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Chu, Angus and Liao, Chih-Hsing

University of Macau, National Central University

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Angus C. Chu Chih-Hsing Liao

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Abstract

Does wealth inequality affect optimal patent policy? This study develops a Schumpeterian growth model with heterogeneous households to explore this question. The model features a general innovation specification that captures two common specifications as special cases: (a) the knowledge-driven specification that uses R&D labor, and (b) the lab-equipment specification that uses final output for R&D. Under the knowledge-driven specification, all households prefer the same level of patent protection. However, under the lab-equipment specification, wealthier households prefer stronger patent protection, and higher wealth inequality reduces the optimal level of patent protection and economic growth. Under the general innovation specification, strengthening patent protection has an inverted-U effect on innovation, in contrast to the positive effect under the two special cases. Furthermore, wealthier households continue to prefer stronger patent protection, and wealth inequality also reduces optimal patent protection. Therefore, all households preferring the same level of patent protection under the knowledge-driven specification is due to a knife-edge parameter condition. Calibrating the model to US data, we find that eliminating wealth inequality raises the optimal level of patent protection and economic growth.

JEL classification: O30, O40
Keywords: patent policy, innovation, wealth inequality, economic growth

Chu: anguscccc@gmail.com. Department of Economics, University of Macau, Macau.
Liao: chihhsingliao@gmail.com. Department of Economics, National Central University, Taoyuan, Taiwan.

1 Introduction

The seminal study by Solow (1956) shows that economic growth is ultimately driven by technological progress. Therefore, innovation policies, such as R&D subsidies and patent protection, are crucial for stimulating economic growth and technological progress. For example, according to the Royal Swedish Academy of Sciences (2018), "Romer showed that unregulated markets will produce technological change, but tend to underprovide R&D and the new goods created by it. Addressing this under-provision requires well-designed government interventions, such as R&D subsidies and patent regulation. His analysis says that such policies are vital to long-run growth". However, most growth-theoretic studies on optimal patent policy are based on growth models that feature a representative household without considering wealth inequality. Therefore, this study asks the following question: does the wealth distribution affect the optimal design of patent policy?

To explore the above question, we develop a Schumpeterian growth model with heterogeneous households. Interestingly, we find that whether the wealth distribution affects optimal patent policy depends on the underlying innovation specification. A novelty of our Schumpeterian growth model is that it features a general innovation specification that captures two commonly used innovation specifications as special cases: (a) the knowledge-driven innovation specification that uses labor as R&D input, and (b) the lab-equipment innovation specification that uses final output as R&D input. Within this growth-theoretic framework, we obtain the following results.

Under our general innovation specification, strengthening patent protection has an inverted-U effect on innovation, whereas the effect of patent protection on innovation is positive under the two special cases. Intuitively, stronger patent protection reallocates labor from production to R&D and leads to a reduction in production, which in turn decreases the amount of final output for R&D whenever R&D requires both labor and final output as inputs under our general innovation specification. Under the knowledge-driven innovation specification, the effect of patent protection on innovation (which requires only R&D labor) is positive, and all households prefer the same level of patent protection. Therefore, in this case, the optimal level of patent protection does not depend on the wealth distribution. Under the lab-equipment innovation specification, the effect of patent protection on innovation (which requires only final output for R&D) is also positive, but wealthier households prefer a higher level of patent protection. Therefore, the wealth distribution affects optimal patent policy in this case. Specifically, higher wealth inequality reduces the optimal level of patent protection and economic growth. Our general innovation specification captures these two specifications as special cases and shows that the surprising result of all households preferring the same level of patent protection under the knowledge-driven specification is due to a knife-edge parameter condition.

The intuition of the above finding can be explained as follows. The optimal level of patent protection is determined by a tradeoff between innovation and monopolistic distortion. In our general-equilibrium setting, the monopolistic distortionary effect is represented by a reduction in the level of consumption. Whether this effect affects all households equally depends on the aggregate consumption-asset ratio. If this ratio decreases, then less wealthy households suffer a larger reduction in consumption relative to wealthier households; in this case, less wealthy households prefer a lower level of patent protection than wealthier households. So, does the aggregate consumption-asset ratio depend on the level of patent protection? Under the knowledge-driven specification, it does not because innovation uses only labor as R&D input. However, whenever innovation uses also final output as R&D input (under both the general and lab-equipment specifications), an increase in the level of patent protection reallocates some final output from consumption to R&D and reduces the aggregate consumption-asset ratio, which in turn affects the optimal level of patent protection for heterogeneous households. Finally, calibrating the model to data for a quantitative analysis, we find that eliminating wealth inequality in the US raises the optimal level of patent protection and leads to a quantitatively significant increase in economic growth.

This study relates to the literature on innovation and economic growth. In this literature, the seminal study by Romer (1990) develops the first R&D-based growth model, in which innovation is driven by the creation of new products. Then, Aghion and Howitt (1992) develop the Schumpeterian growth model, in which innovation is driven by the development of higher-quality products; see also Grossman and Helpman (1991) and Segerstrom *et al.* (1990) for other early studies. Subsequent studies apply the Schumpeterian growth model to explore the effects of innovation policies, such as R&D subsidies and patent protection. This study provides a contribution to this literature by exploring optimal patent policy in a Schumpeterian growth model with heterogeneous households and a general innovation specification.

Therefore, this study also relates to the literature on patent policy and innovation-driven growth. The seminal study on optimal patent protection is by Nordhaus (1969), who uses a partial-equilibrium model. Judd (1985) is the first study that explores optimal patent protection in a dynamic general-equilibrium model. Since the development of the innovation-driven growth model by Romer (1990) and Aghion and Howitt (1992), subsequent studies have used the innovation-driven growth model to explore the effects of patent policy; see Cozzi (2001), Li (2001), Goh and Olivier (2002) and Iwaisako and Futagami (2003) for early studies and Chu (2022) for a recent survey of the subsequent theoretical and empirical studies in this literature.[1] Unlike previous studies, we consider a general innovation specification and show that an inverted-U effect of patent protection on innovation emerges via a novel mechanism;[2] see Lerner (2009) and Qian (2007) for empirical evidence for this inverted-U effect. Recent studies explore the effects of patent policy on income inequality and innovation in the presence of heterogeneous households; see for example, Chu (2010), Chu and Cozzi (2018), Chu *et al.* (2021, 2022) and Kiedaisch (2021). This study contributes to this branch of the literature by showing that the innovation specification and heterogeneous households have the following implication: whether the wealth distribution affects optimal patent policy depends on the underlying innovation specification.

The rest of this study is organized as follows. Section 2 presents the Schumpeterian growth model with heterogeneous households. Section 3 explores the effects of wealth inequality on optimal patent policy under different innovation specifications. Section 4 concludes.

[1] A recent study by Ohki (2023) develops a tractable endogenous growth model to examine heterogeneous incumbents' current technology-switching behavior and then examines the growth effects of subsidy and patent policies.

[2] See Horii and Iwaisako (2007), Furukawa (2007), Chu *et al.* (2012) and Chu and Pan (2013) for earlier studies that also identify an inverted-U effect via other mechanisms.

2 A Schumpeterian model with wealth inequality

The seminal study by Aghion and Howitt (1992) develops the Schumpeterian growth model. As in Romer (1990), they assume that R&D uses labor as input, which is known as the knowledge-driven innovation specification in the literature. Rivera-Batiz and Romer (1991) instead assume that R&D uses final output as input, which is known as the lab-equipment innovation specification. We consider a general innovation process that uses both labor and final output as factor inputs and captures these two commonly used specifications as special cases. Furthermore, we introduce heterogeneous households to the Schumpeterian model as in Chu (2010) and Chu and Cozzi (2018).

2.1 Heterogeneous households

There is a unit continuum of households $i \in [0, 1]$. They have identical preferences but differ in their levels of wealth. Household h has the following utility function:

$$u(h) = \int_0^{\infty} e^{-\rho t} \ln c_t(h) dt, \tag{1}$$

where the parameter $\rho > 0$ is the subjective discount rate and $c_t(h)$ is the consumption of household h at time t. The household maximizes utility subject to the following asset-accumulation equation:

$$\dot{a}_t(h) = r_t a_t(h) + w_t - c_t(h), \tag{2}$$

where $a_t(h)$ is the value of assets owned by household h and r_t is the real interest rate. Each household supplies one unit of labor to earn wage income w_t.

From standard dynamic optimization, household h's consumption path is given by

$$\frac{\dot{c}_t(h)}{c_t(h)} = r_t - \rho, \tag{3}$$

which shows that the growth rate of consumption is the same across all households such that $\dot{c}_t(h)/c_t(h) = \dot{c}_t/c_t$ for all $h \in [0, 1]$, where $c_t \equiv \int_0^1 c_t(h)dh$ denotes aggregate consumption. Therefore, the growth rate of aggregate consumption is also given by

$$\frac{\dot{c}_t}{c_t} = r_t - \rho. \tag{4}$$

2.2 Final output

Final output y_t is produced by competitive firms using the following production function that aggregates a unit continuum of intermediate goods into the final good:

$$y_t = \exp\left(\int_0^1 \ln x_t(i) di\right), \tag{5}$$

where $x_t(i)$ denotes intermediate good $i \in [0,1]$. From profit maximization, the conditional demand function for $x_t(i)$ is

$$x_t(i) = \frac{y_t}{p_t(i)}, \tag{6}$$

where $p_t(i)$ is the price of $x_t(i)$.

