all $i \in \{1, \dots, d-1\}$. Finally, A is positive definite at the point $(\theta x_1 + (1-\theta)x_d, \dots, \theta x_{d-1} + (1-\theta)x_d, x_d) \in U^d$, hence the last term on the right hand side of (42) is nonnegative. Thus (42) shows that $D(x_1, \dots, x_{d-1}, x_d) \geq 0$, which implies that M is smaller than N on U^d. □

Remark 8. We note that, for the sufficiency part of the theorem, the standard 2nd-order sufficient condition for the local minimum cannot be applied. The reason is that the matrix

$$\big(\partial_i \partial_j N(\boldsymbol{x}_0) - \partial_i \partial_j M(\boldsymbol{x}_0)\big)_{i,j=1}^{d} \tag{43}$$

can never be positive definite. Indeed, if M is locally smaller than N at x_0, then M is locally smaller than N at every x in a neighborhood U of x_0 and hence (40) holds for all $x \in U$ and $i \in \{1, \dots, d\}$. Differentiating (40) with respect to x at x_0, we obtain, for all $i \in \{1, \dots, d\}$, that

$$\sum_{j=1}^{d} \partial_j \partial_i N(\boldsymbol{x}_0) = \sum_{j=1}^{d} \partial_j \partial_i M(\boldsymbol{x}_0).$$

This shows that the sum of the columns of the matrix in (43) is the zero vector. Therefore, the determinant of this matrix is zero, showing that this matrix is not positive definite.

Corollary 9. *Let $(f, g), (h, k) \in \mathcal{C}_1(I)$, let $m \in \mathcal{C}_1(I^d \times T)$ and $n \in \mathcal{C}_1(I^d \times S)$ be measurable families of means, and let μ and ν be probability measures on the measurable spaces $(T, \mathcal{A})$ and $(S, \mathcal{B})$, respectively. Suppose that $M_{f,g,m;\mu}$ is locally smaller than $M_{h,k,n;\nu}$ at $x_0 \in I$. Then, there exists a neighborhood $U \subseteq I$ of x_0 such that for $x \in U$ and for all $i \in \{1, \dots, d\}$,*

$$\langle \partial_i m, 1 \rangle_\mu (x, \dots, x) = \langle \partial_i n, 1 \rangle_\nu (x, \dots, x). \tag{44}$$

If, in addition, $(f, g), (h, k) \in \mathcal{C}_2(I)$, $m \in \mathcal{C}_2(I^d \times T)$, and $n \in \mathcal{C}_2(I^d \times S)$, then the $(d-1) \times (d-1)$-matrix whose (i, j)th entry is given by

$$\begin{aligned} &\big(\langle \partial_i n, \partial_j n \rangle_\nu - \langle \partial_i n, 1 \rangle_\nu \langle \partial_j n, 1 \rangle_\nu\big)(\boldsymbol{x}_0) \frac{\partial_1^2 \mathcal{D}_{h,k}(x_0, x_0)}{\partial_1 \mathcal{D}_{h,k}(x_0, x_0)} + \langle \partial_i \partial_j n, 1 \rangle_\nu (\boldsymbol{x}_0) \\ &- \big(\langle \partial_i m, \partial_j m \rangle_\mu - \langle \partial_i m, 1 \rangle_\mu \langle \partial_j m, 1 \rangle_\mu\big)(\boldsymbol{x}_0) \frac{\partial_1^2 \mathcal{D}_{f,g}(x_0, x_0)}{\partial_1 \mathcal{D}_{f,g}(x_0, x_0)} - \langle \partial_i \partial_j m, 1 \rangle_\mu (\boldsymbol{x}_0) \end{aligned} \tag{45}$$

for $i, j \in \{1, \dots, d-1\}$ is positive semidefinite.

On the other hand, if $(f, g), (h, k) \in \mathcal{C}_2(I)$, $m \in \mathcal{C}_2(I^d \times T)$, $n \in \mathcal{C}_2(I^d \times S)$, and (44) *holds for all*

$i \in \{1, \dots, d\}$ *and for all* x *in a neighborhood of* x_0 *and the* $(d-1) \times (d-1)$*-matrix whose* (i, j)*th entry is given by* (45) *is positive definite, then* $M_{f,g,m;\mu}$ *is locally smaller than* $M_{h,k,n;\nu}$ *at* $x_0 \in I$.

Proof. If $(f, g), (h, k) \in \mathcal{C}_k(I)$, $m \in \mathcal{C}_k(I^d \times T)$, and $n \in \mathcal{C}_k(I^d \times S)$, then Theorem 4 implies that $M_{f,g,m;\mu}$ and $M_{h,k,n;\nu}$ are k-times continuously differentiable on I^d in the cases $k \in \{1, 2\}$.

