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Optimization and stability problems for eigenvalues of linear and non linear operators

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Introduction

One of the classical questions in spectral geometry is the problem of minimizing or maximizing under some geometrical constraints one of the eigenvalues of the Laplace operator with different boundary conditions. The first conjecture in this field goes back to the end of the 19th Century and can be found in the famous book of Lord Rayleigh, *The Theory of Sound* [114]. The author conjectured that, among all planar sets with fixed area, the disk minimizes the first Dirichlet-Laplacian eigenvalue, that can be physically interpreted as the principal frequency of a membrane fixed at its boundary. This conjecture was proved 50 years later by two simultaneous but independent works, one by Faber [59] and one by Krahn [89], and it was completely solved later with the work of Pólya and Szegö [113]. Let $\Omega \subseteq \mathbb{R}^n$, with $n \geqslant 2$, be an open set with finite Lebesgue measure, the first Dirichlet-Laplacian eigenvalue is the least positive λ such that

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

admits non-trivial solutions in $H_0^1(\Omega)$. The classical result of Faber and Krahn for the first Dirichlet eigenvalue $\lambda_1(\Omega)$ states that, among measurable domains with fixed measure, $\lambda_1(\cdot)$ is minimized by a ball; in other words, the following scaling invariant inequality holds:

$$\lambda_1(\Omega)V(\Omega)^{2/n} \geqslant \lambda_1(B)V(B)^{2/n}, \tag{2}$$

where by $V(\cdot)$ we denote the volume of a measurable set and by B a ball in $\mathbb{R}^n$. Moreover, equality holds in (2) if and only if Ω is equivalent to a ball.

On the other hand, when considering the Laplacian eigenvalue problem with Neumann boundary condition, it makes sense to deal with a maximization problem. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, open and Lipschitz domain; the problem is

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where we denote by $\partial u/\partial \nu$ the outer normal derivative of u on $\partial\Omega$. In this case the first eigenvalue μ_1 is always zero and the associated eigenfunctions are the constant functions. The following inequality was proved by Szegö in the plane [122] and then generalized in higher dimension by Weinberger [131]. The so called Szegö-Weinberger inequality states that the first non-zero Neumann eigenvalue $\mu_2(\Omega)$ is maximized by a ball among domains with fixed measure, that is equivalent to say that the following scaling invariant inequality holds:

$$\mu_2(\Omega)V(\Omega)^{2/n} \leqslant \mu_2(B)V(B)^{2/n}. \tag{4}$$

Inequalities (1) and (3) are two examples of isoperimetric inequalities. Recently, stability results concerning the above problems have been obtained. The fact that balls can be characterized as the only sets for which equality holds leads to ask if these inequalities are stable, in other words we want to improve them by adding a remainder term that measures the deviation of a set Ω from the spherical symmetry. Since the quantitative isoperimetric inequality proved in [69], several spectral quantitative isoperimetric inequalities were proved, as for example the Faber-Krahn [26] and the Szegö-Weinberger [25] inequalities.

The aim of this thesis is to obtain analogous results in these directions for the eigenvalue problem with different boundary conditions and for some operators of linear and non linear type. In particular, we focus our study on Steklov and Robin boundary conditions, obtaining isoperimetric inequalities as (1) and (3) and the relative stability results with different hypothesis on the class of sets considered. A stability result in terms of the perimeter is also obtained for the first Dirichlet eigenvalue of the Laplacian operator.

In the first part of this thesis we focus on the Steklov boundary condition problem, introduced by the Russian mathematician V. A. Steklov [121]. Let $\Omega \subset \mathbb{R}^n$, with $n \geqslant 2$, be a bounded, connected, open set with Lipschitz boundary. A real $\sigma \geqslant 0$ is called a Steklov eigenvalue if there exists $u \in H^1(\Omega)$ with $u \neq 0$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial\Omega. \end{cases} \tag{5}$$

The Steklov eigenvalues can be interpreted as the eigenvalues of the *Dirichlet-to-Neumann operator* $\mathcal{D} : H^{1/2}(\Omega) \to H^{-1/2}(\Omega)$ which maps a function $f \in H^{1/2}(\Omega)$ to $\mathcal{D}f = \frac{\partial Hf}{\partial n}$, where Hf is the harmonic extension of f to Ω. For a survey concerning this topic we refer to [82]. As usual, problem (5) is considered in the weak sense, that is, for every $v \in H^1(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \sigma \int_{\partial\Omega} u \, v \, d\mathcal{H}^{n-1}, \tag{6}$$

where $\cdot$ denotes the standard Euclidean scalar product and $\mathcal{H}^{n-1}$ denotes the $(n-1)-$dimensional Hausdorff measure in $\mathbb{R}^n$. In this framework, since the trace operator $H^1(\Omega) \to L^2(\partial\Omega)$ is compact (see [99], Theorem 6.2), it is known that the Steklov spectrum consists of a discrete sequence diverging at infinity

$$0 = \sigma_1(\Omega) \leqslant \sigma_2(\Omega) \leqslant \sigma_3(\Omega) \leqslant \cdots \nearrow +\infty. \tag{7}$$

In particular, we deal with the first non-trivial Steklov eigenvalue of Ω, that has the following variational characterization:

$$\sigma_2(\Omega) = \min \left\{ \frac{\displaystyle\int_\Omega |\nabla v|^2 \, dx}{\displaystyle\int_{\partial\Omega} v^2 \, d\mathcal{H}^{n-1}} \; : \; v \in H^1(\Omega) \backslash \{0\}, \; \int_{\partial\Omega} v \, d\mathcal{H}^{n-1} = 0 \right\}. \tag{8}$$

If we take $\Omega = B_R(x)$, where $B_R(x)$ is the ball of radius R centered at the point x, then

$$\sigma_2(B_R(x)) = \frac{1}{R}. \tag{9}$$

Moreover, we know that $\sigma_2(B_R(x))$ has multiplicity n and the corresponding eigenfunctions are $u_i(x) = x_i$, with $i = 1, \dots, n$. Let us focus now our attention on shape optimization problems concerning the first non trivial Steklov eigenvalue. In [132] the author considers the problem of maximizing $\sigma_2(\Omega)$ in the plane, keeping the perimeter of Ω fixed. If $\Omega \subseteq \mathbb{R}^2$ is a Lipschitz, simply connected open set, the following inequality, known as Weinstock inequality, is proved

$$\sigma_2(\Omega) P(\Omega) \leqslant \sigma_2(B_R(x)) P(B_R(x)), \tag{10}$$

where $P(\Omega)$ denotes the Euclidean perimeter of Ω. In other words, inequality (10) states that, among all planar, simply connected, open sets with prescribed perimeter, $\sigma_2(\Omega)$ is maximum for the disk. Moreover, in [75], it is proved that (10) fails to be true in general in dimension $n > 2$. If we consider indeed the annulus $A_\epsilon = B_1(x)\backslash\overline{B_\epsilon(x)}$, having that $B_R(x)$ is the ball of radius R centered at x, with $\epsilon \approx 0$, that is a simply connected set, the following reverse inequality holds,

$$\sigma(A_\epsilon)P(A_\epsilon)^{\frac{1}{n-1}} > \sigma(B_R(x))P(B_R(x)))^{\frac{1}{n-1}}.$$

In [31], the authors generalize the Weinstock inequality (10) in any dimension, when restricting to the class of convex sets. More precisely, if $\Omega \subseteq \mathbb{R}^n$ is an open, bounded, convex set, then

$$\sigma_2(\Omega)P(\Omega)^{\frac{1}{n-1}} \leqslant \sigma_2(B_R(x))P(B_R(x))^{\frac{1}{n-1}} \tag{11}$$

and equality holds only if Ω is a ball. In order to prove (11), the authors prove the following weighted isoperimetric inequality, involving the boundary momentum $M(\Omega)$, defined as

$$M(\Omega) = \int_{\partial\Omega} |x|^2 \, d\mathcal{H}^{n-1}, \tag{12}$$

that is

$$\frac{M(\Omega)}{P(\Omega)\, V(\Omega)^{\frac{2}{n}}} \geqslant \frac{M(B)}{P(B)\, V(B)^{\frac{2}{n}}} = \omega_n^{\frac{-2}{n}}, \tag{13}$$

where ω_n is the measure of the n-dimensional unit ball in $\mathbb{R}^n$ and equality holds if and only if Ω is a ball centered at the origin

Let us recall now the results concerning the volume constraint. In [28] the author proves that the first non-trivial Steklov eigenvalue is maximized by balls, among sets with the same volume. More precisely, if $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, is an open bounded set with Lipschitz boundary, then

$$\sigma_2(\Omega)V(\Omega)^{\frac{1}{n}} \leqslant \sigma_2(B_R(x))V(B_R(x))^{\frac{1}{n}}, \tag{14}$$

where $V(\Omega)$ denotes the Lebesgue measure of Ω and equality holds if and only if Ω is a ball. Actually, he proves the following more general inequality, known as Brock inequality,

$$\sum_{i=2}^{n+1} \frac{1}{\sigma_i(\Omega)} \geqslant nR, \tag{15}$$

where R is the radius of the ball having the same volume as Ω. We also observe that (11) and the classical isoperimetric inequality imply (14) for convex sets; so, inequality (14) is weaker than (11) because it contains the volume, but it is more general because it holds without geometric restrictions.

Recently, concerning the stability issue, in [25] the authors prove the following quantitative version of inequality (15):

$$\frac{1}{V(\Omega)^{1/n}} \sum_{i=2}^{n+1} \frac{1}{\sigma_i(\Omega)} \geqslant \frac{n}{\omega_n^{1/n}} \left[1 + c_n \mathcal{A}_{\mathcal{F}}(\Omega)^2\right], \tag{16}$$

where $\mathcal{A}_{\mathcal{F}}(\Omega)$ is the so-called Fraenkel asymmetry and c_n is an explicit constant which depends only on the dimension. The Fraenkel asymmetry is an index of asymmetry, i.e. it measures how much a set differs from the ball in the L^1 norm and it is defined as follows

$$\mathcal{A}_F(\Omega) := \min_{x\in\mathbb{R}^n} \left\{ \frac{V(\Omega\Delta B_R(x))}{V(B_R(x))} \ , \ V(B_R(x)) = V(\Omega) \right\}, \tag{17}$$

where Δ denotes the symmetric difference between two sets. The quantitative result (16) is obtained as a consequence of having proved a quantitative version of a weighted isoperimetric inequality proved in [16] and that was used in [28] in order to prove (14). More precisely, in [16] it is proved that, if $\Omega \subseteq \mathbb{R}^n$ is a bounded, open and Lipschitz set, then

$$\frac{M(\Omega)}{V(\Omega)^{\frac{n+1}{n}}} \geqslant \frac{M(B)}{V(B)^{\frac{n+1}{n}}} = n\omega_n^{-1/n} \tag{18}$$

and equality holds for any ball centered at the origin. In particular, inequality (18) implies that, among sets with fixed volume, the boundary momentum is minimal on balls centered at the origin.

The first part of the thesis, that is Section 2.1, deals with the study of a quantitative version of the Weinstock inequality (11). Since we are working with convex sets, we consider the following asymmetry functional

$$\mathcal{A}_{\mathcal{H}}(\Omega) := \min_{x\in\mathbb{R}^n} \left\{ \left(\frac{d_{\mathcal{H}}\left(\Omega, B_R(x)\right)}{R} \right),\, P(B_R(x)) = P(\Omega) \right\}, \tag{19}$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded, open, convex set. Our main result, contained in [73], is stated in the following theorem.

Theorem. *Let $n \geqslant 2$. There exists $\bar{\delta} > 0$ such that for every $\Omega \subset \mathbb{R}^n$ bounded, convex open set with $\sigma_1(B_R(x)) \leqslant (1+\bar{\delta})\,\sigma_1(\Omega)$, where $B_R(x)$ is a ball with $P(B_R(x)) = P(\Omega)$, then*

$$\frac{\sigma_2(B_R(x)) - \sigma_2(\Omega)}{\sigma_2(\Omega)} \geqslant \begin{cases} \frac{16}{9\pi}\left(\mathcal{A}_{\mathcal{H}}(\Omega)\right)^{\frac{5}{2}} & \text{if } n = 2 \\ \frac{2}{3}\sqrt{\pi}\, g\left(\left(\dfrac{\mathcal{A}_{\mathcal{H}}(\Omega)}{\beta}\right)^2\right) & \text{if } n = 3 \\ \frac{(n\omega_n)^{\frac{1}{n-1}}}{n}\left(\dfrac{\mathcal{A}_{\mathcal{H}}(\Omega)}{\beta_n}\right)^{\frac{n+1}{2}} & \text{if } n \geqslant 4, \end{cases} \tag{20}$$

where β and β_n are defined in (2.17) and g is the inverse function of $f(t) = t\log\left(\frac{1}{t}\right)$, for $0 < t < e^{-1}$.

Moreover, the result is sharp, in the sense that the quantitative inequality becomes asymptotically an equality, at least for particular shapes having small deficits. The key point to prove the previous Theorem is a quantitative version of the weighted isoperimetric inequality (13), obtained using Fuglede's technique [67]. We also recall a recent result, proved in [35], where it is proved that the Weinstock inequality is not stable among simply connected sets in the plane.

The second part of Chapter 2, that is Section 2.2, deals with a different shape optimization problem involving the Steklov boundary condition. Let $\Omega_0 \subseteq \mathbb{R}^n$, $n \geqslant 2$, be an open, bounded, connected set, with Lipschitz boundary such that $B_r \Subset \Omega_0$, where B_r is the open ball of radius $r > 0$ centered at the origin. Let us set $\Omega := \Omega_0 \backslash \overline{B_r}$; then we study the following Steklov-Dirichlet boundary eigenvalue problem for the Laplacian:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial B_r, \\ \dfrac{\partial u}{\partial \nu} = \sigma^{DS}(\Omega) u & \text{on } \partial\Omega_0. \end{cases} \tag{21}$$

The study of the first eigenvalue of problem (45) leads to the following minimization problem:

$$\sigma_1^{DS}(\Omega) = \min_{\substack{w \in H^1_{\partial B_r}(\Omega) \\ w \not\equiv 0}} \frac{\displaystyle\int_\Omega |\nabla w|^2 \, dx}{\displaystyle\int_{\partial \Omega_0} w^2 \, d\mathcal{H}^{n-1}}, \tag{22}$$

where $H^1_{\partial B_r}(\Omega)$ is the set of Sobolev functions on Ω that vanish on ∂B_r. Notice also that the value $\sigma_1^{DS}(\Omega)$ is the optimal constant in the Sobolev-Poincaré trace inequality:

$$\sigma_1^{DS}(\Omega) ||w||_{L^2(\partial\Omega_0)} \leqslant ||\nabla w||_{H^1_{\partial B_r}(\Omega)}. \tag{23}$$

We treat the following shape optimization issue:

Which sets maximize $\sigma_1^{DS}(\cdot)$ among sets of the form $\Omega = \Omega_0 \backslash \overline{B_r}$, where Ω_0 contains the fixed ball B_r and Ω has prescribed volume?

In the class of sets of the form $B_R(x_0)\backslash\overline{B}_r$ with $B_R(x_0)$ being a ball containing B_r, the maximizer of σ_1^{DS} is the spherical shell, that is the annulus when the balls are concentric (see [65]). This is also proved in [128] and for more general spaces in [119]. We partially solve the problem of the optimality of σ_1^{DS}, restricting our study to nearly spherical sets, that are sets whose boundary can be parametrized on the sphere by means of a Lipschitz function with a small $W^{1,\infty}$-norm; see Definition 1.6 in Chapter 1. Our result is the following and is contained in [106].

Theorem. *Let $\Omega = \Omega_0 \backslash \overline{B_r}$, with Ω_0 a nearly spherical set. Then*

$$\sigma_1^{DS}(\Omega) \leqslant \sigma_1^{DS}(A_{r,R}), \tag{24}$$

where $A_{r,R} = B_R \backslash \overline{B_r}$, with $R > r > 0$, is the spherical shell with the same volume as Ω. Moreover the equality in (24) holds if and only if Ω is a spherical shell.

So, we study the optimal shape for $\sigma_1^{SD}(\Omega)$ when both the volume of the domain and the radius of the internal ball are fixed. We also find some counterexamples showing that when only a volume constraint holds, then σ_1^{DS} is not upper bounded, hence we cannot speak about optimality. In order to prove the Theorem, we find $K = K(n, |\Omega|) > 0$, such that

$$\sigma_1^{DS}(A_{r,R}) \geqslant \sigma_1^{DS}(\Omega) \left(1 + K(n, |\Omega|) \int_{\mathbb{S}^{n-1}} v^2(\xi) \, d\mathcal{H}^{n-1} \right).$$

When $r = 0$ and Ω is connected, the problem becomes the classical Steklov eigenvalue problem .

Chapter 3 deals with Steklov boundary condition in the anisotropic case. Firstly, in Section 3.1, we prove an anisotropic generalization of the inequality (13). Let $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, be an open bounded and convex set, let F be a Finsler norm (see Section 1.3.1), i.e. a convex positive C^2 function, let F^o be its dual norm and let us fix a real number $p > 1$. We define the anisotropic $p-$boundary momentum as

$$M_{F,p}(\Omega) := \int_{\partial\Omega} [F^o(x)]^p \, F(\nu(x)) \, d\mathcal{H}^{n-1}(x),$$

where ν is the outer normal on $\partial\Omega$, the anisotropic perimeter as

$$P_F(\Omega) := \int_{\partial\Omega} F(\nu(x)) \, d\mathcal{H}^{n-1}(x)$$

and we consider the scaling invariant functional

$$\mathcal{F}_{F,p}(\Omega) = \frac{M_{F,p}(\Omega)}{P_F(\Omega)V(\Omega)^{\frac{p}{n}}}.$$

Moreover, we define the Wulff shape of radius r centered at the point x_0 as

$$\mathcal{W}(x_0, r) = \{\xi \in \mathbb{R}^n : F^o(\xi - x_0) < r\}$$

and we denote by κ_n the volume of the Wulff shape of radius 1 centered at the origin. By adapting the arguments in [31], we are able to prove in [108] the non linear counterpart of (13).

Theorem. *Let Ω be a bounded, open convex set of $\mathbb{R}^n$. Then*

$$\mathcal{F}_{F,p}(\Omega) \geqslant \kappa_n^{-\frac{p}{n}} = \mathcal{F}_{F,p}(\mathcal{W}), \tag{25}$$

and equality holds only for Wulff shapes centered at the origin.

A fundamental tool that we use is the inverse anisotropic mean curvature flow (we refer to [135] for details). Roughly speaking, the smooth boundary $\partial\Omega$ of an open set $\Omega = \Omega(0)$ flows by anisotropic inverse mean curvature if there exists a time dependent family $(\partial\Omega(t))_{t\in[0,T)}$, $T > 0$, of smooth boundaries such that the anisotropic normal velocity at any point $x \in \partial\Omega(t)$ is equal to the inverse of the anisotropic mean curvature of $\partial\Omega(t)$ at x. We make also use of the following anisotropic version of the Heintze-Karcher inequality

$$\int_{\partial\Omega} \frac{F(\nu)}{H_F} d\mathcal{H}^{n-1} \geqslant \frac{n}{n-1} V(\Omega),$$

see [115] for the Euclidean case and [136] for its anisotropic analogous, recalled in Lemma 1.18.

The previous result (25) is mainly motivated by the following application to the study of the Steklov spectrum problem for the orthotropic p-Laplacian, see Section 3.2. Let Ω be an open, bounded and convex set in $\mathbb{R}^n$, with $n \geqslant 2$, and let $p > 1$. We consider the following non linear operator, called the orthotropic $p-$Laplace operator,

$$\widetilde{\Delta}_p u = \sum_{j=1}^{n} \left(|u_{x_i}|^{p-2} u_{x_i}\right)_{x_i} \tag{26}$$

and we study the limit problem, as $p \to \infty$, of the Steklov problem associated to it, that is

$$\begin{cases} -\widetilde{\Delta}_p u = 0 & \text{on } \Omega \\ \sum_{j=i}^{N} |u_{x_j}|^{p-2} u_{x_j} \nu_j = \sigma |u|^{p-2} u \rho_p & \text{on } \partial\Omega, \end{cases} \tag{27}$$

where u_{x_j} is the partial derivative of u with respect to x_j, $\nu = (\nu_1, \ldots, \nu_n)$ is the outer normal of $\partial\Omega$, $\rho_p(x) = \|\nu(x)\|_{\ell^{p'}}$, p' is the coniugate exponent of p and

$$\|x\|_{\ell^p}^p = \sum_{j=1}^{n} |x_j|^p. \tag{28}$$

The real number σ is called orthotropic Steklov eigenvalue. In particular, problem (27) has been investigated in [27], where it is proved that these eigenvalues form at least a countably infinite sequence of positive numbers diverging at infinity where the first eigenvalue is 0 and corresponds to constant eigenfunctions. Denoting by $\Sigma_p^p(\Omega)$ the first non-trivial eigenvalue of (27), the following variational characterization is showed in [27]:

$$\Sigma_p^p(\Omega) = \min\left\{ \frac{\int_\Omega \|\nabla u\|_{\ell^p}^p \, dx}{\int_{\partial\Omega} |u|^p \rho_p(x) d\mathcal{H}^{n-1}}, \; u \in W^{1,p}(\Omega), \; \int_{\partial\Omega} |u|^{p-2} u \rho_p(x) d\mathcal{H}^{n-1} = 0 \right\}. \tag{29}$$

Let us observe that the value $\Sigma_p^p(\Omega)$ represents the optimal constant in the weighted trace-type inequality

$$\int_\Omega \|\nabla u\|_{\ell^p}^p \, dx \geqslant \Sigma_p^p(\Omega) \int_{\partial\Omega} |u|^p \rho_p d\mathcal{H}^{n-1}$$

in the class of Sobolev functions $u \in W^{1,p}(\Omega)$, such that

$$\int_{\partial\Omega} |u|^{p-2} u \rho_p d\mathcal{H}^{n-1} = 0.$$

By the way we recall that the orthotropic Laplacian, sometimes also called pseudo $p-$Laplacian, was introduced in [94, 130, 129]; for $p = 2$ it coincides with the Laplacian, but for $p \neq 2$ it differs from the usual $p-$Laplacian, that is defined as $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$. Let us recall that for this operator an isoperimetric inequality concerning the first Dirichlet eigenvalue has been discussed in the planar case in [17, 18]. The orthotropic $p-$ Laplacian can be considered indeed as an anisotropic operator, associated to the Finsler norm (28). In the second part of Chapter 3 we focus our attention on the limit operator $\lim_{p\to\infty} \widetilde{\Delta}_p u = \widetilde{\Delta}_\infty u$, the so-called orthotropic ∞-Laplace operator, that can also be defined, see for example [15], as

$$\widetilde{\Delta}_\infty u = \sum_{j \in I(\nabla u(x))} u_{x_j}^2 u_{x_j, x_j}, \tag{30}$$

where

$$I(x) := \{ j \leqslant n : \; |x_j| = \|x\|_{\ell^\infty} \}$$

and

$$\|x\|_{\ell^\infty} = \max_{j=1,\dots,n} |x_j|.$$

We are inspired by the results given in [72], where the authors study the Steklov eigenvalue problem for the $\infty-$Laplacian Δ_∞, given by

$$\Delta_\infty u = \sum_{i,j=1}^n u_{x_j} u_{x_i} u_{x_j x_i}.$$

This operator was also studied for example in [57], with Neumann boundary conditions, [116] for mixed Dirichlet and Robin boundary conditions. In particular we find a limit eigenvalue problem of (27) that is satisfied in a viscosity sense and we show that we can pass to the limit in the variational caracterization (29). We observe that, since the first eigenvalue of (27) is 0 with constant eigenfunction, we can trivially pass to the limit and obtain that the first eigenvalue is also 0 with constant associated eigenfunction.

We want to prove Brock-Weinstock and Weinstock type inequalities for the orthotropic $p-$Laplacian, possibly with $p = \infty$. We will use the following notation to denote respectively the unit ball and the anisotropic perimeter with respect to the ℓ^p norm, for $p \in [1, \infty]$,

$$\mathcal{W}_p = \{ x \in \mathbb{R}^n \mid \|x\|_{\ell^p} \leqslant 1 \},$$

$$\mathcal{P}_p(\Omega) := \int_{\partial\Omega} \rho_p(x) d\mathcal{H}^{n-1}(x).$$

In [27] it is proved a Brock-Weinstock type inequality of the form

$$\Sigma_p^p(\Omega) \leqslant \left(\frac{V(\mathcal{W}_p)}{V(\Omega)}\right)^{\frac{p-1}{n}}. \tag{31}$$

Let us recall that (up to our knowledge) we cannot write inequality (31) in a fully scaling invariant form, except for $p = 2$, since it is still an open problem to determine whether $\Sigma_p^p(\mathcal{W}_p) = 1$ or not for $p \neq 2$, as conjectured in [27].

Using (25) with the anisotropy given by (28), we prove in [8] the following result.

Theorem. *For any bounded convex open set $\Omega \subseteq \mathbb{R}^n$, for $n \geqslant 2$, it holds*

$$\Sigma_\infty(\Omega)V(\Omega)^{1/n} \leqslant \Sigma_\infty(\mathcal{W}_1)V(\mathcal{W}_1)^{1/n}. \tag{32}$$

Equality holds if and only if Ω is equivalent to $\mathcal{W}_1$ up to translations and scalings. Moreover, for any open bounded convex set $\Omega \subseteq \mathbb{R}^2$ it holds

$$\Sigma_\infty(\Omega)P_\infty(\Omega) \leqslant \Sigma_\infty(\mathcal{W}_1)P_\infty(\mathcal{W}_1). \tag{33}$$

and equality holds if and only if Ω is of constant width.

