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Short communication

Experiments versus distributions of posteriors[☆]

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ABSTRACT

A fundamental result in Bayesian persuasion and information design states that a distribution of posterior beliefs can be induced by an experiment if and only if the posterior beliefs average to the prior belief. We present a general version of this result that applies to infinite state and signal spaces.

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1. Introduction

An agent must choose an action without knowing a payoff-relevant state of the world. The agent has access to an experiment, which draws a signal conditionally on the state. Observing the signal allows to update the prior to a posterior belief. Because the signal is stochastic, an experiment induces a distribution of posteriors. The distribution must average to the prior to be consistent with Bayesian updating. A fundamental result states that the converse is also true: for any distribution of posteriors that is Bayes plausible in this sense, there exists an experiment that induces it. This result often simplifies problems in Bayesian persuasion and information design in which the experiment is a choice variable.[1] Because experiments and Bayes-plausible distributions of posteriors are equivalent, one can directly work with the latter rather than deducing posteriors from experiments.

In their seminal paper, Kamenica and Gentzkow (2011) prove the result for a finite state space and experiments with finitely many signals (see our literature review below for earlier versions of the result obtained in other contexts). The proof follows from Bayes' Theorem. Specifically, just as the posterior probability of a state can be computed from the prior and the conditional distributions of signals, one can reverse this computation and derive conditional distributions of signals that result in a given Bayes-plausible distribution of posteriors.

The purpose of this paper is to demonstrate the equivalence of experiments and Bayes-plausible distributions of posteriors for general, possibly infinite state and signal spaces, for which there seems to be no published proof in the literature.[2] With general spaces, posteriors resulting from a given experiment, as well as an experiment that induces a given distribution of posteriors, are obtained as regular conditional distributions. Although the logic is similar to the one for finite spaces, there are a number of technical issues that do not arise in the finite case. As a consequence, one needs assumptions on the state space and the signal space as highlighted below.

Versions of the fundamental result have also been obtained in other contexts. In his comparison of experiments, Blackwell (1951) demonstrates for a finite state space that every experiment induces a Bayes-plausible distribution of posteriors and every such distribution can be induced by an experiment. Specifically, the "standard experiments" in Blackwell (1951) correspond to Bayes-plausible distributions of posteriors with respect to a uniform prior, but uniformity is unimportant for his observation. In the textbook by Togersen (1991) on comparison of experiments, the result appears as Proposition 7.2.1 assuming a finite state space. The result is also known as the "splitting lemma" in the context of repeated games with incomplete information, where players signal information through their actions (Aumann and Maschler, 1995).[3] In the textbook by Mertens et al. (2015), Proposition V.1.2 shows for a finite state space that to every distribution of posteriors averaging to the prior, there exists a joint

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1 See Kamenica (2019) and Bergemann and Morris (2019) for literature surveys on Bayesian persuasion and information design.

2 See Section 3 of the Online Appendix of Kamenica and Gentzkow (2011) for a discussion of Bayesian persuasion beyond the finite case. See also our literature review below.

3 See Forges (2020) for an overview of conceptual relationships between repeated games and Bayesian persuasion or information design.

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distribution of posteriors and states that induces the given distribution of posteriors. Said joint distribution can be interpreted as an experiment. A different version of the splitting lemma for a particular infinite state space is included in Proposition V.3.40 in Mertens et al. (2015). As it is tailored to repeated games, some work would be required to adapt it to the setting of this paper.[4]

In the literature on Bayesian persuasion and information design, working with an infinite state space (e.g., an interval on the real line) often proves convenient for other aspects of the analysis (see, e.g., Guo and Shmaya, 2019; Dworczak and Martini, 2019; Lipnowski and Ravid, 2020; Ravid et al., 2022). For instance, in Terstiege and Wasser (2020), we study experiments on an infinite state space for a buyer under the constraint that the seller must not add information. We identify optimal experiments that draw a deterministic signal conditionally on the state, resulting in a partition of the state space. With a finite state space, by contrast, such optimal experiments would draw a stochastic signal. Moreover, also models with a finite state space may require an infinite signal space and thus may call for the technical apparatus considered here. For example, in Terstiege and Wasser (2022), we study a game in which bidders disclose information to an auctioneer with the aim of influencing the auction design. As we show, an equilibrium may only exist for an infinite signal space even though the state space is finite. Boleslavsky and Cotton (2015) consider another game in Bayesian persuasion with a finite state space that only has equilibria with an infinite signal space.