Assume that $M_{f,g,m;\mu}$ is locally smaller than $M_{h,k,n;\nu}$ at $x_0 \in I$. Then, by Theorem 7, there exists a neighborhood $U \subseteq I$ of x_0 such that for $x \in U$ and for all $i \in \{1, \dots, d\}$,

$$\partial_i M_{f,g,m;\mu}(x, \dots, x) = \partial_i M_{h,k,n;\nu}(x, \dots, x).$$

Applying formula (27) of Theorem 4, the necessity of condition (44) follows.

In addition, if the second-order regularity assumptions are satisfied, then, by Theorem 7, the $(d-1) \times (d-1)$-matrix whose (i, j)th entry is given by

$$\big(\partial_i \partial_j M_{h,k,n;\nu}(x_0, \dots, x_0) - \partial_i \partial_j M_{f,g,m;\mu}(x_0, \dots, x_0)\big)_{i,j=1}^{d-1}$$

for $i, j \in \{1, \dots, d-1\}$ is positive semidefinite. Now the application of formula (28) of Theorem 4 yields the necessity of condition (45).

Now, under the second-order regularity assumptions suppose that (44) holds for all $i \in \{1, \dots, d\}$ and for all x in a neighborhood of x_0 and the $(d-1) \times (d-1)$-matrix whose (i, j)th entry is given by (45) is positive definite. Since $M_{f,g,m;\mu}$ and $M_{h,k,n;\nu}$ are twice continuously differentiable and by Theorem 7, we have $M_{f,g,m;\mu}$ is locally smaller than $M_{h,k,n;\nu}$ at $x_0 \in I$. □

In the special setting when $T = [0, 1]$, $d = 2$, m is given by $m(\boldsymbol{x}, t) := tx_1 + (1-t)x_2$, the above Corollary 9 simplifies to the result of [20, Theorem 5]. Now we consider the particular case when the families of means m and n as well as the measures μ and ν coincide.

Corollary 10. *Let* $(f, g), (h, k) \in \mathcal{C}_2(I)$*, let* $m \in \mathcal{C}_2(I^d \times T)$ *be a measurable family of means, and let* μ *be a probability measure on the measurable space* $(T, \mathcal{A})$*. Let* $x_0 \in I$ *and assume that there exists* $i \in \{1, \dots, d-1\}$ *such that, the map* $t \mapsto \partial_i m(\boldsymbol{x_0}, t)$ *is not* μ*-almost everywhere constant on* T*. If* $M_{f,g,m;\mu}$ *is locally smaller than* $M_{h,k,m;\mu}$ *at* $x_0 \in I$*, then*

$$\frac{\partial_1^2 \mathcal{D}_{f,g}(x_0, x_0)}{\partial_1 \mathcal{D}_{f,g}(x_0, x_0)} \le \frac{\partial_1^2 \mathcal{D}_{h,k}(x_0, x_0)}{\partial_1 \mathcal{D}_{h,k}(x_0, x_0)}. \tag{46}$$

On the other hand, if the functions

$$t \mapsto \partial_i m(\boldsymbol{x}_0, t) - \langle \partial_i m, 1 \rangle_\mu(\boldsymbol{x}_0) \qquad (i \in \{1, \dots, d-1\})$$

are μ*-linearly independent and* (46) *holds with strict inequality, then* $M_{f,g,m;\mu}$ *is locally smaller than* $M_{h,k,m;\mu}$ *at* $x_0 \in I$.

Proof. Assume that $M_{f,g,m;\mu}$ is locally smaller than $M_{h,k,m;\mu}$ at $x_0 \in I$. Then, by Corollary 9, the $(d-1) \times (d-1)$-matrix whose (i, j)th entry is given by

$$\big(\langle \partial_i m, \partial_j m \rangle_\mu - \langle \partial_i m, 1 \rangle_\mu \langle \partial_j m, 1 \rangle_\mu\big)(\boldsymbol{x}_0) \cdot \left(\frac{\partial_1^2 \mathcal{D}_{h,k}(x_0, x_0)}{\partial_1 \mathcal{D}_{h,k}(x_0, x_0)} - \frac{\partial_1^2 \mathcal{D}_{f,g}(x_0, x_0)}{\partial_1 \mathcal{D}_{f,g}(x_0, x_0)} \right) \tag{47}$$