As far as concerned with the Robin boundary conditions, in Chapter 4 we obtain some results both in the linear and the non linear case. We start by recalling the Robin eigenvalue problem for the Laplacian. Let Ω be a bounded, open subset of $\mathbb{R}^n$, $n \geqslant 2$, with Lipschitz boundary; its Robin eigenvalues related to the Laplacian are the real numbers λ such that

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} + \alpha u = 0 & \text{on } \partial\Omega \end{cases} \tag{34}$$

admits non trivial $W^{1,2}(\Omega)$ solutions; α is an arbitrary real constant, which will be referred to as boundary parameter of the Robin problem. We observe that for $\alpha = 0$ we obtain the Neumann problem, for $\alpha = +\infty$ we formally obtain the Dirichlet problem and for $\lambda = 0$ the Steklov problem; for this reason it can be considered as the most general eigenvalue problem for the Laplace operator. For each fixed Ω and α there is a sequence of eigenvalues

$$\lambda_1(\alpha, \Omega) \leqslant \lambda_2(\alpha, \Omega) \leqslant \cdots \to +\infty$$

which depend on α. In particular, the first non trivial Robin eigenvalue of Ω is characterized by the expression

$$\lambda_1(\alpha, \Omega) = \min_{\substack{u \in W^{1,2}(\Omega) \\ u \neq 0}} \frac{\displaystyle\int_\Omega |Du|^2 \, dx + \alpha \int_{\partial\Omega} |u|^2 \, d\mathcal{H}^1}{\displaystyle\int_\Omega |u|^2 \, dx}.$$

We refer to [87] for a collection of properties of the Robin Laplacian eigenvalues and the related proofs. From the monotonicity of the Rayleigh quotient, we can deduce the following, for $0 < t \leqslant 1$,

$$\lambda_1(\alpha, t\Omega) = \frac{1}{t^2}\lambda_1(t\alpha, \Omega) \leqslant \frac{1}{t}\lambda_1(\alpha, \Omega) \leqslant \lambda_1(\alpha, \Omega). \tag{35}$$

Firstly, let us assume $\alpha > 0$. We have the following Faber-Krahn type inequality, that was proved in [19] in the planar case and was then generalized in [49] in any dimension. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and Lipschitz domain. Then,

$$\lambda_1(\alpha, \Omega) \geqslant \lambda_1(\alpha, B), \tag{36}$$

where B is a ball such that $V(B) = V(\Omega)$. Equality holds if and only if Ω is a ball. The generalization to the p-Laplacian is given in [48] and in [29]; this result was also shown to hold on general open sets of finite measure, see [32].

Let us now assume that $\alpha < 0$ and $\Omega \subseteq \mathbb{R}^n$ is a bounded and Lipschitz domain. If we put a constant function as a test function in the Rayleigh quotient (4.2), we have that the first eigenvalue is always negative:

$$\lambda_1(\alpha, \Omega) < \alpha \frac{P(\Omega)}{V(\Omega)}. \tag{37}$$

Moreover, if we choose in (37) a sequence of domains Ω_n of fixed volume and such that $P(\Omega_n) \to +\infty$, we have that $\lambda_1(\Omega_n, \alpha) \to -\infty$. This tells us that it has not sense to seek for a minimizer of $\lambda_1(\Omega, \alpha)$, if we fix the volume, and the upper bound (37) suggests to look for a maximizer. In 1977 Bareket conjectured that the maximizer was a ball [10]. Evidence to this conjecture was provided in [62] , where it is proved that the ball is a local maximizer among bounded open and Lipschitz sets with fixed volume that are closed to a ball in a L^∞ sense. However in [64] the authors disproved Bareket's conjecture, showing that the first Robin-Laplacian computed on a spherical shell is asymptotically greater than the one computed on a ball with the same volume. In [88, 103] this was clarified by showing that for $\Omega \subseteq \mathbb{R}^n$ of class $C^{1,1}$, it holds

$$\lambda_1(\alpha, \Omega) = -\alpha^2 - (n-1)\alpha \sup_{\partial\Omega} H + o(\alpha^{2/3}), \tag{38}$$

as $\alpha \to -\infty$, where H is the mean curvature of the boundary. Still in [64], it is proved that Bareket's conjecture holds for α negative small enough in absolute value. More precisely, the authors proved that, for bounded planar domains of class C^2 and fixed area, there exists a negative number α_*, depending only on the area, such that

$$\lambda_1(\alpha, \Omega) \leqslant \lambda_1(\alpha, B^\sharp), \tag{39}$$

holds for all $\alpha \in [\alpha_*, 0]$, where $B^\sharp$ is the ball with the same area. This fact is proved by applying the method of interior parallel, introduced by Makai [96] and Pólya [112] and used by Weinberger in [110]. We remark that the problem of maximizing the first Robin-Laplacian for $\alpha < 0$ and $n \geqslant 3$ is still open.

If, instead of the volume, we keep the perimeter fixed, the authors in [6] prove that for any bounded planar domains of class C^2, if $\alpha < 0$, then

$$\lambda_1(\alpha, \Omega) \leqslant \lambda_1(\alpha, B^*), \tag{40}$$

where B^* is a disk with the same perimeter as Ω. Moreover, in [30] the authors show that, among all bounded, open and convex sets with given perimeter the ball is a maximizer for the fist Robin-Laplacian eigenvalue for any negative value of α and for every dimension.

In the first part of this Chapter 4, Section 4.1, we find analogous of inequalities (39) and (3.39) in the anisotropic case. Let F be a Finsler norm and let us consider the anisotropic version of problem (34), that is

$$\begin{cases} -\mathrm{div}\,(F(\nabla u)F_\xi(\nabla u)) = \lambda_F(\alpha, \Omega)u & \text{in } \Omega \\ \langle F(\nabla u)F_\xi(\nabla u), \nu_{\partial\Omega}\rangle + \alpha F(\nu_{\partial\Omega})u = 0 & \text{on } \partial\Omega \end{cases}$$

with the following variational characterization of the first eigenvalue:

$$\lambda_{1,F}(\alpha,\Omega) = \min_{\substack{u\in W^{1,2}(\Omega)\\ u\neq 0}} \frac{\displaystyle\int_\Omega F^2(\nabla u)\,dx + \alpha\int_{\partial\Omega}|u|^2 F(\nu_{\partial\Omega})\,d\mathcal{H}^1(x)}{\displaystyle\int_\Omega |u|^2\,dx}.$$

This problem is studied for instance in [51, 52, 53, 74]. Using the method of interior parallel, adapted to the anisotropic case, we prove in [109] the two results. The first one is a generalization of (39) to the anisotropic case.

Theorem. *For bounded planar domains of class C^2 and fixed area, there exists a negative number α_*, depending only on the area, such that the following inequality holds $\forall\alpha\in[\alpha_*,0]$:*

$$\lambda_{1,F}(\alpha,\Omega) \leqslant \lambda_{1,F}(\alpha,\mathcal{W}_\Omega^\sharp),$$

where $\mathcal{W}_\Omega^\sharp$ is the Wulff shape of the same area as Ω.

The second result is a generalization in the anisotropic case of inequality (3.39).

Theorem. *Let $\alpha \leqslant 0$. For bounded planar domains of class C^2, we have*

$$\lambda_{1,F}(\alpha,\Omega) \leqslant \lambda_{1,F}(\alpha,\mathcal{W}_\Omega^*),$$

where $\mathcal{W}_\Omega^*$ is the Wulff shape with the same perimeter as Ω.

Since (38) holds, another interesting problem is the minimization of the maximal curvature in classes of domains of given volume with an additional topological constraint. In [102] the author proves that, if $\Omega\subset\mathbb{R}^2$ is a bounded and simply connected domain, then

$$||k_{\partial\Omega}||_{L^\infty(\partial\Omega)} \geqslant ||k_{\partial B_R(x)}||_{L^\infty(\partial B_R(x))}, \tag{41}$$

where $k_{\partial\Omega}$ is the curvature relative to $\partial\Omega$ and $B_R(x)$ is a ball with the same volume as Ω. Moreover equality holds if and only if Ω is a ball. This result was obtained by the use of the curve shortening flow ([70, 76]). In Section 4.2, we find the analogous result of (41) in the anisotropic case. More precisely, let $F:\mathbb{R}^2\to[0,+\infty)$ be a Finsler norm. We denote by $k^F_{max}(\partial\Omega)$ the anisotropic maximum curvature over $\partial\Omega$, that is

$$k^F_{max}(\partial\Omega) := ||k^F_{\partial\Omega}||_{L^\infty(\partial\Omega)},$$

where k^F is the F-anisotropic curvature, that will be properly defined in Section 1.3.4. The main result of [104] is the following.

Theorem. *Let $\Omega\subseteq\mathbb{R}^2$ such that $\gamma:=\partial\Omega$ is a smooth Jordan curve. Then,*

$$k^F_{\max}(\partial\Omega) \geqslant k^F_{\max}(\partial\mathcal{W}^\sharp), \tag{42}$$

where $\mathcal{W}^\sharp$ is the Wulff shape having the same volume as Ω. Moreover, equality holds if and only if Ω coincides with a Wulff shape.

The proof is based on the use of the anisotropic mean curvature flow, for some reference see for example [4, 12, 43, 97]. We will reduce our study to the case in which the curve is convex and we will use the so called Wulff- Gage inequality, true for convex sets, proved in [77], that states

$$\int_{\partial\Omega}(k^F_{\partial\Omega}(x)^2 F(\nu(x)))\,d\mathcal{H}^1(x) \geqslant \frac{\kappa P_F(\Omega)}{A(\Omega)}, \tag{43}$$

where $P_F(\Omega)$ is the anisotropic perimeter. The isotropic version of this inequality was proved in [70] for convex sets of the plane and generalized in [34, 60, 61] for non convex sets, whose boundary is simply connected.

In Section 4.3 we take into consideration the p-Laplacian operator

$$-\Delta_p u := -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

defined on a convex set Ω of $\mathbb{R}^n$, $n \geqslant 2$, that contains holes; more precisely we are considering sets Ω of the form $\Omega = \Omega_0 \backslash \overline{\Theta}$, where $\Omega_0 \subseteq \mathbb{R}^n$ is an open bounded and convex set and $\Theta \subset\subset \Omega_0$ is a finite union of sets, each of one homeomorphic to a ball. In this setting, we study the eigenvalue problem and the torsion problem for the p-Laplacian operator with boundary conditions of Robin type on the exterior boundary $\Gamma_0 := \partial\Omega_0$ and of Neumann type on the interior boundary $\Gamma_1 := \partial\Theta$. The first quantity we deal with is

$$\lambda_p^{NR}(\alpha, \Omega) = \min_{\substack{w \in W^{1,p}(\Omega)\\ w \not\equiv 0}} \frac{\displaystyle\int_\Omega |\nabla w|^p \, dx + \alpha \int_{\Gamma_0} |w|^p \, d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |w|^p \, dx}. \tag{44}$$

This minimization problem is a variational characterization of the first eigenvalue, i.e. the lowest eigenvalue, of the following problem:

$$\begin{cases} -\Delta_p u = \lambda_p^{NR}(\alpha, \Omega)|u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta |u|^{p-2}u = 0 & \text{on } \Gamma_0 \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1, \end{cases} \tag{45}$$

where $\alpha \in \mathbb{R}\backslash\{0\}$ is the boundary parameter. Moreover, we will only consider non zero values of the boundary parameter α, since the case $\alpha = 0$ is trivial, being the first eigenvalue identically zero and the relative eigenfunctions constant. In [107] we have proved the following result.

Theorem. *Let Ω be of the form $\Omega = \Omega_0 \backslash \overline{\Theta}$, where $\Omega_0 \subseteq \mathbb{R}^n$ is an open bounded and convex set and $\Theta \subset\subset \Omega_0$ is a finite union of sets, each of one homeomorphic to a ball Then,*

$$\lambda_p^{NR}(\alpha, \Omega) \leqslant \lambda_p^{NR}(\alpha, A_{r_1,r_2}), \tag{46}$$

where $A_{r_1,r_2} = B_{r_2} \backslash \overline{B_{r_1}}$ is the annulus such that $V(A_{r_1,r_2}) = V(\Omega)$ and $P(B_{r_2}) = P(\Omega_0)$.

In particular, when $\alpha \to +\infty$, this Theorem gives an answer to the Open Problem 5 in [81, Chap. 3], restricted to convex sets with holes. In this Section, we generalize in any dimension the method of interior parallels as used by Payne and Weinberger in [110] to study the Laplacian eigenvalue problem with external Robin boundary condition and with Neumann internal boundary condition in the plane. More precisely, our proof is based on the use of the web functions, particular test functions used e.g. in [22, 30, 45] and on the study of their level sets.

Similarly, but only for positive values of α, we also study the p-torsional rigidity type problem:

$$\frac{1}{T_p^{NR}(\alpha, \Omega)} = \min_{\substack{w \in W^{1,p}(\Omega)\\ w \not\equiv 0}} \frac{\displaystyle\int_\Omega |\nabla w|^p \, dx + \alpha \int_{\Gamma_0} |w|^p \, d\mathcal{H}^{n-1}}{\left|\displaystyle\int_\Omega w \, dx\right|^p};$$

in particular, this problem leads to, up to a suitable normalization,

$$\begin{cases} -\Delta_p u = 1 & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \alpha |u|^{p-2}u = 0 & \text{on } \Gamma_0 \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1. \end{cases}$$

The second main result of this Section is the following.

Theorem. *Let Ω be of the form $\Omega = \Omega_0 \backslash \overline{\Theta}$ as defined in the previous Theorem. Then,*

$$T_p^{NR}(\alpha, \Omega) \geqslant T_p^{NR}(\alpha, A_{r_1,r_2}), \tag{47}$$

where $A_{r_1,r_2} = B_{r_2} \backslash \overline{B_{r_1}}$ is the annulus such that $V(A_{r_1,r_2}) = V(\Omega)$ and $P(B_{r_2}) = P(\Omega_0)$.

Equation (47), when $\Theta = \varnothing$, $p = 2$ and Dirichlet boundary condition holds on the whole boundary, is the Saint-Venant inequality, by the name of the authors that first conjectured that the ball in the plane (under area constraint) gives the maximum in quantity (47). This is a relevant problem in the elasticity theory of beams [120, 32, Sec.35]. It is known that the ball maximizes the torsional rigidity with Robin boundary conditions [33] among bounded open sets with Lipschitz boundary and given measure. Related results for the spectral optimization problems involving the rigidity are obtained also, for example, in [126, 127, 23].We recall that in [54] the authors prove that the first eigenvalue of the p-Laplacian with external Neumann and internal Robin boundary conditions is maximum on spherical shells when the volume and the internal $(n-1)$-quermassintegral are fixed, generalizing a result cointened in [84] for the planar case and $p = 2$. They also prove that with these boundary conditions the spherical shell minimizes the p torsional rigidity among domains with volume and $(n-1)$-quermassintegral fixed.

Finally, in Chapter 5, we consider the eigenvalue problem for the Laplacian with Dirichlet boundary condition and we work with the class of admissible sets

$$\mathcal{C}_n := \{\Omega \subseteq \mathbb{R}^n \mid \Omega \text{ convex},\ V(\Omega) = 1\}.$$

Our starting point is the following conjecture, that is stated in [66]. Here the authors conjecture that, if $\Omega \in \mathcal{C}_2$, then

$$\lambda_1(\Omega) - \lambda_1(B) \geqslant \beta \left(P(\Omega) - P(B)\right)^{3/2}, \tag{48}$$

where $B \subseteq \mathbb{R}^2$ is a ball of area 1, $\beta := \frac{4 \cdot 3^{3/2}\, \zeta(3)}{\pi^{11/4}}$ and $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$ is the Riemann zeta function. This conjecture is supported by numerical and analytical results. In particular, the first analytic result we refer to can be found in [79, 78]. The authors prove that, if P_k^* is the regular polygon with k edges and area equal to 1, then, as k goes to $+\infty$,

$$\lambda_1(P_k^*) - \lambda_1(B) \sim \beta \left(P(P_k^*) - P(B)\right)^{3/2}. \tag{49}$$

The method used in [79] to prove this fact comes from differential geometry and relays on the calculus of moving surfaces. This result was also proved in [98], using the Schwartz-Christoffel mappings, that are useful tools to express the Laplace-Dirichlet eigenvalue of a polygon as a series expansion, relating each expansion term to a summation over Bessel functions. By the way, we recall that the fundamental tone of the Dirichlet Laplacian on polygons has been widely investigated and nevertheless many questions are still unsolved. See, for example, the Polyá-Szegö conjecture [113], stating that among all the k-gons of given area the regular one achieves the least possible λ_1 and that has been settled only for $k = 3$ and $k = 4$, that are the only cases for which it is possible to use the Steiner symmetrization.

The conjectured inequality (48) is also supported by numerical observations, linked to the plot of the Blaschke-Santaló diagram for the triplet $(P(\cdot),\, \lambda_1(\cdot),\, V(\cdot))$, that is the sets of points

$$\{(P(\Omega), \lambda_1(\Omega)) \mid V(\Omega) = 1, \Omega \in \mathcal{C}_2\}.$$

In particular, in [66], the authors generate random polygons and find out that regular polygons lay on the lower part of the diagram.

In our work [105] we are not able to prove the conjectured inequality (48). Instead, we prove the following less strong result, that is a step forward in its resolution.

Theorem. *Let $n \geqslant 2$. There exists a constant $c > 0$ depending only on n such that for every $\Omega \in \mathcal{C}_n$ it holds*

$$\lambda_1(\Omega) - \lambda_1(B) \geqslant c\,(P(\Omega) - P(B))^2\,. \tag{50}$$

In order to obtain this inequality, we prove an intermediate result: there exists a constant $C = C(n) > 0$ such that, for every $\Omega \in \mathcal{C}_n$, it holds

$$\mathcal{A}_F(\Omega) \geqslant C\,(P(\Omega) - P(B))\,, \tag{51}$$

provided that the Fraenkel asymmetry of Ω is small. Since, by definition, it holds $\mathcal{A}_F(\Omega) \in [0, 2)$, we have that inequality (51) is not true when Ω is a long and flat domain. However, in this case, inequality (50) can be proved directly, using an estimate in terms of the diameter of the set contained in [56]. We prove (50) combing (51) with the sharp stability result for the Faber-Krahn inequality, proved in [26], that states that there exists a constant $\bar{C} > 0$ such that for every open set Ω with $V(\Omega) = 1$, the following inequality holds

$$\lambda_1(\Omega) - \lambda_1(B) \geqslant \bar{C}\mathcal{A}_F(\Omega)^2. \tag{52}$$

Unfortunately, as we will show providing a class of counter-examples in the bidimensional case, inequality (51) is not true when the difference of perimeter has exponent 3/4. This is the reason for which we cannot use this strategy to prove the conjectured inequality (48).

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Chapter 1

Preliminaries

1.1 Notations

Throughout this thesis, $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$ and $\cdot$ is the standard Euclidean scalar product for $n \geqslant 2$. We denote by $V(\cdot)$ the Lebesgue measure $\mathcal{L}^n$ in $\mathbb{R}^n$ and by $\mathcal{H}^k$, for $k \in [0, n)$, the $k-$dimensional Hausdorff measure in $\mathbb{R}^n$. In the planar case the volume of $E \subseteq \mathbb{R}^2$ will be called sometimes $A(E)$, i.e. the area of the set E. Moreover, we use the following notation: $B_R(x)$ is the ball of $\mathbb{R}^n$ with radius R and centered at x, B is a generic ball such that $V(B) = 1$ and $A_{r_1,r_2}(x)$ is the open annulus $B_{r_2}(x)\backslash\overline{B}_{r_1}(x)$, where $\overline{B}_{r_1}(x)$ is the closed ball such that $r_1 < r_2$. Moreover, we define ω_n as the Lebesgue measure in $\mathbb{R}^n$ of the ball of radius 1, so that $\mathcal{L}^n(B_R(x)) = \omega_n R^n$ and we denote by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$.

If $\Omega \subseteq \mathbb{R}^n$, $\mathrm{Lip}(\partial\Omega)$ (resp. $\mathrm{Lip}(\partial\Omega; \mathbb{R}^n)$) is the class of all Lipschitz functions (resp. vector fields) defined on $\partial\Omega$. If Ω has Lipschitz boundary, for $\mathcal{H}^{n-1}-$ almost every $x \in \partial\Omega$, we denote by $\nu_{\partial\Omega}(x)$ the outward unit Euclidean normal to $\partial\Omega$ at x and by $T_x(\partial\Omega)$ the tangent hyperplane to $\partial\Omega$ at x. Sometimes, when there is no possibility of confution, in order to simplify the notation, we will use ν instead of ν_Ω.

1.2 General facts

1.2.1 Basic definitions

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, open set and let $E \subseteq \mathbb{R}^n$ be a measurable set. We recall now the definition of the perimeter of E in Ω, that is

$$P(E;\Omega) = \sup\left\{ \int_E \mathrm{div}\varphi \, dx : \ \varphi \in C_c^\infty(\Omega;\mathbb{R}^n), \ ||\varphi||_\infty \leqslant 1 \right\}.$$

The perimeter of E in $\mathbb{R}^n$ will be denoted by $P(E)$ and, if $P(E) < \infty$, we say that E is a set of finite perimeter. Some references for results relative to the sets of finite perimeter are for example [95, 3]. We observe that a remarkable feature of this definition is that in this way the perimeter is not affected by modifications on sets of measure 0. Moreover, if E has Lipschitz boundary, we have that

$$P(E) = \mathcal{H}^{n-1}(\partial E). \tag{1.1}$$

In order to deduce properties, it is often very useful to approximate sets of finite perimeter with smooth sets. Therefore, we give the following notion of convergence.

Definition 1.1. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, open set, let $(E_j)_j$ be a sequence of measurable sets in $\mathbb{R}^n$ and let $E \subseteq \mathbb{R}^n$ be a measurable set. We say that $(E_j)_j$ converges in measure in Ω to E, and we write $E_j \to E$, if $\chi_{E_j} \to \chi_E$ in $L^1(\Omega)$, or in other words, if $\lim_{j\to\infty} V((E_j \Delta E) \cap \Omega) = 0$.

We also recall that the perimeter is lower semicontinuous with respect to the local convergence in measure, that means, if the sequence of sets (E_j) converges in measure in Ω to E, then

$$P(E;\Omega) \leqslant \liminf_{j\to\infty} P(E_j;\Omega).$$

As a consequence of the Rellich-Kondrachov theorem, the following compactness result holds and its proof can be found for instance in [3, Theorem 3.39].

Proposition 1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, open set and let $(E_j)_j$ be a sequence of measurable sets of $\mathbb{R}^n$, such that $\sup_j P(E_j;\Omega) < \infty$. Then, there exists a subsequence $(E_{j_k})_k$ converging in measure in Ω to a set E, such that*

$$P(E;\Omega) \leqslant \liminf_{k\to\infty} P(E_{j_k};\Omega).$$

Another useful property concerning the sets of finite perimeter is stated in the next approximation result, see [3, Theorem 3.42].

Proposition 1.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, open set and let E be a set of finite perimeter in Ω. Then, there exists a sequence of smooth, bounded open sets $(E_j)_j$ converging in measure in Ω and such that $\lim_{j\to\infty} P(E_j;\Omega) = P(E;\Omega)$.*

Moreover, if E has Lipschitz boundary, we denote by

$$M(E) = \int_{\partial E} |x|^2 \, d\mathcal{H}^{n-1}$$

its boundary momentum. By their respectively definitions, we have that $P(E)$, $M(E)$ and $V(E)$ satisfy the following scaling properties, for $t > 0$,

$$P(tE) = t^{n-1}P(E), \qquad V(tE) = t^n V(E), \qquad M(tE) = t^{n+1}M(E).$$

We conclude by recalling the classical isoperimetric inequality. We refer the reader, for example, to [100, 36, 40, 123] and to the original paper by De Giorgi [50].

Theorem 1.3. *Let $E \subseteq \mathbb{R}^n$, $n \geqslant 2$, a Borel set with finite Lebesgue measure, then*

$$n\omega_n^{1/n} V(E)^{(n-1)/n} \leqslant P(E) \tag{1.2}$$

and equality holds if and only if E is a ball.

1.2.2 First variation of the Euclidean perimeter

For the content of this Section we refer mainly to [11] and [95]. Let us start from recalling the definition of tangential gradient.

Definition 1.2. Let Ω be an open, bounded subset of $\mathbb{R}^n$ with C^2 boundary and let $u : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function. We can define the tangential gradient of u for almost every $x \in \partial\Omega$ as

$$\nabla^{\partial\Omega} u(x) = \nabla u(x) - \langle \nabla u(x), \nu_{\partial\Omega}(x)\rangle \nu(x),$$

whenever ∇u exists at x.

If we consider a vector field $T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$, we can also define the tangential divergence of T on $\partial\Omega$ by the formula

$$\mathrm{div}^{\partial\Omega} T = \mathrm{div} T - \langle \nabla T\, \nu, \nu\rangle.$$

The following theorem is an extention to hypersurfaces in $\mathbb{R}^n$ of Gauss-Green theorem.