2. Review of the finite case

We first review the equivalence of experiments and Bayes-plausible distributions of posteriors for finitely many states and signals (Kamenica and Gentzkow, 2011).

Let Ω be a finite set of possible states of the world. Let μ_0 be a probability distribution on Ω that represents the prior belief. That is, under the prior, state $\omega \in \Omega$ is assumed to obtain with probability $\mu_0(\omega)$. Without loss of generality, let $\mu_0(\omega) > 0$ for all ω. An experiment consists of a finite set of signals S and, for each state $\omega \in \Omega$, a probability distribution $\sigma(\cdot|\omega)$ on S. The interpretation is that signal $s \in S$ is drawn with probability $\sigma(s|\omega)$ if state ω obtains. Denote the unconditional probability of signal s by

$$\bar{\sigma}(s) = \sum_{\omega\in\Omega} \sigma(s|\omega)\mu_0(\omega).$$

The posterior belief after any possible signal is determined by Bayes' Theorem: if signal s with $\bar{\sigma}(s) > 0$ was drawn, the posterior belief is represented by the distribution $\rho(\cdot|s)$ on Ω that assigns to state ω the probability

$$\rho(\omega|s) = \frac{\sigma(s|\omega)\mu_0(\omega)}{\bar{\sigma}(s)}.$$

Because the signal is stochastic, the experiment induces a distribution τ of posteriors (which has finite support $\mathrm{Supp}(\tau)$ as the set of signals S is finite). Specifically, the experiment results in posterior μ with probability

$$\tau(\mu) = \sum_{s\in S:\rho(\cdot|s)=\mu} \bar{\sigma}(s).$$

Simple algebra shows that τ averages to the prior: for all $\omega \in \Omega$, it holds that

$$\sum_{\mu\in\mathrm{Supp}(\tau)} \mu(\omega)\tau(\mu) = \sum_{s\in\mathrm{Supp}(\bar{\sigma})} \rho(\omega|s)\bar{\sigma}(s) = \sum_{s\in S} \sigma(s|\omega)\mu_0(\omega) = \mu_0(\omega),$$

an immediate consequence of Bayesian updating. This property of τ is called "Bayes plausibility".

We now reverse the reasoning. Let τ be any (finitely supported) distribution of posteriors that satisfies Bayes plausibility, that is,

$$\sum_{\mu\in\mathrm{Supp}(\tau)} \mu(\omega)\tau(\mu) = \mu_0(\omega)$$

for all states $\omega \in \Omega$. Then we can define an experiment that draws signal $s = \mu$ from the set of signals $S = \mathrm{Supp}(\tau)$ with probability

$$\sigma(\mu|\omega) = \frac{\mu(\omega)\tau(\mu)}{\mu_0(\omega)}$$

if state ω obtains. Accordingly, signal μ is drawn with unconditional probability $\bar{\sigma}(\mu) = \tau(\mu)$ and, by Bayes' Theorem, results in posterior $\rho(\cdot|\mu) = \mu$. The experiment thus induces the distribution of posteriors τ.

3. The general case

We now show the equivalence of experiments and Bayes-plausible distributions of posteriors for general, possibly infinite state and signal spaces. Posteriors resulting from a given experiment, as well as the experiment that induces a given distribution of posteriors, are then obtained as regular conditional distributions, and Bayes' Theorem is replaced by theorems on existence and uniqueness of a regular conditional distribution. We first introduce the mathematical concepts.