for $i, j \in \{1, \dots, d-1\}$ is positive semidefinite at $x_0 \in I$. This implies that all the diagonal elements of this matrix are nonnegative, i.e., for all $i \in \{1, \dots, d-1\}$,

$$\big(\langle \partial_i m, \partial_i m \rangle_\mu - \langle \partial_i m, 1 \rangle_\mu \langle \partial_i m, 1 \rangle_\mu\big)(\boldsymbol{x}_0) \cdot \left(\frac{\partial_1^2 \mathcal{D}_{h,k}(x_0, x_0)}{\partial_1 \mathcal{D}_{h,k}(x_0, x_0)} - \frac{\partial_1^2 \mathcal{D}_{f,g}(x_0, x_0)}{\partial_1 \mathcal{D}_{f,g}(x_0, x_0)} \right) \ge 0. \tag{48}$$

If, for some $i \in \{1, \dots, d-1\}$, the map $t \mapsto \partial_i m(\boldsymbol{x_0}, t)$ is not μ-almost everywhere constant, then

$$\mu(\{t \mid \partial_i m(\boldsymbol{x}_0, t) \ne \langle \partial_i m, 1 \rangle_\mu(\boldsymbol{x}_0)\}) > 0,$$

whence

$$\big(\langle\partial_i m,\partial_i m\rangle_\mu-\langle\partial_i m,1\rangle_\mu\langle\partial_i m,1\rangle_\mu\big)(\boldsymbol{x}_0)=\int_T\big(\partial_i m(\boldsymbol{x_0},t)-\langle\partial_i m,1\rangle_\mu(\boldsymbol{x}_0)\big)^2\,\mathrm{d}\mu(t)>0. \tag{49}$$

Inequality (49), combined with (48), implies that

$$\frac{\partial_1^2\mathcal{D}_{h,k}(x_0,x_0)}{\partial_1\mathcal{D}_{h,k}(x_0,x_0)}-\frac{\partial_1^2\mathcal{D}_{f,g}(x_0,x_0)}{\partial_1\mathcal{D}_{f,g}(x_0,x_0)}\geq 0,$$

i.e, the inequality (46) holds.

Now assume that the functions

$$t\mapsto\partial_i m(\boldsymbol{x}_0,t)-\langle\partial_i m,1\rangle_\mu(\boldsymbol{x}_0)\qquad(i\in\{1,\dots,d-1\}) \tag{50}$$

are μ-linearly independent and (46) holds with strict inequality. It is clear that the $(d-1)\times(d-1)$-matrix whose (i,j)th entry is given by

$$\begin{aligned}\big(\langle\partial_i m,\partial_j m\rangle_\mu-&\langle\partial_i m,1\rangle_\mu\langle\partial_j m,1\rangle_\mu\big)(\boldsymbol{x}_0)\\&=\int_T\big(\partial_i m(\boldsymbol{x_0},t)-\langle\partial_i m,1\rangle_\mu(\boldsymbol{x}_0)\big)\big(\partial_j m(\boldsymbol{x_0},t)-\langle\partial_j m,1\rangle_\mu(\boldsymbol{x}_0)\big)\,\mathrm{d}\mu(t)\end{aligned} \tag{51}$$

for $i,j\in\{1,\dots,d-1\}$ is a so-called Gram matrix which is always positive semmidefinite (see [6]). Since the functions (50) are μ-linearly independent it follows that the Gram matrix with entries given by (51) is positive definite. This result, combined with the strict inequality (46), implies that the $(d-1)\times(d-1)$-matrix whose (i,j)th entry is given by (47) is positive definite at $x_0\in I$. Hence, by Corollary 9, $M_{f,g,m;\mu}$ is locally smaller than $M_{h,k,m;\mu}$ at $x_0\in I$. □

Now we formulate a particular case concerning generalized Gini means when the partial derivatives can be calculated more explicitly. Indeed, if, for given $p,q\in\mathbb{R}$, the functions f and g are given by equations (10), then

$$\mathcal{D}_{f,g}(x,y)=\Delta_{p,q}(x,y):=y^{p+q}\delta_{p,q}\Big(\frac{x}{y}\Big)\qquad(x,y\in\mathbb{R}_+), \tag{52}$$

where

$$\delta_{p,q}(t):=\begin{cases}\dfrac{t^p-t^q}{p-q} & \text{if } p\neq q\\ t^p\ln t & \text{if } p=q\end{cases}\qquad(t\in\mathbb{R}_+). \tag{53}$$

Then, one can easily get that $\delta'_{p,q}(1)=1$ and $\delta''_{p,q}(1)=p+q-1$, whence

$$\partial_1\Delta_{p,q}(x,x)=x^{p+q-1}\qquad\text{and}\qquad\partial_1^2\Delta_{p,q}(x,x)=(p+q-1)x^{p+q-2}.$$

