

CONNECTIONS BETWEEN CENTRALITY AND LOCAL MONOTONICITY OF CERTAIN FUNCTIONS ON C^* -ALGEBRAS

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ABSTRACT. We introduce a quite large class of functions (including the exponential function and the power functions with exponent greater than one), and show that for any element f of this function class, a self-adjoint element a of a C^* -algebra is central if and only if $a \leq b$ implies $f(a) \leq f(b)$. That is, we characterize centrality by local monotonicity of certain functions on C^* -algebras. Numerous former results (including works of *Ogasawara*, *Pedersen*, *Wu*, and *Molnár*) are apparent consequences of our result.

1. INTRODUCTION

Connections between the commutativity of a C^* -algebra \mathcal{A} and the monotonicity of some functions defined on some subsets of \mathcal{A} have been investigated widely. The first result related to this topic is due to *Ogasawara* who showed in 1955 that a C^* -algebra \mathcal{A} is commutative if and only if the square function is monotone on the positive cone of \mathcal{A} [7]. It was observed later by *Pedersen* that the above statement remains true for any power function with exponent greater than one [8]. *Wu* proved a similar result for the exponential function in 2001 [10]. *Ji* and *Tomiya* showed in 2003 that for any function f which is monotone but not matrix monotone of order 2, a C^* -algebra \mathcal{A} is commutative if and only if f is monotone on the positive cone of \mathcal{A} [2]. The reader is advised to consult the papers [9] and [6] for other closely related results.

Very recently, *Molnár* proved a local theorem, namely, that a self-adjoint element a of a C^* -algebra \mathcal{A} is central if and only if $a \leq b$ implies $\exp a \leq \exp b$ [5].

Motivated by the work of *Molnár*, we show the following. If $I = (\gamma, \infty)$ is a real interval and f is a continuously differentiable function on I such

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that the derivative of f is positive, strictly monotone increasing and logarithmically concave, then a self-adjoint element a of a C^* -algebra \mathcal{A} with spectrum in I is central if and only if $a \leq b$ implies $f(a) \leq f(b)$, that is, f is locally monotone at the point a . This result easily implies the results of Ogasawara, Pedersen, Wu, and Molnár.

2. THE MAIN THEOREM

The precise formulation of our main result reads as follows (here and throughout, the symbol \mathcal{A}_s stands for the set of the self-adjoint elements of a C^* -algebra \mathcal{A}).

Theorem. *Let $I = (\gamma, \infty)$ for some $\gamma \in \mathbb{R} \cup \{-\infty\}$ and let $f \in C^1(I)$ be such that*

- (i) $f'(x) > 0 \quad (x \in I)$,
- (ii) $x < y \Rightarrow f'(x) < f'(y) \quad (x, y \in I)$,
- (iii) $\log(f'(tx + (1-t)y)) \geq t \log f'(x) + (1-t) \log f'(y) \quad (x, y \in I, t \in [0, 1])$.

Let \mathcal{A} be a unital C^ -algebra and let $a \in \mathcal{A}$ be a self-adjoint element with $\sigma(a) \subset I$. The followings are equivalent.*

- (1) a is central, that is, $ab = ba \quad (b \in \mathcal{A})$,
- (2) f is locally monotone at the point a , that is, $a \leq b \Rightarrow f(a) \leq f(b) \quad (b \in \mathcal{A}_s)$.

Example. We enumerate the most important examples of intervals and functions satisfying the conditions given in the Theorem:

- $I = (0, \infty)$, $f(x) = x^p \quad (p > 1)$,
- $I = (-\infty, \infty)$, $f(x) = e^x$.

3. THE PROOF OF THE THEOREM

Notation. If φ and ψ are elements of some Hilbert space \mathcal{H} , then the symbol $\varphi \otimes \psi$ denotes the linear map $\mathcal{H} \ni \xi \mapsto \langle \xi, \psi \rangle \varphi \in \mathcal{H}$.

The following proposition is a key step of the proof.