2.3 Intermediate goods

Each intermediate good i is produced by an industry leader, who acts as a monopolist. The production function of the leader in industry i is

$$x_t(i) = z^{n_t(i)} l_{x,t}(i), \tag{7}$$

where the parameter $z > 1$ is the step size of each quality improvement and $n_t(i)$ is the number of quality improvements that have occurred in industry i as of time t. Given the productivity level $z^{n_t(i)}$, the industry leader employs production labor $l_{x,t}(i)$ and faces the marginal cost function $w_t/z^{n_t(i)}$. From Bertrand competition between the current industry leader and the previous industry leader, the profit-maximizing price for the current industry leader is:

$$p_t(i) = \mu \frac{w_t}{z^{n_t(i)}}, \tag{8}$$

where the markup ratio $\mu > 1$ is a patent policy parameter as in Li (2001).[3] The amount of monopolistic profit in industry i is

$$\pi_t(i) = p_t(i)x_t(i) - w_t l_{x,t}(i) = \frac{\mu - 1}{\mu} y_t, \tag{9}$$

and the wage payment in industry i is

$$w_t l_{x,t}(i) = \frac{1}{\mu} p_t(i)x_t(i) = \frac{1}{\mu} y_t. \tag{10}$$

2.4 R&D

From (9), we see that $\pi_t(i) = \pi_t$. Therefore, in a symmetric equilibrium, the value of inventions is also equal across industries such that $v_t(i) = v_t$ for $i \in [0,1]$.[4] The no-arbitrage condition that determines v_t is

$$r_t = \frac{\pi_t + \dot{v}_t - \lambda_t v_t}{v_t}, \tag{11}$$

where λ_t is the arrival rate of innovation. Intuitively, (11) equates the interest rate r_t to the rate of return on v_t for which the latter is given by the sum of monopolistic profit π_t, capital

[3] Here we follow Dinopoulos and Segerstrom (2010) to assume that new industry leaders are able to charge the markup μ (even when it is above the quality step size z) because the closest competitors choose to immediately exit the market in equilibrium; see Dinopoulos and Segerstrom (2010) for a detailed discussion.

[4] See Cozzi *et al.* (2007) for a justification for the symmetric equilibrium in the Schumpeterian model.

gain $\dot{v}_t$ and expected capital loss $\lambda_t v_t$. The last term captures the situation in which the current technology becomes obsolete when the next innovation arrives.[5]

Competitive entrepreneurs devote R_t units of final output and employ $l_{r,t}$ units of labor to conduct innovation. The arrival rate of innovation λ_t is given by the following specification:

$$\lambda_t = \varphi \left(\frac{R_t}{Z_t} \right)^{\alpha} (l_{r,t})^{1-\alpha}, \tag{12}$$

where $\varphi > 0$ is a productivity parameter and Z_t is the aggregate level of technology, which captures an increasing-difficulty effect of R&D. The parameter $\alpha \in (0,1)$ determines the intensity of final output R_t in the R&D process relative to R&D labor $l_{r,t}$ and nests the knowledge-driven specification ($\alpha \to 0$) and the lab-equipment specification ($\alpha \to 1$) in the literature as special cases. The profit-maximizing conditions of R&D are as follows:

$$\alpha \lambda_t v_t = R_t, \tag{13}$$

$$(1-\alpha)\lambda_t v_t = w_t l_{r,t}. \tag{14}$$

2.5 Decentralized equilibrium

The equilibrium is a time path of allocations $\{c_t(h), a_t(h), y_t, x_t(i), l_{x,t}(i), l_{r,t}, R_t\}$ and a time path of prices $\{w_t, r_t, p_t(i), v_t\}$. Also, at each instance of time, the following conditions hold:

- households $h \in [0,1]$ maximize utility taking $\{w_t, r_t\}$ as given;
- competitive firms produce final good y_t to maximize profit taking $p_t(i)$ as given;
- monopolistic firm i produces intermediate good $x_t(i)$ and chooses $\{l_{x,t}(i), p_t(i)\}$ to maximize profit taking w_t as given;
- competitive R&D entrepreneurs choose R_t and $l_{r,t}$ to maximize expected profit taking $\{w_t, v_t\}$ as given;
- the market-clearing condition for labor holds such that $l_{r,t} + \int_0^1 l_{x,t}(i)di = 1$;
- the market-clearing condition for the final good holds such that $\int_0^1 c_t(h)dh + R_t = y_t$;
- the total value of household assets equals the value of all monopolistic firms such that $\int_0^1 a_t(h)dh = \int_0^1 v_t(i)di$.

[5] See Cozzi (2007) for a discussion on this Arrow replacement effect.

2.6 Aggregate economy

We define aggregate technology Z_t as follows:

$$Z_t \equiv \exp\left(\int_0^1 n_t(i)di \ln z\right) = \exp\left(\int_0^t \lambda_\omega d\omega \ln z\right), \tag{15}$$

which uses the law of large numbers. Differentiating the log of Z_t in (15) with respect to time yields the growth rate of technology given by

$$g_t \equiv \frac{\dot{Z}_t}{Z_t} = \lambda_t \ln z. \tag{16}$$

Substituting (7) into (5) yields the aggregate production function as follows:

$$y_t = Z_t l_{x,t}, \tag{17}$$

where $l_{x,t} = l_{x,t}(i)$ for all $i \in [0, 1]$. Lemma 1 shows that the aggregate economy jumps to a balanced growth path with a constant growth rate g and a stationary allocation of labor $\{l_x, l_r\}$.

Lemma 1 *The aggregate economy always jumps a unique and stable balanced growth path.*

Proof. See Appendix A. ■

2.7 Economic growth

Combining (13) and (14) yields

$$\frac{\alpha}{1-\alpha} = \frac{R_t}{w_t l_{r,t}}. \tag{18}$$

Then, substituting $w_t = Z_t/\mu$ from (10) and (17) into (18) yields

$$\frac{R_t}{Z_t} = \frac{\alpha}{1-\alpha}\frac{l_{r,t}}{\mu}, \tag{19}$$

which can then be substituted into (12) and (16) to derive the growth rate of technology as

$$g_t = \lambda_t \ln z = \varphi\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^\alpha l_{r,t} \ln z. \tag{20}$$

Lemma 2 shows that the steady-state equilibrium R&D labor l_r is increasing in the level of patent protection μ.

Lemma 2 *The steady-state equilibrium level of R&D labor l_r is increasing in μ.*

Proof. See Appendix A. ■

This result originates from Li (2001), who however focuses on the knowledge-driven specification captured by $\alpha \to 0$ in (20), which then implies a positive effect of μ on g. Here, we consider a general innovation specification with $\alpha \in (0, 1)$ under which the steady-state equilibrium growth rate g depends on both R&D labor l_r and final output R. In this case, (20) shows that the level of patent protection μ has both positive and negative effects on the steady-state equilibrium growth rate g. Intuitively, stronger patent protection increases R&D labor l_r and decreases production labor l_x, which in turn decreases the amount of output available for R&D. These positive and negative effects together generate an inverted-U effect on innovation. Proposition 1 summarizes this result.

Proposition 1 *The steady-state equilibrium growth rate g is an inverted-U function in μ.*

Proof. See Appendix A. ■

2.8 Wealth distribution

From (2), the law of motion for the aggregate value of assets is given by

$$\dot{a}_t = r_t a_t + w_t - c_t, \tag{21}$$

where $a_t = \int_0^1 a_t(h)dh$. We define the initial share of wealth owned by household h as $\theta_{a,0}(h) \equiv a_0(h)/a_0$, which is exogenously given at time 0. We consider a general distribution function of initial wealth share with a mean of one and a standard deviation of $\sigma_a > 0$. Taking the log of wealth share $\theta_{a,t}(h) \equiv a_t(h)/a_t$ at time t and differentiating the resulting expression with respect to time yield

$$\frac{\dot{\theta}_{a,t}(h)}{\theta_{a,t}(h)} = \frac{\dot{a}_t(h)}{a_t(h)} - \frac{\dot{a}_t}{a_t} = \frac{c_t - w_t}{a_t} - \frac{c_t(h) - w_t}{a_t(h)}, \tag{22}$$

which uses (2) and (21). Then, (22) can be re-expressed as

$$\dot{\theta}_{a,t}(h) = \frac{c_t - w_t}{a_t}\theta_{a,t}(h) - \frac{\theta_{c,t}(h)c_t - w_t}{a_t}, \tag{23}$$

where $\theta_{c,t}(h) \equiv c_t(h)/c_t$ is the share of consumption by household h at time t. Taking the log of $\theta_{c,t}(h)$ and differentiating the resulting expression with respect to time yield

$$\frac{\dot{\theta}_{c,t}(h)}{\theta_{c,t}(h)} = \frac{\dot{c}_t(h)}{c_t(h)} - \frac{\dot{c}_t}{c_t}. \tag{24}$$

Given that (3) and (4) imply $\dot{c}_t(h)/c_t(h) = \dot{c}_t/c_t$, (24) becomes $\dot{\theta}_{c,t}(h) = 0$ for all t, which in turn implies $\theta_{c,t}(h)$ must jump to its steady-state value $\theta_c(h)$ at any time t.