Therefore,

$$\frac{\partial_1^2\mathcal{D}_{f,g}(x,x)}{\partial_1\mathcal{D}_{f,g}(x,x)}=\frac{\partial_1^2\Delta_{p,q}(x,x)}{\partial_1\Delta_{p,q}(x,x)}=\frac{(p+q-1)x^{p+q-2}}{x^{p+q-1}}=(p+q-1)\frac{1}{x}\qquad(x\in\mathbb{R}_+). \tag{54}$$

Corollary 11. *Assume that $I\subseteq\mathbb{R}_+$. Let $p,q,r,s\in\mathbb{R}$, let $m\in\mathcal{C}_2(I^d\times T)$ and $n\in\mathcal{C}_2(I^d\times S)$ be measurable families of means, and let μ and ν be probability measures on the measurable spaces $(T,\mathcal{A})$ and $(S,\mathcal{B})$, respectively. Suppose that $G_{p,q,m;\mu}$ is locally smaller than $G_{r,s,n;\nu}$ at $x_0\in I$. Then, there exists a neighborhood $U\subseteq I$ of x_0 such that for $x\in U$, for all $i\in\{1,\dots,d\}$,* (44) *holds. In addition, the $(d-1)\times(d-1)$-matrix whose (i,j)th entry is given by*

$$\begin{aligned}&\big(\langle\partial_i n,\partial_j n\rangle_\nu-\langle\partial_i n,1\rangle_\nu\langle\partial_j n,1\rangle_\nu\big)(\boldsymbol{x}_0)(p+q-1)\frac{1}{x_0}+\langle\partial_i\partial_j n,1\rangle_\nu(\boldsymbol{x}_0)\\&-\big(\langle\partial_i m,\partial_j m\rangle_\mu-\langle\partial_i m,1\rangle_\mu\langle\partial_j m,1\rangle_\mu\big)(\boldsymbol{x}_0)(r+s-1)\frac{1}{x_0}-\langle\partial_i\partial_j m,1\rangle_\mu(\boldsymbol{x}_0)\end{aligned} \tag{55}$$

for $i,j\in\{1,\dots,d-1\}$ *is positive semidefinite.*

On the other hand, if (44) *holds for all* $i\in\{1,\dots,d\}$ *and for all* x *in a neighborhood of* x_0 *and the* $(d-1)\times(d-1)$*-matrix whose* (i,j)*th entry is given by* (55) *is positive definite, then* $G_{p,q,m;\mu}$ *is locally smaller than* $G_{r,s,n;\nu}$ *at* $x_0\in I$.

Proof. The proof is a direct consequence of Corollary 9 and formula (54). □

Corollary 12. *Assume that* $I\subseteq\mathbb{R}_+$. *Let* $p,q,r,s\in\mathbb{R}$, *let* $m\in\mathcal{C}_2(I^d\times T)$ *be a measurable family of means, and let* μ *be a probability measure on the measurable space* $(T,\mathcal{A})$. *Let* $x_0\in I$ *and assume that there exists* $i\in\{1,\dots,d-1\}$ *such that, the map* $t\mapsto\partial_i m(\boldsymbol{x_0},t)$ *is not* μ*-almost everywhere constant on* T. *If* $G_{p,q,m;\mu}$ *is locally smaller than* $G_{r,s,m;\mu}$ *at* $x_0\in I$, *then*

$$p+q\le r+s. \tag{56}$$

On the other hand, if $x_0\in I$, *the functions*

$$t\mapsto\partial_i m(\boldsymbol{x}_0,t)-\langle\partial_i m,1\rangle_\mu(\boldsymbol{x}_0)\qquad(i\in\{1,\dots,d-1\})$$

are μ*-linearly independent and* (56) *holds with strict inequality, then* $G_{p,q,m;\mu}$ *is locally smaller than* $G_{r,s,m;\mu}$ *at* $x_0\in I$.

Proof. Applying Corollary 10 and using formula (54), the result follows immediately. □

4. Necessary and sufficient conditions for global comparison of means

In the rest of the paper, we consider the case when $\mu=\nu$ and $m=n$. In what follows, we give a condition containing two independent variables for (12) which does not involve the measure μ and assumes first-order continuous differentiability of the Chebyshev system. In the special setting when $T=[0,1]$, $d=2$, m is given by $m(\boldsymbol{x},t):=tx_1+(1-t)x_2$, the following theorem simplifies to the result of [20, Theorem 6].