Theorem 1.4. *Let Ω be a subset of $\mathbb{R}^n$ with C^2 boundary. Then there exists a continuous scalar function $H_{\partial\Omega} : \partial\Omega \to \mathbb{R}$ such that, for every $\varphi \in C^1_c(\mathbb{R}^n)$,*

$$\int_{\partial\Omega} \nabla^{\partial\Omega}\varphi(x)\, d\mathcal{H}^{n-1}(x) = \int_{\partial\Omega} \varphi(x) H_{\partial\Omega}(x)\nu(x)\, d\mathcal{H}^{n-1}(x).$$

The scalar function $H_{\partial\Omega} : \partial\Omega \to \mathbb{R}$ is the so-called mean curvature. If we define the Gaussian map associated to Ω as the map

$$u_\Omega : \partial\Omega \to \mathbb{S}^{n-1},$$

that maps $x \in \partial\Omega$ to the external unit normal to $\partial\Omega$ in x, we can observe that $u_{\partial\Omega}$ is of class C^1. The differential of u_Ω in x is a linear application that maps $T_x\Omega$ in itself and that is usually denoted by

$$W_x := d(u_\Omega)_x : T_x\Omega \to T_x\Omega,$$

called Weingarten map. The bilinear form defined on $T_x\Omega$ by

$$\Pi_x(v, w) := (W_x v, w),$$

for every $v, w \in T_x\Omega$ is called second fundamental form associated to $\partial\Omega$ in x and it is symmetric. The eigenvalue of the Weingarten map W_x are called principal curvature of Ω in x and we have that

$$H_{\partial\Omega}(x) = \frac{1}{n-1}\sum_{i=1}^{n-1} \kappa_i, \tag{1.3}$$

where k_i are the principal curvatures.

Remark 1.5. Using the definition of tangential divergence, the Gauss-Green theorem can be reformulated in the following way:

$$\int_{\partial\Omega} \mathrm{div}^{\partial\Omega} T(x)\, d\mathcal{H}^{n-1}(x) = \int_{\partial\Omega} H_{\partial\Omega}(x)\langle T(x), \nu(x)\rangle\, d\mathcal{H}^{n-1}(x),$$

for every $T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$.

A $1-$parameter family of diffeomorphisms of $\mathbb{R}^n$ is a smooth function

$$(x, t) \in \mathbb{R}^n \times (-\epsilon, \epsilon) \mapsto \phi(x, t),$$

for $\epsilon > 0$ such that, for each fixed $|t| < \epsilon$, $\phi(\cdot, t)$ is a diffeomorphism. We consider here a particular class of $1-$parameter family of diffeomorphisms such that $\phi(x, t) = x + tT(x) + O(t^2)$, with $T \in C^0_c(\mathbb{R}^n;\mathbb{R}^n)$. In [95] the following theorem is proved.

Theorem 1.6. *Let Ω be a bounded, open set of $\mathbb{R}^n$ with C^2 boundary and let $\phi(x, t)$ be a $1-$parameter family of diffeomorphisms as previously defined. If we denote by $\Omega(t)$ the image of Ω through $\phi(\cdot, t)$, then*

$$P(\Omega(t)) = P(\Omega) + t\int_{\partial\Omega} \mathrm{div}^{\partial\Omega} T(x)\, d\mathcal{H}^{n-1}(x) + o(t).$$

Using now the Gauss-Green theorem and this last theorem, we obtain the following expression for the first variation of the perimeter of an open set with C^2 boundary:

$$\frac{d}{dt}P(\Omega(t))|_{t=0} = \int_{\partial\Omega} H_{\partial\Omega}(x)\langle T(x), \nu(x)\rangle\, d\mathcal{H}^{n-1}(x).$$

1.2.3 Some properties of convex sets

We recall here some properties of convex sets that we will use in this thesis. We start by recalling the definition of Hausdorff distance between two non-empty compact sets $C, K \subseteq \mathbb{R}^n$, that is (see for instance [118]):

$$d_{\mathcal{H}}(C, K) = \inf \{\varepsilon > 0 \ : \ C \subset K + B_\varepsilon, \ K \subset C + B_\varepsilon\}. \tag{1.4}$$

Note that, in the case C and K are convex sets, we have that $d_{\mathcal{H}}(C, K) = d_{\mathcal{H}}(\partial C, \partial K)$ and that the following rescaling property holds

$$d_{\mathcal{H}}(tC, tK) = t\, d_{\mathcal{H}}(C, K), \quad t > 0.$$

We give now the definition of support function of a convex set.

Definition 1.3. Let K be an open, bounded and convex set of $\mathbb{R}^n$. The support function associated to K is defined as, for every $y \in \mathbb{R}^n$,

$$h_K(y) = \max_{x \in K} (x \cdot y).$$

It is easy to see that the support function associated to a ball of radius R is constantly equal to R.

Remark 1.7. Let K, C be two open, convex and bounded sets of $\mathbb{R}^n$; the following relation holds:

$$d_{\mathcal{H}(C,K)} = ||h_C - h_K||_{L^\infty(\mathbb{S}^{n-1})}$$

For completeness, we give the proof of the following Lemma (see as a reference [3]).

Lemma 1.8. *Let $(K_j)_j$ be a sequence of convex sets of $\mathbb{R}^n$ such that $K_j \to B$ in measure, then $\lim_{j\to\infty} P(K_j) = P(B)$.*

Proof. Since, in the case of convex sets, the convergence in measure implies the Hausdorff convergence, we have that $\lim_{j\to\infty} d_{\mathcal{H}}(K_j, B) = 0$ (see for instance [56]). Thus, for j large enough, there exists ε_j going to 0 as $j \to \infty$, such that

$$(1 - \varepsilon_j)B \subset K_j \subset (1 + \varepsilon_j)B.$$

Being the perimeter monotone with respect to the inclusion of convex sets, we have

$$(1 - \varepsilon_j)^{n-1} P(B) \leqslant P(K_j) \leqslant (1 + \varepsilon_j)^{n-1} P(B)$$

and, if we let j go to infinity, we have the thesis. □

We conclude this paragraph by recalling the following result, that is proved in [56], which gives an upper bound of the diameter of a convex set K, that will be denoted by $\operatorname{diam}(K)$ and it is defined as

$$\operatorname{diam}(K) = \sup\{|x - y| \mid x, y \in K\}. \tag{1.5}$$

Lemma 1.9. *Let $K \subseteq \mathbb{R}^n$, $n \geqslant 2$, be a bounded, open, convex set. There exists a positive constant $C(n)$ such that*

$$\operatorname{diam}(K) \leqslant C(n) \frac{P(K)^{n-1}}{V(K)^{n-2}}. \tag{1.6}$$

1.2.4 Quermassintegrals: definition and properties

For the content of this section we refer, for instance, to [118]. Let $\emptyset \neq K \subseteq \mathbb{R}^n$ be an open, compact and convex set. We define the outer parallel body of K at distance ρ as the Minkowski sum

$$K + \rho B_1 = \{x + \rho y \in \mathbb{R}^n \mid x \in K,\ y \in B_1\}.$$

The Steiner formula asserts that

$$V(K + \rho B_1) = \sum_{i=0}^{n} \binom{n}{i} W_i(K)\rho^i. \tag{1.7}$$

and

$$P(K + \rho B_1) = n \sum_{i=0}^{n-1} \binom{n}{i} W_{i+1}(K)\rho^i, \tag{1.8}$$

where the coefficients $W_i(K)$ are known as quermassintegrals. Some of them have an easy interpretation:

$$W_0(K) = V(K); \qquad nW_1(K) = P(K); \qquad W_n(K) = \omega_n. \tag{1.9}$$

Moreover, we have that

$$W_{n-1}(K) = \frac{\omega_n}{2}\, \omega(K), \tag{1.10}$$

where $\omega(K)$ is called mean width of the convex body K, it is defined as

$$\omega(K) = \frac{1}{n\omega_n} \int_{\mathbb{S}^{n-1}} \left(h_K(x) + h_K(-x)\right)\ d\mathcal{H}^{n-1}(x) \tag{1.11}$$

and it represents the mean value over all possible directions of the distance between parallel supporting hyperplanes to K. Furthermore, we have that

$$\lim_{\rho \to 0^+} \frac{P(K + \rho B_1) - P(K)}{\rho} = n(n-1)W_2(K). \tag{1.12}$$

We recall also the Aleksandrov-Fenchel inequalities

$$\left(\frac{W_j(K)}{\omega_n}\right)^{\frac{1}{n-j}} \geqslant \left(\frac{W_i(K)}{\omega_n}\right)^{\frac{1}{n-i}}, \tag{1.13}$$

for $0 \leqslant i < j < n$, with equality if and only if K is a ball. If we put in the last inequality $i = 0$ and $j = 1$, we obtain the classical isoperimetric inequality, that is:

$$P(K)^{\frac{n}{n-1}} \geqslant n^{\frac{n}{n-1}} \omega_n^{\frac{1}{n-1}} V(K).$$

In particular, we will use the case in (1.13) as $i = 1$ and $j = 2$:

$$W_2(K) \geqslant n^{-\frac{n-2}{n-1}} \omega_n^{\frac{1}{n-1}} P(K)^{\frac{n-2}{n-1}}. \tag{1.14}$$

We denote by $d_e(x)$ the distance function from the boundary of K and we use the following notations:

$$K_t = \{x \in K\ :\ d_e(x) > t\}, \qquad t \in [0, r_K],$$

where r_K is the inradius of K:

$$r_K = \sup_{x \in K} \inf_{y \in \partial K} |x - y|. \tag{1.15}$$

We state now the following two lemmas, whose proofs can be found in [22] and [30].

Lemma 1.10. *Let K be a bounded, convex, open set in $\mathbb{R}^n$. Then, for almost every $t \in (0, r_K)$, we have*

$$-\frac{d}{dt}P(K_t) \geqslant n(n-1)W_2(K_t)$$

and equality holds if K is a ball.

By simply applying the chain rule formula and recalling that $|Dd_e(x)| = 1$ almost everywhere, it remains proved the following.

Lemma 1.11. *Let $f : [0, +\infty) \to [0, +\infty)$ be a non decreasing C^1 function and let $\tilde{f} : [0, +\infty) \to [0, +\infty)$ a non increasing C^1 function. We define $u(x) := f(d_e(x))$, $\tilde{u}(x) := \tilde{f}(d_e(x))$ and*

$$E_{0,t} := \{x \in K \ : \ u(x) > t\},$$
$$\tilde{E}_{0,t} := \{x \in K \ : \ \tilde{u}(x) < t\}.$$

Then,

$$-\frac{d}{dt}P(E_{0,t}) \geqslant n(n-1)\frac{W_2(E_{0,t})}{|Du|_{u=t}}, \tag{1.16}$$

and

$$\frac{d}{dt}P(\tilde{E}_{0,t}) \geqslant n(n-1)\frac{W_2(\tilde{E}_{0,t})}{|D\tilde{u}|_{\tilde{u}=t}}. \tag{1.17}$$

1.2.5 Definitions of some kind of asymmetries

First of all we give the definition of the Fraenkel asymmetry, which is a L^1 distance between sets.

Definition 1.4. Let $\Omega \subseteq \mathbb{R}^n$ be an open set. We define the Fraenkel asymmetry of Ω as

$$\mathcal{A}_F(\Omega) = \inf_{x\in\mathbb{R}^n}\left\{\frac{V(\Omega\Delta B_R(x))}{V(\Omega)} \ : \ B_R(x) \text{ is a ball s.t. } V(B_R(x)) = V(\Omega)\right\} \tag{1.18}$$

We observe that if $V(B) = V(\Omega)$, then $V(\Omega\Delta B) = 2V(\Omega\backslash B) = 2V(B\backslash\Omega)$. Moreover, the infimum in (1.18) is actually a minimum and it is a bounded quantity, since $\mathcal{A}_F(\Omega) \in [0, 2)$.

We can give now the following definition, introduced in [67] as spherical deviation for a convex set when the volume is fixed, that we will denote by $\widetilde{\mathcal{A}}_{\mathcal{H}}(\Omega)$ and that we adapt in the case of fixed perimeter.

Definition 1.5. Let $\Omega \subseteq \mathbb{R}^n$ a bounded open convex set. Then, we define the following asymmetry functional

$$\mathcal{A}_{\mathcal{H}}(\Omega) = \inf_{x\in\mathbb{R}^n}\left\{\frac{d_{\mathcal{H}}(\Omega, B_R(x))}{V(\Omega)} \ : \ B_R(x) \text{ is a ball s.t. } P(B_R(x)) = P(\Omega)\right\}. \tag{1.19}$$

1.2.6 Definition of nearly spherical sets and main properties

In this section we give the definition of nearly spherical sets and we recall some of their basic properties (see for instance [24, 67, 68]). The usual definition is the following.

Definition 1.6. Let $n \geqslant 2$. An open, bounded set $E \subseteq \mathbb{R}^n$ with the origin contained in E is said a nearly spherical set parametrized by v if there exists $v \in W^{1,\infty}(\mathbb{S}^{n-1})$ such that

$$\partial E = \left\{y \in \mathbb{R}^n \colon y = Rx(1 + v(x)),\, x \in \mathbb{S}^{n-1}\right\}, \tag{1.20}$$

where R is the radius of the ball having the same measure of E and $||v||_{W^{1,\infty}(\mathbb{S}^{n-1})} \leqslant 1/2$.

Since in Section 3.1 we fix the perimeter, we will use the following definition of nearly spherical set.

Definition 1.7. Let $n \geqslant 2$. An open, bounded set $E \subseteq \mathbb{R}^n$ with $P(E) = P(B)$ is said a nearly spherical set parametrized by v, if there exists $v \in W^{1,\infty}(\mathbb{S}^{n-1})$ such that

$$\partial E = \left\{ y \in \mathbb{R}^n \colon y = x(1 + v(x)),\, x \in \mathbb{S}^{n-1} \right\}, \tag{1.21}$$

with $||v||_{W^{1,\infty}(\mathbb{S}^{n-1})} \leqslant 1/2$..

Note also that $||v||_{L^\infty} = d_{\mathcal{H}}(E, B)$. In the following, for simplicity, we denote by $\nabla_\tau := \nabla^{\mathbb{S}^{n-1}}$. The perimeter, the volume and the boundary momentum of a nearly spherical set are given by

$$P(E) = \int_{\mathbb{S}^{n-1}} (1 + v(x))^{n-2} \sqrt{(1 + v(x))^2 + |\nabla_\tau v(x)|^2}\, d\mathcal{H}^{n-1}, \tag{1.22}$$

$$V(E) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} (1 + v(x))^n \, d\mathcal{H}^{n-1}, \tag{1.23}$$

$$M(E) = \int_{\mathbb{S}^{n-1}} (1 + v(x))^n \sqrt{(1 + v(x))^2 + |\nabla_\tau v(x)|^2}\, d\mathcal{H}^{n-1}. \tag{1.24}$$

Finally, we recall two lemmas that we will use later. The first one is an interpolation result; for its proof we refer for instance to [67, 68].

Lemma 1.12. *If* $v \in W^{1,\infty}(\mathbb{S}^{n-1})$ *and* $\int_{\mathbb{S}^{n-1}} v \, d\mathcal{H}^{n-1} = 0$*, then*

$$||v||^{n-1}_{L^\infty(\mathbb{S}^{n-1})} \leqslant \begin{cases} \pi \|\nabla_\tau v\|_{L^2(\mathbb{S}^1)} & n = 2 \\ 4||D_\tau v||^2_{L^2(\mathbb{S}^2)} \log \dfrac{8e||D_\tau v||^2_{L^\infty(\mathbb{S}^2)}}{||\nabla_\tau v||^2_{L^2(\mathbb{S}^2)}} & n = 3 \\ C_n ||\nabla_\tau v||^2_{L^2(\mathbb{S}^{n-1})} ||\nabla_\tau v||^{n-3}_{L^\infty(\mathbb{S}^{n-1})} & n \geqslant 4 \end{cases} \tag{1.25}$$

For this second lemma see for instance [68].

Lemma 1.13. *Let* $n \geqslant 2$*. There exists an universal* $\varepsilon_0 < \frac{1}{8}$ *such that, if* E *is a convex, nearly spherical set with* $V(E) = V(B)$ *and* $||v||_{W^{1,\infty}} \leqslant \varepsilon_0$*, then*

$$||\nabla_\tau v||^2_{L^\infty} \leqslant 8 ||v||_{L^\infty}. \tag{1.26}$$

Finally, we prove the following

Lemma 1.14. *Let* $n \geqslant 2$ *and let* $E \subseteq \mathbb{R}^n$ *be a bounded, convex, nearly spherical set with* $||v||_{W^{1,\infty}} \leqslant \varepsilon_0$*, then*

$$d_{\mathcal{H}}(E, E^*) \leqslant \left(16 \left(\frac{9}{8} \right)^n + n + 1 \right) d_{\mathcal{H}}(E, E^\sharp), \tag{1.27}$$

where E^* *and* $E^\sharp$ *are the balls centered at the origin having, respectively, the same perimeter and the same volume as* E*.*

Proof. By the properties of the Hausdorff distance, we get

$$\begin{aligned} d_{\mathcal{H}}(E, E^*) \leqslant d_{\mathcal{H}}(E, E^\sharp) + d_{\mathcal{H}}(E^*, E^\sharp) &= d_{\mathcal{H}}(E, E^\sharp) + \left(\frac{P(E)}{n\omega_n} \right)^{\frac{1}{n-1}} - \left(\frac{V(E)}{\omega_n} \right)^{\frac{1}{n}} \\ &= d_{\mathcal{H}}(E, E^\sharp) + \left(\frac{V(E)}{\omega_n} \right)^{\frac{1}{n}} \left[\left(\frac{P(E)}{n\omega_n^{\frac{1}{n}} V(E)^{\frac{n-1}{n}}} \right)^{\frac{1}{n-1}} - 1 \right]. \end{aligned} \tag{1.28}$$

We stress that, in the square brackets, we have the isoperimetric deficit of E, which is scaling invariant. Let $F \subseteq \mathbb{R}^n$ be a convex, nearly spherical set parametrized by v_F, with $||v_F||_{W^{1,\infty}} \leqslant \varepsilon_0$ and $V(F) = V(B)$, where ε_0 is the universal constant defined in the previous Lemma. Being F nearly spherical and $||v_F||_{W^{1,\infty}} \leqslant \varepsilon_0$, from the isoperimetric inequality, (1.22), Lemma 1.13, and recalling that $\varepsilon_0 < \frac{1}{8}$ we get

$$\begin{aligned}\left(\frac{P(F)}{n\omega_n^{\frac{1}{n}} V(F)^{\frac{n-1}{n}}}\right)^{\frac{1}{n-1}} - 1 &\leqslant \frac{P(F)}{n\omega_n} - 1 \\ &= \frac{1}{n\omega_n}\int_{\mathbb{S}^{n-1}} \left((1+v_F(x))^{n-2}\sqrt{(1+v_F(x))^2 + |\nabla_\tau v_F(x)|^2} - 1\right) \leqslant \\ &\leqslant \left(n + 8\left(\frac{9}{8}\right)^n\right)||v_F||_{L^\infty} + \left(\frac{9}{8}\right)^{n-2}\|\nabla_\tau v_F\|^2_{L^\infty} \leqslant \left(16\left(\frac{9}{8}\right)^n + n\right)||v_F||_{L^\infty}. \end{aligned} \tag{1.29}$$

As a consequence, recalling that $||v_F||_{L^\infty} = d_{\mathcal{H}}(F, B)$,

$$\left(\frac{V(E)}{\omega_n}\right)^{\frac{1}{n}} \left[\left(\frac{P(E)}{n\omega_n^{\frac{1}{n}} V(E)^{\frac{n-1}{n}}}\right)^{\frac{1}{n-1}} - 1\right] \leqslant \left(16\left(\frac{9}{8}\right)^n + n\right) d_{\mathcal{H}}(E, E^\sharp).$$

Using this inequality in (1.28), we get the claim. □

1.3 Anisotropy: basic facts

1.3.1 Definition of Finsler norm

Let F be a convex, even, 1-homogeneous and non negative function defined in $\mathbb{R}^n$. In particular, F is a convex function such that

$$F(t\xi) = |t|F(\xi), \quad t \in \mathbb{R},\ \xi \in \mathbb{R}^n, \tag{1.30}$$

and such that

$$a|\xi| \leqslant F(\xi), \quad \xi \in \mathbb{R}^n, \tag{1.31}$$

for some constant $a > 0$. The hypotheses on F imply that there exists $b \geqslant a$ such that

$$F(\xi) \leqslant b|\xi|, \quad \xi \in \mathbb{R}^n.$$

Moreover, throughout the paper we will assume that $F \in C^2(\mathbb{R}^n \backslash \{0\})$, and

$$[F^p]_{\xi\xi}(\xi) \text{ is positive definite in } \mathbb{R}^n \backslash \{0\},$$

for any $1 < p < +\infty$. The polar function $F^o \colon \mathbb{R}^n \to [0, +\infty[$ of F is defined as

$$F^o(v) = \sup_{\xi \neq 0} \frac{\langle \xi, v\rangle}{F(\xi)}.$$

It is easy to verify that also F^o is a convex function and satisfies properties (1.30) and (1.31). F and F^o are usually called Finsler norm. Furthermore,

$$F(v) = \sup_{\xi \neq 0} \frac{\langle \xi, v\rangle}{F^o(\xi)},$$

that means that the polarity is an involution. The above property implies the following anisotropic version of the Cauchy Schwartz inequality

$$|\langle \xi, \eta \rangle| \leqslant F(\xi) F^o(\eta), \qquad \forall \xi, \eta \in \mathbb{R}^n.$$

We denote by

$$\mathcal{W} = \{\xi \in \mathbb{R}^n \colon F^o(\xi) < 1\},$$

the Wulff shape centered at the origin and we put $\kappa_n = V(\mathcal{W})$. We denote by $\mathcal{W}_r(x_0)$ the set $r\mathcal{W} + x_0$, that is the Wulff shape centered at x_0 with measure $\kappa_n r^n$ and we set $\mathcal{W}_r(0) = \mathcal{W}_r$.

We conclude this paragraph reporting the following properties of F and F^o:

$$\begin{gathered}
\langle \nabla F(\xi), \xi \rangle = F(\xi), \quad \langle \nabla F^o(\xi), \xi \rangle = F^o(\xi), \qquad \forall \xi \in \mathbb{R}^n \backslash \{0\}; \\
F(\nabla F^o(\xi)) = F^o(\nabla F(\xi)) = 1, \quad \forall \xi \in \mathbb{R}^n \backslash \{0\}; \\
F^o(\xi) \nabla F(\nabla F^o(\xi)) = F(\xi) \nabla F^o\left(\nabla F(\xi)\right) = \xi \qquad \forall \xi \in \mathbb{R}^n \backslash \{0\}.
\end{gathered}$$

1.3.2 Anisotropic perimeter and its first variation

Let Ω be a bounded open convex set of $\mathbb{R}^n$; in the following we are fixing a Finsler norm F.

Definition 1.8. Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary. The anisotropic perimeter of Ω is defined as

$$P_F(\Omega) = \int_{\partial\Omega} F(\nu(x))\, d\mathcal{H}^{n-1}(x).$$

Clearly, the anisotropic perimeter of Ω is finite if and only if the usual Euclidean perimeter of Ω, that we denote by $P(\Omega)$, is finite. Indeed, by the quoted properties of F, we obtain that

$$aP(\Omega) \leqslant P_F(\Omega) \leqslant bP(\Omega).$$

Moreover, the following isoperimetric inequality is proved for the anisotropic perimeter, see for istance [2, 47, 63, 38].

Theorem 1.15. *Let Ω be a subset of $\mathbb{R}^n$ with finite perimeter. Then*

$$P_F(\Omega) \geqslant n \kappa_n^{\frac{1}{n}} V(\Omega)^{1-\frac{1}{n}}$$

and the equality holds if and only if Ω is homothetic to a Wulff shape.

Moreover, if K is a bounded convex subset of $\mathbb{R}^2$, and $\delta > 0$, the following Steiner formulas hold (see [4, 118]):

$$V(K + \delta\mathcal{W}) = V(K) + P_F(K)\delta + \kappa\delta^2; \tag{1.32}$$

$$P_F(K + \delta\mathcal{W}) = P_F(K) + 2\kappa\delta. \tag{1.33}$$

Let Ω be a bounded open set of $\mathbb{R}^n$. The anisotropic distance of a point $x \in \Omega$ to the boundary $\partial\Omega$ is defined as

$$d_F(x, \partial\Omega) = \inf_{y \in \partial\Omega} F^o(x - y).$$

By the properties of the Finsler norm F, the distance function satisfies

$$F(Dd_F(x)) = 1 \quad \text{a.e. in } \Omega. \tag{1.34}$$

For the properties of the anisotropic distance function we refer to [46]. We can define also the anisotropic inradius of Ω as

$$r_F(\Omega) = \sup\{d_F(x, \partial\Omega),\ x \in \Omega\}.$$

We denote by

$$\widetilde{\Omega}_t = \{x \in \Omega \mid d_F(x, \partial\Omega) > t\},$$

with $t \in [0, r_F(\Omega)]$. The general Brunn-Minkowski theorem (see [118]) and the concavity of the anisotropic distance function give that the function $P_F(\tilde{\Omega}_t)$ is concave in $[0, r_F(\Omega)]$, hence it is decreasing and absolutely continuous. In [53] the following result is stated.