Balanced growth of the aggregate economy implies that

$$\frac{\dot{a}_t}{a_t} = \frac{\dot{c}_t}{c_t} = r_t - \rho, \tag{25}$$

which also uses (4). Substituting (25) into (21) yields

$$\frac{c_t - w_t}{a_t} = \rho. \tag{26}$$

Substituting $\theta_{c,t}(h) = \theta_c(h)$ and (26) into (23) yields

$$\dot{\theta}_{a,t}(h) = \rho\left[\theta_{a,t}(h) - 1\right] - \left[\theta_c(h) - 1\right]\frac{c_t}{a_t}, \tag{27}$$

where the aggregate consumption-asset ratio can be derived as[6]

$$\frac{c_t}{a_t} = \frac{c}{a} = \frac{1}{\mu - 1}\left(\mu - \frac{\alpha}{1-\alpha}\frac{l_r}{1-l_r}\right)\left[\rho + \varphi\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha} l_r\right] \tag{28}$$

for all t. Equation (27) implies that the only solution that is consistent with the long-run stability of the state variable $\theta_{a,t}(h)$ is $\dot{\theta}_{a,t}(h) = 0$ for all t. Therefore, the wealth distribution is stationary and exogenously given at time 0 (i.e., $\theta_{a,t}(h) = \theta_{a,0}(h)$ for all t). Finally, imposing $\dot{\theta}_{a,t}(h) = 0$ on (27) yields the steady-state value of the consumption share $\theta_{c,t}(h)$:

$$\theta_{c,t}(h) = \theta_c(h) = 1 - \frac{\rho\left[1 - \theta_{a,0}(h)\right]}{c/a}, \tag{29}$$

which changes whenever the consumption-asset ratio c/a changes.

3 Optimal patent policy

We impose balanced growth on (1) to derive the welfare function of household h as

$$u(h) = \frac{1}{\rho}\left[\ln c_0(h) + \frac{g}{\rho}\right], \tag{30}$$

where $c_0(h)$ is the level of household h's consumption at time 0. Substituting $c_0(h) = \theta_c(h)\, c_0$ into (30) yields

$$u(h) = \frac{1}{\rho}\left[\ln \theta_c(h) + \ln c_0 + \frac{g}{\rho}\right], \tag{31}$$

where the initial level of aggregate consumption c_0 can be derived as[7]

$$c_0 = \frac{\rho + (1-\alpha)\varphi\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha}}{\varphi\mu(1-\alpha)\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha}}. \tag{32}$$

[6] See Appendix A.

[7] See Appendix A.

Lemma 3 shows that the initial level of aggregate consumption c_0 is decreasing in the level of patent protection μ.

Lemma 3 *The initial level of aggregate consumption c_0 is decreasing in μ.*

Proof. See Appendix A. ■

Therefore, the condition that determines the utility-maximizing level of patent protection for household h is, in general, given by

$$\rho \frac{\partial u(h)}{\partial \mu} = \underbrace{\frac{\partial \ln \theta_c(h)}{\partial \mu}}_{?} + \underbrace{\frac{\partial \ln c_0}{\partial \mu}}_{-} + \frac{1}{\rho} \underbrace{\frac{\partial g}{\partial \mu}}_{+/-}, \tag{33}$$

where $\partial g / \partial \mu$ is given by the inverted-U effect of patent protection on innovation from Proposition 1, whereas $\partial \ln c_0 / \partial \mu < 0$ from Lemma 3 captures the negative distortionary effect of patent protection on aggregate consumption. Whether the optimal level of patent protection is the same or different across households depends on $\theta_c(h)$, which in turn depends on the aggregate consumption-asset ratio c/a as shown in (29). Before we discuss the general case, we first consider the two commonly used special cases in the literature.

3.1 Knowledge-driven innovation specification

We first consider the knowledge-driven innovation specification, which is given by $\alpha \to 0$ in (12). Under the knowledge-driven specification, the arrival rate of innovation simplifies to $\lambda_t = \varphi l_{r,t}$, which originates from the seminal study by Aghion and Howitt (1992) and is commonly used in the literature. In this case, the steady-state equilibrium growth rate g is given by

$$g = \left[\varphi \left(\frac{\mu - 1}{\mu} \right) - \frac{\rho}{\mu} \right] \ln z,$$

which becomes increasing in patent protection μ under the knowledge-driven specification. More importantly, the resource constraint on the final good becomes $y_t = c_t$, and the aggregate consumption-asset ratio simplifies to $c/a = \rho + \varphi$, which is independent of the level of patent protection. Therefore, the optimal level of patent protection is the same across all households h because (29) implies that $\theta_c(h)$ is independent of μ (i.e., $\partial \ln \theta_c(h) / \partial \mu = 0$ in (33)). Proposition 2 derives the optimal level of patent protection, which is the same across all households h. However, in the next sections, we will show that this result is due to the knife-edge parameter condition $\alpha = 0$ and does not hold whenever $\alpha > 0$.

Proposition 2 *Under the knowledge-driven innovation specification, the optimal level of patent protection is given by*

$$\mu^* = \left(1 + \frac{\varphi}{\rho} \right) \ln z. \tag{34}$$

Proof. See Appendix A. ■

3.2 Lab-equipment innovation specification

We now consider the lab-equipment innovation specification, which is given by $\alpha \to 1$ in (12). Under the lab-equipment specification, the arrival rate of innovation simplifies to $\lambda_t = \varphi R_t / Z_t$, which uses final output instead of labor as R&D input and is also often used in the literature. In this case, the steady-state equilibrium growth rate g is given by

$$g = \left[\varphi\left(\frac{\mu - 1}{\mu}\right) - \rho\right] \ln z,$$

which is also increasing in patent protection μ under the lab-equipment specification. The resource constraint on the final good becomes $y_t = c_t + R_t$. As for the aggregate consumption-asset ratio, it simplifies to $c/a = \rho + \varphi/\mu$, which is now decreasing in the level of patent protection. Therefore, the optimal level of patent protection is different across households because (29) implies that $\theta_c(h)$ is decreasing (increasing) in μ for less wealthy (wealthier) households; i.e., $\partial \ln \theta_c(h) / \partial \mu < 0$ for $\theta_{a,0}(h) < 1$ ($\partial \ln \theta_c(h) / \partial \mu > 0$ for $\theta_{a,0}(h) > 1$) in (33). Proposition 3 derives the utility-maximizing level of patent protection for household h and shows that it is increasing in the household's wealth share $\theta_{a,0}(h)$. Therefore, wealthier households prefer a higher level of patent protection than less wealthy households.

Proposition 3 *Under the lab-equipment innovation specification, the utility-maximizing level of patent protection for household h is increasing in its wealth share $\theta_{a,0}(h)$ and given by*[8]

$$\mu^*(h) = \frac{\varphi}{\rho} \frac{\ln z}{1 - \theta_{a,0}(h) \ln z}. \tag{35}$$

Proof. See Appendix A. ■

Given that different households prefer different levels of patent protection, we need to specify a social welfare function in order to derive the optimal level of patent protection. For simplicity, we consider a linear aggregate of households' utility functions given by

$$U \equiv \int_0^1 u(h)\, dh = \frac{1}{\rho}\left[\int_0^1 \ln \theta_c(h)\, dh + \ln c_0 + \frac{g}{\rho}\right]. \tag{36}$$

Then, the condition that determines the optimal level of patent protection μ is given by

$$\rho \frac{\partial U}{\partial \mu} = \int_0^1 \underbrace{\frac{\partial \ln \theta_c(h)}{\partial \mu}}_{-/+} dh + \underbrace{\frac{\partial \ln c_0}{\partial \mu}}_{-} + \frac{1}{\rho} \underbrace{\frac{\partial g}{\partial \mu}}_{+}. \tag{37}$$

The first term on the right-hand side of (37) is given by

$$\int_0^1 \frac{\partial \ln \theta_c(h)}{\partial \mu} dh = -\rho \underbrace{\frac{\partial a/c}{\partial \mu}}_{+} \int_0^1 \frac{1 - \theta_{a,0}(h)}{\theta_c(h)} dh = -\rho \underbrace{\frac{\partial a/c}{\partial \mu}}_{+} \int_0^1 \left[\frac{1}{1 - \theta_{a,0}(h)} - \frac{\rho}{\rho + \varphi/\mu}\right]^{-1} dh, \tag{38}$$

[8] Here we assume $\theta_{a,0}(h) < 1/\ln z$ for all $h \in [0,1]$ to ensure an interior solution for $\mu^*(h)$.

where $\theta_c(h)$ is given by (29) and $a/c = (\rho + \varphi/\mu)^{-1}$ is increasing in μ. From Jensen's inequality, we have[9]

$$\int_0^1 \left[\frac{1}{1-\theta_{a,0}(h)} - \frac{\rho}{\rho+\varphi/\mu}\right]^{-1} dh > \left[\frac{1}{\int_0^1 \left[1-\theta_{a,0}(h)\right] dh} - \frac{\rho}{\rho+\varphi/\mu}\right]^{-1} = 0, \tag{39}$$

which together with (38) ensures that

$$\int_0^1 \frac{\partial \ln \theta_c(h)}{\partial \mu} dh < 0. \tag{40}$$

Therefore, the presence of wealth inequality (i.e., $\theta_{a,0}(h) \neq 1$ for some h) reduces the optimal level of patent protection. Proposition 4 summarizes this result. Intuitively, the lower consumption share $\theta_c(h)$ of the less wealthy households implies that the stronger negative effect of patent protection on their consumption carries more weight (due to their higher marginal utility of consumption) in the social welfare function than the weaker negative effect on the wealthier households.