Proposition. *Suppose that $I = (\gamma, \infty)$ for some $\gamma \in \mathbb{R} \cup \{-\infty\}$ and $f \in C^1(I)$ satisfies the conditions (i), (ii) and (iii) given in the Theorem. Let \mathcal{K} be a two-dimensional Hilbert space, let $\{u, v\} \subset \mathcal{K}$ be an orthonormal basis. Let $x, y \in I$ and set $A := xu \otimes u + yv \otimes v$. The followings are equivalent.*

- (I) $x \neq y$,
- (II) *there exist $\lambda, \mu \in \mathbb{C}$ with $|\lambda|^2 + |\mu|^2 = 1$ and $t_0 > 0$ such that using the notation $B = (u + v) \otimes (u + v)$ and $w = \lambda u + \mu v$ we have*

$$\langle f(A)w, w \rangle - \langle f(A + t_0 B)w, w \rangle > 0.$$

Notation. For any fixed interval $I = (\gamma, \infty)$ and function $f \in C^1(I)$ with the properties (i), (ii) and (iii), and different numbers $x, y \in I$, the above Proposition provides a positive number $\langle f(A)w, w \rangle - \langle f(A + t_0B)w, w \rangle$. Let us introduce

$$\delta := \langle f(A)w, w \rangle - \langle f(A + t_0B)w, w \rangle.$$

Proof of the Proposition. The direction (II) \Rightarrow (I) is easy to see (by contradiction). To verify the direction (I) \Rightarrow (II) we recall the following useful formula for the derivative of a matrix function (see [1, Thm. 3.25] and also [1, Thm. 3.33]). If $A = xu \otimes u + yv \otimes v$, then for any self-adjoint $C \in \mathcal{B}(\mathcal{H})$ we have

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{1}{t} (f(A + tC) - f(A)) &= f'(x) \langle Cu, u \rangle u \otimes u + \\ &+ \frac{f(x) - f(y)}{x - y} \langle Cv, u \rangle u \otimes v + \frac{f(y) - f(x)}{y - x} \langle Cu, v \rangle v \otimes u + f'(y) \langle Cv, v \rangle v \otimes v. \end{aligned}$$

This means that for $B = (u + v) \otimes (u + v)$ we have

$$\begin{aligned} (1) \quad L &:= \lim_{t \rightarrow 0} \frac{1}{t} (f(A + tB) - f(A)) \\ &= f'(x)u \otimes u + \frac{f(x) - f(y)}{x - y} u \otimes v + \frac{f(y) - f(x)}{y - x} v \otimes u + f'(y)v \otimes v. \end{aligned}$$

The determinant of the matrix

$$[L] = \begin{bmatrix} f'(x) & \frac{f(x) - f(y)}{x - y} \\ \frac{f(y) - f(x)}{y - x} & f'(y) \end{bmatrix}$$

is negative as $\text{Det}[L] < 0 \Leftrightarrow f'(x)f'(y) < \left(\frac{f(x) - f(y)}{x - y}\right)^2 \Leftrightarrow$

$$\Leftrightarrow \log f'(x) + \log f'(y) < 2 \log \left(\int_0^1 f'(tx + (1 - t)y) dt \right).$$

This latter inequality is true as

$$\begin{aligned} \log f'(x) + \log f'(y) &= 2 \cdot \int_0^1 t \log f'(x) + (1 - t) \log f'(y) dt \\ &\leq 2 \int_0^1 \log(f'(tx + (1 - t)y)) dt < 2 \log \left(\int_0^1 f'(tx + (1 - t)y) dt \right). \end{aligned}$$

In the above computation, the first inequality holds because of the log-concavity of f' and the second (strict) inequality holds because the logarithm function is strictly concave and f' is strictly monotone increasing.