Lemma 1.16. *For almost every $t \in (0, r_F(\Omega))$,*

$$-\frac{d}{dt} V\left(\tilde{\Omega}_t\right) = P_F(\tilde{\Omega}_t).$$

We give now the following definition.

Definition 1.9. Let Ω be a subset of $\mathbb{R}^n$ with C^∞ boundary. At each point of $\partial\Omega$ we define the F-normal vector

$$\nu^F_{\partial\Omega}(x) = \nabla F(\nu(x)),$$

sometimes called the Cahn-Hoffman field.

We observe that, by the properties of F, we have

$$F^o(\nu^F_{\partial\Omega}) = 1. \tag{1.35}$$

Definition 1.10. Let Ω be a subset of $\mathbb{R}^n$ with C^∞ boundary. For every $x \in \partial\Omega$, we define the F-mean curvature

$$H^F_{\partial\Omega}(x) = \operatorname{div}^{\partial\Omega}\left(\nu^F_{\partial\Omega}(x)\right).$$

In [13, Theorem 3.6] we find the computation of the first variation of the anisotropic perimeter. For more details on this part the reader is referred to [135] and [13].

Theorem 1.17. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^∞ boundary. For $t \in \mathbb{R}$, let $\phi(\cdot, t) : \mathbb{R}^n \to \mathbb{R}^n$ be a family of diffeomorphisms such that $\phi(\cdot, 0) = Id$ and $\phi(\cdot, t) - Id$ has compact support in $\mathbb{R}^n$. Set $\Omega(t)$ the image of Ω through $\phi(\cdot, t)$. Then*

$$\frac{d}{dt} P_F(\Omega(t))|_{t=0} = \int_{\partial\Omega} H^F_{\partial\Omega}(x)\langle\nu(x), g(x)\rangle d\mathcal{H}^{n-1}(x), \tag{1.36}$$

where $g(x) := \dfrac{\partial\phi(x,t)}{\partial t}|_{t=0}$.

1.3.3 Inverse Anisotropic curvature flow in dimension n

Let be $T > 0$; we choose, as in [136],

$$\varphi(x) = \frac{1}{H^F_{\partial\Omega}(x)},$$

and we have that

$$\frac{\partial}{\partial t}\phi(x, t) = \frac{\nu^F_{\partial\Omega}(x)}{H^F_{\partial\Omega}(x)},$$

for every $t \in [0, T]$. This one parameter family of diffeomorphisms gives rise to the so called inverse anisotropic mean curvature flow (IAMCF). Concerning this family of flows, local and global existence and uniqueness have been studied in [136, 85, 115].

Definition 1.11. Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^∞ boundary; Ω is called F-mean convex if its anisotropic mean curvature is strictly positive and, in this case, we say that $\Omega \in C_F^{\infty,+}$.

In [136] is proved that, if $\Omega(0) = \Omega \in C_F^{\infty,+}$, then there exists an unique smooth solution $\phi(\cdot,t)$ of the inverse mean curvature flow in $[0,+\infty]$. Moreover the surface $\phi(\cdot,t) = \Omega(t)$, for every $t > 0$, is the boundary of a smooth convex set in $C_F^{\infty,+}$ that asymptotically converges to a Wulff shape as $t \to +\infty$.

Let Ω be a subset of $\mathbb{R}^n$ with C^∞ boundary. We consider the following transformations:

$$\phi(x,t) = x + t\varphi(x)\nu^F_{\partial\Omega}(x), \tag{1.37}$$

where $\varphi \in C_c^\infty(\Omega)$ and $\nu^F_{\partial\Omega}(x) = \nabla F(\nu_{\partial\Omega}(x))$ is the anisotropic normal. We recall that

$$\Omega(t) := \{x + t\varphi(x)\ \nu^F_{\partial\Omega}(x) \mid x \in \Omega\}.$$

From (1.36), we have that

$$\begin{aligned}\frac{d}{dt}P_F(\Omega(t))|_{t=0} &= \int_{\partial\Omega} H^F_{\partial\Omega}(x)\langle \nu(x), \varphi(x)\nu^F_{\partial\Omega}(x)\rangle\, d\mathcal{H}^{n-1}(x) = \\ &= \int_{\partial\Omega} H^F_{\partial\Omega}(x)\varphi(x)\langle \nu_{\partial\Omega}(x), \nabla F(\nu(x))\rangle\, d\mathcal{H}^{n-1}(x) = \int_{\partial\Omega} H^F_{\partial\Omega}(x)\varphi(x)F(\nu(x))\, d\mathcal{H}^{n-1}(x),\end{aligned}$$

where the last equality holds true because of the properties of a Finsler norm. We recall also the variation of the volume of a set:

$$\frac{d}{dt}V(\Omega(t))|_{t=0} = \int_{\partial\Omega} \varphi(x)F(\nu(x))\, d\mathcal{H}^{n-1}(x).$$

We recall a lemma (see [136]), which will be used in the following. This is the anisotropic version of the Heintze-Karcher inequality, whose proof in the Euclidean case can be found in [115].

Lemma 1.18. *Let Ω be a bounded, open convex set of $\mathbb{R}^n$, then*

$$\int_{\partial\Omega} \frac{F(\nu_{\partial\Omega}(x))}{H^F_{\partial\Omega}(x)}\, d\mathcal{H}^{n-1}(x) \geqslant \int_{\partial\mathcal{W}} \frac{F(\nu_{\partial\mathcal{W}}(x))}{H^F_{\partial\mathcal{W}}(x)}\, d\mathcal{H}^{n-1}(x) \tag{1.38}$$

where $\mathcal{W}$ is a Wulff such that $V(\mathcal{W}) = V(\Omega)$.

1.3.4 Anisotropic curvature flow in the plane

In this final paragraph will restrict to the planar case. We give the definitions of anisotropic curvature and of anisotropic maximum curvature.

Definition 1.12. Let $\Omega \subseteq \mathbb{R}^2$ be an open, bounded set with C^2 boundary. For every $x \in \partial\Omega$, we define the F-anisotropic curvature as

$$k^F_{\partial\Omega}(x) = \operatorname{div}\left(\nu^F_{\partial\Omega}(x)\right).$$

Moreover, we denote by $k^F_{max}(\partial\Omega)$ its maximum over $\partial\Omega$, that is

$$k^F_{max}(\partial\Omega) := ||k^F_{\partial\Omega}||_{L^\infty(\partial\Omega)}.$$

Remark 1.19. We recall that for a Wulff shape of the form $\frac{1}{\lambda}\mathcal{W} \subset \mathbb{R}^2$, with $\lambda > 0$, we have that (for the details of the computation see [12]) for every $x \in \partial\left(\frac{1}{\lambda}\mathcal{W}\right)$

$$k^F_{\partial K}(x) = \lambda.$$

Moreover, Wulff shapes are the only sets with constant anisotropic curvature (see, for example, [101, 134]).

We will need the following result concerning the anisotropic curvature of a convex set, whose proof can be found in [77].

Proposition 1.20 (Wulff-Gage inequality)**.** *Let $K \subseteq \mathbb{R}^2$ be a bounded convex set with C^2 boundary. Then,*

$$\int_{\partial K} (k^F_{\partial K}(x)^2 F(\nu(x)))\, d\mathcal{H}^1(x) \geqslant \frac{\kappa P_F(K)}{A(K)} \tag{1.39}$$

and there is equality if and only if K is the Wulff shape.

We will use the following notations. We consider a family of closed curves $u = u(s,t) : \mathbb{S}^1 \times [0,T] \to \mathbb{R}^2$, where s the arc-length parameter and we use the conventional notation $\partial_s(u(s,t)) = u_s(s,t)$. Moreover, $\tau(s,t) = u_s(s,t) = (\sin(\theta(s,t)), -\cos(\theta(s,t)))$ will be the unit tangent and $\nu(s,t) = (\cos(\theta(s,t)), \sin(\theta(s,t)))$ the unit normal of u; $\theta = \theta(s,t)$ is called the normal angle (determined modulo 2π) and we may use it to parametrize the curve $u(\cdot,t)$. The classical Frenet formulas assert that

$$u_{ss}(s,t) = \tau_s(s,t) = k(s,t)\nu(s,t), \tag{1.40}$$

$$\nu_s(s,t) = -k(s,t)\tau(s,t), \tag{1.41}$$

where k is the scalar curvature. Another usefull relation is the following

$$k(s,t) = \theta_s(s,t). \tag{1.42}$$

Now we give the definition of the anisotropic flow; for more details and for the proofs of the properties below see for istance [43]. In the following, whenever no confution is possible, we shall write τ, ν and k as referred to u, using a notation that will not account for the choice of the curve, otherwise we will specify the curve to which they are referred.

Definition 1.13. The family $u : \mathbb{S}^1 \times [0,T] \to \mathbb{R}^2$ of smooth Jordan curves evolves by anisotropic curvature flow if

$$\frac{\partial u(s,t)}{\partial t} = \left(F(\nu(s,t))\, k^F(s,t)\right) \nu(s,t). \tag{1.43}$$

Remark 1.21. We observe that, since the curve u is smooth and the anisotropy F is elliptic, then we can write the anisotropic curvature as

$$k^F(s,t) = \left(\nabla^2 F(\nu(s,t))\tau(s,t) \cdot \tau(s,t)\right) k(s,t). \tag{1.44}$$

Consequently we have that the anisotropic curvature is controlled from above and from below by the Euclidean curvature. In the following remark we recall some important properties related to the anisotropic curvature flow.

Remark 1.22. If we consider a family of curves $u(\cdot,t)$ flowing by the anisotropic curvature flow, we have that the limiting shape is a round point and there exists a time $\bar{t} \in [0,T)$ such that $u(\cdot,t)$ is convex for $t \in [\bar{t},T)$, even though the initial curve is not convex. For a proof of this fact see for istance [43, 42, 71].

As observed in [97], we can rewrite the anisotropic flow as follows. For semplicity of notation in the following formulas we will not mention the dependence from s and t. So, let us define

$$\phi(\theta) := F(\nu) = F(\cos\theta, \sin\theta)$$

and we observe that by the divergence theorem $k^F = \left(\nabla^2_\xi \left(F^o(\nu)\right)\tau\cdot\tau\right) k$. Since it holds $F^o(\theta) + \left(F^o(\theta)\right)'' = \nabla^2_\xi \left(F^o(\nu)\right)\tau\cdot\tau$, then we have

$$u_t = \psi(\theta) k\nu, \tag{1.45}$$

where

$$\psi(\theta) := \phi(\theta)\left(\phi(\theta) + \phi''(\theta)\right). \tag{1.46}$$

The proof of the following result can be found in [97] (proof of Proposition 1).

Proposition 1.23. *It holds*

$$\left(\partial_t - \psi\partial_{ss}\right)\frac{\left(k^F\right)^2}{2} \leqslant \left(3kh\phi' + h'k\phi\right)\partial_s (k^F)^2 + (k^F)^4, \tag{1.47}$$

where $h = \phi + \phi''$.

In [42] the authors compute the first derivative of the area enclosed by a family of curves that flows by anisotropic curvature flow (see the following Proposition). It is proved simply integrating by parts the formula that gives the area enclosed by a curve γ, that is

$$A(\gamma) = -\frac{1}{2}\int_\gamma \langle \gamma, \nu\rangle \, ds.$$

Proposition 1.24. *Let* $u : \mathbb{S}^1 \times [0,T] \to \mathbb{R}^2$ *a family of smooth Jordan curvan satisfying* (1.43). *If we denote by* $u_t(\cdot) := u(\cdot, t)$ *and by* $A(t)$ *the area enclosed by* u_t, *then we have*

$$\frac{dA(t)}{dt} = -\int_{u_t} F(\nu_{u_t}(s,t)) k^F_{u_t}(s,t) ds, \tag{1.48}$$

where ν_{u_t} *and* $k^F_{u_t}$ *are respectively the unit normal and the anisotropic curvature of the curve* u_t.

Chapter 2

Results about Steklov type problems in the linear case

In the first part of this Chapter we prove a quantitative version of the Weinstock inequality in higher dimension, that states that the ball maximizes the first non trivial Steklov eigenvalue among convex sets with fixed perimeter.

The second part deals with the study of the first Steklov-Laplacian eigenvalue with an internal spherical obstacle. In particular, we prove, via a stability result, that the spherical shell locally maximizes the first eigenvalue among nearly spherical sets, when both the volume and the internal ball are fixed.

2.1 The quantitative Weinstock inequality

2.1.1 Stability of a particular isoperimetric inequality

An isoperimetric inequality for a functional involving the quantities $P(\cdot)$, $M(\cdot)$ and $V(\cdot)$ is proved in [132] in the planar case and then in [31] in any dimension, restricting to the class of convex sets. More precisely, if $E \subseteq \mathbb{R}^n$ is a bounded, open, convex set, it is proved that

$$\mathcal{J}(E) = \frac{M(E)}{P(E)\, V(E)^{\frac{2}{n}}} \geqslant \frac{M(B)}{P(B)\, V(B)^{\frac{2}{n}}} = \omega_n^{\frac{-2}{n}} = \mathcal{J}(B) \tag{2.1}$$

where equality holds only on balls centered at the origin. In the same spirit, if $E \subset \mathbb{R}^n$ is a bounded, open, convex set, we define the following functional

$$I(E) = \frac{M(F)}{V(E)P(E)^{\frac{1}{n-1}}} \tag{2.2}$$

and we prove that the following isoperimetric inequality holds.

Proposition 2.1. *Let $n \geqslant 2$. For every bounded, open, convex set $E \subset \mathbb{R}^n$, it holds*

$$I(E) \geqslant \frac{n}{(n\omega_n)^{\frac{1}{n-1}}} = I(B). \tag{2.3}$$

Equality holds only for balls centered at the origin.

Proof. The proof follows easily by using inequality (2.1), the standard isoperimetric inequality and observing that

$$I(E) = \mathcal{J}(E)\left(\frac{P(E)}{V(E)^{1-\frac{1}{n}}}\right)^{\frac{n-2}{n-1}}.$$

□

Our aim is to prove a quantitative version of (2.3). From now on, we will use the following notation

$$\mathcal{D}(E) = I(E) - \frac{n}{(n\omega_n)^{\frac{1}{n-1}}} = I(E) - I(B). \tag{2.4}$$

Stability for nearly spherical sets

Following Fuglede's approach (see [67]), we first prove a quantitative version of (2.3) for nearly spherical sets as in Definition 1.7, when $n \geqslant 3$.

Theorem 2.2. *Let $n \geqslant 3$ and B the unit ball of $\mathbb{R}^n$ centered at the origin. There exists $\varepsilon = \varepsilon(n) > 0$, such that if $E \subseteq \mathbb{R}^n$ is a nearly spherical set with $P(E) = P(B)$ and $||v||_{W^{1,\infty}(\mathbb{S}^{n-1})} \leqslant \varepsilon$, then*

$$\frac{3^n}{n\omega_n}||v||_{W^{1,1}(\mathbb{S}^{n-1})} \geqslant \mathcal{D}(E) \geqslant \frac{n-2}{4(n-1)}||v||^2_{W^{1,2}(\mathbb{S}^{n-1})}. \tag{2.5}$$

Proof. Setting $v = tu$, with $||u||_{W^{1,\infty}} = 1/2$, we have $||v||_{W^{1,\infty}} = t||u||_{W^{1,\infty}} = t/2$. Thus, using the expressions of $P(E)$ and $M(E)$ given in (1.22) and (1.24), we get

$$\begin{aligned}\mathcal{D}(E) &= \frac{n}{P(B)^{\frac{1}{n-1}}}\left(\frac{\displaystyle\int_{\mathbb{S}^{n-1}}(1+tu(x))^n\sqrt{(1+tu(x))^2+t^2|\nabla_\tau u(x)|^2}\,d\mathcal{H}^{n-1}}{\displaystyle\int_{\mathbb{S}^{n-1}}(1+tu(x))^n\,d\mathcal{H}^{n-1}} - 1\right) \qquad (2.6)\\ &= \frac{n}{P(B)^{\frac{1}{n-1}}}\left(\frac{\displaystyle\int_{\mathbb{S}^{n-1}}(1+tu(x))^n\left(\sqrt{(1+tu(x))^2+t^2|\nabla_\tau u(x)|^2}-1\right)d\mathcal{H}^{n-1}}{nV(E)}\right).\end{aligned}$$

Now we prove the lower bound in (2.5). Firstly we take into account the numerator in (2.6). Let $f_k(t) = (1+tu)^k\sqrt{(1+tu)^2+t^2|\nabla_\tau u|^2}$. An elementary calculation shows that

$$\begin{aligned}f_k(0) = 1, \qquad f_k'(0) = (k+1)u, \qquad f_k''(0) &= (k+1)ku^2 + |\nabla_\tau u|^2\\ f_k'''(\tau) &\leqslant 2(k+2)(k+1)k\left(|u|^3 + |u||\nabla_\tau u|^2\right)\end{aligned} \tag{2.7}$$

for any $\tau \in (0,t)$. Thus, since the numerator of (2.6) is given by $f_n(t) - (1+tu)^n$, using the Lagrange expression of the remainder term, we can Taylor expand up to the third order, obtaining

$$\begin{aligned}&\int_{\mathbb{S}^{n-1}}(1+tu(x))^n\left(\sqrt{(1+tu(x))^2+t^2|\nabla_\tau u(x)|^2}-1\right)d\mathcal{H}^{n-1}\\ &\qquad\geqslant t\int_{\mathbb{S}^{n-1}}u\,d\mathcal{H}^{n-1} + nt^2\int_{\mathbb{S}^{n-1}}u^2\,d\mathcal{H}^{n-1} + \frac{1}{2}t^2\int_{\mathbb{S}^{n-1}}|\nabla_\tau u|^2\,d\mathcal{H}^{n-1}\\ &\qquad\qquad - C_1(n)\varepsilon t^2\int_{\mathbb{S}^{n-1}}\left(u^2+|\nabla_\tau u|^2\right)d\mathcal{H}^{n-1}.\end{aligned} \tag{2.8}$$

Since $P(E) = P(B)$, we have

$$\int_{\mathbb{S}^{n-1}} (1+tu(x))^{n-2}\sqrt{(1+tu(x))^2+t^2|\nabla_\tau u(x)|^2}\,d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} 1 d\mathcal{H}^{n-1}. \tag{2.9}$$

Using (2.7) for f_{n-2}, we infer

$$\begin{aligned} t\int_{\mathbb{S}^{n-1}} u d\mathcal{H}^{n-1} \geqslant -\frac{n-2}{2}t^2\int_{\mathbb{S}^{n-1}} u^2 d\mathcal{H}^{n-1} - \frac{t^2}{2(n-1)}\int_{\mathbb{S}^{n-1}} |\nabla_\tau u|^2 d\mathcal{H}^{n-1} \\ - C_2(n)\varepsilon t^2\int_{\mathbb{S}^{n-1}} \left(u^2+|\nabla_\tau u|^2\right) d\mathcal{H}^{n-1}. \end{aligned} \tag{2.10}$$

Since $n \geqslant 3$, using inequality (2.10) in (2.8), we get

$$\begin{aligned} &\int_{\mathbb{S}^{n-1}} (1+tu(x))^n \left(\sqrt{(1+tu(x))^2+t^2|\nabla_\tau u(x)|^2}-1\right) d\mathcal{H}^{n-1} \\ &\geqslant \left(\frac{n+2}{2}-C_3(n)\varepsilon\right)t^2\int_{\mathbb{S}^{n-1}} u^2 d\mathcal{H}^{n-1} + \left(\frac{n-2}{2(n-1)}-C_3(n)\varepsilon\right)t^2\int_{\mathbb{S}^{n-1}} |\nabla_\tau u|^2 d\mathcal{H}^{n-1} \\ &\geqslant \left(\frac{n-2}{2(n-1)}-C_3(n)\varepsilon\right)t^2\int_{\mathbb{S}^{n-1}} u^2+|\nabla_\tau u|^2 d\mathcal{H}^{n-1}, \end{aligned} \tag{2.11}$$

where $C_3(n) = C_1(n) + C_2(n)$. Choosing $\varepsilon = \frac{n-2}{4C_3(n-1)}$, we obtain

$$\mathcal{D}(E) \geqslant \frac{n-2}{4(n-1)}||tu||^2_{W^{1,2}(\mathbb{S}^{n-1})} = \frac{n-2}{4(n-1)}||v||^2_{W^{1,2}(\mathbb{S}^{n-1})},$$

which is the lower bound in (2.5). Then, recalling that $\|v\|_\infty \leqslant \frac{1}{2}$ we have

$$\begin{aligned} \frac{M(E)}{nV(E)} - 1 &= \frac{\int_{\mathbb{S}^{n-1}} (1+v(x))^n \left(\sqrt{(1+v(x))^2+|D_\tau v(x)|^2}-1\right) d\mathcal{H}^{n-1}}{nV(E)} \\ &\leqslant \left(\frac{3}{2}\right)^n \frac{\int_{\mathbb{S}^{n-1}} \left(\sqrt{(1+|v(x)|)^2+|D_\tau v(x)|^2}-1\right) d\mathcal{H}^{n-1}}{nV(E)} \\ &\leqslant \left(\frac{3}{2}\right)^n \frac{\int_{\mathbb{S}^{n-1}} \left(\sqrt{(1+|v(x)|+|D_\tau v(x)|)^2}-1\right) d\mathcal{H}^{n-1}}{nV(E)} \\ &\leqslant \left(\frac{3}{2}\right)^n \frac{\int_{\mathbb{S}^{n-1}} (|v(x)|+|D_\tau v(x)|)\; d\mathcal{H}^{n-1}}{nV(E)} \leqslant \frac{3^n}{n\omega_n}||v||_{W^{1,1}(\mathbb{S}^{n-1})}, \end{aligned} \tag{2.12}$$

where last inequality follows from the following estimate

$$nV(E) = \int_{\mathbb{S}^{n-1}} (1+v(x))^n \; d\mathcal{H}^{n-1} \geqslant n\omega_n \left(\frac{1}{2}\right)^n.$$

□

Remark 2.3. Observe that the proof of the lower bound in (2.5) does not seem to work in the planar case. The reason is that for $n = 2$ the coefficient of $\|D_\tau u\|_{L^2}$ in (2.11) could be negative.

Stability for convex sets

Before completing the proof of the quantitative version of the inequality (2.3), we need the following useful technical lemmas.

Lemma 2.4. *Let $n \geqslant 2$. There exists $M > 0$ such that, if $K \subseteq \mathbb{R}^n$ is an open, convex set with finite perimeter and $I(K) \leqslant \dfrac{2n}{(n\omega_n)^{\frac{1}{n-1}}}$, then $K \subset Q_M$, where Q_M is the hypercube centered at the origin with edge M.*

Proof. Since the functional is scaling invariant, we can assume $V(F) = 1$. Let $L > 1$, we have

$$\begin{aligned} M(K) = \int_{\partial K} |x|^2 d\mathcal{H}^{n-1} &= \int_{(\partial K)\cap Q_L} |x|^2 d\mathcal{H}^{n-1} + \int_{\partial K \setminus Q_L} |x|^2 d\mathcal{H}^{n-1} \\ &\geqslant \int_{\partial K \cap Q_L} |x|^2 d\mathcal{H}^{n-1} + L^2 P(K; C(Q_L)), \end{aligned}$$

where by $C(Q_L)$ we denote the complementary set of Q_L in $\mathbb{R}^n$. Since K is convex, also $K \cap Q_L$ is convex and then

$$P(K) \leqslant P(K; C(Q_L)) + P(K; Q_L) \leqslant P(K; C(Q_L)) + 2nL^{n-1}, \tag{2.13}$$

by the monotonicity of the perimeter. Suppose $P(K) > L^n$; then, equation (2.13) gives $P(K; C(Q_L)) \geqslant L^n - 2nL^{n-1}$ and, as a consequence,

$$I(K) \geqslant \frac{\displaystyle\int_{\partial K \cap Q_L} |x|^2 d\mathcal{H}^{n-1} + L^2 P(K; C(Q_L))}{(P(K; C(Q_L)) + 2nL^{n-1})^{\frac{1}{n-1}}} > \frac{L^{n+2} - L^{n+1}}{L^{\frac{n}{n-1}}}. \tag{2.14}$$

The previous inequality leads to a contradiction for L large enough, since we are assuming $I(K) < \dfrac{2n}{(n\omega_n)^{\frac{1}{n-1}}}$, while the last term of the above inequality diverges when $L \to \infty$. Thus, there exists L_0 such that, for every convex set K with $I(K) \leqslant \dfrac{2n}{(n\omega_n)^{\frac{1}{n-1}}}$, we have $P(K) < L_0^n$. Since $V(K) = 1$ and $P(K) \leqslant L_0^n$, using (1.6), we get

$$\operatorname{diam}(K) \leqslant C(n) L_0^{n(n-1)}.$$

The last inequality proves (2.13), if we choose $M = C(n) L_0^{n(n-1)}$. □

Lemma 2.5. *Let $(K_j) \subseteq \mathbb{R}^n$, $n \geqslant 2$, be a sequence of convex sets such that $I(K_j) \leqslant \dfrac{2n}{(n\omega_n)^{\frac{1}{n-1}}}$ and $P(K_j) = P(B)$. Then, there exists a convex set $K \subseteq \mathbb{R}^n$ with $P(K) = P(B)$ and such that, up to a subsequence,*

$$V(K_j \Delta F) \to 0 \quad \textit{and} \quad I(K) \leqslant \liminf I(K_j). \tag{2.15}$$

Proof. The existence of the limit set K comes from the proof of Lemma 4.12: since $I(K_j) < \dfrac{2n}{(n\omega_n)^{\frac{1}{n-1}}}$, there exists $M > 0$ such that $K_j \subset Q_M$ and $P(K_j) = P(B)$ for every $i \in \mathbb{N}$. Thus, the sequence $\{\chi_{K_j}\}_{j\in\mathbb{N}}$ is precompact in $BV(Q_M)$ and so there exists a subsequence and a set K such that

$V(K\Delta K_j) \to 0$. Moreover, from Lemma 1.8, we have that $P(K) = P(B)$. Note that we can write

$$M(K) = \sup\left\{\int_K \operatorname{div}\left(|x|^2\phi(x)\right)dx, \quad \phi \in C_c^1(Q_M, \mathbb{R}^n), \quad ||\phi||_\infty \leqslant 1\right\}.$$

Observing that

$$\int_K |\operatorname{div}\left(|x|^2\phi(x)\right)|dx \leqslant M||\operatorname{div}\phi||_\infty + M^2,$$

using the dominate convergence theorem, we have that the functional

$$K \to \int_K \operatorname{div}\left(|x|^2\phi(x)\right)dx$$

is continuous with respect to the L^1 convergence. Hence, since $M(K)$ is obtained by taking the supremum of continuous functionals, it is lower semicontinuous. As a consequence, we obtain inequality (2.15). $\square$

The next result allows us to reduce the study of the stability issue to nearly spherical sets.