Proposition 4 *Under the lab-equipment innovation specification, wealth inequality reduces the optimal level of patent protection.*

Proof. Proven in text. ■

Suppose we consider the following simple parametric example: $\theta_{a,0}(h) = 1-\varepsilon$ for $h \in [0, 0.5]$ and $\theta_{a,0}(h) = 1+\varepsilon$ for $h \in [0.5, 1]$, where the parameter $\varepsilon \in (0,1)$ measures the degree of wealth inequality. In this case, (38) becomes

$$\int_0^1 \frac{\partial \ln \theta_c(h)}{\partial \mu} dh = -\underbrace{\frac{\partial a/c}{\partial \mu}}_{+} \frac{\frac{(\varepsilon\rho)^2}{\rho+\varphi/\mu}}{1-\left(\frac{\varepsilon\rho}{\rho+\varphi/\mu}\right)^2} < 0, \tag{41}$$

which is strictly negative (unless $\varepsilon = 0$) and decreasing in ε. Therefore, a higher degree of wealth inequality (i.e., a larger ε) strengthens the negative effect of patent protection. Proposition 5 derives the condition for the optimal level of patent protection and shows that it is decreasing in the degree of wealth inequality.

Proposition 5 *Under the lab-equipment innovation specification and the parametric example in which $\theta_{a,0}(h) = 1-\varepsilon$ for $h \in [0, 0.5]$ and $\theta_{a,0}(h) = 1+\varepsilon$ for $h \in [0.5, 1]$, the optimal level of patent protection μ^* is determined by*[10]

$$\frac{1}{\rho^2}\left[\left(\frac{\varphi}{\mu^*}+\rho\right)^2 - \frac{\rho}{\ln z}\left(\frac{\varphi}{\mu^*}+\rho\right)\right] = \varepsilon^2, \tag{42}$$

and it is decreasing in the degree of wealth inequality ε.

Proof. See Appendix A. ■

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[9] Recall that $\int_0^1 \theta_{a,0}(h) dh = 1$.

[10] Note that (42) simplifies to $\mu^* = \frac{\varphi}{\rho}\frac{\ln z}{1-\ln z}$ under $\varepsilon = 0$ as in $\theta_{a,0}(h) = 1$ for all $h \in [0,1]$ in (35).

3.3 General innovation specification

Finally, we consider our general innovation specification given by $\alpha \in (0, 1)$ in (12). In this case, the optimal level of patent protection is determined by the condition in (33), in which $\theta_c(h)$ is given in (29) and depends on c/a. Lemma 4 shows that the aggregate consumption-asset ratio c/a in (28) is decreasing in the level of patent protection μ.

Lemma 4 *Under the general innovation specification, c/a in (28) is decreasing in μ.*

Proof. See Appendix A. ■

Therefore, the optimal level of patent protection under the general innovation specification is different across households because $\theta_c(h)$ is once again decreasing (increasing) in μ for less wealthy (wealthier) households; i.e., $\partial \ln \theta_c(h) / \partial \mu < 0$ for $\theta_{a,0}(h) < 1$ ($\partial \ln \theta_c(h) / \partial \mu > 0$ for $\theta_{a,0}(h) > 1$) in (33). Proposition 6 shows that wealthier households prefer a higher level of patent protection than less wealthy households and that wealth inequality reduces the optimal level of patent protection. These implications are the same as in the lab-equipment specification but differ from the knowledge-driven specification, under which patent policy does not affect the aggregate consumption-asset ratio c/a because innovation uses only labor. However, when innovation uses also final output for R&D under the general innovation specification (and also under the lab-equipment specification), an increase in the level of patent protection reallocates some final output from consumption to R&D and reduces the aggregate consumption-asset ratio, which in turn affects the negative effect of patent protection on consumption differently across heterogeneous households. As a result, an unequal distribution of wealth across households reduces the optimal level of patent protection under the general innovation specification.

Proposition 6 *Under the general innovation specification, the utility-maximizing level of patent protection for household h is increasing in the household's wealth share $\theta_{a,0}(h)$. Furthermore, wealth inequality reduces the optimal level of patent protection.*

Proof. See Appendix A. ■

3.4 Quantitative analysis

In this section, we calibrate the model to see if an unequal distribution of wealth has a quantitatively significant effect on the optimal level of patent protection. In order to perform a more realistic quantitative analysis, we generalize the utility function to an isoelastic form as follows:[11]

$$u(h) = \int_0^\infty e^{-\rho t} \frac{[c_t(h)]^{1-\sigma} - 1}{1 - \sigma} dt, \tag{43}$$

which captures the log utility function in (1) as a special case when $\sigma \to 1$. In this case, the model features the following set of parameters $\{\sigma, \rho, \alpha, \mu, z, \varphi\}$. We follow the empirical

[11]See Appendix B for the detailed derivations under this generalized utility function.

estimate in Cashin and Unayama (2016) to set the intertemporal elasticity of substitution to 0.2 (i.e., $\sigma = 5$). We set the discount rate ρ to a conventional value of 0.05 and the degree of labor intensity $1-\alpha$ in the R&D process to an empirical value 0.184 (i.e., $\alpha = 0.816$) computed by Chu and Cozzi (2019). Then, we follow Jones and Williams (2000) to set the markup parameter μ to an empirical value of 1.25. Finally, we calibrate the remaining parameters $\{z, \varphi\}$ by targeting a long-run annual GDP growth rate g of 3% in the US and an arrival rate of innovation of 1/3 as in Acemoglu and Akcigit (2012). Table 1 summarizes the calibrated parameter values.

Table 1: Calibration

σ	ρ	α	μ	z	φ
5	0.050	0.816	1.250	1.095	3.989

Given the parameter values in Table 1, we simulate the optimal markup level μ^* under different degrees of wealth inequality. Once again, we consider the following simple parametric example: $\theta_{a,0}(h) = 1 - \varepsilon$ for $h \in [0, 0.5]$ and $\theta_{a,0}(h) = 1 + \varepsilon$ for $h \in [0.5, 1]$. In the US, the bottom 50% of population owns roughly 3% of total wealth, which corresponds to a value of 0.94 for ε (i.e., $(1 - \varepsilon)/2 = 0.03$). We consider the entire range of values $\varepsilon \in [0, 1]$ to explore how the degree of wealth inequality affects optimal patent protection. Figure 1 presents the simulation results. In the case of a completely equal wealth distribution (i.e., $\varepsilon = 0$), the optimal markup level is $\mu^*_{\varepsilon=0} = 1.292$. As ε increases, the optimal markup level decreases. At $\varepsilon = 0.94$, the optimal markup level decreases to $\mu^*_{\varepsilon=0.94} = 1.250$, which corresponds to the empirical markup in Table 1. Figure 2 presents the equilibrium growth rate g under different values for the markup and shows a significant decrease in the equilibrium growth rate of 0.44% from $\mu^*_{\varepsilon=0} = 1.292$ to $\mu^*_{\varepsilon=0.94} = 1.250$. In other words, moving from the current degree of wealth inequality in the US to a completely equal society would lead to an increase in the growth rate of almost 0.5%. Therefore, this simple numerical exercise shows that wealth inequality can have a quantitatively significant effect on optimal patent protection, innovation and economic growth.

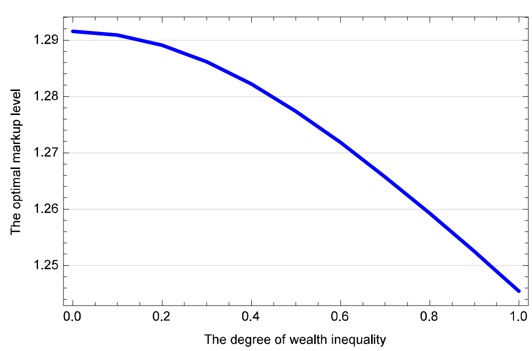

Figure 1: Markup and inequality

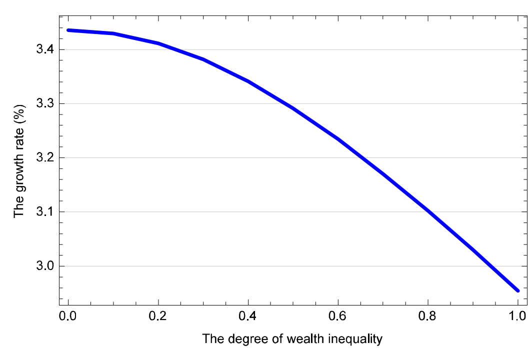

Figure 2: Growth and inequality

4 Conclusion

In this study, we have developed a Schumpeterian growth model with heterogeneous households to explore the conditions under which wealth inequality affects the optimal level of patent protection. Our results can be summarized as follows. Under the knowledge-driven specification, all households prefer the same level of patent protection. In contrast, under the lab-equipment specification, we find that wealthier households prefer stronger patent protection than less wealthy households and that higher wealth inequality reduces the optimal level of patent protection. To explore which of these two results are more robust, we also consider a general innovation specification that captures the two specifications as special cases. In this case, we continue to find that wealthier households prefer stronger patent protection and that wealth inequality reduces optimal patent protection. Therefore, all households preferring the same level of patent protection under the knowledge-driven specification is due to a knife-edge parameter condition.