Theorem 13. *Let* $(f,g),(h,k)\in\mathcal{C}_1(I)$ *be Chebyshev systems, let* T *be a compact and connected topological space and let* $m:I^d\times T\to\mathbb{R}$ *be a continuous family of* d*-variable means. Define the set* U_m *by*

$$U_m:=\{(u,v)\mid\exists\boldsymbol{x}\in I^d:u,v\in[\underline{m}(\boldsymbol{x}),\overline{m}(\boldsymbol{x})]\}=\bigcup_{\boldsymbol{x}\in I^d}[\underline{m}(\boldsymbol{x}),\overline{m}(\boldsymbol{x})]^2. \tag{57}$$

The following three assertions are equivalent:

(i) for all Borel probability measures μ *on* T,

$$M_{f,g,m;\mu}(\boldsymbol{x})\le M_{h,k,m;\mu}(\boldsymbol{x})\qquad(\boldsymbol{x}\in I^d); \tag{58}$$

(ii) there exists a nullsequence (γ_j) *of positive numbers in* $[0,1]$ *such that, for all* $t_0,t\in T$ *and for all* $j\in\mathbb{N}$,

$$M_{f,g,m;(1-\gamma_j)\delta_{t_0}+\gamma_j\delta_t}(\boldsymbol{x})\le M_{h,k,m;(1-\gamma_j)\delta_{t_0}+\gamma_j\delta_t}(\boldsymbol{x})\qquad(\boldsymbol{x}\in I^d); \tag{59}$$

(iii) for all $(u,v)\in U_m$,

$$\frac{\mathcal{D}_{f,g}(u,v)}{\partial_1\mathcal{D}_{f,g}(v,v)}\le\frac{\mathcal{D}_{h,k}(u,v)}{\partial_1\mathcal{D}_{h,k}(v,v)}. \tag{60}$$

Proof. The implication (i)⟹(ii) is obvious.

To prove (ii)⟹(iii), let $(u,v)\in U_m$. Then there exists $\boldsymbol{x}\in I^d$ such that $u,v\in[\underline{m}(\boldsymbol{x}),\overline{m}(\boldsymbol{x})]$. Due to the compactness and connectedness of T, we have that (6) holds. Therefore, there exits $t_0,t\in T$

such that $u = m(\boldsymbol{x}, t)$ and $v = m(\boldsymbol{x}, t_0)$. Applying Lemma 6 twice with the measure sequence $\mu_j := (1-\gamma_j)\delta_{t_0} + \gamma_j\delta_t$ and using inequality (58), we get

$$\frac{\mathcal{D}_{f,g}(u,v)}{\partial_1\mathcal{D}_{f,g}(v,v)} = \frac{\mathcal{D}_{f,g}(m(\boldsymbol{x},t), m(\boldsymbol{x},t_0))}{\partial_1\mathcal{D}_{f,g}(m(\boldsymbol{x},t_0), m(\boldsymbol{x},t_0))} = \lim_{j\to\infty}\frac{1}{\gamma_j}\big[M_{f,g,m;\mu_j}(\boldsymbol{x}) - m(\boldsymbol{x},t_0)\big]$$
$$\leq \lim_{j\to\infty}\frac{1}{\gamma_j}\big[M_{h,k,m;\mu_j}(\boldsymbol{x}) - m(\boldsymbol{x},t_0)\big] = \frac{\mathcal{D}_{h,k}(m(\boldsymbol{x},t), m(\boldsymbol{x},t_0))}{\partial_1\mathcal{D}_{h,k}(m(\boldsymbol{x},t_0), m(\boldsymbol{x},t_0))} = \frac{\mathcal{D}_{h,k}(u,v)}{\partial_1\mathcal{D}_{h,k}(v,v)},$$

which proves (60).

For the proof of (iii)$\Longrightarrow$(i), let $\boldsymbol{x} \in I^d$ be arbitrarily fixed. In view of the inclusion $M_{h,k,m;\mu}(\boldsymbol{x}) \in [\underline{m}(\boldsymbol{x}), \overline{m}(\boldsymbol{x})]$ and the equality (6), there exits $t_0 \in T$ such that

$$m(\boldsymbol{x}, t_0) := M_{h,k,m;\mu}(\boldsymbol{x}).$$

Taking any $t \in T$ and applying inequality (60) for $u := m(\boldsymbol{x}, t)$ and $v := m(\boldsymbol{x}, t_0)$, we get

$$\frac{\mathcal{D}_{f,g}(m(\boldsymbol{x},t), m(\boldsymbol{x},t_0))}{\partial_1\mathcal{D}_{f,g}(m(\boldsymbol{x},t_0), m(\boldsymbol{x},t_0))} \leq \frac{\mathcal{D}_{h,k}(m(\boldsymbol{x},t), m(\boldsymbol{x},t_0))}{\partial_1\mathcal{D}_{h,k}(m(\boldsymbol{x},t_0), m(\boldsymbol{x},t_0))}.$$