Lemma 2.6. *Let $n \geqslant 2$. For every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that, if $E \subseteq \mathbb{R}^n$ is a bounded, open, convex set with $P(E) = P(B)$ and $\mathcal{D}(E) < \delta_\varepsilon$, with $\mathcal{D}(E)$ defined as in* (2.4)*, then there exists a Lipschitz function $v \in W^{1,\infty}(\mathbb{S}^{n-1})$ such that E is a nearly spherical set parametrized by v and $\|v\|_{W^{1,\infty}} \leqslant \varepsilon$.*

Proof. Firstly, we prove that $d_{\mathcal{H}}(E, B) < \varepsilon$. Suppose by contraddiction that there exists $\varepsilon_0 > 0$ such that, for every $j \in \mathbb{N}$, there exists a convex set E_j with $I(E_j) - \frac{n}{(n\omega_n)^{\frac{1}{n-1}}} < \frac{1}{j}$, $d_{\mathcal{H}}(E_j, B) \geqslant \varepsilon_0$ and $P(E_j) = P(B)$. By Lemma 2.5, we have that there exists a convex set E such that E_j converges to E in measure and $P(E) = P(B)$. From the semicontinuouity of $M(E)$, we have that $I(E) \leqslant \liminf I(E_j) \leqslant \frac{n}{(n\omega_n)^{\frac{1}{n-1}}}$. Since B is the only minimizer of the functional I, we obtain the contradiction. Then, since E is convex and $d_{\mathcal{H}}(E, B) \leqslant \varepsilon$, E contains the origin and so there exists a Lipschitz function $v \in L^\infty(\mathbb{S}^{n-1})$, with $||v||_\infty < \varepsilon$, such that

$$\partial E = \{x(1 + v(x)),\, x \in \mathbb{S}^{n-1}\}.$$

Now, in order to complete the proof, we have only to show that $\|v\|_{W^{1,\infty}}$ is small when $\mathcal{D}(E)$ is small. This is a consequence of Lemma 1.13. $\square$

Now we can prove the stability result for the inequality (2.3). We first consider the case $n \geqslant 3$. The two dimensional case will be discussed separately in the next section.

Theorem 2.7. *Let $n \geqslant 3$. There exists $\delta > 0$ such that if $E \subseteq \mathbb{R}^n$ is a bounded, open, convex set with $\mathcal{D}(E) \leqslant \delta$, then*

$$\left(\frac{n\omega_n}{P(E)}\right)^{1/(n-1)} d_{\mathcal{H}}(E, E^*) \leqslant \begin{cases} \beta\sqrt{\mathcal{D}(E)\log\frac{1}{\mathcal{D}(E)}} & n = 3 \\ \beta_n\left(\mathcal{D}(E)\right)^{\frac{2}{n+1}} & n \geqslant 4, \end{cases} \tag{2.16}$$

where $\mathcal{D}(E)$ is defined in (2.4) *and E^* is the ball centered at the origin with $P(E^*) = P(E)$ and*

$$\beta = \frac{2^{-\frac{29}{6}}}{9}, \quad \beta_n = \left(\frac{n-2}{4(n-1)C_n^{\frac{1}{n-1}}}2^{-\frac{5n-7}{2}}\right)^{\frac{2}{n+1}}\left(16\left(\frac{9}{8}\right)^n + n + 1\right)^{-1} \tag{2.17}$$

Remark 2.8. We observe that inequality (2.16) implies the following

$$\mathcal{A}_{\mathcal{H}}(E) \leqslant \begin{cases} \beta\sqrt{\mathcal{D}(E)\log\frac{1}{\mathcal{D}(E)}} & n = 3 \\ \beta_n\left(\mathcal{D}(E)\right)^{\frac{2}{n+1}} & n \geqslant 4, \end{cases} \tag{2.18}$$

where $\mathcal{A}_{\mathcal{H}}(E)$ is the asymmetry defined in (1.19). We emphasize that (2.16) and (2.18) are not equivalent, because $\mathcal{A}_{\mathcal{H}}(E)$ is in general different from $d_{\mathcal{H}}(E, E^*)$, since one does not know where is centered the optimal ball for (1.19). For istance, if E is a ball not centered at the origin, we have that $\mathcal{A}_{\mathcal{H}}(E) = 0$, but $d_{\mathcal{H}}(E, E^*) > 0$. On the other hand, since the functional $I(\cdot)$ is not translational invariant, it admits a very unique minimizer once a value of the perimeter is fixed, that is the ball centered at the origin and with the right radius. Thus, it seems more reasonable to use $d_{\mathcal{H}}(E, E^*)$ in (2.16), since it measures how different is the set E from *the* minimizer of $I(\cdot)$.

Proof. Since the functional I is scaling invariant, we can suppose that E is a convex set with $P(E) = P(B)$. We fix now $\varepsilon > 0$. Using Lemma 2.6, we can suppose that there exists $v \in W^{1,\infty}(\mathbb{S}^{n-1})$ with $||v||_{W^{1,\infty}} < \varepsilon$ such that

$$\partial E = \{x(1 + v(x)),\, x \in \mathbb{S}^{n-1}\}.$$

Then, if we take ε small enough, by Theorem 2.19, we obtain

$$\mathcal{D}(E) \geqslant \frac{n-2}{4(n-1)}||v||^2_{W^{1,2}(\mathbb{S}^{n-1})}.$$

Let $F = \lambda E$, with λ such that $V(F) = V(B)$. From the isoperimetric inequality, it follows that $\lambda > 1$. Since the quantity $I(E)$ is scaling invariant, we have that $I(F) = I(E)$ and, from the definition of F, that

$$\partial F = \{\lambda x(1 + v(x)), x \in \mathbb{S}^{n-1}\} = \{x(1 + (\lambda - 1 + \lambda v(x))), x \in \mathbb{S}^{n-1}\}. \tag{2.19}$$

Using the definition of λ , we obtain

$$\lambda^n - 1 = \frac{V(B)}{V(E)} - 1 = \frac{\sum_{k=1}^n \binom{n}{k} \displaystyle\int_{\mathbb{S}^{n-1}} v^k \, d\mathcal{H}^{k-1}}{V(E)}$$

and, as a consequence,

$$\lambda - 1 = \frac{\sum_{k=1}^n \binom{n}{k} \displaystyle\int_{\mathbb{S}^{n-1}} v^k \, d\mathcal{H}^{k-1}}{V(E)\sum_0^{n-1}\lambda^k}. \tag{2.20}$$

Let now $h(x) = \lambda - 1 + \lambda v(x)$. Note that $||h||_{W^{1,\infty}} < 2^n||v||_{W^{1,\infty}}$ and that $\lambda^n \in (1, 2)$. Moreover, using Hölder inequality, it is easy to check that

$$||h||^2_{L^2(\mathbb{S}^{n-1})} \leqslant 2^{n+2}||v||^2_{L^2(\mathbb{S}^{n-1})} \quad \text{and} \quad ||\nabla_\tau h||^2_{L^2(\mathbb{S}^{n-1})} \leqslant 2^{1/n}||D_\tau v||^2_{L^2(\mathbb{S}^{n-1})}.$$

Thus,

$$\mathcal{D}(F) = \mathcal{D}(E) \geqslant \frac{n-2}{4(n-1)}||v||^2_{W^{1,2}(\mathbb{S}^{n-1})} \geqslant 2^{-n-1}\frac{n-2}{4(n-1)}||h||^2_{W^{1,2}(\mathbb{S}^{n-1})}. \tag{2.21}$$

Let $g = (1 + h)^n - 1$. Then, since $V(F) = V(B)$, we have $\int_{\mathbb{S}^{n-1}} g \, d\mathcal{H}^{n-1} = 0$ and, from the smallness assumption on u, we immediately have $\frac{1}{2}|h| \leqslant |g| \leqslant 2|h|$ and $\frac{1}{2}|\nabla h| \leqslant |\nabla g| \leqslant 2|\nabla h|$.

Now we have to distinguish the cases $n = 3$ and $n \geqslant 4$, since we are going to apply the interpolation Lemma 1.12 to g. In the case $n \geqslant 4$, recalling that C_n is the constant given by the Sobolev embedding in Lemma 1.12, we get

$$||h||_\infty \leqslant 2||g||_\infty \leqslant 2C_n||D_\tau g||_{L^2(\mathbb{S}^{n-1})}^{\frac{2}{n-1}}||\nabla_\tau g||_{L^\infty(\mathbb{S}^{n-1})}^{\frac{n-3}{n-1}}$$
$$\leqslant C_n||\nabla_\tau h||_{L^2(\mathbb{S}^{n-1})}^{\frac{2}{n-1}}||\nabla_\tau h||_{L^\infty(\mathbb{S}^{n-1})}^{\frac{n-3}{n-1}} \leqslant 8^{\frac{n-3}{2(n-1)}}C_n||D_\tau h||_{L^2(\mathbb{S}^{n-1})}^{\frac{2}{n-1}}||h||_{L^\infty(\mathbb{S}^{n-1})}^{\frac{n-3}{2(n-1)}},$$

where in the last inequality we use (1.26). From the above chain of inequalities we deduce

$$||h||_{L^\infty}^{\frac{n+1}{2}} \leqslant 8^{\frac{n-3}{2}}C_n^{\frac{1}{n-1}}||\nabla_\tau h||_{L^2(\mathbb{S}^{n-1})}^2$$

and finally, recalling that $F = \lambda E$ and $V(F) = V(B)$, we get

$$\mathcal{D}(E) \geqslant 2^{-n-1}\frac{n-2}{4(n-1)}||D_\tau h||_{L^2(\mathbb{S}^{n-1})}^2 \geqslant \gamma_n||h||_{L^\infty}^{\frac{n+1}{2}} = \gamma_n d_{\mathcal{H}}(F,B)^{\frac{n+1}{2}} = \gamma_n\left(\frac{d_{\mathcal{H}}(E,E^\sharp)}{V(E)^{\frac{1}{n}}}\right)^{\frac{n+1}{2}}, \tag{2.22}$$

where $\gamma_n = \dfrac{n-2}{4(n-1)C_n^{\frac{1}{n-1}}}2^{-\frac{5n-7}{2}}$. So, using (1.27) and the isoperimetric inequality, we obtain the desired result (2.16) in the case $n \geqslant 4$. We proceed in an analogous way in the case $n = 3$. Firstly we observe that, by definition of h it is quickly checked that $||v||_{W^{1,1}(\mathbb{S}^2)} \leqslant ||h||_{W^{1,1}(\mathbb{S}^2)}$. Then, the upper bound in (2.16) in terms of h, can be written as follows

$$\mathcal{D}(E) = \mathcal{D}(F) \leqslant \bar{C}||h||_{W^{1,1}(\mathbb{S}^2)}, \tag{2.23}$$

with $\bar{C}$ positive costant depending on the dimension. Applying Lemma 1.12 to g and using Lemma 1.13, we obtain:

$$||h||_\infty^2 \leqslant 4||g||_\infty^2 \leqslant 16||D_\tau g||_{L^2(\mathbb{S}^2)}^2 \log\left[\frac{8e||D_\tau g||_\infty^2}{||\nabla_\tau g||_{L^2(\mathbb{S}^2)}^2}\right]$$
$$\leqslant 64||D_\tau h||_{L^2(\mathbb{S}^2)}^2 \log\left[\frac{2^7e||D_\tau h||_\infty^2}{||\nabla_\tau h||_{L^2(\mathbb{S}^2)}^2}\right] \leqslant 64||D_\tau h||_{L^2(\mathbb{S}^2)}^2 \log\left[\frac{2^{10}e\ ||v||_\infty}{||\nabla_\tau v||_{L^2(\mathbb{S}^2)}^2}\right].$$

Choosing now $||h||_\infty$ small enough, from the upper bound in (2.5), we have

$$||h||_\infty^2 \leqslant 64||\nabla h||_{L^2(\mathbb{S}^2)}^2 \log\left[\frac{1}{\mathcal{D}(E)}\right], \tag{2.24}$$

and, as a consquence, using (2.5) and (2.24),

$$\mathcal{D}(E)\log\left(\frac{1}{\mathcal{D}(E)}\right) \geqslant \frac{1}{8}||\nabla_\tau v||_{L^2(\mathbb{S}^2)}^2 \log\left(\frac{1}{\mathcal{D}(E)}\right) \geqslant 2^{-\frac{29}{3}}||h||_\infty^2 \frac{\log\left(\frac{1}{\mathcal{D}(E)}\right)}{\log\left(\frac{1}{\mathcal{D}(E)}\right)} = 2^{-\frac{29}{3}}||h||_{L^\infty(\mathbb{S}^2)}^2.$$

□

2.1.2 Optimality issue.

In this Section we will show the sharpness of inequality (2.16) and, as a consequence, the sharpness for the exponent in inequality (2.16). We start by taking into exam the case $n = 3$.

Theorem 2.9. *Let $n = 3$. There exists a family of convex sets $\{E_\alpha\}_{\alpha>0}$ such that for every α*

$$\mathcal{D}(E_\alpha) \to 0, \quad \text{when } \alpha \to 0$$

and

$$d_{\mathcal{H}}(E_\alpha, E_\alpha^*) = C\sqrt{\mathcal{D}(E_\alpha)\log\frac{1}{\mathcal{D}(E_\alpha)}} \tag{2.25}$$

where C is a suitable positive constant independent of α.

Proof. We follow the idea contained in [67] (Example 3.1) and recall it here for the convenience of the reader. Let $\alpha \in (0, \pi/2)$ and consider the following function $\omega = \omega(\varphi)$ defined over $\mathbb{S}^2$ and depending only on the spherical distance φ, with $\varphi \in [0, \pi]$, from a prescribed north pole $\xi^* \in \mathbb{S}^2$:

$$\omega = \omega(\varphi) = \begin{cases} -\sin^2\alpha\log(\sin\alpha) + \sin\alpha\,(\sin\alpha - \sin\varphi) & \text{for } \sin\varphi \leqslant \sin\alpha \\ -\sin^2(\alpha)\log(\sin\varphi) & \text{for } \sin\varphi \geqslant \sin\alpha. \end{cases} \tag{2.26}$$

Let $g := \omega - \bar{\omega}$, with $\bar{\omega}$ the mean value of ω, i.e.

$$\bar{\omega} = \int_0^{\pi/2} \omega(\varphi)\sin\varphi\, d\varphi = (1 - \log 2)\,\alpha^2 + O(\alpha^3),$$

when α goes to 0, and let

$$R := (1 + 3g)^{1/3} = 1 + h.$$

The C^1 function $R = R(\varphi)$ determines in polar coordinates (R, φ) a planar curve. We rotate this curve about the line $\xi^*\mathbb{R}$, determining in this way the boundary of a convex and bounded set, that we call E_α. We can observe that h and g are the same fuctions cointained in the proof of Theorem 2.7. The set E_α is indeed a nearly spherical set, which has h as a representative function and with $V(E_\alpha) = V(B)$. Therefore, taking into account the computations contained in the proof of Theorem 2.7 relative to the functions h and g and the ones contained in [67] combined with (2.5), we have

$$||g||_\infty = \alpha^2\log\frac{1}{\alpha} + O(\alpha^2), \tag{2.27}$$

$$||h||_\infty \geqslant \frac{1}{2}||g||_\infty = \frac{1}{2}\alpha^2\log\frac{1}{\alpha} + O(\alpha^2), \tag{2.28}$$

and

$$||\nabla h||_2^2 = ||\nabla g||_2^2 = \alpha^4\log\left(\frac{1}{\alpha}\right) + O(\alpha^4).$$

Using (2.23), we obtain:

$$\mathcal{D}(E_\alpha) = O\left(\alpha^4\log\frac{1}{\alpha}\right) \tag{2.29}$$

Consequently

$$\mathcal{D}(E_\alpha)\log\left(\frac{1}{\mathcal{D}(E_\alpha)}\right) = O\left(\alpha^2\log\frac{1}{\alpha}\right)^2. \tag{2.30}$$

So, we have that $\mathcal{D}(E_\alpha) \to 0$ as α goes to 0 and, combining (2.28) with (2.30), we have the validity of (2.25). $\square$

We show now the sharpness of the quantitative Weinstock inequality in dimension $n \geqslant 4$.

Theorem 2.10. *Let $n \geqslant 4$. There exists a family of convex sets $\{P_\alpha\}_{\alpha>0}$ such that*

$$\mathcal{D}(P_\alpha) \to 0, \quad \text{when } \alpha \to 0$$

and

$$d_{\mathcal{H}}(P_\alpha, P_\alpha^*) \geqslant C(n) \left(\mathcal{D}(P_\alpha)\right)^{2/(n+1)},$$

where $C(n)$ is a suitable positive constant.

Proof. In this proof we follow the costruction given in [67] (Example 3.2). Let $\alpha \in]0, \pi/2[$ and let P_α be the convex hull of $B \cup \{-p, p\}$, where $p \in \mathbb{R}^n$ is given by

$$|p| = \frac{1}{\cos\alpha}.$$

We have that

$$V(P_\alpha) = \omega_n + \frac{2}{n(n+1)}\omega_{n-1}\alpha^{n+1} + O(\alpha^{n+3})$$

and

$$P(P_\alpha) = nV(P_\alpha).$$

We provide here the computation of the boundary momentum, that is

$$M(P_\alpha) = \frac{2\omega_{n-1}}{n(n+1)} \frac{(\sin(\alpha))^{(n-1)}}{\cos(\alpha)} \left(n^2 + n + 2\tan^2(\alpha)\right) + 2(n-1) \left[\frac{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}{2\Gamma\left(\frac{n}{2}\right)} - \int_0^\alpha \sin^{n-2}(\theta)\, d\theta \right]. \tag{2.31}$$

Since $n > 2$, we have

$$(n\omega_n)^{\frac{1}{n-1}} V(P_\alpha) P(P_\alpha)^{\frac{1}{n-1}} \mathcal{D}(P_\alpha) = (n\omega_n)^{\frac{1}{n-1}} \frac{2\omega_{n-1}}{n+1} \frac{(n-2)}{n(n-1)} \alpha^{n+1} + o(\alpha^{n+3}).$$

Since $d_{\mathcal{H}}(P_\alpha, P_\alpha^*)$ behaves asimptotically as α^2, we have proved the desired claim. □

2.1.3 The planar case

In this section we discuss the stability of the isoperimetric inequality (2.3) in the plane. This case is treated in a different way since the proof given in the previous section does not seem to be adapted to the planar case, as explained in Remark 2.3. Moreover, we observe that, in two dimension, the inequality (2.1) contained in [31] and the inequality (2.3) are proved by Weinstock in the technical report [133], using the representation of a two dimensional convex set via its support function. The section containing this result in the convex case was removed from the published work of Weinstock [132] and discovered later by the authors in [31], who reported it the Appendix of their work.

Let $E \subset \mathbb{R}^2$ be an open, convex set in the plane containing the origin and let $h(\theta)$ be the support function of E with $\theta \in [0, 2\pi]$. Weinstock proved in [132] the following inequality (see also [31] for details)

$$\pi M(E) - P(E)V(E) \geqslant \frac{P(E)}{2} \int_0^{2\pi} p^2(\theta)\, d\theta \geqslant 0, \tag{2.32}$$

where, for every $\theta \in [0, 2\pi]$, $p(x)$ is defined by

$$h(\theta) = \frac{P(E)}{2\pi} + p(\theta).$$

By the definition of support function, it holds

$$\int_0^{2\pi} h(\theta)\, d\theta = P(E). \tag{2.33}$$

Moreover, since E is convex, we have

$$h(\theta) + h''(\theta) \geqslant 0, \tag{2.34}$$

where h'' has to be understood in the distributional sense. Then, the function p verifies

$$\int_0^{2\pi} p(\theta)\, d\theta = 0,$$

and

$$\frac{P(E)}{2\pi} + p(\theta) + p''(\theta) \geqslant 0. \tag{2.35}$$

We observe that

$$\|p\|_{L^\infty([0,2\pi])} = d_{\mathcal{H}}(E, E^*), \tag{2.36}$$

where E^* is the disc centered at the origin having the same perimeter as E. Consider $\theta_0 \in [0, 2\pi]$ such that $\|p\|_{L^\infty} = p(\theta_0)$. By using property (2.35), it is not difficult to prove the following result.

Proposition 2.11. *Let p be as above, then*

$$p(\theta) \geqslant \gamma(\theta), \tag{2.37}$$

where $\gamma(\theta) := p(\theta_0) - \frac{1}{2}\left(\frac{P(E)}{2\pi} + p(\theta_0)\right)(\theta - \theta_0)^2$ *is a parabola which vanishes at the following points*

$$\theta_{1,2} = \theta_0 \pm \sqrt{\frac{2p(\theta_0)}{\frac{P(E)}{2\pi} + p(\theta_0)}}.$$

Proof. By property (2.35), we obtain

$$\begin{aligned} p(\theta) = p(\theta_0) + \int_{\theta_0}^{\theta} p'(t)\, dt = p(\theta_0) + \int_{\theta_0}^{\theta}\int_{\theta_0}^{t} p''(s)\, ds\, dt \\ \geqslant p(\theta_0) + \int_{\theta_0}^{\theta}\int_{\theta_0}^{t} -\left(\frac{P(E)}{2\pi} + p(s)\right) ds\, dt \\ \geqslant p(\theta_0) - \left(\frac{P(E)}{2\pi} + p(\theta_0)\right)\frac{(\theta - \theta_0)^2}{2}, \end{aligned} \tag{2.38}$$

which is the claim. Then, p is above the parabola γ, that attains its zeros at the following points:

$$\theta_{1,2} = \theta_0 \pm \sqrt{\frac{2p(\theta_0)}{\frac{P(E)}{2\pi} + p(\theta_0)}}.$$

This concludes the proof. $\square$

Inequality (2.32) implies Weinstock inequality but it hides also a stability result. Indeed, by using the previous Proposition, we get the following quantitative Weinstock inequality in the plane.

Theorem 2.12. *There exist $\delta > 0$ such that, if $E \subseteq \mathbb{R}^2$ is a bounded, open, convex with $\mathcal{D}(E) \leqslant \delta$, then*

$$\frac{16}{9\pi^2}\left(2\pi\frac{d_{\mathcal{H}}(E,E^*)}{P(E)}\right)^{\frac{5}{2}} \leqslant \mathcal{D}(E),$$

where $\mathcal{D}(E)$ is defined in (2.4). Moreover, the exponent $\frac{5}{2}$ is sharp.