References

[1] Acemoglu, D., and Akcigit, U., 2012. Intellectual property rights policy, competition and innovation. *Journal of the European Economic Association*, 10, 1-42.

[2] Aghion, P., and Howitt, P., 1992. A model of growth through creative destruction. *Econometrica*, 60, 323-351.

[3] Cashin, D., and Unayama, T., 2016. Measuring intertemporal substitution in consumption: Evidence from a VAT increase in Japan. *Review of Economics and Statistics*, 98, 285-297.

[4] Chu, A., 2010. Effects of patent policy on income and consumption inequality in an R&D-based growth model. *Southern Economic Journal*, 77, 336-350.

[5] Chu, A., 2022. Patent policy and economic growth: A survey. *Manchester School*, 90, 237-254.

[6] Chu, A., and Cozzi, G., 2018. Effects of patents versus R&D subsidies on income inequality. *Review of Economic Dynamics*, 29, 68-84.

[7] Chu, A., and Cozzi, G., 2019. Growth: Scale or market-size effects? *Economics Letters*, 178, 13-17.

[8] Chu, A., Cozzi, G., and Galli, S., 2012. Does intellectual monopoly stimulate or stifle innovation? *European Economic Review*, 56, 727-746.

[9] Chu, A., Furukawa, Y., Mallick, S., Peretto, P., and Wang, X., 2021. Dynamic effects of patent policy on innovation and inequality in a Schumpeterian economy. *Economic Theory*, 71, 1429-1465.

[10] Chu, A., Kou, Z., and Wang, X., 2022. Class struggle in a Schumpeterian economy. MPRA Paper 114907.

[11] Chu, A., and Pan, S., 2013. The escape-infringement effect of blocking patents on innovation and economic growth. *Macroeconomic Dynamics*, 17, 955-969.

[12] Cozzi, G., 2001. Inventing or spying? Implications for growth. *Journal of Economic Growth*, 6, 55-77.

[13] Cozzi, G., 2007. The Arrow effect under competitive R&D. *The B.E. Journal of Macroeconomics* (Contributions), 7, Article 2.

[14] Cozzi, G., Giordani, P., and Zamparelli, L., 2007. The refoundation of the symmetric equilibrium in Schumpeterian growth models. *Journal of Economic Theory*, 136, 788-797.

[15] Dinopoulos, E., and Segerstrom, P., 2010. Intellectual property rights, multinational firms and economic growth. *Journal of Development Economics*, 92, 13-27.

[16] Furukawa, Y., 2007. The protection of intellectual property rights and endogenous growth: Is stronger always better? *Journal of Economic Dynamics and Control*, 31, 3644-3670.

[17] Goh, A.-T., and Olivier, J., 2002. Optimal patent protection in a two-sector economy. *International Economic Review*, 43, 1191-1214.

[18] Grossman, G., and Helpman, E., 1991. Quality ladders in the theory of growth. *Review of Economic Studies*, 58, 43-61.

[19] Horii, R., and Iwaisako, T., 2007. Economic growth with imperfect protection of intellectual property rights. *Journal of Economics*, 90, 45-85.

[20] Iwaisako, T., and Futagami, K., 2003. Patent policy in an endogenous growth model. *Journal of Economics*, 78, 239-258.

[21] Jones, C., and Williams, J. 2000. Too much of a good thing? The economics of investment in R&D. *Journal of Economic Growth*, 5, 65-85.

[22] Judd, K., 1985. On the performance of patents. *Econometrica*, 53, 567-586.

[23] Kiedaisch, C., 2021. Growth and welfare effects of intellectual property rights when consumers differ in income. *Economic Theory*, 72, 1121-1170.

[24] Lerner, J., 2009. The empirical impact of intellectual property rights on innovation: Puzzles and clues. *American Economic Review*, 99, 343-348.

[25] Li, C.-W., 2001. On the policy implications of endogenous technological progress. *Economic Journal*, 111, C164-C179.

[26] Nordhaus, W., 1969. *Invention, Growth, and Welfare*. Cambridge, MA: The MIT Press.

[27] Ohki, K., 2023. Disruptive innovation by heterogeneous incumbents and economic growth: When do incumbents switch to new technology? *Journal of Mathematical Economics*, forthcoming.

[28] Qian, Y., 2007. Do national patent laws stimulate domestic innovation in a global patenting environment? A cross-country analysis of pharmaceutical patent protection, 1978-2002. *Review of Economics and Statistics*, 89, 436-453.

[29] Rivera-Batiz, L., and Romer, P., 1991. Economic integration and endogenous growth. *Quarterly Journal of Economics*, 106, 531-555.

[30] Romer, P., 1990. Endogenous technological change. *Journal of Political Economy*, 98, S71-S102.

[31] Royal Swedish Academy of Sciences, 2018. *Popular Science Background on the Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel: William Nordhaus and Paul Romer's Contribution to Integrating Nature and Knowledge into Economics.*

[32] Segerstrom, P., Anant, T., and Dinopoulos, E., 1990. A Schumpeterian model of the product life cycle. *American Economic Review*, 80, 1077-91.

[33] Solow, R., 1956. A contribution to the theory of economic growth. *Quarterly Journal of Economics*, 70, 65-94.

Appendix A: Proofs

Proof of Lemma 1. Taking the log of the profit-maximizing condition of R&D $(1-\alpha)\lambda_t v_t = w_t l_{r,t}$ and then differentiating it with respect to time yields

$$\frac{\dot{\lambda}_t}{\lambda_t} + \frac{\dot{v}_t}{v_t} = \frac{\dot{w}_t}{w_t} + \frac{\dot{l}_{r,t}}{l_{r,t}}. \tag{A1}$$

Using (10) and (17) yields $w_t = Z_t/\mu$. Then, we combine $\alpha/(1-\alpha) = R_t/(w_t l_{r,t})$ from (13) and (14) to obtain $R_t/Z_t = \alpha l_{r,t}/[\mu(1-\alpha)]$ as shown in (19). Substituting (19) into (12) yields

$$\lambda_t = \varphi \left(\frac{1}{\mu} \frac{\alpha}{1-\alpha} \right)^{\alpha} l_{r,t}. \tag{A2}$$

Taking the log of (A2) and then differentiating it with respect to time yields $\dot{\lambda}_t/\lambda_t = \dot{l}_{r,t}/l_{r,t}$. Substituting this condition into (A1), we obtain

$$\frac{\dot{v}_t}{v_t} = \frac{\dot{w}_t}{w_t} = \frac{\dot{Z}_t}{Z_t}, \tag{A3}$$

where the second equality uses $\dot{w}_t/w_t = \dot{Z}_t/Z_t$ from $w_t = Z_t/\mu$. Based on (A2) and the no-arbitrage condition $r_t v_t = \pi_t + \dot{v}_t - \lambda_t v_t$, (A3) can be rewritten as

$$r_t + \varphi \left(\frac{1}{\mu} \frac{\alpha}{1-\alpha} \right)^{\alpha} l_{r,t} - \frac{\pi_t}{v_t} = \frac{\dot{Z}_t}{Z_t}. \tag{A4}$$

Using (9) and (14) yields $\pi_t/v_t = (\mu-1)(1-\alpha)\lambda_t y_t/(\mu w_t l_{r,t})$ and then combining (10) obtains $\pi_t/v_t = (\mu-1)(1-\alpha)\lambda_t l_{x,t}/l_{r,t}$. Substituting this condition into (A4) and using (A2), we obtain

$$r_t + \varphi \left(\frac{1}{\mu} \frac{\alpha}{1-\alpha} \right)^{\alpha} l_{r,t} - \varphi(\mu-1)(1-\alpha) \left(\frac{1}{\mu} \frac{\alpha}{1-\alpha} \right)^{\alpha} (1-l_{r,t}) = \frac{\dot{Z}_t}{Z_t}, \tag{A5}$$

where we have used the resource constraint on labor $l_{x,t} = 1 - l_{r,t}$. We define a transformed variable $s_t \equiv c_t/Z_t$. Then, differentiating s_t with respect to time yields $\dot{s}_t/s_t = \dot{c}_t/c_t - \dot{Z}_t/Z_t$ and combining (4) obtains $r_t = \dot{s}_t/s_t + \dot{Z}_t/Z_t + \rho$. Substituting this condition into (A5) yields

$$\frac{\dot{s}_t}{s_t} = \varphi(\mu-1)(1-\alpha) \left(\frac{1}{\mu} \frac{\alpha}{1-\alpha} \right)^{\alpha} - \rho - \varphi[1+(\mu-1)(1-\alpha)] \left(\frac{1}{\mu} \frac{\alpha}{1-\alpha} \right)^{\alpha} l_{r,t}. \tag{A6}$$