So, the operator L (defined in eq. (1)) has a negative eigenvalue, that is, there exist $\lambda, \mu \in \mathbb{C}$ (with $|\lambda|^2 + |\mu|^2 = 1$) such that with $w = \lambda u + \mu v$ we have

$$\langle Lw, w \rangle = \left\langle \lim_{t \rightarrow 0} \frac{1}{t} (f(A + tB) - f(A)) w, w \right\rangle < 0.$$

Therefore,

$$\lim_{t \rightarrow 0} \frac{1}{t} (\langle f(A + tB) w, w \rangle - \langle f(A) w, w \rangle) < 0,$$

and so there exists some $t_0 > 0$ such that $0 < \langle f(A) w, w \rangle - \langle f(A + t_0 B) w, w \rangle$. \square

Proof of the Theorem. The direction (1) \Rightarrow (2) is easy to verify, because if f is continuous and monotone increasing as a function of one real variable, then the map $x \mapsto f(x)$ preserves the order of commuting self-adjoint elements of a unital C^* -algebra.

To see the contrary, assume that $a \in \mathcal{A}_s$, $\sigma(a) \subset I$ and $aa' - a'a \neq 0$ for some $a' \in \mathcal{A}$. Then, by [4, 10.2.4. Corollary], there exists an irreducible representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ such that $\pi(aa' - a'a) \neq 0$, that is, $\pi(a)\pi(a') \neq \pi(a')\pi(a)$. Let us fix this irreducible representation π . So, $\pi(a)$ is a non-central self-adjoint (and hence normal) element of $\mathcal{B}(\mathcal{H})$ with $\sigma(\pi(a)) \subset I$ (as a representation do not increase the spectrum). By the non-centrality, $\sigma(\pi(a))$ has at least two elements, and by the normality, every element of $\sigma(\pi(a))$ is an approximate eigenvalue [3, 3.2.13. Lemma]. Let x and y be two different elements of $\sigma(\pi(a))$, and let $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ and $\{v_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ satisfy

$$\lim_{n \rightarrow \infty} \pi(a)u_n - xu_n = 0, \quad \lim_{n \rightarrow \infty} \pi(a)v_n - yv_n = 0, \quad \text{and} \quad \langle u_m, v_n \rangle = 0 \quad m, n \in \mathbb{N}.$$

(As $x \neq y$, the approximate eigenvectors can be chosen to be orthogonal.) Set $\mathcal{K}_n := \text{span}\{u_n, v_n\}$ and let E_n be the orthoprojection onto the closed subspace $\mathcal{K}_n^\perp \subset \mathcal{H}$. Let

$$\psi_n(a) := xu_n \otimes u_n + yv_n \otimes v_n + E_n \pi(a) E_n.$$

We intend to show that

$$\lim_{n \rightarrow \infty} \psi_n(a) = \pi(a)$$

in the operator norm topology. Let h be an arbitrary non-zero element of \mathcal{H} and consider the orthogonal decompositions $h = h_1^{(n)} + h_2^{(n)}$, where $h_1^{(n)} \in \mathcal{K}_n$ and $h_2^{(n)} \in \mathcal{K}_n^\perp$ for any $n \in \mathbb{N}$. Let us introduce the symbols $\varepsilon_{u,n} := \pi(a)u_n - xu_n$ and $\varepsilon_{v,n} := \pi(a)v_n - yv_n$ and recall that $\lim_{n \rightarrow \infty} \varepsilon_{u,n} = 0$ and $\lim_{n \rightarrow \infty} \varepsilon_{v,n} = 0$ in the standard topology of the Hilbert space \mathcal{H} . Now,

$$\frac{1}{\|h\|} \left\| (\pi(a) - \psi_n(a)) h \right\|$$

$$\leq \frac{1}{\|h\|} \left\| (\pi(a) - \psi_n(a)) h_1^{(n)} \right\| + \frac{1}{\|h\|} \left\| (\pi(a) - \psi_n(a)) h_2^{(n)} \right\|.$$