3.1. Preliminaries

We endow any topological space H with the Borel σ-algebra, denoted by $\mathcal{B}(H)$.[5] We denote the set of all Borel probability measures on H by ΔH, and we endow ΔH with the weak* topology. If H is compact and metrizable, then so is ΔH (Aliprantis and Border, 2006, Thm. 15.11).

Given two topological spaces E and F, a map $\kappa : E \times \mathcal{B}(F) \to [0, 1]$ is called a *Markov kernel* from E to F if

(i) $e \mapsto \kappa(e, B)$ is measurable for any $B \in \mathcal{B}(F)$;
(ii) $B \mapsto \kappa(e, B)$ is an element of ΔF for any $e \in E$.

Let $(A, \mathcal{B}(A), \mathbb{P})$ be a probability space. Let X and Y be random variables on $(A, \mathcal{B}(A), \mathbb{P})$ with values in E and F, respectively. Let $\chi \in \Delta E$ be the distribution of X. A Markov kernel κ from E to F is called a *regular conditional distribution* of Y given X if

$$\mathbb{P}[X^{-1}(B_E) \cap Y^{-1}(B_F)] = \int_{B_E} \kappa(e, B_F)\mathrm{d}\chi(e) \quad \forall B_E \in \mathcal{B}(E), \forall B_F \in \mathcal{B}(F).$$

A regular conditional distribution exists, and $\kappa(e, \cdot)$ is unique for χ-almost all e, if F is a Polish space (see, e.g., Klenke, 2020, Thms. 8.37 and 8.38), and so in particular if F is compact and metrizable.

3.2. Experiments versus distributions of posteriors

Let Ω be the state space, which we require to be compact and metrizable. The prior belief is denoted by $\mu_0 \in \Delta\Omega$. That is, under the prior belief, the state is assumed to lie in $B_\Omega \in \mathcal{B}(\Omega)$ with probability $\mu_0(B_\Omega)$. An *experiment*, denoted by (S, σ), consists of a compact, metrizable signal space S and a Markov kernel σ from Ω to S. Thus, an experiment draws a signal from the probability measure $\sigma(\omega, \cdot) \in \Delta S$ if the state is $\omega \in \Omega$.

[4] Proposition V.3.40 in Mertens et al. (2015) shows that for certain maps from priors to distributions of posteriors that average to the prior, there is a corresponding map from priors to experiments that induce the distributions of posteriors. The proof refers to an exercise.

[5] In Section 3.2, we will focus on compact metrizable spaces.

Denote the corresponding unconditional probability measure by $\bar{\sigma} \in \Delta S$, where

$$\bar{\sigma}(B_S) = \int_{\Omega} \sigma(\omega, B_S) \mathrm{d}\mu_0(\omega) \quad \forall B_S \in \mathcal{B}(S).$$

The posterior belief after any signal is determined by a regular conditional distribution. Given any experiment (S, σ), consider the probability space $(\Omega \times S, \mathcal{B}(\Omega \times S), \mathbb{P})$ with $\mathbb{P}$ such that

$$\mathbb{P}[B_\Omega \times B_S] = \int_{B_\Omega} \sigma(\omega, B_S) \mathrm{d}\mu_0(\omega) \quad \forall B_\Omega \in \mathcal{B}(\Omega), \forall B_S \in \mathcal{B}(S).$$

Since Ω and S are compact and metrizable, $\mathbb{P}$ is uniquely defined by its values on $\mathcal{B}(\Omega) \times \mathcal{B}(S)$ (see Aliprantis and Border, 2006, Thms. 4.44 and 10.10). Let Y be the projection map from $\Omega \times S$ to Ω, and let X be the projection map from $\Omega \times S$ to S. Note that the marginal of $\mathbb{P}$ on Ω (i.e., the distribution of Y) is μ_0 and the marginal of $\mathbb{P}$ on S (i.e., the distribution of X) is $\bar{\sigma}$. Because Ω is compact and metrizable, there exists a regular conditional distribution ρ of Y given X, which is unique almost everywhere. Hence,

$$\int_{B_S} \rho(s, B_\Omega) \mathrm{d}\bar{\sigma}(s) = \int_{B_\Omega} \sigma(\omega, B_S) \mathrm{d}\mu_0(\omega) \quad \forall B_\Omega \in \mathcal{B}(\Omega), \forall B_S \in \mathcal{B}(S). \tag{1}$$

The probability measure $\rho(s, \cdot) \in \Delta\Omega$ represents the posterior belief after signal s.