Integrating this inequality with respect to the variable $t \in T$, we get

$$\frac{\int_T \mathcal{D}_{f,g}\big(m(\boldsymbol{x},t), M_{h,k,m;\mu}(\boldsymbol{x})\big)\,\mathrm{d}\mu(t)}{\partial_1\mathcal{D}_{f,g}\big(M_{h,k,m;\mu}(\boldsymbol{x}), M_{h,k,m;\mu}(\boldsymbol{x})\big)} \leq \frac{\int_T \mathcal{D}_{h,k}\big(m(\boldsymbol{x},t), M_{h,k,m;\mu}(\boldsymbol{x})\big)\,\mathrm{d}\mu(t)}{\partial_1\mathcal{D}_{h,k}\big(M_{h,k,m;\mu}(\boldsymbol{x}), M_{h,k,m;\mu}(\boldsymbol{x})\big)}. \tag{61}$$

By the definition of the value $M_{h,k,m;\mu}(\boldsymbol{x})$, the numerator of the right hand side of this inequality is zero, whence we obtain

$$\frac{\int_T \mathcal{D}_{f,g}\big(m(\boldsymbol{x},t), M_{h,k,m;\mu}(\boldsymbol{x})\big)\,\mathrm{d}\mu(t)}{\partial_1\mathcal{D}_{f,g}\big(M_{h,k,m;\mu}(\boldsymbol{x}), M_{h,k,m;\mu}(\boldsymbol{x})\big)} \leq 0.$$

If $\partial_1\mathcal{D}_{f,g}(x,x) < 0$ for all $x \in I$, then

$$\int_T \mathcal{D}_{f,g}\big(m(\boldsymbol{x},t), M_{h,k,m;\mu}(\boldsymbol{x})\big)\,\mathrm{d}\mu(t) \geq 0 \tag{62}$$

and also (f,g) is a positive Chebyshev system, hence, by Lemma 1, the above inequality implies that (58) holds. In the other possible case, i.e., when $\partial_1\mathcal{D}_{f,g}(x,x) > 0$ for all $x \in I$, then inequality (62) is reversed but (f,g) is a negative Chebyshev system, thus by Lemma 1 again, inequality (58) follows. $\square$

Having a look at the proof, one can see that the compactness and connectedness of T was only used to prove implication (ii)$\Longrightarrow$(iii).

Corollary 14. *Assume that $I \subseteq \mathbb{R}_+$ and $p,q,r,s \in \mathbb{R}$. Let T be a compact and connected topological space and let $m : I^d \times T \to \mathbb{R}$ be a continuous family of means. Define the constant $m^* \in [1,+\infty]$ by*

$$m^* := \sup_{\boldsymbol{x}\in I^d}\frac{\overline{m}(\boldsymbol{x})}{\underline{m}(\boldsymbol{x})}.$$

The following three assertions are equivalent:

(i) for all Borel probability measures μ on T,

$$G_{p,q,m;\mu}(\boldsymbol{x}) \leq G_{r,s,m;\mu}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in I^d); \tag{63}$$

(ii) there exists a null sequence (γ_j) of positive numbers in $[0,1]$ such that, for all $t_0, t \in T$ and for all $j \in \mathbb{N}$,

$$G_{p,q,m;(1-\gamma_j)\delta_{t_0}+\gamma_j\delta_t}(\boldsymbol{x}) \leq G_{r,s,m;(1-\gamma_j)\delta_{t_0}+\gamma_j\delta_t}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in I^d); \tag{64}$$

(iii)

$$\delta_{p,q}(t) \leq \delta_{r,s}(t) \qquad \big(t \in \,\big](m^*)^{-1}, m^*\big[\big); \tag{65}$$

(iv) In the case $m^* = +\infty$,

$$\min(p, q) \le \min(r, s) \qquad \textit{and} \qquad \max(p, q) \le \max(r, s), \tag{66}$$

while in the case $m^* < +\infty$,

$$\delta_{p,q}\big((m^*)^{-1}\big) \le \delta_{r,s}\big((m^*)^{-1}\big), \qquad \delta_{p,q}\big(m^*\big) \le \delta_{r,s}\big(m^*\big), \qquad \textit{and} \qquad p + q \le r + s. \tag{67}$$