Both the first and the second term of the right hand side of the above inequality are bounded by the term $\|\varepsilon_{u,n}\| + \|\varepsilon_{v,n}\|$ because

$$\begin{aligned} \frac{1}{\|h\|} \left\| (\pi(a) - \psi_n(a)) h_1^{(n)} \right\| &= \frac{1}{\|h\|} \left\| (\pi(a) - \psi_n(a)) (\alpha_n u_n + \beta_n v_n) \right\| \\ &= \frac{1}{\|h\|} \left\| \alpha_n x u_n + \alpha_n \varepsilon_{u,n} - x \alpha_n u_n + \beta_n y v_n + \beta_n \varepsilon_{v,n} - y \beta_n v_n \right\| \\ &\leq \frac{|\alpha_n|}{\|h\|} \|\varepsilon_{u,n}\| + \frac{|\beta_n|}{\|h\|} \|\varepsilon_{v,n}\| \leq \|\varepsilon_{u,n}\| + \|\varepsilon_{v,n}\| \end{aligned}$$

as the sequences $\{|\alpha_n|\}$ and $\{|\beta_n|\}$ are obviously bounded by $\|h\|$, and

$$\begin{aligned} \frac{1}{\|h\|} \left\| (\pi(a) - \psi_n(a)) h_2^{(n)} \right\| &= \frac{1}{\|h\|} \left\| (I_{\mathcal{H}} - E_n) \pi(a) h_2^{(n)} \right\| \\ &= \frac{1}{\|h\|} \left\| (u_n \otimes u_n + v_n \otimes v_n) \pi(a) h_2^{(n)} \right\| = \frac{1}{\|h\|} \left\| \left\langle \pi(a) h_2^{(n)}, u_n \right\rangle u_n + \left\langle \pi(a) h_2^{(n)}, v_n \right\rangle v_n \right\| \\ &= \frac{1}{\|h\|} \left\| \left\langle h_2^{(n)}, \pi(a) u_n \right\rangle u_n + \left\langle h_2^{(n)}, \pi(a) v_n \right\rangle v_n \right\| \\ &\leq \frac{1}{\|h\|} \left| \left\langle h_2^{(n)}, x u_n + \varepsilon_{u,n} \right\rangle \right| + \frac{1}{\|h\|} \left| \left\langle h_2^{(n)}, y v_n + \varepsilon_{v,n} \right\rangle \right| \\ &= \frac{1}{\|h\|} \left| \left\langle h_2^{(n)}, \varepsilon_{u,n} \right\rangle \right| + \frac{1}{\|h\|} \left| \left\langle h_2^{(n)}, \varepsilon_{v,n} \right\rangle \right| \\ &\leq \frac{\|h_2^{(n)}\|}{\|h\|} \|\varepsilon_{u,n}\| + \frac{\|h_2^{(n)}\|}{\|h\|} \|\varepsilon_{v,n}\| \leq \|\varepsilon_{u,n}\| + \|\varepsilon_{v,n}\|. \end{aligned}$$

We used that a is self-adjoint, hence so is $\pi(a)$.

So, we found that

$$\sup \left\{ \frac{1}{\|h\|} \left\| (\pi(a) - \psi_n(a)) h \right\| \mid h \in \mathcal{H} \setminus \{0\} \right\} \leq 2 (\|\varepsilon_{u,n}\| + \|\varepsilon_{v,n}\|) \rightarrow 0,$$

which means that $\psi_n(a)$ tends to $\pi(a)$ in the operator norm topology.

We have fixed I, f, x and y . By the Proposition, we have $\lambda, \mu \in \mathbb{C}$ (with $|\lambda|^2 + |\mu|^2 = 1$) and $t_0 > 0$ such that using the notation $B_n := (u_n + v_n) \otimes (u_n + v_n)$ and $w_n := \lambda u_n + \mu v_n$, we have

$$(2) \quad \langle f(\psi_n(a)) w_n, w_n \rangle - \langle f(\psi_n(a) + t_0 B_n) w_n, w_n \rangle = \delta > 0$$

for any $n \in \mathbb{N}$. That is, the left hand side of (2) is independent of n .