Proof. Since the functional $\mathcal{D}$ is scaling invariant, we can assume that E is a convex set of finite measure with $P(E) = P(B) = 2\pi$. From Lemma 2.6, if we take a sufficiently small ε, there exists $\delta > 0$ such that, if $\mathcal{D}(E) \leqslant \delta$, then E contains the origin, its boundary can be parametrized as above by means the support function and, by (2.36),

$$d := \|p\|_{L^\infty([0,2\pi])} \leqslant \varepsilon.$$

Under these assumptions, since in particular $|d| < \frac{1}{2}$, Proposition 2.11 gives

$$p(\theta) \geqslant d - \left(\frac{1+d}{2}\right)(\theta-\theta_0)^2 \geqslant d - \frac{(\theta-\theta_0)^2}{4}. \tag{2.39}$$

Denoting by $\theta_{1,2}$ the zeros of the parabola $d - \frac{(\theta-\theta_0)^2}{4}$, that are

$$\theta_{1,2} = \theta_0 \pm 2\sqrt{d},$$

by using (2.32), the isoperimetric inequality, Hölder inequality and (2.39), we get

$$\begin{aligned}\mathcal{D}(E) = \frac{M(E)}{P(E)V(E)} - \frac{1}{\pi} = \frac{\pi M(E) - P(E)V(E)}{\pi P(E)V(E)} &\geqslant \frac{1}{2\pi^2}\int_0^{2\pi} p^2(\theta)\,d\theta \\ > \frac{1}{2\pi^2}\int_{\theta_1}^{\theta_2} p^2(\theta)\,d\theta \geqslant \frac{1}{2\pi^2(\theta_2-\theta_1)}\left(\int_{\theta_1}^{\theta_2} p(\theta)\,d\theta\right)^2 &> \frac{16}{9\pi^2}d^{\frac{5}{2}}. \end{aligned}\tag{2.40}$$

By (2.36) and (1.19), being $P(E) = 2\pi$, we get the claim. In order to conclude the proof, we have to show the sharpness of the exponent. We construct a family of convex sets E_ε, with $P(E_\varepsilon) = 2\pi$, such that

$$\mathcal{D}(E_\varepsilon) \to 0 \text{ for } \varepsilon \to 0,$$

and

$$\|p\|_{L^\infty([0,2\pi])} = \varepsilon + o\left(\varepsilon^{\frac{3}{2}}\right)$$

Let us consider the convex set E having the following support function:

$$h(\theta) = 1 + p(\theta), \quad \theta \in [0, 2\pi],$$

where the function p is the following

$$p(\theta) = \begin{cases} b & \text{if } \theta \in [0, \pi-\alpha] \\ c - \frac{(\theta-\pi)^2}{4} & \text{if } \theta \in [\pi-\alpha, \pi+\alpha] \\ b & \text{if } \theta \in [\pi+\alpha, 2\pi]. \end{cases}$$

Here the parameters α, b and c are

$$\alpha = 2\sqrt{\varepsilon}, \quad b = -\frac{4}{3\pi}\varepsilon^{\frac{3}{2}}, \quad c = \varepsilon - \frac{4}{3\pi}\varepsilon^{\frac{3}{2}}.$$

By construction, we have that

$$P(E_\varepsilon) = 2\pi \quad \text{and} \quad \int_0^{2\pi} p(\theta)\, d\theta = 0.$$

We recall that (see for instance [132, 31])

$$\begin{cases} V(E_\varepsilon) = \dfrac{1}{2}\displaystyle\int_0^{2\pi} \left(h^2(\theta) + h(\theta)h''(\theta)\right) d\theta \\ M(E_\varepsilon) = \displaystyle\int_0^{2\pi} \left(h^3(\theta) + \frac{1}{2}h^2(\theta)h''(\theta)\right) d\theta. \end{cases}$$

Arguing as in the proof of Weinstock inequality, a simple calculation gives

$$\begin{aligned} \pi M(E_\varepsilon) - P(E_\varepsilon)V(E_\varepsilon) &= \pi \int_0^{2\pi} p^2(\theta)\left(2 + p(\theta) + \frac{1}{2}p''(\theta)\right) d\theta \\ &= 2\pi \int_0^{2\pi} p^2(\theta)\, d\theta + \pi \int_0^{2\pi} p^3(\theta)\, d\theta + \frac{\pi}{2}\int_0^{2\pi} p^2(\theta)p''(\theta)\, d\theta = C\varepsilon^{\frac{5}{2}} + O(\varepsilon^3), \end{aligned} \tag{2.41}$$

where C is a positive constant. This concludes the proof. □

In the following Figure it is represented E_ε, as defined in the previous proof, for a fixed value of $\varepsilon > 0$.

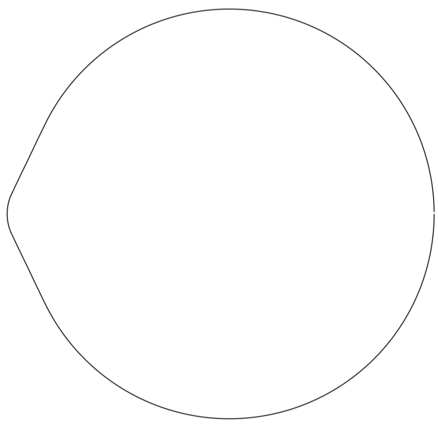

Figure 2.1: Sharpness in the bidimensional case

2.1.4 Main Theorem

In this paragraph we state and prove the main Theorem of this Section, which is a stability result for the Weinstock inequality restricted to the case of convex sets.

Theorem 2.13. *Let $n \geqslant 2$. There exists $\bar{\delta} > 0$ such that for every $\Omega \subset \mathbb{R}^n$ bounded, convex open set with $\sigma_2(B_R) \leqslant (1+\bar{\delta})\,\sigma_2(\Omega)$, where B_R is a ball with $P(B_R) = P(\Omega)$, then*

$$\frac{\sigma_2(B_R)-\sigma_2(\Omega)}{\sigma_2(\Omega)} \geqslant \begin{cases} \frac{16}{9\pi}\left(\mathcal{A}_{\mathcal{H}}(\Omega)\right)^{\frac{5}{2}} & \text{if } n=2 \\ \frac{2}{3}\sqrt{\pi}\, g\left(\left(\frac{\mathcal{A}_{\mathcal{H}}(\Omega)}{\beta}\right)^2\right) & \text{if } n=3 \\ \frac{(n\omega_n)^{\frac{1}{n-1}}}{n}\left(\frac{\mathcal{A}_{\mathcal{H}}(\Omega)}{\beta_n}\right)^{\frac{n+1}{2}} & \text{if } n \geqslant 4, \end{cases}$$

where β and β_n are defined in (2.17) and g is the inverse function of $f(t) = t\log\left(\frac{1}{t}\right)$, for $0 < t < e^{-1}$.

Proof. The proof is a consequence of Theorems 2.7 and 2.12. Since all the quantities involved are invariant under translations, we can assume that $\partial\Omega$ has the origin as barycenter. Under this assumption in [31] it is proved that

$$\sigma_2(\Omega) \leqslant \frac{nV(\Omega)}{M(\Omega)}.$$

It holds

$$\sigma_2(B_R) = \frac{1}{R} = \left[\frac{n\omega_n}{P(\Omega)}\right]^{1/(n-1)}$$

then, using the previous inequality and (2.2), we have

$$\frac{\sigma_2(B_R)-\sigma_2(\Omega)}{\sigma_2(\Omega)} = \frac{\sigma_2(B_R)}{\sigma_2(\Omega)} - 1 \geqslant \frac{M(\Omega)}{n\,|\Omega|}\left(\frac{n\omega_n}{P(\Omega)}\right)^{1/(n-1)} - 1 = \frac{(n\omega_n)^{\frac{1}{n-1}}}{n}\mathcal{D}(\Omega).$$

Let δ be as in Theorem 2.7. Then if Ω is such that $\sigma_2(B_R) \leqslant (1+\bar{\delta})\sigma_2(\Omega)$, with $\bar{\delta} = \frac{(n\omega_n)^{\frac{1}{n-1}}}{n}\delta$ then $\mathcal{D}(\Omega) \leqslant \delta$ and, for $n \geqslant 4$ from (2.16) in Theorem 2.7, we get

$$\frac{\sigma_2(B_R)-\sigma_2(\Omega)}{\sigma_2(\Omega)} \geqslant \frac{(n\omega_n)^{\frac{1}{n-1}}}{n}\left(\frac{\mathcal{A}_{\mathcal{H}}(\Omega)}{\beta_n}\right)^{\frac{n+1}{2}}.$$

If $n = 3$, we can conclude a similar way, observing that $f(t) = t\log\left(\frac{1}{t}\right)$ is invertible for $0 < t < e^{-1}$. Thus, being $D(\Omega)$ small, we can explicit it in (2.16), obtaining the thesis. The result in two dimension follows from Theorem 2.12. $\square$

2.2 Steklov-Dirichlet type problem on a perforated domain for nearly spherical sets

2.2.1 Notations

Let $R > r > 0$, throughout this section we denote by $B_r := \{x \in \mathbb{R}^n \ : \ |x| < r\}$ the ball centered at the origin with radius $r > 0$, by $A_{r,R}$ the spherical shell $B_R \backslash \overline{B}_r$ and we define

$$\mathcal{A}_r := \left\{ \begin{array}{l} \Omega = \Omega_0 \backslash \overline{B_r} \ : \ \Omega_0 \subset \mathbb{R}^n \text{ open, bounded, connected,} \\ \qquad\qquad\qquad\qquad\qquad \text{with Lipschitz boundary, s.t.} B_r \Subset \Omega_0 \end{array} \right\}.$$

Since we are studying a Steklov eigenvalue problem with a spherical obstacle, we need to introduce the definition of a closed subspace of $H^1(\Omega)$ that incorporates the Dirichlet boundary condition on ∂B_r. We denote the set of Sobolev functions on Ω that vanish on ∂B_r by

$$H^1_{\partial B_r}(\Omega),$$

that is (see [55]) the closure in $H^1(\Omega)$ of the set of test functions

$$C^\infty_{\partial B_r}(\Omega) := \{u|_\Omega \ : \ u \in C^\infty_0(\mathbb{R}^n), \ \operatorname{spt}(u) \cap \partial B_r = \varnothing\}.$$

2.2.2 Main properties of the Steklov-Dirichlet problems

Eigenvalues and Eigenfunctions

We are dealing with the following boundary eigenvalue problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial B_r, \\ \dfrac{\partial u}{\partial \nu} = \sigma^{DS}(\Omega)\, u & \text{on } \partial \Omega_0 \end{cases} \tag{2.42}$$

where ν is the outer normal to $\partial\Omega_0$. We give now the definitions and some geometric properties of eigenvalues and eigenfunctions of problem (2.42).

Definition 2.1. The real number $\sigma^{DS}(\Omega)$ and the function $u \in H^1_{\partial B_r}(\Omega)$ are, respectively, called eigenvalue of (2.42) and eigenfunction associated to $\sigma(\Omega)$, if and only if

$$\int_\Omega \nabla u \nabla \varphi \, dx = \sigma^{DS}(\Omega) \int_{\partial\Omega_0} u\varphi \, d\mathcal{H}^{n-1}(x)$$

for every $\varphi \in H^1_{\partial B_r}(\Omega)$.

Furthermore, the first eigenvalue is variationally characterized by

$$\sigma_1^{DS}(\Omega) = \min_{\substack{w \in H^1_{\partial B_r}(\Omega) \\ w \not\equiv 0}} J[w], \tag{2.43}$$

where

$$J[w] := \frac{\displaystyle\int_\Omega |\nabla w|^2 \, dx}{\displaystyle\int_{\partial\Omega_0} w^2 \, d\mathcal{H}^{n-1}}. \tag{2.44}$$

We point out that the condition of being orthogonal to constants in $L^2(\partial\Omega)$ is not required, unlike the classical Steklov eigenvalue (when $r = 0$). The following ensures the existence of minimizers of problem (2.43).

Proposition 2.14. *Let $r > 0$ and $\Omega \in \mathcal{A}_r$, then there exists a function $u \in H^1_{\partial B_r}(\Omega)$ achieving the minimum in* (2.43) *and satisfying problem* (2.42).

Proof. Let $u_k \in H^1_{\partial B_r}(\Omega)$ be a minimizing sequence of (2.43) such that $||u_k||_{L^2(\partial\Omega_0)} = 1$. Since the minimum in (2.43) is positive, then there exists a constant $C > 0$ such that $J[u_k] \leqslant C$ for every $k \in \mathbb{N}$ and therefore $||Du_k||_{L^2(\Omega)} \leqslant C$. Moreover, a Poincaré inequality in $H^1_{\partial B_r}(\Omega)$ holds and this implies that $\{u_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $H^1_{\partial B_r}(\Omega)$. Therefore, there exist a subsequence, still denoted by u_k, and a function $u \in H^1_{\partial B_r}(\Omega)$ with $||u||_{L^2(\partial\Omega_0)} = 1$, such that $u_k \to u$ strongly in $L^2(\Omega)$, hence also almost everywhere, and $Du_k \rightharpoonup Du$ weakly in $L^2(\Omega)$. By the compactness of the trace operator (see for example [91, Cor. 18.4]), u_k converges strongly to u in $L^2(\partial\Omega)$ and almost everywhere on $\partial\Omega$ to u. Then, by weak lower semicontinuity we have

$$\lim_{k\to+\infty} J[u_k] \geqslant J[u].$$

Hence, the existence of a minimizer $u \in H^1_{\partial B_r}(\Omega)$ follows. Moreover, u is harmonic in Ω and so, by strong maximum principle, it has constant sign on Ω. □

Now we state the simplicity of the first eigenvalue of (2.42), following the idea in [58, Section 6.5.1].

Proposition 2.15. *Let $r > 0$ and $\Omega \in \mathcal{A}_r$, then the first eigenvalue $\sigma_1^{DS}(\Omega)$ of* (2.42) *is simple, that is all the associated eigenfunctions are scalar multiple of each other.*

Proof. Let $u, \tilde{u}$ be two non trivial weak solutions of the problem (2.42). Since, by Proposition 2.14, we can assume that $\tilde{u}$ is positive in Ω, then it is clear that

$$\int_\Omega \tilde{u}\,dx \neq 0.$$

So, we can find a real constant χ such that

$$\int_\Omega (u - \chi\tilde{u})\,dx = 0. \tag{2.45}$$

Since $u - \chi\tilde{u}$ is still a solution of problem (2.42), then it is also non-negative (or non-positive) in Ω. Therefore, (2.45) implies that $u \equiv \chi\tilde{u}$ in Ω and the simplicity of $\sigma_1^{DS}(\Omega)$ follows.

□

It is worth noticing that the first nontrivial eigenvalue for the classical Steklov-Laplacian problem (when $r = 0$) on B_R is $1/R$ and the corresponding eigenfunctions are the coordinate axis x_i, for $i = 1, .., n$. This means that the first nontrivial eigenvalue has multiplicity n and this makes a significant difference with problem (2.42), for which we proved that the simplicity holds. On the other hand, it is easy to verify that both have the same scaling property:

$$\sigma_1^{DS}(t\Omega) = \frac{1}{t}\sigma_1^{DS}(\Omega), \quad \forall t > 0. \tag{2.46}$$

The first attempts to study the optimal shape of problem (2.42) has been done on spherical shells, i.e. when $\Omega_0 = B_R$, for $R > r > 0$. We recall from [128], the explicit expression of the first eigenfunction on the spherical shell $A_{r,R}$:

$$z(\rho) = \begin{cases} \ln\rho - \ln r & \text{for } n = 2 \\ \left(\dfrac{1}{r^{n-2}} - \dfrac{1}{\rho^{n-2}}\right) & \text{for } n \geqslant 3, \end{cases} \tag{2.47}$$

with $\rho = |x|$. This function is radial, positive, strictly increasing and it is associated to the following eigenvalue:

$$\sigma_1^{DS}(A_{r,R}) = \begin{cases} \frac{1}{R\log\left(\frac{R}{r}\right)} & \text{for } n = 2 \\ \frac{n-2}{R\left[\left(\frac{R}{r}\right)^{n-2}-1\right]} & \text{for } n \geqslant 3. \end{cases} \tag{2.48}$$

It is worth noting that, since problem (2.42) and the classical Steklov ($r = 0$) have the same scaling property (2.46), then the shape functional $\Omega \to V(\Omega)^{\frac{1}{N}}\sigma_1^{DS}(\Omega)$ is scaling invariant, as in the classical case.

A first upper bound

We show an upper bound for σ_1^{DS} depending only by the dimension n, by the measure of Ω and by the radius of the internal ball r.

Proposition 2.16. *Let $r > 0$ and $\Omega \in \mathcal{A}_r$, then*

$$\sigma_1^{DS}(\Omega) \leqslant \frac{2}{n\omega_n^{\frac{1}{n}}\left(\left(\dfrac{V(\Omega)}{2\omega_n} + r^n\right)^{1/n} - r\right)^2} V(\Omega)^{1/n}.$$

Proof. Let $\bar{R} > 0$ be such that $V(A_{r,\bar{R}}) = V(\Omega)/2$, then $\bar{R}$ depends only by the dimension n, the measure $V(\Omega)$ and r, that is

$$\bar{R} = \left(\frac{V(\Omega)}{2\omega_n} + r^n\right)^{1/n}.$$

Consider the function

$$\varphi(x) = \begin{cases} |x| - r & \text{if } r \leqslant |x| \leqslant \bar{R}; \\ \bar{R} - r & \text{if } |x| \geqslant \bar{R}. \end{cases} \tag{2.49}$$

We distinguish now two cases. Firstly, we assume that $B_{\bar{R}} \Subset \Omega_0$, i.e. $d := \mathrm{dist}(\partial B_{\bar{R}}, \partial\Omega_0) > 0$. By using (2.49) as test function in the Rayleigh quotient (2.44) and by the isoperimetric inequality, we obtain

$$\sigma_1^{DS}(\Omega) \leqslant \frac{V(\Omega)}{\left(\bar{R} - r\right)^2 P(\Omega_0)} \leqslant \frac{1}{n\omega_n^{\frac{1}{n}}\left(\bar{R} - r\right)^2} V(\Omega)^{\frac{1}{n}}. \tag{2.50}$$

We consider now the case $d = 0$, that is when the ball $B_{\bar{R}}$ is not strictly contained in Ω_0. Therefore, we divide the boundary of Ω_0 in the two sets $\partial^{int}\Omega_0$ and $\partial^{ext}\Omega_0$ that live, respectively, inside and outside of $B_{\bar{R}}$. Using the test function (2.49) in the Raylegh quotient (2.44), we have

$$\sigma_1^{DS}(\Omega) \leqslant \frac{V(\Omega)}{\int_{\partial\Omega_0} |\varphi|^2 \, d\mathcal{H}^{n-1}} \leqslant \frac{V(\Omega)}{(\bar{R} - r)^2 \int_{\partial^{ext}\Omega_0} 1 \, d\mathcal{H}^{n-1}}. \tag{2.51}$$

We recall that a relative isoperimetric inequality with supporting set $B_{\bar{R}}$ holds (see as reference e.g. [37, 44, 41]):

$$\mathcal{H}^{n-1}(\partial^{ext}\Omega_0) \geqslant n\left(\frac{\omega_n}{2}\right)^{1/n}\left(\frac{V(\Omega_0)}{2}\right)^{1-\frac{1}{n}}. \tag{2.52}$$

By using (2.52) in (2.51), we have

$$\sigma_1^{DS}(\Omega) \leqslant \frac{2}{n\omega_n^{\frac{1}{n}}(\bar{R}-r)^2}V(\Omega)^{\frac{1}{n}}. \tag{2.53}$$

The conclusion follows by observing that the upper bound (2.53) is greater than (2.50). □

We remark that, when a volume constraint for Ω holds, then the upper bound is still finite, when $r \to 0$. On the other hand, when $r \to \infty$, the first eigenvalue cannot be upper bounded. This, together with other examples that we are going to illustrate, motivates the study the optimality of σ_1^{DS} when another constraint holds, besides the volume one.

Volume constraint on the spherical shells

In this paper we deal with geometric properties of the first eigenvalue of (2.42). We look for shapes minimizing $\sigma_1^{DS}(\Omega)$, when both ω, the volume of Ω and the radius r of the internal ball are fixed. We show that, even among the spherical shells, σ_1^{DS} cannot be upper bounded when only a volume constraint holds.

Let us consider the spherical shell $A_{r,R}$ with the volume constraint:

$$V(A_{r,R}) = \omega_n(R^n - r^n) = \omega.$$

We show that both in bidimensional case and in higher dimension, σ_1^{DS} is not upper bounded in the class of spherical shells of fixed volume.
Let $n = 2$, then $R = \left(r^2 + \frac{\omega}{\pi}\right)^{\frac{1}{2}}$ and, by (2.48), we have

$$\sigma_1^{DS}(A_{r,R}) = \frac{1}{\left(r^2+\frac{\omega}{\pi}\right)^{\frac{1}{2}}\log\left(1+\frac{\omega}{\pi r^2}\right)^{\frac{1}{2}}} = \frac{2}{r\left(1+\frac{\omega}{\pi r^2}\right)^{\frac{1}{2}}\log\left(1+\frac{\omega}{\pi r^2}\right)}.$$

Hence, for r big enough,

$$\sigma_1^{DS}(A_{r,R}) \approx \frac{2}{r\left(1+\frac{\omega}{2\pi r^2}\right)\frac{\omega}{\pi r^2}} = \frac{2\pi r}{\omega\left(1+\frac{\omega}{2\pi r^2}\right)}$$

and so

$$\lim_{r\to+\infty}\sigma_1^{DS}(A_{r,R}) = +\infty.$$

Let $n \geqslant 3$, then, $R = \left(r^n + \frac{\omega}{\omega_n}\right)^{\frac{1}{n}}$ and

$$\begin{aligned}\sigma_1^{DS}(A_{r,R}) &= \frac{n-2}{r\left(1+\frac{\omega}{\omega_n r^n}\right)^{\frac{1}{n}}\left[\left(1+\frac{\omega}{\omega_n r^n}\right)^{1-\frac{2}{n}}-1\right]} = \\ &= \frac{n-2}{r\left[\left(1+\frac{\omega}{\omega_n r^n}\right)^{1-\frac{1}{n}}-\left(1+\frac{\omega}{\omega_n r^n}\right)^{\frac{1}{n}}\right]}.\end{aligned}$$

Again, if r is big

$$\sigma_1^{DS}(A_{r,R}) \approx \frac{n-2}{r\left[1+\left(1-\frac{1}{n}\right)\frac{\omega}{\omega_n r^n}-1-\frac{1}{n}\frac{\omega}{\omega_n r^n}\right]} = \frac{n\omega_n}{\omega}r^{n-1}$$

and hence again

$$\lim_{r\to+\infty}\sigma_1^{DS}(A_{r,R}) = +\infty. \tag{2.54}$$

Further, it is clear that, in any dimension, we have

$$\lim_{r\to 0^+}\sigma_1^{DS}(A_{r,R}) = 0. \tag{2.55}$$

The limiting results (2.54) and (2.55) motivate the fact that it is not enough to fix the volume to study the first eigenvalue σ_1^{DS}. Indeed, when r is too big, it is not possible to find an upper bound, and, on the other hand, when r is too small, the eigenvalue is trivial. We remark that, in the class of sets of the form $B_R(x_0)\backslash\overline{B}_r$, with $B_R(x_0)$ being a ball containing B_r, the maximizer of σ_1 is the spherical shell (see [65]).

Spherical shell with fixed difference between radii.

It is clear now that we cannot study the shape optimization for σ_1^{DS} when only a volume constraint holds. On the other hand, it could be interesting to understand if we can study the shape optimization for double connected domains, when only one geometric quantity is fixed. Here, for example, we briefly study the behavior of the spherical shell when the distance between the radii is fixed. Let d be a positive real number such that

$$R-r=d,$$

so that $R=r+d$ and $\frac{R}{r}=1+\frac{d}{r}$.
If $n=2$, then for r big enough, we have

$$\sigma_1^{DS}(A_{r,R}) = \frac{1}{(r+d)\log\left(1+\frac{d}{r}\right)} \approx \frac{r}{rd+d^2}$$

and, hence,

$$\lim_{r\to+\infty}\sigma_1^{DS}(A_{r,R}) = \frac{1}{d}.$$

If $n\geqslant 3$, we have

$$\begin{aligned}\sigma_1^{DS}(A_{r,R}) &= \frac{n-2}{(r+d)\left[\left(1+\frac{d}{r}\right)^{n-2}-1\right]}\\ &\approx \frac{n-2}{(r+d)\left[1+(n-2)\frac{d}{r}-1\right]} = \frac{r}{rd+d^2}\end{aligned}$$

and, hence,

$$\lim_{r\to+\infty}\sigma_1^{DS}(A_{r,R}) = \frac{1}{d}.$$

Furthermore, in any dimensions, we have

$$\lim_{r\to 0^+}\sigma_1^{DS}(A_{r,R}) = 0.$$

The case of r small is again trivial. On the other hand, σ_1^{DS} is upper bounded for any value of R by the reciprocal of the difference between the radii d. The fact that an uniform upper bound holds for spherical shells when only the difference between the radii is fixed, suggests that it could be interesting to study the shapes minizing σ_1^{DS} in the class of double connected sets when only the width is fixed.

2.2.3 Main result

In this Section we prove that the spherical shell is a local maximizer for the first eigenvalue of (2.42) among nearly spherical sets with fixed volume, containing B_r, for a fixed value $r > 0$.

We recall that, if Ω_0 is a nearly spherical set, as defined Definition 1.6 in Chapter 1, its volume is given by

$$V(\Omega_0) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} (1+v(\xi))^n \, d\mathcal{H}^{n-1}.$$

The class of nearly spherical sets has a peculiar importance in shape optimization theory, in particular for stability results for spectral inequalities. We are considering sets $\Omega = \Omega_0\backslash\overline{B}_r$ beloging to $\mathcal{A}_r$ with $r > 0$, with Ω_0 nearly spherical, and the main result is the following.

Theorem 2.17. *Let $n \geqslant 2$, $r > 0$, $\omega > 0$ and let $R > r$ be such that $V(A_{r,R}) = \omega$. There exists $\varepsilon = \varepsilon(n, r, \omega) > 0$ such that, for any $\Omega = \Omega_0\backslash\overline{B_r}$ belonging to $\mathcal{A}_r$, with Ω_0 nearly spherical set parametrized by v such that $||v||_{W^{1,\infty}} \leqslant \varepsilon$ and $V(\Omega) = \omega$, then*

$$\sigma_1^{DS}(\Omega) \leqslant \sigma_1^{DS}(A_{r,R}). \tag{2.56}$$

Moreover the equality in (2.56) *holds if and only if Ω is a spherical shell.*

Let us remark that, in order to have $B_r \Subset \Omega_0$, we need to require that $\varepsilon \leqslant 1 - r/R$ to verify that $|y| \geqslant r$, that is $R(1 + v(\xi)) \geqslant r$. Moreover, we observe that, since all the quantities involved are translation invariant, the result in Theorem 2.17 holds also among nearly spherical sets with fixed volume and containing a fixed internal ball.