Proof. Applying Theorem 13 and using notations introduced in (52) and (53) imply that conditions (63) and (64) are equivalent to the inequality

$$\frac{\Delta_{p,q}(v, u)}{\partial_1 \Delta_{p,q}(u, u)} \le \frac{\Delta_{r,s}(v, u)}{\partial_1 \Delta_{r,s}(u, u)} \qquad ((u, v) \in U_m),$$

where the set U_m is defined in (57). This inequality can be rewritten as

$$v\delta_{p,q}\Big(\frac{u}{v}\Big) \le v\delta_{r,s}\Big(\frac{u}{v}\Big) \qquad ((u, v) \in U_m). \tag{68}$$

Observe that

$$\big](m^*)^{-1}, m^*\big[\;\subseteq \Big\{\frac{u}{v} \colon (u, v) \in U_m\Big\} \subseteq \big[(m^*)^{-1}, m^*\big]. \tag{69}$$

Indeed, if $t \in \big](m^*)^{-1}, m^*\big[$ and $t \ge 1$, then $t < m^*$, hence there exits $\boldsymbol{x} \in I^d$ such that $t < \frac{\overline{m}(\boldsymbol{x})}{\underline{m}(\boldsymbol{x})}$. Then, with $v = \underline{m}(\boldsymbol{x})$, $u = t\underline{m}(\boldsymbol{x})$, we have that $t = \frac{u}{v}$ and $u, v \in [\underline{m}(\boldsymbol{x}), \overline{m}(\boldsymbol{x})]$. Therefore t is of the form $\frac{u}{v}$ for some $(u, v) \in U_m$. A similar argument yields for $t \le 1$ a similar representation. This proves the first inclusion.

For the second inclusion, observe that if $(u, v) \in U_m$, then, for some $\boldsymbol{x} \in I^d$, we have $u, v \in [\underline{m}(\boldsymbol{x}), \overline{m}(\boldsymbol{x})]$. Hence

$$(m^*)^{-1} \le \frac{\underline{m}(\boldsymbol{x})}{\overline{m}(\boldsymbol{x})} \le \frac{u}{v} \le \frac{\overline{m}(\boldsymbol{x})}{\underline{m}(\boldsymbol{x})} \le m^*$$

Therefore, in view of the inclusions in (69), inequality (53) is equivalent to condition (65).

To show the equivalence of condition (iv) to the previous ones, we have to distinguish two cases. If $m^* = +\infty$, then $(m^*)^{-1} = 0$, therefore (iii) can be rewritten as

$$\delta_{p,q}(t) \le \delta_{r,s}(t) \qquad \big(t \in]0, \infty[\big).$$

This inequality is known to be equivalent (cf. [10]) to the comparison inequality

$$G_{p,q}(\boldsymbol{x}) \le G_{r,s}(\boldsymbol{x}) \qquad (d \in \mathbb{N},\, \boldsymbol{x} \in \big]0, \infty[^d),$$

of Gini means (with arbitrary many variables over the interval $]0, \infty[$). In view of the result [10, Satz 5], the above inequality is characterized by the condition (66). Therefore (iii) is equivalent to (iv) in this case.

Now consider the case $m^* < +\infty$. Then the inequality in (iii) is equivalent to the comparison inequality

$$G_{p,q}(\boldsymbol{x}) \le G_{r,s}(\boldsymbol{x}) \qquad (d \in \mathbb{N},\, \boldsymbol{x} \in \big]1, m^*[^d),$$

of Gini means (with arbitrary many variables over the interval $]1, m^*[$). Using the results of the papers [15, Theorem 7] or [24], it follows that the above inequality is characterized by (67), which implies that (iii) is equivalent to (iv) also in this case.

This completes the proof of the corollary. □

5. Necessary and sufficient conditions for the local and global comparison of generalized quasi-arithmetic means

In the next result we offer 6 equivalent conditions for the comparison of d-variable generalized quasi-arithmetic means. The interesting feature of this result is the equivalence of the global and local comparability.