We call any probability measure $\tau \in \Delta\Delta\Omega$ a *distribution of posteriors*. Because the signal is stochastic, the experiment induces a distribution of posteriors, which we state next. We have

$$\{s \in S : \rho(s, \cdot) \in B_{\Delta\Omega}\} \in \mathcal{B}(S) \quad \forall B_{\Delta\Omega} \in \mathcal{B}(\Delta\Omega)$$

because Ω and S are compact and metrizable (see Aliprantis and Border, 2006, Thm. 19.7). Hence, the experiment (S, σ) *induces* the distribution of posteriors τ given by

$$\tau(B_{\Delta\Omega}) = \bar{\sigma}(\{s \in S : \rho(s, \cdot) \in B_{\Delta\Omega}\}) \quad \forall B_{\Delta\Omega} \in \mathcal{B}(\Delta\Omega).$$

A distribution of posteriors $\tau \in \Delta\Delta\Omega$ is *Bayes plausible* if it averages to the prior:

$$\int_{\Delta\Omega} \mu(B_\Omega) \mathrm{d}\tau(\mu) = \mu_0(B_\Omega) \quad \forall B_\Omega \in \mathcal{B}(\Omega). \tag{2}$$

Theorem 1. *Let $\tau \in \Delta\Delta\Omega$ be any distribution of posteriors. There exists an experiment that induces τ if and only if τ is Bayes plausible.*

Proof. Let (S, σ) be any experiment, and suppose it induces the distribution of posteriors $\tau \in \Delta\Delta\Omega$. Then, τ is Bayes plausible by (1):

$$\int_{\Delta\Omega} \mu(B_\Omega) \mathrm{d}\tau(\mu) = \int_S \rho(s, B_\Omega) \mathrm{d}\bar{\sigma}(s) = \mathbb{P}[B_\Omega \times S] = \mu_0(B_\Omega) \quad \forall B_\Omega \in \mathcal{B}(\Omega).$$

Conversely, let $\tau \in \Delta\Delta\Omega$ be any Bayes-plausible distribution of posteriors. Because Ω is metrizable, the function $\rho(\cdot, B_\Omega) : \Delta\Omega \to [0, 1]$ given by $\rho(\mu, B_\Omega) = \mu(B_\Omega)$ is measurable for any $B_\Omega \in \mathcal{B}(\Omega)$ (see Aliprantis and Border, 2006, Lem. 15.16). Hence, ρ is a Markov kernel from $\Delta\Omega$ to Ω. For $S = \Delta\Omega$ and $\bar{\sigma} = \tau$, consider the probability space $(\Omega \times S, \mathcal{B}(\Omega \times S), \mathbb{P})$ with $\mathbb{P}$ such that

$$\mathbb{P}[B_\Omega \times B_S] = \int_{B_S} \rho(s, B_\Omega) \mathrm{d}\bar{\sigma}(s) \quad \forall B_\Omega \in \mathcal{B}(\Omega), \forall B_S \in \mathcal{B}(S).$$

Let X and Y be as above. Because $\Delta\Omega$ is compact and metrizable, there exists a regular conditional distribution σ of X given Y, which is unique almost everywhere. Since τ is Bayes plausible, the marginal of $\mathbb{P}$ on Ω is μ_0, so σ satisfies (1). Then, the experiment (S, σ) induces τ. □

Data availability

No data was used for the research described in the article.

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