Recalling the explicit expression (2.47) of the first eigenfunction z on the spherical shell $A_{r,R}$, we define the *weighted volume* and the *weighted perimeter* as:

$$\overline{V}(\Omega) := \int_\Omega |\nabla z|^2 \, dx,$$
$$\overline{P}(\Omega) := \int_{\partial\Omega_0} z^2 \, dx.$$

Furthermore, to simplify the notations, we set, for $n = 2$,

$$h_R(t) = (\ln(tR) - \ln r)^2, \tag{2.57}$$
$$f_R(t) = \frac{h'_R(t)}{2R} = \frac{\sqrt{h_R(t)}}{(tR)} \tag{2.58}$$

and for $n \geqslant 3$

$$h_R(t) = \left(\frac{1}{r^{n-2}} - \frac{1}{(tR)^{n-2}}\right)^2, \tag{2.59}$$
$$f_R(t) = \frac{h'_R(t)}{2R} = \frac{n-2}{(tR)^{n-1}}\left(\frac{1}{r^{n-2}} - \frac{1}{(tR)^{n-2}}\right), \tag{2.60}$$

where R is the radius of the ball with the same volume of Ω_0 and $t \geqslant \frac{r}{R}$. Now, we write the Raylegh quotient (2.44) using the parametrization in (1.21).

Lemma 2.18. *Let $n \geqslant 2$, $r > 0$, $\omega > 0$ and let $R > r$ be such that $V(A_{r,R}) = \omega$. For any $0 < \varepsilon < 1 - r/R$ and for any $\Omega = \Omega_0 \backslash \overline{B_r}$ belonging to $\mathcal{A}_r$, with Ω_0 nearly spherical set parametrized by v such that $||v||_{W^{1,\infty}} \leqslant \varepsilon$ and $V(\Omega) = \omega$, then*

$$\sigma_1^{DS}(\Omega) \leqslant \frac{\overline{V}(\Omega)}{\overline{P}(\Omega)} = \frac{\displaystyle\int_{\mathbb{S}^{n-1}} f_R(1+v(\xi))(1+v(\xi))^{n-1}\, d\mathcal{H}^{n-1}}{\displaystyle\int_{\mathbb{S}^{n-1}} h_R(1+v(\xi))(1+v(\xi))^{n-1}\sqrt{1+\frac{|\nabla v(\xi)|^2}{(1+v(\xi))^2}}\, d\mathcal{H}^{n-1}}. \tag{2.61}$$

Moreover if $\Omega = A_{r,R}$, then equality holds in (2.61) *and* $\sigma_1^{DS}(A_{r,R}) = \dfrac{f_R(1)}{h_R(1)}$.

Proof. From the variational characterization (2.43) of $\sigma_1^{DS}(\Omega)$, we have

$$\sigma_1^{DS}(\Omega) \leqslant \frac{\overline{V}(\Omega)}{\overline{P}(\Omega)} = \frac{\displaystyle\int_{\Omega} |\nabla z|^2\, dx}{\displaystyle\int_{\partial\Omega_0} z^2\, d\mathcal{H}^{n-1}} = \frac{\displaystyle\int_{\partial\Omega_0} \frac{\partial z}{\partial \nu} z\, d\mathcal{H}^{n-1}}{\displaystyle\int_{\partial\Omega_0} z^2\, d\mathcal{H}^{n-1}}.$$

The conclusion follows using the change of variables in (1.21). □

We recall the following result, whose proof can be found in [67].

Lemma 2.19. *Let $n \geqslant 2$ and $R > 0$. There exists a constant $C = C(n) > 0$ such that for any $0 < \varepsilon < 1$ and for any v parametrizing a nearly spherical set Ω_0 such that $||v||_{W^{1,\infty}} \leqslant \varepsilon$ and $V(\Omega_0) = V(B_R)$, then*

$$\left|(1+v)^{n-1} - \left(1 + (n-1)v + (n-1)(n-2)\frac{v^2}{2}\right)\right| \leqslant C\varepsilon v^2 \ \text{ on } \mathbb{S}^{n-1},$$

$$1 + \frac{|\nabla v|^2}{2} - \sqrt{1 + \frac{|\nabla v|^2}{(1+v)^2}} \leqslant C\varepsilon\left(v^2 + |\nabla v|^2\right) \ \text{ on } \mathbb{S}^{n-1},$$

$$\left|\int_{\mathbb{S}^{n-1}} v(\xi)\, d\mathcal{H}^{n-1} + \frac{n-1}{2}\int_{\mathbb{S}^{n-1}} v^2(\xi)\, d\mathcal{H}^{n-1}\right| \leqslant C\varepsilon \|v\|_{L^2}^2.$$

As a consequence of the analyticity of h_R and f_R, defined in (2.57)-(2.58)-(2.59)-(2.60), the following Lemma holds.

Lemma 2.20. *Let $n \geqslant 2$ and $0 < r < R$. There exists $K = K(n, r, R) > 0$ such that for any $0 < \varepsilon < 1$ and for any v parametrizing a nearly spherical set Ω_0 such that $||v||_{W^{1,\infty}} \leqslant \varepsilon$ and $V(\Omega_0) = V(B_R)$, then*

$$\left|h_R(1+v) - h_R(1) - h_R'(1)v - h_R''(1)\frac{v^2}{2}\right| \leqslant K\varepsilon v^2 \ \text{ on } \mathbb{S}^{n-1},$$

$$\left|f_R(1+v) - f_R(1) - f_R'(1)v - f_R''(1)\frac{v^2}{2}\right| \leqslant K\varepsilon v^2 \ \text{ on } \mathbb{S}^{n-1}.$$

Furthermore, this Poincaré inequality holds.

Lemma 2.21. *(Poincaré inequality) Let $n \geqslant 2$ and $R > 0$, then there exists a positive constant $C = C(n)$ such that for any $0 < \varepsilon < 1$ and for any function v parametrizing a nearly spherical set Ω_0 such that $||v||_{W^{1,\infty}} \leqslant \varepsilon$ and $V(\Omega_0) = V(B_R)$, then*

$$\|\nabla v\|_{L^2}^2 \geqslant (n-1)(1 - C\varepsilon)\|v\|_{L^2}^2.$$

Proof. The function $v \in L^2(\mathbb{S}^{n-1})$ admits a harmonic expansion (see e.g. [80, Chap. 3]), in the sense that there exists a family of n-dimensional spherical harmonics $\{H_j(\xi)\}_{j\in\mathbb{N}}$ such that

$$v(\xi) = \sum_{j=0}^{+\infty} c_j H_j(\xi), \quad \xi \in \mathcal{S}^{n-1} \quad \text{with} \quad \|H_j\|_{L^2(\mathbb{S}^{n-1})} = 1,$$

where

$$c_j = \langle v, H_j\rangle_{L^2(\mathbb{S}^{n-1})} = \int_{\mathbb{S}^{n-1}} v(\xi)H_j(\xi)d\mathcal{H}^{n-1}.$$

and H_j satisfying

$$\Delta_{\mathbb{S}^{n-1}} H_j = j(j+n-2)H_j, \quad \forall\, j \in \mathbb{N},$$

where $\Delta_{\mathbb{S}^{n-1}}$ is the Laplace-Beltrami operator. Furthermore the following identities hold true

$$||v||^2_{L^2(\mathbb{S}^{n-1})} = \sum_{j=0}^{\infty} c_j^2, \tag{2.62}$$

$$||\nabla v||^2_{L^2(\mathbb{S}^{n-1})} = \sum_{j=1}^{\infty} j(j+n-2)c_j^2. \tag{2.63}$$

Since $H_0 = (n\omega_n)^{-\frac{1}{2}}$, we have

$$|c_0| = (n\omega_n)^{-\frac{1}{2}} \left| \int_{\mathbb{S}^{n-1}} v(\xi)d\mathcal{H}^{n-1} \right| \leqslant$$
$$(n\omega_n)^{-\frac{1}{2}} \left| \int_{\mathbb{S}^{n-1}} v^2(\xi)d\mathcal{H}^{n-1} \right| \left(\frac{n-1}{2} + C\varepsilon \right) = C\varepsilon \|v\|_{L^2},$$

where the constant C has been renamed. Using this estimate, by (2.62) and (2.63), we have

$$\|v\|_{L^2} = \sum_{j=0}^{\infty} c_j^2 = c_0^2 + \sum_{j=1}^{\infty} c_j^2 \leqslant C\varepsilon\|v\|^2_{L^2} + \sum_{j=1}^{\infty} c_j^2,$$

and

$$\|\nabla v\|_{L^2} = \sum_{j=1}^{\infty} j(j+n-2)c_j^2 \geqslant (n-1)\sum_{j=1}^{\infty} c_j^2 \geqslant (n-1)(1-C\varepsilon)\|v\|^2_{L^2},$$

which concludes the proof. □

Now we give a key estimate for the main Theorem.

Proposition 2.22. *Let $n \geqslant 2$, $r > 0$, $\omega > 0$ and let $R > r$ be such that $V(A_{r,R}) = \omega$. There exist two positive constants $K > 0$ and $0 \leqslant \varepsilon_0 < 1 - r/R$, depending on n, r and ω only, such that for any $0 < \varepsilon < \varepsilon_0$, for any $\Omega = \Omega_0 \backslash \overline{B_r}$ belonging to $\mathcal{A}_r$, with Ω_0 nearly spherical set parametrized by v such that $||v||_{W^{1,\infty}} \leqslant \varepsilon$ and $V(\Omega) = \omega$, then*

$$\begin{aligned} &\frac{\overline{V}(\Omega^\sharp)\overline{P}(\Omega) - \overline{P}(\Omega^\sharp)\overline{V}(\Omega)}{n\omega_n} = \\ &= f_R(1) \int_{\mathbb{S}^{n-1}} h_R(1+v(\xi))(1+v(\xi))^{n-1} \sqrt{1 + \frac{|\nabla v(\xi)|^2}{(1+v(\xi))^2}}\, d\mathcal{H}^{n-1} \\ &\quad - h_R(1) \int_{\mathbb{S}^{n-1}} f_R(1+v(\xi))(1+v(\xi))^{n-1}\, d\mathcal{H}^{n-1} \geqslant K \int_{\mathbb{S}^{n-1}} v^2\, d\mathcal{H}^{n-1}. \end{aligned} \tag{2.64}$$

Proof. Using Lemmata 2.19, 2.20, 2.21, we have

$$
\begin{aligned}
& f_R(1)\int_{\mathbb{S}^{n-1}} h_R(1+v(\xi))(1+v(\xi))^{n-1}\sqrt{1+\frac{|\nabla v(\xi)|^2}{(1+v(\xi))^2}}\,d\mathcal{H}^{n-1} \\
&\quad - h_R(1)\int_{\mathbb{S}^{n-1}} f_R(1+v(\xi))(1+v(\xi))^{n-1}\,d\mathcal{H}^{n-1} \\
&\geqslant \int_{\mathbb{S}^{n-1}} v\left(f_R(1)h_R'(1)-f_R'(1)h_R(1)\right)\,d\mathcal{H}^{n-1} \\
&\quad + \int_{\mathbb{S}^{n-1}} \frac{v^2}{2}[f_R(1)h_R''(1)-f_R''(1)h_R(1)+2(n-1)(f_R(1)h_R'(1)-f_R'(1)h_R(1))]\,d\mathcal{H}^{n-1} \\
&\quad + \int_{\mathbb{S}^{n-1}} f_R(1)h_R(1)\frac{|\nabla v|^2}{2}d\mathcal{H}^{n-1} - \varepsilon K_1\|\nabla v\|^2_{L^2},
\end{aligned} \tag{2.65}
$$

where K_1 is a positive constant. Let us set

$$
\begin{aligned}
Q_1(t) &:= f_R(t)h_R'(t) - f_R'(t)h_R(t),\\
Q_2(t) &:= f_R(t)h_R''(t) - f_R''(t)h_R(t),\\
Q_3(t) &:= f_R(t)h_R(t),
\end{aligned}
$$

In order to show (2.64), we need to prove

1. $Q_1(1) > 0$,
2. $Q_3(1) > 0$,
3. $(n-1)\,[Q_1(1)+Q_3(1)] + Q_2(1) > 0$.

Indeed, when (1), (2), (3) hold, then, by using Lemmata 2.19 and 2.21, the last term in (2.65) can be estimated as

$$
\begin{aligned}
& Q_1(1)\int_{\mathbb{S}^{n-1}} v\,d\mathcal{H}^{n-1} + (2(n-1)Q_1(1)+Q_2(1))\int_{\mathbb{S}^{n-1}}\frac{v^2}{2}\,d\mathcal{H}^{n-1} \\
&\qquad\qquad + Q_3(1)\int_{\mathbb{S}^{n-1}}\frac{|\nabla v|^2}{2}d\mathcal{H}^{n-1} - \varepsilon K_1\|\nabla v\|^2_{L^2} \\
&\geqslant -\frac{n-1}{2}Q_1(1)\int_{\mathbb{S}^{n-1}} v^2\,d\mathcal{H}^{n-1} - \varepsilon K_2\|v\|^2_{L^2} + \left((n-1)Q_1(1)+\frac{Q_2(1)}{2}\right)\int_{\mathbb{S}^{n-1}} v^2\,d\mathcal{H}^{n-1} \\
&\qquad\qquad + \frac{n-1}{2}Q_3(1)\int_{\mathbb{S}^{n-1}} v^2 - \varepsilon K_3\|v\|^2_{L^2} - \varepsilon K_1\|\nabla v\|^2_{L^2} \\
&= \frac{1}{2}\left\{(n-1)[Q_1(1)+Q_3(1)]+Q_2(1)\right\}\|v\|^2_{L^2} \\
&\qquad\qquad - \varepsilon K_2\|v\|^2_{L^2} - \varepsilon K_3\|v\|^2_{L^2} - \varepsilon K_1\|\nabla v\|^2_{L^2} \\
&\geqslant K\|v\|^2_{L^2} - \varepsilon K_4\|v\|^2_{W^{1,2}(\mathbb{S}^{n-1})},
\end{aligned}
$$

where we denoted $K = \frac{1}{2}\{(n-1)\,[Q_1(1)+Q_3(1)]+Q_2(1)\} > 0$ and $K_4 = \max\{K_1, K_2, K_3\}$. The proof concludes by choosing ε small enough.

It remains to prove (1), (2), (3) by distinguishing the bidimensional from the higher dimensional case. We note that

$$
Q_1(t) = f_R^2(t)\left[\frac{h_R(t)}{f_R(t)}\right]' = 2Rf_R^2(t)\left[\frac{h_R(t)}{h_R'(t)}\right]', \tag{2.66}
$$

and

$$Q_2(t) = Q_1'(t) = \left[f_R^2(t)\right]' \left[\frac{h_R(t)}{f_R(t)}\right]' + f_R^2(t) \left[\frac{h_R(t)}{f_R(t)}\right]''. \tag{2.67}$$

Case 1. Let be $n = 2$. We observe that

$$\frac{h_R(t)}{f_R(t)} = Rt(\ln(tR) - \ln r),$$

is positive and strictly increasing, since it is a product of two strictly increasing positive functions. Hence $Q_1(t) > 0$ and in particular

$$Q_1(1) = \frac{h_R(1)}{R} \left(\sqrt{h_R(1)} + 1\right) > 0.$$

Moreover, it is clear that

$$Q_3(1) = \frac{h_R(1)\sqrt{h_R(1)}}{R} > 0.$$

Let us now calculate all the terms in (2.67) and evaluate them for $t = 1$. We have

$$\begin{aligned}
\left[\frac{h_R(t)}{f_R(t)}\right]'_{t=1} &= R\left(\sqrt{h_R(t)} + 1\right)_{t=1} = R\left(\sqrt{h_R(1)} + 1\right) > 0,\\
\left[\frac{h_R(t)}{f_R(t)}\right]''_{t=1} &= \left(\frac{R}{t}\right)_{t=1} = R > 0
\end{aligned}$$

and

$$\begin{aligned}
f_R^2(1) &= \frac{h_R(1)}{R^2} > 0,\\
\left[f_R^2(t)\right]'_{t=1} &= \left[\frac{2R}{(tR)^3}\left(\sqrt{h_R(t)} - h_R(t)\right)\right] = \frac{2}{R^2}\left(\sqrt{h_R(1)} - h_R(1)\right).
\end{aligned}$$

Summing up, estimate (3) follows by

$$\begin{aligned}
Q_1(1) + Q_3(1) + Q_2(1) = {} & \frac{h_R(1)\sqrt{h_R(1)}}{R} + \frac{h_R(1)}{R} + \frac{h_R(1)\sqrt{h_R(1)}}{R} + \\
& 2\frac{\sqrt{h_R(1)}}{R} - 2\frac{h_R(1)\sqrt{h_R(1)}}{R} + \frac{h_R(1)}{R} = \frac{2}{R}(h_R(1) + \sqrt{h_R(1)}) > 0.
\end{aligned}$$

Case 2. For $n \geqslant 3$, from (2.66) we have

$$\frac{h_R(t)}{h_R'(t)} = \frac{(tR)^{n-1}}{2(n-2)R}\left(\frac{1}{r^{n-2}} - \frac{1}{(tR)^{n-2}}\right),$$

that is a strictly increasing function, since it is product of two strictly increasing and positive functions. Hence $Q_1(t) > 0$ and, in particular

$$Q_1(1) = \frac{(n-1)(n-2)}{R^{n-1}} h_R(1)\sqrt{h_R(1)} + \frac{2(n-2)^2}{R^{2n-3}} h_R(1) > 0.$$

Moreover, it is easily seen that

$$Q_3(1) = \frac{n-2}{R^{n-1}} h_R(1)\sqrt{h_R(1)} > 0.$$

Eventually, we have

$$\begin{aligned} Q_2(1) = &\frac{(n-2)^3}{R^{3n-3}}\sqrt{h_R(1)} - \frac{(n-1)^2(n-2)}{R^{n-1}}h_R(1)\sqrt{h_R(1)} \\ &+ \frac{(n-1)(n-2)^2}{R^n}h_R(1)\sqrt{h_R(1)} + \frac{(n-1)(n-2)^2}{R^{2n-2}}h_R(1), \end{aligned}$$

and therefore, it follows that $(n-1)\left[Q_1(1) + Q_3(1)\right] + Q_2(1) > 0$. □

We use the previous result to give a stability result in a quantitative form.

Theorem 2.23. *Let $n \geqslant 2$, $r > 0$, $\omega > 0$ and let $R > r$ be such that $V(A_{r,R}) = \omega$. There exist two positive constants $K > 0$ and $0 \leqslant \varepsilon_0 < 1 - r/R$, depending on n, r and ω only, such that for any $0 < \varepsilon < \varepsilon_0$, for any $\Omega = \Omega_0 \backslash \overline{B_r}$ belonging to $\mathcal{A}_r$, with Ω_0 nearly spherical set parametrized by v such that $||v||_{W^{1,\infty}} \leqslant \varepsilon$, and $V(\Omega) = \omega$, then*

$$\sigma_1^{DS}(A_{r,R}) \geqslant \sigma_1^{DS}(\Omega)\left(1 + K(n,r,\omega)\int_{\mathbb{S}^{n-1}} v^2(\xi)\, d\mathcal{H}^{n-1}\right).$$

Proof. From Proposition 2.22 we know that there exists $K > 0$ such that

$$\overline{P}(A_{r,R})\overline{P}(\Omega)\left(\frac{\overline{V}(A_{r,R})}{\overline{P}(A_{r,R})} - \frac{\overline{V}(\Omega)}{\overline{P}(\Omega)}\right) \geqslant n\omega_n K \int_{\mathbb{S}^{n-1}} v^2\, d\mathcal{H}^{n-1}.$$

Then, we have

$$\begin{aligned} \sigma_1^{DS}(A_{r,R}) = \frac{\overline{V}(A_{r,R})}{\overline{P}(A_{r,R})} &\geqslant \frac{\overline{V}(\Omega)}{\overline{P}(\Omega)} + \frac{n\omega_n K \displaystyle\int_{\mathbb{S}^{n-1}} v^2\, d\mathcal{H}^{n-1}}{P(A_{r,R})P(\Omega)} \\ &= \frac{V(\Omega)}{P(\Omega)}\left(1 + \frac{n\omega_n K \displaystyle\int_{\mathbb{S}^{n-1}} v^2\, d\mathcal{H}^{n-1}}{P(A_{r,R})V(\Omega)}\right) \\ &= \frac{\overline{V}(\Omega)}{\overline{P}(\Omega)}\left(1 + \frac{K\displaystyle\int_{\mathbb{S}^{n-1}} v^2\, d\mathcal{H}^{n-1}}{h_R(1)\displaystyle\int_{\mathcal{S}^{n-1}} f_R(1+v(\xi))(1+v(\xi))^{n-1}\, d\mathcal{H}^{n-1}}\right) \\ &\geqslant \frac{\overline{V}(\Omega)}{\overline{P}(\Omega)}\left(1 + \frac{K\displaystyle\int_{\mathbb{S}^{n-1}} v^2\, d\mathcal{H}^{n-1}}{n\omega_n 2^{n-1} h_R(1) f_R(2)}\right) \geqslant \sigma_1(\Omega)\left(1 + K\int_{\mathbb{S}^{n-1}} v^2\, d\mathcal{H}^{n-1}\right), \end{aligned}$$

where the second inequality follows by the fact that $\|v\|_{W^{1,\infty}(\mathbb{S}^{n-1})} \leqslant \varepsilon < 1$ and by the monotonicity of $f_R(\cdot)$. □

Eventually, the main result (Theorem 2.17) easily follows by Theorem 3.4. Moreover, if $\Omega = A_{r,R}$, then the function v parametrizing the outer boundary is constantly equal to zero and equality in (2.56) holds.

Chapter 3

Study of the Steklov problem in the anisotropic case

In the first part of this Chapter we generalize the isoperimetric inequality proved in [31], that is

$$\frac{\displaystyle\int_{\partial\Omega} |x|^2 \, d\mathcal{H}^{n-1}}{P(\Omega)V(\Omega)^{2/n}} \geqslant \omega_n^{-2/n}, \tag{3.1}$$

to a functional involving the anisotropic $p-$momentum, the anisotropic perimeter and the volume, being Ω an open bounded and convex set of $\mathbb{R}^n$, $n \geqslant 2$. In the second part we focus our attention on the anisotropy $\|x\|_{\ell^p}^p = \sum_{j=1}^n |x^j|^p$ and consider the anisotropic p-Laplace operator associated to this norm, that is

$$\widetilde{\Delta}_p u = \sum_{j=1}^{n} \left(|u_{x_i}|^{p-2} u_{x_i}\right)_{x_i},$$

called the $p-$orthotropic Laplacian. We study the Steklov eigenvalue problem for the $\infty-$orthotropic Laplace operator, considering the limit for $p \to \infty$ of the Steklov problem for the $p-$orthotropic Laplacian. Using the generalization of the isoperimetric inequality (3.1), we prove Brock-Weinstock and Weinstock type inequalities for the first non trivial eigenvalue of the Steklov $\infty-$orthotropic Laplacian among convex sets.

3.1 An isoperimetric inequality involving the volume, the anisotropic perimeter and the anisotropic boundary momentum

Let Ω be a bounded, open set of $\mathbb{R}^n$ with Lipschitz boundary. Let $p > 1$, we consider the following scaling invariant functional:

$$\mathcal{F}_{F,p}(\Omega) = \frac{\displaystyle\int_{\partial\Omega} [F^o(x)]^p \, F(\nu(x)) \, d\mathcal{H}^{n-1}(x)}{\left[\displaystyle\int_{\partial\Omega} F(\nu(x)) \, d\mathcal{H}^{n-1}(x)\right] V(\Omega)^{\frac{p}{n}}},$$

where $\nu(x)$ is the unit outer normal to $\partial\Omega$ at $x \in \partial\Omega$ and F is a Finsler norm as defined in Section 3.1 . We define the anisotropic p-boundary momentum of Ω as

$$M_{F,p}(\Omega) = \int_{\partial\Omega} [F^o(x)]^p \, F(\nu(x)) \, d\mathcal{H}^{n-1}(x)$$

and we recall Definition 1.8, where we have defined the anisotropic perimeter as

$$P_F(\Omega) = \int_{\partial\Omega} F(\nu(x)) \, d\mathcal{H}^{n-1}(x).$$

The main result of this Section is the following. We recall that κ_n is the volume of the unitary Wulff shape

$$\mathcal{W} = \{\xi \in \mathbb{R}^n : F^o(\xi) < 1\}.$$

Theorem 3.1. *Let Ω be a bounded, open, convex set of $\mathbb{R}^n$. The following inequality holds true:*

$$\mathcal{F}_{F,p}(\Omega) \geqslant \kappa_n^{-\frac{p}{n}} = \mathcal{F}_{F,p}(\mathcal{W})$$

and equality holds only for Wulff shapes centered at the origin.

In the rest of this Section, for simplicity, we will write $\mathcal{F}$ instead of $\mathcal{F}_{F,p}$ and $M_F(\cdot)$ instead of $M_{F,p}(\cdot)$.