Theorem 15. *Let $f,h : I \to \mathbb{R}$ be twice continuously differentiable functions with non-vanishing first derivatives, and let $m \in \mathcal{C}_2(I^d \times T)$ be a measurable family of d-variable means. Let μ_0 be a probability measure such that, for all $x_0 \in I$, there exists $i \in \{1,\dots,d-1\}$ such that $t \mapsto \partial_i m(\boldsymbol{x}_0,t)$ is not μ_0-almost everywhere constant on T. The following assertions are equivalent:*

(i) for all Borel probability measures μ on T,

$$M_{f,1,m;\mu}(\boldsymbol{x}) \le M_{h,1,m;\mu}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in I^d); \tag{70}$$

(ii)

$$M_{f,1,m;\mu_0}(\boldsymbol{x}) \le M_{h,1,m;\mu_0}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in I^d);$$

(iii) for all $x_0 \in I$, there exists a neighborhood $U \subseteq I$ of x_0 such that

$$M_{f,1,m;\mu_0}(\boldsymbol{x}) \le M_{h,1,m;\mu_0}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in U^d);$$

(iv) for all $x \in I$,

$$\frac{f''(x)}{f'(x)} \le \frac{h''(x)}{h'(x)}; \tag{71}$$

(v) the function $h \circ f^{-1}$ is convex (concave) on $f(I)$ provided that f is increasing (decreasing);
(vi) for all $(u,v) \in I^2$,

$$\frac{f(u)-f(v)}{f'(v)} \le \frac{h(u)-h(v)}{h'(v)}. \tag{72}$$

Proof. The implications (i)$\Longrightarrow$(ii) and (ii)$\Longrightarrow$(iii) are trivial.

To prove (iii)$\Longrightarrow$(iv), will apply Corollary 10. Let $x_0 \in I$ be arbitrary. Then (iii) asserts that $M_{f,1,m;\mu_0}$ is locally smaller than $M_{h,1,m;\mu_0}$ at x_0 and we also have an index $i \in \{1,\dots,d-1\}$ such that $t \mapsto \partial_i m(\boldsymbol{x}_0,t)$ is not μ_0-almost everywhere constant on T. Therefore, by Corollary 10, inequality (46) follows with the functions $g := k := 1$. It is immediate to see that (46) implies (71) for $x = x_0$.

Now assume (iv) and that f is increasing (the nondecreasing case is analogous). Then $g := h \circ f^{-1}$ is twice differentiable on $f(I)$. By (71), the ratio $\frac{h'}{f'}$ is a nondecreasing function. Therefore, $g' = \frac{h' \circ f^{-1}}{f' \circ f^{-1}}$ is also nondecreasing, which proves that $g'' \ge 0$. Hence g must be convex on $f(I)$, i.e., (v) holds.

If (v) is valid and f is increasing, then the convexity of $g := h \circ f^{-1}$ implies that

$$g(y) + g'(y)(x-y) \le g(x)$$

for all $x,y \in f(I)$. With the substitution $x = f(u)$, $y = f(v)$, where $u,v \in I$, the above inequality reduces to (72), proving (vi).

Finally, assume that (vi) holds. Observe that then (60) is valid for all $(u,v) \in U_m$ with the functions $g := k := 1$. Thus, the condition (iii) of Theorem 13 is satisfied, whence it follows that the mean $M_{f,1,m;\mu_0}$ is (globally) smaller than $M_{h,1,m;\mu_0}$ on I^d, i.e., (70) holds. □

As an immediate consequence, we obtain the characterization of the comparison among generalized Hölder means.

Corollary 16. *Let $I \subseteq \mathbb{R}_+$, $p,q \in \mathbb{R}$, and let $m \in \mathcal{C}_2(I^d \times T)$ be a measurable family of d-variable means. Let μ_0 be a probability measure such that, for all $x_0 \in I$, there exists $i \in \{1,\dots,d-1\}$ such that $t \mapsto \partial_i m(\boldsymbol{x}_0,t)$ is not μ_0-almost everywhere constant on T. The following assertions are equivalent:*

(i) for all Borel probability measures μ on T,

$$G_{p,0,m;\mu}(\boldsymbol{x}) \le G_{q,0,m;\mu}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in I^d);$$

(ii)

$$G_{p,0,m;\mu_0}(\boldsymbol{x}) \le G_{q,0,m;\mu_0}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in I^d);$$

(iii) for all $x_0 \in I$, there exists a neighborhood $U \subseteq I$ of x_0 such that

$$G_{p,0,m;\mu_0}(\boldsymbol{x}) \le G_{q,0,m;\mu_0}(\boldsymbol{x}) \qquad (\boldsymbol{x} \in U^d);$$

(iv) $p \le q$.

Proof. By taking $f(x) := x^p$ if $p \neq 0$ and $f(x) := \log(x)$ if $p = 0$ and $h(x) := x^q$ if $q \neq 0$ and $h(x) := \log(x)$ if $q = 0$ and applying Theorem 15 the result follows immediately because conditions (i), (ii) and (iii) are equivalent to the same conditions of Theorem 15, and $p \le q$ is equivalent to condition (iv) of Theorem 15. □

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