Based on the market-clearing condition for final goods, we obtain $s_t = (y_t - R_t)/Z_t$. Substituting (17) and (19) into this condition, we obtain the following relationship between s_t and $l_{r,t}$:

$$s_t = 1 - \frac{\mu(1-\alpha)+\alpha}{\mu(1-\alpha)} l_{r,t}. \tag{A7}$$

Differentiating (A7) with respect to time yields $\dot{s}_t = -[\mu(1-\alpha)+\alpha]\dot{l}_{r,t}/[\mu(1-\alpha)]$ and then substituting it into (A6) obtains

$$\dot{l}_{r,t} = \frac{\mu(1-\alpha)s_t}{\mu(1-\alpha)+\alpha} \left\{ \varphi[1+(\mu-1)(1-\alpha)] \left(\frac{1}{\mu} \frac{\alpha}{1-\alpha} \right)^{\alpha} l_{r,t} - \varphi(\mu-1)(1-\alpha) \left(\frac{1}{\mu} \frac{\alpha}{1-\alpha} \right)^{\alpha} + \rho \right\}, \tag{A8}$$

which is a one-dimensional differential equation in $l_{r,t}$. Drawing $\dot{l}_{r,t}$ as a function of $l_{r,t}$ on phase diagram, one can easily show that the dynamics of $l_{r,t}$ is characterized by saddle-point stability such that $l_{r,t}$ must jump to the unique steady-state value l_r:

$$l_r = \frac{\varphi\left(\mu-1\right)\left(1-\alpha\right)\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha}-\rho}{\varphi\left[1+\left(\mu-1\right)\left(1-\alpha\right)\right]\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha}}. \tag{A9}$$

■

Proof of Lemma 2. Differentiating (A9) with respect to μ yields

$$\frac{dl_r}{d\mu} = \frac{\left(1-\alpha\right)\left[\rho+\varphi\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha}\right]-\rho\left[1+\left(\mu-1\right)\left(1-\alpha\right)\right]\frac{\alpha}{\mu}}{\varphi\left[1+\left(\mu-1\right)\left(1-\alpha\right)\right]^2\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha}}. \tag{A10}$$

From (A9), we obtain

$$l_r > 0 \Longleftrightarrow \varphi\left(1-\alpha\right)\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha} > \frac{\rho}{\left(\mu-1\right)}. \tag{A11}$$

Substituting (A11) into (A10) yields

$$\frac{dl_r}{d\mu} > \frac{\rho}{\varphi\mu\left(\mu-1\right)\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha}} > 0. \tag{A12}$$

Equation (A12) shows that l_r is increasing in μ. ■

Proof of Proposition 1. Substituting (A9) into (20) and then differentiating it with respect to μ yields

$$\frac{dg}{d\mu} = \frac{\left(1-\alpha\right)\ln z}{\left[1+\left(\mu-1\right)\left(1-\alpha\right)\right]^2}\left\{\rho-\underbrace{\varphi\left[\alpha\mu^2-\left(1+\alpha\right)\mu-\alpha^2(\mu-1)^2\right]\left(\frac{\alpha}{1-\alpha}\right)^{\alpha}\left(\frac{1}{\mu}\right)^{1+\alpha}}_{\equiv\Theta(\mu)}\right\}. \tag{A13}$$

Note the following properties: (a) $\Theta(1) = -\varphi\left[\alpha/\left(1-\alpha\right)\right]^{\alpha}$; (b) $\lim_{\mu\to\infty}\Theta\left(\mu\right)\to\infty$; (c) $\Theta\left(\mu\right)$ is a strictly increasing function, i.e.,

$$\frac{d\Theta\left(\mu\right)}{d\mu} = \alpha\varphi\left(\frac{\alpha}{1-\alpha}\right)^{\alpha}\left(\frac{1}{\mu}\right)^{2+\alpha}\left\{\left[\mu-\alpha^2(\mu-1)\right]+\mu\left(1-\alpha\right)\left[\mu\left(1-\alpha\right)+\alpha\right]+\alpha\right\} > 0.$$

Using these properties, we can graphically show that $\Theta\left(\mu\right)$ intersects ρ from below only once at some point $\mu > 1$, below (above) which $dg/d\mu > 0(< 0)$; see Figure 3. This result shows that g is an inverted-U function in μ.

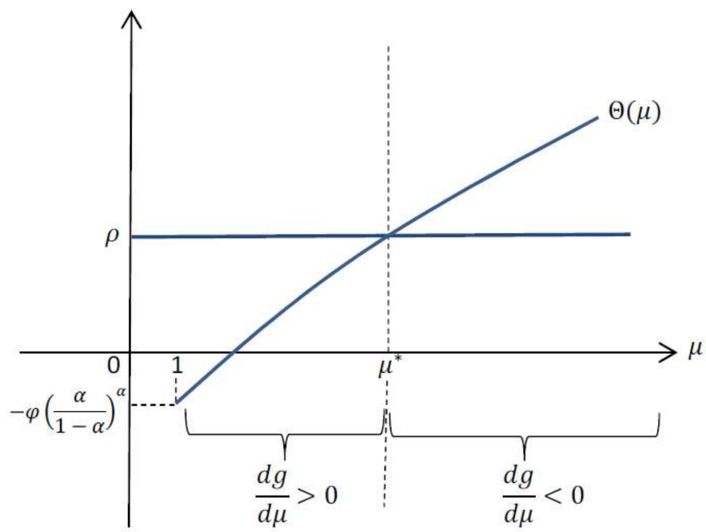

Figure 3

■

Proof of Lemma 3. Substituting (A9) into (A7), we obtain the initial level of aggregate consumption c_0 as

$$c_0 \equiv s_0 Z_0 = \frac{\rho + (1-\alpha)\,\varphi\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha}}{\varphi\mu\,(1-\alpha)\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha}}, \tag{A14}$$

where Z_0 is normalized to unity. Equation (A14) is identical to (32) in text. Differentiating (A14) with respect to μ yields

$$\frac{dc_0}{d\mu} = -\frac{\rho + \varphi\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha}}{\varphi\mu^2\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha}} < 0. \tag{A15}$$

Equation (A15) shows that c_0 is decreasing in μ. ■

Proof of Proposition 2. In this proof, we make use of the parameter $\alpha \to 0$, which renders the general innovation specification degenerate. First, using (A9), the steady-state equilibrium level of R&D labor l_r is given by

$$\lim_{\alpha\to 0} l_r = \frac{\mu-1}{\mu} - \frac{1}{\mu}\frac{\rho}{\varphi}\exp\left[\lim_{\alpha\to 0} -\frac{\ln\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)}{\frac{1}{\alpha}}\right] = \frac{\mu-1}{\mu} - \frac{1}{\mu}\frac{\rho}{\varphi}, \tag{A16}$$

where the second equality uses the L'Hôpital's rule. Equation (A16) shows that l_r is also increasing in μ as in the case of the general innovation specification. Using (A2) and (A9) derives the arrival rate of innovation:

$$\lim_{\alpha \to 0} \lambda = \varphi(\frac{\mu - 1}{\mu}) \exp\left[\lim_{\alpha \to 0} \frac{\ln\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)}{\frac{1}{\alpha}}\right] - \frac{\rho}{\mu} = \varphi\left(\frac{\mu - 1}{\mu}\right) - \frac{\rho}{\mu}, \tag{A17}$$

where the second equality also uses the L'Hôpital's rule. Substituting (A17) into (16), under the knowledge-driven innovation specification, the steady-state growth rate g is given by

$$g = \left[\varphi\left(\frac{\mu - 1}{\mu}\right) - \frac{\rho}{\mu}\right] \ln z, \tag{A18}$$

Equation (A18) shows that g is increasing in μ. Using (A14) and the L'Hôpital's rule, we can derive the initial level of aggregate consumption c_0 is given by:

$$\lim_{\alpha \to 0} c_0 = \frac{1}{\mu}\left\{1 + \frac{\rho}{\varphi}\exp\left[\lim_{\alpha \to 0} - \frac{\ln\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)}{\frac{1}{\alpha}}\right]\right\} = \frac{1}{\mu}\left(1 + \frac{\rho}{\varphi}\right). \tag{A19}$$

Equation (A19) shows that c_0 is also decreasing in μ as in the case of the general innovation specification. As for the aggregate consumption-asset ratio, using (28) and the L'Hôpital's rule yields

$$\lim_{\alpha \to 0} \frac{c}{a} = \frac{\mu}{\mu - 1}\left[\rho + \lim_{\alpha \to 0} \underbrace{\varphi\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha} l_r}_{=\lambda}\right] = \rho + \varphi. \tag{A20}$$

Combining (A20) and (29) yields $\theta_c(h) = 1 - \rho\left[1 - \theta_{a,0}(h)\right] / (\rho + \varphi)$. Substituting this condition, (A18) and (A19) into (31) and then differentiating it with respect to μ yields

$$\rho\frac{\partial u(h)}{\partial \mu} = \frac{1}{\mu^2}\left[\underbrace{\left(1 + \frac{\varphi}{\rho}\right)\ln z - \mu}_{\equiv \Phi}\right]. \tag{A21}$$

The utility-maximizing level of patent protection for household h requires $\Phi = 0$. Then, we can derive

$$\mu^*(h) = \left(1 + \frac{\varphi}{\rho}\right)\ln z. \tag{A22}$$

Equation (A22) shows that $\mu^*(h) = \mu^*$ across all households h because it is independent of $\theta_{a,0}(h)$. As a result, $\mu^*(h) = \mu^*$ is also the optimal level of patent protection. ■