The operator B_n is a self-adjoint element of $\mathcal{B}(\mathcal{H})$ and \mathcal{K}_n is a finite dimensional subspace of \mathcal{H} , hence by Kadison's transitivity theorem [4, 10.2.1. Theorem], there exists a self-adjoint $b_n \in \mathcal{A}$ such that

$$\pi(b_n)|_{\mathcal{K}_n} = B_n|_{\mathcal{K}_n}.$$

Observe that $B_n \mathcal{K}_n \subseteq \mathcal{K}_n$ and so $\pi(b_n) \mathcal{K}_n \subseteq \mathcal{K}_n$. On the other hand, $\pi(b_n)$ is self-adjoint as b_n is self-adjoint, hence it follows that $\pi(b_n) \mathcal{K}_n^\perp \subseteq \mathcal{K}_n^\perp$. Therefore, the fact $B_n = \frac{1}{2} B_n^2$ implies that

$$\pi \left(\frac{1}{2} b_n^2 \right)_{|\mathcal{K}_n} = \left(\frac{1}{2} \pi(b_n)^2 \right)_{|\mathcal{K}_n} = \frac{1}{2} (\pi(b_n)|_{\mathcal{K}_n})^2 = \frac{1}{2} B_{n|\mathcal{K}_n}^2 = B_{n|\mathcal{K}_n}.$$

So, we can rewrite (2) as

$$(3) \quad \langle f(\psi_n(a)) w_n, w_n \rangle - \left\langle f \left(\psi_n(a) + t_0 \pi \left(\frac{1}{2} b_n^2 \right) \right) w_n, w_n \right\rangle = \delta > 0$$

A standard continuity argument which is based on the fact that $\psi_n(a)$ tends to $\pi(a)$ in the operator norm topology shows that

$$(4) \quad \lim_{n \rightarrow \infty} \|f(\psi_n(a)) - f(\pi(a))\| = 0.$$

Moreover, by Kadison's transitivity theorem, the sequence $\pi \left(\frac{1}{2} b_n^2 \right)$ is bounded (for details, the reader should consult the proof of [3, 5.4.3. Theorem]), and hence

$$(5) \quad \lim_{n \rightarrow \infty} \left\| f \left(\psi_n(a) + t_0 \pi \left(\frac{1}{2} b_n^2 \right) \right) - f \left(\pi(a) + t_0 \pi \left(\frac{1}{2} b_n^2 \right) \right) \right\| = 0$$

also holds. By (4) and (5), for any $\delta > 0$ one can find $n_0 \in \mathbb{N}$ such that for $n > n_0$ we have

$$\|f(\psi_n(a)) - f(\pi(a))\| < \frac{1}{4} \delta$$

and

$$\left\| f \left(\psi_n(a) + t_0 \pi \left(\frac{1}{2} b_n^2 \right) \right) - f \left(\pi \left(a + \frac{t_0}{2} b_n^2 \right) \right) \right\| < \frac{1}{4} \delta.$$

Therefore, by (3), for $n > n_0$, the inequality

$$\langle f(\pi(a)) w_n, w_n \rangle - \left\langle f \left(\pi \left(a + \frac{t_0}{2} b_n^2 \right) \right) w_n, w_n \right\rangle > \frac{1}{2} \delta > 0$$

holds. In other words,

$$f(\pi(a)) \not\leq f \left(\pi \left(a + \frac{t_0}{2} b_n^2 \right) \right),$$

or equivalently,

$$\pi(f(a)) \not\leq \pi \left(f \left(a + \frac{t_0}{2} b_n^2 \right) \right).$$

Any representation of a C^* -algebra preserves the semidefinite order, hence this means that

$$f(a) \not\leq f \left(a + \frac{t_0}{2} b_n^2 \right),$$

despite the fact that $a \leq a + \frac{t_0}{2} b_n^2$. The proof is done. \square

Remark. Note that our theorem generalizes Molnár's result, and — as every "local" theorem easily implies its "global" counterpart — we recover the theorems of Ogasawara, Pedersen, and Wu, as well.

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