Proof of Proposition 3. In this proof, we make use of the parameter $\alpha \to 1$, which renders the general innovation specification degenerate. First, using (A9), the steady-state equilibrium level of R&D labor l_r is given by

$$\lim_{\alpha \to 1} l_r = 0. \tag{A23}$$

Using (A2) and (A9) derives the arrival rate of innovation:

$$\lim_{\alpha\to 1}\lambda = \varphi(\frac{\mu-1}{\mu})\exp\left[\lim_{\alpha\to 1}\frac{\ln\left(\frac{\alpha}{1-\alpha}\right)}{\frac{1}{1-\alpha}}\right] - \rho = \varphi\left(\frac{\mu-1}{\mu}\right) - \rho, \tag{A24}$$

where the second equality uses the L'Hôpital's rule. Substituting (A24) into (16), under the lab-equipment innovation specification, the steady-state growth rate g is given by

$$g = \left[\varphi\left(\frac{\mu-1}{\mu}\right) - \rho\right]\ln z, \tag{A25}$$

Equation (A25) shows that g is increasing in μ. Using (A14) and the L'Hôpital's rule, we can derive the initial level of aggregate consumption c_0 is given by:

$$\lim_{\alpha\to 1} c_0 = \frac{1}{\mu} + \frac{\rho}{\varphi}\exp\left[\lim_{\alpha\to 1}\frac{\ln\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)}{\frac{1}{1-\alpha}}\right] = \frac{1}{\mu} + \frac{\rho}{\varphi}. \tag{A26}$$

Equation (A26) shows that c_0 is also decreasing in μ as the general innovation specification. As for the aggregate consumption-asset ratio, using (28) and the L'Hôpital's rule yields

$$\lim_{\alpha\to 1}\frac{c}{a} = \varphi\left\{\frac{1}{\mu} + \frac{\rho}{\varphi}\exp\left[\lim_{\alpha\to 1}\frac{\ln\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)}{\frac{1}{1-\alpha}}\right]\right\} = \rho + \frac{\varphi}{\mu}. \tag{A27}$$

Combining (A27) and (29) yields $\theta_c(h) = 1 - \rho\left[1-\theta_{a,0}(h)\right]/\left(\rho+\varphi/\mu\right)$. Substituting this condition, (A25) and (A26) into (31) and differentiating it with respect to μ yields

$$\rho\frac{\partial u(h)}{\partial\mu} = \frac{\varphi}{\mu^2\left\{\left(\rho+\frac{\varphi}{\mu}\right) - \rho\left[1-\theta_{a,0}(h)\right]\right\}}\left\{\underbrace{1 - \left[\frac{\varphi}{\rho}\left(\frac{1}{\mu}+\frac{\rho}{\varphi}\right) - \left[1-\theta_{a,0}(h)\right]\right]\ln z}_{\equiv\Omega}\right\}. \tag{A28}$$

The utility-maximizing level of patent protection for household h requires $\Omega = 0$. Then, we can derive

$$\mu^*(h) = \frac{\varphi}{\rho}\frac{\ln z}{1-\theta_{a,0}(h)\ln z}. \tag{A29}$$

■

Proof of Proposition 5. There are two types of households. Type 1 has $\theta_{a,0}(h) = 1-\varepsilon$ for $h \in [0, 0.5]$ whereas type 2 has $\theta_{a,0}(h) = 1+\varepsilon$ for $h \in [0.5, 1]$. As a result, (29) can be rewritten as $\theta_c(h) = 1 - \varepsilon\rho/\left(\rho+\varphi/\mu\right)$ for $h \in [0, 0.5]$ and $\theta_c(h) = 1 + \varepsilon\rho/\left(\rho+\varphi/\mu\right)$ for $h \in [0.5, 1]$. Substituting these condition into (31), we obtain the welfare functions of two types respectively:

$$\rho u(h) = \ln c_0 + \ln\left(1 - \frac{\varepsilon\rho}{\rho+\frac{\varphi}{\mu}}\right) + \frac{g}{\rho} \text{ for } h \in [0, 0.5], \tag{A30}$$

$$\rho u(h) = \ln c_0 + \ln\left(1 + \frac{\varepsilon\rho}{\rho + \frac{\varphi}{\mu}}\right) + \frac{g}{\rho} \text{ for } h \in [0.5, 1]. \tag{A31}$$

We substitute (A30) and (A31) into (36) to derive the social welfare function:

$$\rho U = \ln c_0 + 0.5\ln\left(1 - \frac{\varepsilon\rho}{\rho + \frac{\varphi}{\mu}}\right) + 0.5\ln\left(1 + \frac{\varepsilon\rho}{\rho + \frac{\varphi}{\mu}}\right) + \frac{g}{\rho}. \tag{A32}$$

Substituting (A25) and (A26) into (A32) and then differentiating it with respect to μ yields

$$\rho\frac{\partial U}{\partial \mu} = \frac{\varphi \ln z}{\rho\mu^2\left[\left(\rho + \frac{\varphi}{\mu}\right)^2 - (\varepsilon\rho)^2\right]}\left\{(\varepsilon\rho)^2 - \left[\left(\rho + \frac{\varphi}{\mu}\right)^2 - \frac{\rho}{\ln z}\left(\rho + \frac{\varphi}{\mu}\right)\right]\right\}. \tag{A33}$$

Based on (A33), we know that the optimal level of patent protection μ^* is determined by

$$\frac{1}{\rho^2}\left[\underbrace{\left(\frac{\varphi}{\mu^*} + \rho\right)^2}_{\equiv\chi^2} - \frac{\rho}{\ln z}\underbrace{\left(\frac{\varphi}{\mu^*} + \rho\right)}_{\equiv\chi}\right] = \varepsilon^2. \tag{A34}$$

The left-hand side (LHS) of (A34) is increasing in χ because $\varepsilon > 0 \Longleftrightarrow \chi > \rho/\ln z$ whereas the right-hand side (RHS) of (A34) ε^2 is independent of χ. Therefore, we can find the optimal level of χ, which is increasing in ε. Based on $\chi \equiv \varphi/\mu^* + \rho$, we know μ^* is decreasing in χ. As a result, μ^* is decreasing in ε. ■

Proof of Lemma 4. The market-clearing condition for final goods is $y_t = c_t + R_t$. Using this condition, one can derive the following aggregate consumption-asset ratio:

$$\frac{c_t}{a_t} = \frac{\frac{y_t}{Z_t}}{\frac{a_t}{Z_t}} - \frac{\frac{R_t}{Z_t}}{\frac{a_t}{Z_t}} = \frac{l_{x,t}}{\frac{a_t}{Z_t}} - \frac{\frac{\alpha}{1-\alpha}\frac{l_{r,t}}{\mu}}{\frac{a_t}{Z_t}}, \tag{A35}$$

where the second equality uses (17) and (19). We know that the value of assets equals the value of inventions such that $a_t = v_t$. The balanced-growth values of an innovation is $v_t = \pi_t/(\rho + \lambda)$ and the combining (9) and (A2) yields

$$\frac{a_t}{Z_t} = \frac{a}{Z} = \frac{\frac{\mu-1}{\mu}l_x}{\rho + \varphi\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^\alpha l_r}. \tag{A36}$$

Substituting (A36) into (A35) yields

$$\frac{c_t}{a_t} = \frac{c}{a} = \frac{1}{\mu - 1}\left(\mu - \frac{\alpha}{1-\alpha}\frac{l_r}{1-l_r}\right)\left[\rho + \varphi\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^\alpha l_r\right]. \tag{A37}$$

Equation (A37) is identical to (28). Substituting (A9) into (A37) and differentiating it with respect to μ yields

$$\frac{d\left(\frac{c}{a}\right)}{d\mu}=-\frac{\alpha\left(\frac{c}{a}\right)}{\mu\left[\frac{\rho}{1-\alpha}+\varphi\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha}\right]}<0. \tag{A38}$$

Equation (A38) shows that c/a is decreasing in μ. ■

Proof of Proposition 6. The condition that determines the utility-maximizing level of patent protection for household h is given by

$$\rho\frac{\partial u\left(h\right)}{\partial\mu}=-\rho\underbrace{\frac{\partial a/c}{\partial\mu}}_{+}\frac{1-\theta_{a,0}(h)}{\theta_{c}\left(h\right)}+\underbrace{\frac{\partial\ln c_{0}}{\partial\mu}}_{-}+\frac{1}{\rho}\underbrace{\frac{\partial g}{\partial\mu}}_{+/-}, \tag{A39}$$

where $\partial(a/c)/\partial\mu$ is positive from Lemma 4. Therefore, the first term on the right-hand side of (A39) is negative for less wealthy households (i.e., $\theta_{a,0}(h)<1$) and positive for wealthier households (i.e., $\theta_{a,0}(h)>1$), implying that wealthier households prefer a stronger level of patent protection. As before, we consider a linear aggregate of the households' utility functions given by

$$U\equiv\int_{0}^{1}u\left(h\right)dh=\frac{1}{\rho}\left[\int_{0}^{1}\ln\theta_{c}\left(h\right)dh+\ln c_{0}+\frac{g}{\rho}\right]. \tag{A40}$$

Then, the condition that determines the optimal level of patent protection μ is given by

$$\rho\frac{\partial U}{\partial\mu}=\int_{0}^{1}\frac{\partial\ln\theta_{c}\left(h\right)}{\partial\mu}dh+\underbrace{\frac{\partial\ln c_{0}}{\partial\mu}}_{-}+\frac{1}{\rho}\underbrace{\frac{\partial g}{\partial\mu}}_{+/-}. \tag{A41}$$

The first term on the right-hand side of (A41) is given by

$$\int_{0}^{1}\frac{\partial\ln\theta_{c}\left(h\right)}{\partial\mu}dh=-\rho\underbrace{\frac{\partial a/c}{\partial\mu}}_{+}\int_{0}^{1}\frac{1-\theta_{a,0}(h)}{\theta_{c}\left(h\right)}dh=-\rho\underbrace{\frac{\partial a/c}{\partial\mu}}_{+}\int_{0}^{1}\left[\frac{1}{1-\theta_{a,0}(h)}-\rho\frac{a}{c}\right]^{-1}dh, \tag{A42}$$

where $\theta_{c}\left(h\right)$ is given by (29) and a/c is given by (28) and increasing in μ from Lemma 4. Finally, from Jensen's inequality, we have

$$\int_{0}^{1}\left[\frac{1}{1-\theta_{a,0}(h)}-\rho\frac{a}{c}\right]^{-1}dh>\left[\frac{1}{\int_{0}^{1}[1-\theta_{a,0}(h)]dh}-\rho\frac{a}{c}\right]^{-1}=0, \tag{A43}$$

which together with (A42) implies that

$$\int_{0}^{1}\frac{\partial\ln\theta_{c}\left(h\right)}{\partial\mu}dh<0 \tag{A44}$$

unless $\theta_{a,0}(h)=1$ for all h. Therefore, wealth inequality gives rise to an additional negative effect of patent protection on social welfare. ■

Appendix B: The generalized utility function

This appendix presents the key equilibrium conditions for the model under the isoelastic utility function in (43). Equation (3) can be revised as follows:

$$\frac{\dot{c}_t(h)}{c_t(h)} = \frac{1}{\sigma}\left(r_t - \rho\right). \tag{B1}$$

Therefore, the growth rate of aggregate consumption is given by

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma}\left(r_t - \rho\right). \tag{B2}$$

Appendix A shows $\dot{v}_t/v_t = \dot{c}_t/c_t$ on the balanced-growth path. Substituting this condition into (11) and using (B2), we obtain

$$\frac{\rho}{\sigma} + \left(\frac{\sigma-1}{\sigma}\right) r_t + \lambda_t = \frac{\pi_t}{v_t}. \tag{B3}$$

Combining (9), (10) and (14) yields $\pi_t/v_t = (\mu-1)(1-\alpha)\lambda_t l_{x,t}/l_{r,t}$. Using this condition and $\lambda_t = \varphi\left\{\alpha/\left[\mu(1-\alpha)\right]\right\}^{\alpha} l_{r,t}$ from (20), (B3) can be rewritten as

$$\frac{\rho}{\sigma} + \left(\frac{\sigma-1}{\sigma}\right) r_t + \varphi\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha} l_{r,t} = (\mu-1)(1-\alpha)\varphi\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha}(1-l_{r,t}), \tag{B4}$$

where we have used the resource constraint on labor $l_{x,t} = 1 - l_{r,t}$. Moreover, on the balanced-growth path, $\dot{c}_t/c_t = \dot{Z}_t/Z_t$ implies that

$$r_t = \sigma\varphi\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha} l_{r,t}\ln z + \rho, \tag{B5}$$

where we have used (20) and (B2). We substitute (B5) into (B4) to derive the equilibrium l_r under the generalized utility function:

$$l_r = \frac{\varphi(\mu-1)(1-\alpha)\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha} - \rho}{\varphi\left[1+(\sigma-1)\ln z+(\mu-1)(1-\alpha)\right]\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha}}. \tag{B6}$$

As for wealth distribution, we firstly substitute (B2) into (21) by considering $\dot{a}_t/a_t = \dot{c}_t/c_t$ and then (26) can be revised as follows

$$\frac{c_t - w_t}{a_t} = \left(\frac{\sigma-1}{\sigma}\right) r_t + \frac{\rho}{\sigma}. \tag{B7}$$

Substituting (B5) into (B7) and using (B6), (B7) can be rewritten as

$$\frac{c_t - w_t}{a_t} = \frac{\varphi(\sigma-1)(\mu-1)(1-\alpha)\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha}\ln z + \left[1+(\mu-1)(1-\alpha)\right]\rho}{1+(\sigma-1)\ln z+(\mu-1)(1-\alpha)}. \tag{B8}$$

Under this generalized utility function, we know that $\theta_{c,t}(h) = \theta_c(h)$ still holds for all t. Given this condition and using (B8), (27) can be revised as follows

$$\dot{\theta}_{a,t}(h)=\frac{\varphi\left(\sigma-1\right)\left(\mu-1\right)\left(1-\alpha\right)\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha}\ln z+\left[1+\left(\mu-1\right)\left(1-\alpha\right)\right]\rho}{1+\left(\sigma-1\right)\ln z+\left(\mu-1\right)\left(1-\alpha\right)}\left[\theta_{a,t}(h)-1\right]-\left[\theta_{c}(h)-1\right]\frac{c_t}{a_t}, \tag{B9}$$

where c_t/a_t can be derived as

$$\frac{c_t}{a_t}=\frac{c}{a}=\frac{1}{\mu-1}\left(\mu-\frac{\alpha}{1-\alpha}\frac{l_r}{1-l_r}\right)\left[\rho+\varphi\left[1+\left(\sigma-1\right)\ln z\right]\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha}l_r\right], \tag{B10}$$

for all t. As a result, we know $\dot{\theta}_{a,t}(h)=0$ for all t with long-run stability. Imposing $\dot{\theta}_{a,t}(h)=0$ on (B9) yields the steady-state value of $\theta_{c,t}(h)$ given by

$$\theta_{c,t}(h)=\theta_{c}(h)=1-\frac{\varphi\left(\sigma-1\right)\left(\mu-1\right)\left(1-\alpha\right)\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha}\ln z+\left[1+\left(\mu-1\right)\left(1-\alpha\right)\right]\rho}{1+\left(\sigma-1\right)\ln z+\left(\mu-1\right)\left(1-\alpha\right)}\frac{\left[1-\theta_{a,0}(h)\right]}{c/a}. \tag{B11}$$

Finally, we impose balanced growth on (43) to derive the welfare function of household h as

$$u(h)=\frac{1}{1-\sigma}\left\{\frac{\left[c_0\ \theta_c(h)\right]^{1-\sigma}}{\rho-\left(1-\sigma\right)g}-\frac{1}{\rho}\right\}, \tag{B12}$$

where we have used $c_0\left(h\right)=\theta_c\left(h\right)c_0$. Then, we assume $\rho>(1-\sigma)g$ to ensure that utility is bounded. The market-clearing condition for final goods implies $y_t/Z_t=\left(c_t+R_t\right)/Z_t$. Using this condition, (17) and (19), we obtain the initial level of aggregate consumption c_0 as

$$c_0=\frac{\left(1-\alpha\right)\left[\mu\left(1-\alpha\right)+\alpha+\mu\left(\sigma-1\right)\ln z\right]\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha}+\left[\mu\left(1-\alpha\right)+\alpha\right]\rho}{\mu\varphi\left(1-\alpha\right)\left[1+\left(\sigma-1\right)\ln z+\left(\mu-1\right)\left(1-\alpha\right)\right]\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha}}, \tag{B13}$$

where we have used (B6) and Z_0 is normalized to unity. Similarly, we consider two types of households: type 1 has $\theta_{a,0}(h)=1-\varepsilon$ for $h\in[0,0.5]$ and type 2 has $\theta_{a,0}(h)=1+\varepsilon$ for $h\in[0.5,1]$. The social welfare function is given by

$$U=\frac{0.5}{1-\sigma}\left\{\frac{\left[c_0\ \theta_{1c}(h)\right]^{1-\sigma}}{\rho-\left(1-\sigma\right)g}-\frac{1}{\rho}\right\}+\frac{0.5}{1-\sigma}\left\{\frac{\left[c_0\ \theta_{2c}(h)\right]^{1-\sigma}}{\rho-\left(1-\sigma\right)g}-\frac{1}{\rho}\right\}, \tag{B14}$$

where

$$\theta_{1c}(h)=1-\frac{\varphi\left(\sigma-1\right)\left(\mu-1\right)\left(1-\alpha\right)\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha}\ln z+\left[1+\left(\mu-1\right)\left(1-\alpha\right)\right]\rho}{1+\left(\sigma-1\right)\ln z+\left(\mu-1\right)\left(1-\alpha\right)}\frac{\varepsilon}{c/a}\text{ for }h\in\left[0,0.5\right],$$

$$\theta_{2c}(h)=1+\frac{\varphi\left(\sigma-1\right)\left(\mu-1\right)\left(1-\alpha\right)\left(\frac{1}{\mu}\frac{\alpha}{1-\alpha}\right)^{\alpha}\ln z+\left[1+\left(\mu-1\right)\left(1-\alpha\right)\right]\rho}{1+\left(\sigma-1\right)\ln z+\left(\mu-1\right)\left(1-\alpha\right)}\frac{\varepsilon}{c/a}\text{ for }h\in\left[0.5,1\right].$$