Remark 3.2. We observe that from this last theorem follows a particular case of the inequality proved in [16] that we have recalled in (18). Indeed, if we take $F(x) = |x|$, we obtain

$$\left(\int_{\partial\Omega} |x|^p \, d\mathcal{H}^{n-1}(x) \right)^n \geqslant n^n \omega_n^{1-p} V(\Omega)^{n+p-1}.$$

In what follows we will need the following definitions:

- $r^F_{\max}(\Omega) := \max \left\{ F^o(x) \mid x \in \bar{\Omega} \right\}$;
- $x^F_{\max}(\Omega) \in \partial\Omega$ is such that $F^o(x^F_{\max}(\Omega)) = r^F_{\max}(\Omega)$;
- the anisotropic p-excess function $E_F(\Omega) := (r^F_{\max}(\Omega))^{p-1} - \dfrac{M_{F,p}(\Omega)}{nV(\Omega)}$.

The general way of proceeding to prove our main Theorem is analogous to the one presented in [31].

First variation of the p momentum in the smooth case

We recall the Definition of Cahn-Hoffman vector $\nu^F_{\partial\Omega}$, given in

Proposition 3.3. *Let Ω and $\Omega(t)$ be the subsets of $\mathbb{R}^n$ defined in Section 1.3.3 with C^∞ boundary. Then*

$$\begin{aligned}
&\frac{d}{dt} M_F(\Omega(t))|_{t=0} = \\
&= p \int_{\partial\Omega} (F^o(x))^{p-1} \, \langle \nabla F^o(x), \varphi(x) \, \nu^F_{\partial\Omega}(x) \rangle F(\nu(x)) \, d\mathcal{H}^{n-1}(x) + \\
&+ \int_{\partial\Omega} [F^o(x)]^p \, F(\nu(x)) \, H^F_{\partial\Omega}(x) \varphi(x) \, d\mathcal{H}^{n-1}(x).
\end{aligned}$$

Proof. Considering the change of variables given by (4.9), i.e. $y = \phi(x,t)$, we have that

$$
\begin{aligned}
&\frac{d}{dt} M_F(\Omega(t))|_{t=0} = \\
&= \int_{\partial\Omega} \frac{d}{dt}\left([F^o(\phi(x,t))]^p\right) F(\nu(\phi(x,t)))\, d\mathcal{H}^{n-1}(\phi(x,t))|_{t=0} + \\
&+ \int_{\partial\Omega} (F^o(\phi(x,t)))^p \frac{d}{dt}\left[F(\nu(\phi(x,t)))\, d\mathcal{H}^{n-1}(\phi(x,t))\right]|_{t=0}.
\end{aligned}
$$

We observe that

$$
\begin{aligned}
&\int_{\partial\Omega} \frac{d}{dt}\left([F^o(\phi(x,t))]^p\right) F(\nu(\phi(x,t)))\, d\mathcal{H}^{n-1}(\phi(x,t))|_{t=0} \\
&= \int_{\partial\Omega} p\,(F^o(\phi(x,t)))^{p-1} \langle \nabla F^o(\phi(x,t)), \varphi(x)\nu^F_{\partial\Omega}(x)\rangle F(\nu(\phi(x,t))\, d\mathcal{H}^{n-1}(\phi(x,t))|_{t=0}.
\end{aligned}
$$

Moreover, from the first variation of the perimeter (1.36), we can say that

$$
\frac{d}{dt}\left[F(\nu(\phi(x,t)))\, d\mathcal{H}^{n-1}(\phi(x,t))\right]|_{t=0} = H^F_{\partial\Omega}(x)\varphi(x)F(\nu(x)).
$$

The thesis follows.

□

Considering now the derivative of the quotient, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{F}((\Omega(t))|_{t=0} = \\
&= \frac{1}{P_F(\Omega)^2 V(\Omega)^{\frac{p}{n}}} \Bigg[p \int_{\partial\Omega} \Big[(F^o(x))^{p-1} \langle \nabla F^o(x), \nu^F_{\partial\Omega}(x)\rangle F(\nu(x)) \\
&\qquad - \frac{M_F(\Omega)))}{nV(\Omega)} F(\nu(x))\Big] \varphi(x)\, d\mathcal{H}^{n-1}(x) + \\
&\qquad + \int_{\partial\Omega} \Big[(F^o(x))^p - \frac{M_F(\Omega)}{P_F(\Omega)}\Big] H^F_{\partial\Omega}(x)\, F(\nu(x))\varphi(x)\, d\mathcal{H}^{n-1}(x) \Bigg].
\end{aligned}
$$

Let be $T > 0$; we choose,

$$
\varphi(x) = \frac{1}{H^F_{\partial\Omega}(x)},
$$

and we have that

$$
\frac{\partial}{\partial t}\phi(x,t) = \frac{\nu^F_{\partial\Omega}(x)}{H^F_{\partial\Omega}(x)},
$$

for every $t \in [0,T]$. This one parameter family of diffeomorphisms gives rise to the inverse anisotropic mean curvature flow (IAMCF), see for a reference [135] and Section 1.3.3 for the result of existence and its properties. Substituting this φ in the derivative of the quotient and taking in account the fact that

$$
\int_{\partial\Omega} \left[(F^o(x))^p - \frac{M_F(\Omega)}{P_F(\Omega)}\right] F(\nu(x))\, d\mathcal{H}^{n-1}(x) = 0, \tag{3.2}
$$

we obtain

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{F}((\Omega(t))|_{t=0} = \qquad (3.3)\\
&\frac{p}{P_F(\Omega)^2 V(\Omega)^{\frac{p}{n}}}\int_{\partial\Omega}\left[(F^o(x))^{p-1}\langle\nabla F^o(x),\nu^F_{\partial\Omega}(x)\rangle F(\nu(x)) - \frac{M_F(\Omega)}{nV(\Omega)}F(\nu_{\partial\Omega}(x))\right]\frac{d\mathcal{H}^{n-1}(x)}{H^F_{\partial\Omega}(x)} =\\
&= \frac{p}{P_F(\Omega)^2 V(\Omega)^{\frac{p}{n}}}\int_{\partial\Omega}\left[(F^o(x))^{p-1}\langle\nabla F^o(x),\nu^F_{\partial\Omega}(x)\rangle - \frac{M_F(\Omega)}{nV(\Omega)}\right]\frac{F(\nu(x))}{H^F_{\partial\Omega}(x)}\,d\mathcal{H}^{n-1}(x).
\end{aligned}
$$

Existence of minimizers (Step 1)

Proposition 3.4. *There exists a convex set minimizing $\mathcal{F}(\cdot)$.*

Proof. Given a convex set Ω, up to a translation, we can consider that $\int_{\partial\Omega} xF(\nu_{\partial\Omega}(x))d\sigma_x = 0$. Since the anisotropic perimeter and volume do not change, while the anisotropic boundary p-momentum does not increase, we can assume that a possible minimum has the center of gravity in 0. Therefore, we can take a minimizing sequence $(\Omega_i)_i$, having the same volume of Ω and satisfying $\int_{\partial\Omega_i} xF(\nu_{\partial\Omega_i}(x))d\sigma_x = 0$. In particular, this implies that the origin has to be inside Ω_i for every i.
By Blaschke selection Theorem in [118, Theorem 1.8.7], it is enough to show that the Ω_i's have equibounded anisotropic diameters. For sake of simplicity, we suppose that $V(\Omega_i) = \kappa_n$ and, since any Wulff $\mathcal{W}$ with center of gravity in the origin is such that $\mathcal{F}(\mathcal{W}) = \kappa_n^{-\frac{p}{n}}$, we have that

$$\lim_{i\to+\infty}\mathcal{F}(\Omega_i) \leqslant \kappa_n^{-\frac{p}{n}},$$

and consequently

$$\lim_{i\to+\infty}\frac{M_F(\Omega_i)}{P_F(\Omega_i)} \leqslant 1.$$

Arguing by contradiction, if we assume that $\lim_{i\to+\infty}\operatorname{diam}_F(\Omega_i) = +\infty$, from convexity follows easily that $\lim_{i\to+\infty} P_F(\Omega_i) = +\infty$. Thereafter, if $\mathcal{W}_2$ is the Wulff of anisotropic radius 2 centered at the origin, it is enough to observe that

$$\lim_{i\to+\infty}\frac{\int_{\partial\Omega_i\cap\mathcal{W}_2} F(\nu(x)\,d\mathcal{H}^{n-1}(x)}{\int_{\partial\Omega_i\setminus\mathcal{W}_2} F(\nu(x))\,d\mathcal{H}^{n-1}(x)} = 0$$

and

$$\lim_{i\to+\infty}\frac{M_F(\Omega_i)}{P_F(\Omega_i)} \geqslant \lim_{i\to+\infty}\frac{2^p}{1+\frac{\int_{\partial\Omega_i\cap\mathcal{W}_2} F(\nu(x))\,d\mathcal{H}^{n-1}(x)}{\int_{\partial\Omega_i\setminus\mathcal{W}_2} F(\nu(x))\,d\mathcal{H}^{n-1}(x)}} = 2^p,$$

which gives a contradiction. □

A minimizer cannot have negative Excess (Step 2)

Remark 3.5. There exist sets with negative anisotropic p-Excess. We prove this fact in dimension 2 and for $p = 2$. We consider the elliptic metric

$$F(x,y) = \sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}};$$

we know that its polar is this elliptic norm

$$F^o(x,y) = \sqrt{a^2x^2 + b^2y^2}$$

and we consider now the following convex domain:

$$R_\epsilon = \left\{ (x,y) \in \mathbb{R}^2 \ : \ |x| \leqslant \frac{1}{\epsilon}, \ \ |y| \leqslant \epsilon \right\}.$$

From the computations we obtain that $V(R_\epsilon) = 4$, $r^F_{\max}(R_\epsilon) = a/\epsilon + O(\epsilon^3)$ and $M_F(R_\epsilon) = (4a^2/3b)(1/\epsilon^3) + 4a/\epsilon + O(\epsilon)$.

Lemma 3.6. *Let Ω be a bounded, open convex set of $\mathbb{R}^n$. Then*

$$(F^o(x))^{p-1} \langle \nabla F^o(x), \nu^F_{\partial\Omega}(x) \rangle - \frac{M_F(\Omega)}{nV(\Omega)} \leqslant E_F(\Omega). \tag{3.4}$$

Proof. We observe that

$$\langle \nabla F^o(x), \nu^F_{\partial\Omega}(x) \rangle = \langle \nabla F^o(x), \nabla F(\nu(x)) \rangle \leqslant F(\nabla F^o(x)) F^o(\nabla F(\nu(x))) = 1,$$

for the properties of the Finsler norm F. □

Lemma 3.7. *Let Ω be a bounded, open convex set of $\mathbb{R}^n$. Then*

$$\int_{\partial\Omega} \Big[(F^o(x))^{p-1} \langle \nabla F^o(x), \nu^F_{\partial\Omega}(x) \rangle - \frac{M_F(\Omega)}{nV(\Omega)} \Big] d\mathcal{H}^{n-1}(x) \leqslant 0.$$

Proof. We notice that it is enough to prove that

$$\int_{\partial\Omega} \Big[(F^o(x))^{p-1} \langle \nabla F^o(x), \nu^F_{\partial\Omega}(x) \rangle - \frac{M_F(\Omega)}{nV(\Omega)} \Big] F(\nu(x)) \, d\mathcal{H}^{n-1}(x) \leqslant 0, \tag{3.5}$$

since there exists $\alpha > 0$ such that $\alpha \leqslant F(\nu_{\partial\Omega}(x))$. In order to prove (3.5), we observe that

$$\begin{aligned}
&\int_{\partial\Omega} \Big[(F^o(x))^{p-1} \langle \nabla F^o(x), \nu^F_{\partial\Omega}(x) \rangle F(\nu(x)) - \frac{M_F(\Omega)}{nV(\Omega)} F(\nu(x)) \Big] d\mathcal{H}^{n-1}(x) = \\
&\int_{\partial\Omega} \Big[(F^o(x))^{p-1} \langle \nabla F^o(x), \nabla F(\nu(x)) \rangle F(\nu(x)) - \frac{M_F(\Omega)}{nV(\Omega)} F(\nu(x)) \Big] d\mathcal{H}^{n-1}(x) \\
&\leqslant \int_{\partial\Omega} [(F^o(x))^{p-1} F(\nu(x))] \, d\mathcal{H}^{n-1}(x) - \frac{M_F(\Omega)}{nV(\Omega)} P_F(\Omega) \\
&\leqslant \int_{\partial\Omega} [(F^o(x))^{p-1} F(\nu(x))] \, d\mathcal{H}^{n-1}(x) - \frac{M_F(\Omega) P_F(\Omega)}{\int_{\partial\Omega} F^o(x) F(\nu(x)) \, d\mathcal{H}^{n-1}(x)}
\end{aligned}$$

and the last inequality holds since

$$nV(\Omega) = \int_{\partial\Omega} \langle x, \nu(x) \rangle \, d\mathcal{H}^{n-1}(x) \leqslant \int_{\partial\Omega} F^o(x) F(\nu(x)) \, d\mathcal{H}^{n-1}(x),$$

for the properties of the Finsler norms. Using now Hölder inequality, we obtain

$$\begin{aligned}
&\int_{\partial\Omega} (F^o(x))^{p-1} F(\nu(x)) \, d\mathcal{H}^{n-1}(x) \\
&\leqslant \left[\int_{\partial\Omega} \Big[(F^o(x))^{p-1} \Big]^{\frac{p}{p-1}} F(\nu(x)) \, d\mathcal{H}^{n-1}(x) \right]^{\frac{p-1}{p}} (P_F(\Omega))^{\frac{1}{p}} \\
&= \left[\int_{\partial\Omega} (F^o(x))^p F(\nu(x)) \, d\mathcal{H}^{n-1} \right]^{\frac{p-1}{p}} (P_F(\Omega))^{\frac{1}{p}}
\end{aligned}$$

and

$$\int_{\partial\Omega} F^o(x)F(\nu(x))\, d\mathcal{H}^{n-1}(x) \leqslant \left[\int_{\partial\Omega} (F^o(x))^p\, F(\nu(x))\, d\mathcal{H}^{n-1}(x)\right]^{\frac{1}{p}} \,(P_F(\Omega))^{\frac{p-1}{p}}.$$

Finally, from these last two inequalities, follows that

$$\left(\int_{\partial\Omega} [(F^o(x))^{p-1}\, F(\nu(x))]\, d\mathcal{H}^{n-1}(x)\right)\left(\int_{\partial\Omega} F^o(x)F(\nu(x))\, d\mathcal{H}^{n-1}(x)\right) \leqslant M_F(\Omega)P_F(\Omega).$$

□

Proposition 3.8. *Let Ω be a bounded, open convex set of $\mathbb{R}^n$ such then*

$$E_F(\Omega) < 0, \tag{3.6}$$

then Ω is not a minimizer of $\mathcal{F}(\cdot)$.

Proof. We firstly assume that $\Omega \in C_F^{\infty,+}$. Since $E_F(\Omega) \neq 0$, Ω is not a Wullf shape centered at the origin. Then, from (3.4) and (3.3), we have

$$\mathcal{F}'(\Omega) \leqslant \frac{p}{P_F(\Omega)^2 V(\Omega)^{\frac{p}{n}}} E_F(\Omega) \int_{\partial\Omega} \frac{d\mathcal{H}^{n-1}(x)}{H^F_{\partial\Omega}(x)} < 0.$$

We suppose now that $\Omega \notin C_F^{\infty,+}$ and we assume by contradiction that Ω minimizer the functional $\mathcal{F}(\cdot)$. We can find a decreasing (in the sense of inclusion) sequence of sets $(\Omega_k)_{k\in\mathbb{N}} \subset C_F^{\infty,+}$ that converges to Ω in the Hausdorff sense. We have that

$$\lim_{k\to+\infty} V(\Omega_k) = V(\Omega); \qquad \lim_{k\to+\infty} P_F(\Omega_k) = P_F(\Omega);$$

$$\lim_{k\to+\infty} M_F(\Omega_k) = M_F(\Omega); \qquad \lim_{k\to+\infty} r^F_{\max}(\Omega_k) = r^F_{\max}(\Omega).$$

We now consider the IAMCF (inverse anisotropic mean curvature flow) for every Ω_k and we denote by $\Omega_k(t)$, for $t \geqslant 0$, the family generated in this way. We let $\Omega_k(0) = \Omega_k$. Using Hadamard formula (see [83]), we obtain:

$$\frac{d}{dt} V(\Omega_k(t)) = \int_{\partial\Omega_k(t)} \frac{F(\nu(x))}{H^F_{\partial\Omega_k(t)}} d\mathcal{H}^{n-1}(x); \tag{3.7}$$

$$\frac{d}{dt} P_F(\Omega_k(t)) = P_F(\Omega_k(t)). \tag{3.8}$$

We have also that

$$\frac{d}{dt} r^F_{\max}(\Omega_k(t)) \leqslant \frac{r^F_{\max}(\Omega_k(t))}{n-1}. \tag{3.9}$$

We prove now this last inequality. From definition of $x^F_{\max}(\Omega(t))$ and (4.9) in the IAMCF case, we have that

$$r^F_{\max}(\Omega(t)) = F^o(x^F_{\max}(\Omega(t)));$$

$$x^F_{\max}(\Omega(t)) = x^F_{\max}(\Omega) + \frac{t\nu^F_{\partial\Omega}}{H^F_{\partial\Omega}(x^F_{\max}(\Omega))}.$$

Then

$$
\begin{aligned}
\partial_t r^F_{\max}(\Omega(t)) &= \partial_t F^o(x^F_{\max}(\Omega(t))) = \langle \nabla F^o(x^F_{\max}(\Omega(t)), \frac{\nu^F_{\partial\Omega}(x^F_{\max}(\Omega))}{H^F_{\partial\Omega}(x^F_{\max}(\Omega))}\rangle \leqslant \\
&\leqslant F(\nabla F^o(x^F_{\max}(\Omega(t))))F^o(\nu^F_{\partial\Omega}(x^F_{\max}(\Omega)))\frac{1}{H^F_{\partial\Omega}(x^F_{\max}(\Omega))} = \\
&= F(\nabla F^o(x^F_{\max}(\Omega(t))))F^o(\nabla F(\nu_{\partial\Omega}(x^F_{\max}(\Omega))))\frac{1}{H^F_{\partial\Omega}(x^F_{\max}(\Omega))} \leqslant \\
&= \frac{1}{H_F(x^F_{\max}(\Omega))} = \frac{r^F_{\max}(\Omega)}{n-1},
\end{aligned}
$$

since F is a Finsler norm and therefore it is true that $F(\nabla F^o(x)) = 1 = F^o(\nabla F(x)) = 1$. We can then repeat this last inequality for every Ω_k. From (3.9) follows that

$$
r^F_{\max}(\Omega_k(t)) \leqslant r^F_{\max}(\Omega_k)e^{\frac{t}{(n-1)}}, \text{ for } t > 0. \tag{3.10}
$$

Analogous computations to the ones reported in [31, Proposition 2.4] lead to a contradiction with the minimality of Ω. Let us see that in details. Using the minimality of Ω, we have that $\mathcal{F}(\Omega) \leqslant \mathcal{F}(\Omega_k(t))$. Then, using the monotonicity of the perimeter with respect to the inclusion of convex set, (3.7) and (3.8), we obtain that $P_F(\Omega_k(t)) \geqslant P(\Omega_k(0)) = P(\Omega_k)$. From (3.2), (3.3), (3.4), (3.10), it holds for every $t \geqslant 0$, setting $\alpha_k = P_F^2(\Omega_k(t))V(\Omega_k(t))^{p/n}/p$,

$$
\begin{aligned}
\alpha_k\, \mathcal{F}'(\Omega_k(t)) &\leqslant \int_{\partial\Omega_k(t)} \left(F^o(x)^{p-1} r^F_{\max}(\Omega_k(t)) - \frac{M_F(\Omega_k(t))}{nV(\Omega_k(t))}\right) \frac{F(\nu(x))}{H^F_{\partial\Omega}(x)}\, d\mathcal{H}^{n-1}. \\
&\leqslant V'(\Omega_k(t))\left(\frac{1}{a^{p-1}} r^F_{\max}(\Omega_k)e^{\frac{t}{(n-1)}} - \frac{\mathcal{F}(\Omega)P_F(\Omega)}{nV(\Omega_k(t))^{1-n/p}}\right).
\end{aligned}
$$

Now we integrate in the interval $[0, T]$ and, using (3.7) and (3.8), we obtain $V(\Omega) \leqslant V(\Omega_k) \leqslant V(\Omega_k(t)) \leqslant V(\Omega_k(T))$. Consequently,

$$
\alpha_k\left(\mathcal{F}(\Omega_k(T)) - \mathcal{F}(\Omega_k)\right) \leqslant [V(\Omega_k(T)) - V(\Omega_k)]\left(\frac{1}{a^{p-1}} r^F_{\max}(\Omega_k)e^{\frac{T}{(n-1)}} - \frac{\mathcal{F}(\Omega)P_F(\Omega)}{nV(\Omega_k(T))^{1-n/p}}\right).
$$

Since we are supposing (3.6), there exist $\delta > 0$ and $T > 0$ small enough and $k_0 >> 1$ such that $V(\Omega_{k_0}(T)) < V(\Omega) + \delta$ and

$$
\left(\frac{1}{a^{p-1}} r^F_{\max}(\Omega_k)e^{\frac{T}{(n-1)}} - \frac{\mathcal{F}(\Omega)P_F(\Omega)}{nV(\Omega_k(T))^{1-n/p}}\right) < 0. \tag{3.11}
$$

Moreover, since the AIMCF preserves the inclusion, we have $\Omega_k(T) \subset \Omega_{k_0}(T)$ for $k \geqslant k_0$ which implies $V(\Omega_k(T)) < V(\Omega) + \delta$ for $k \geqslant k_0$. So, for $k \geqslant k_0$,

$$
\alpha_k\left(\mathcal{F}(\Omega_k(T)) - \mathcal{F}(\Omega_k)\right) \leqslant \tag{3.12}
$$

$$
[V(\Omega_k(T)) - V(\Omega_k)]\left(\frac{1}{a^{p-1}} r^F_{\max}(\Omega_k)e^{\frac{T}{(n-1)}} - \frac{\mathcal{F}(\Omega)P_F(\Omega)}{n\,[V(\Omega) + \delta]^{1-n/p}}\right). \tag{3.13}
$$

Using the anisotropic Heintze-Karcher inequality (1.38) in Lemma 1.18, for $t \geqslant 0$;

$$
V'(\Omega_k(t)) \geqslant n(n-1)V(\Omega_k(t))
$$

and so

$$V(\Omega_k(T)) \geqslant V(\Omega_k)e^{n(n-1)T}.$$

By (3.11) and (3.12), passing to the limit,

$$\alpha_k\left(\left[\lim_{k\to\infty}\mathcal{F}(\Omega_k(T))-\mathcal{F}(\Omega)\right]\right)\leqslant$$
$$V(\Omega)\left(e^{n(n-1)T}-1\right)\left(\frac{1}{a^{p-1}}r^F_{\max}(\Omega_k)e^{\frac{T}{(n-1)}}-\frac{\mathcal{F}(\Omega)P_F(\Omega)}{n\left[V(\Omega)+\delta\right]^{1-n/p}}\right)<0.$$

Since for $k \geqslant k_0$, it holds $\Omega \subset \Omega_k(T) \subset \Omega_{k_0}(T)$, there exists a convex set $\tilde{\Omega}$ such that $\lim_{k\to\infty}\mathcal{F}(\Omega_k(T)) = \mathcal{F}(\tilde{\Omega})$, so $\mathcal{F}(\tilde{\Omega}) < \mathcal{F}(\Omega)$, that contradicts the minimality of Ω.

□

A minimizer cannot have positive Excess.

We start by observing that there exist sets with positive excess.

Remark 3.9. We consider the case $n = 2$ and $p = 2$. The norm that we take into consideration is

$$F(x,y)=\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}},$$

and its polar is:

$$F^o(x,y)=\sqrt{a^2x^2+b^2y^2}.$$

We define

$$\mathcal{E}_\epsilon=\{(x,y)\in\mathbb{R}^2 \mid a^2(1-\epsilon)^2x^2+b^2(1+\epsilon)^2y^2\}.$$

We have that

$$r^F_{\max}(\mathcal{E}_\epsilon)=1+\epsilon+o(\epsilon)$$

and

$$V(R_\epsilon)=\frac{\pi}{ab}(1+\epsilon^2+o(\epsilon)).$$

Computing the momentum, we find that

$$M_F(R_\epsilon)=\frac{2}{ab(1-\epsilon)^2(1+\epsilon)^2}\left(\pi+\epsilon\int_0^\pi\cos(2t)\,dt\right)+o(\epsilon)=\frac{2}{ab(1-\epsilon)^2(1+\epsilon)^2}\left(\pi+o(\epsilon)\right)$$

and so it results that $E_F = \epsilon + o(\epsilon)$.

Following [31], for every $\epsilon > 0$, we consider the halfspace T_ϵ that has outer unit normal pointing in the direction $x^F_{\max}(\Omega)$ and that intersects Ω at a distance ϵ from $x^F_{\max}(\Omega)$. We define the sets:

$$\Omega_\epsilon := \Omega \cap T_\epsilon,$$
$$A_\epsilon := \partial\Omega_\epsilon \cap \partial T_\epsilon,$$

and the following quantitities, that vanish as ϵ goes to 0:

$$\Delta M_F := M_{F,p}(\Omega_\epsilon) - M_{F,p}(\Omega),$$
$$\Delta V := V(\Omega_\epsilon) - V(\Omega),$$
$$\Delta P_F := P_F(\Omega_\epsilon) - P_F(\Omega).$$

Lemma 3.10. *There exists a positive constant $C(\Omega)$ such that for all $\epsilon > 0$ small enough, we have that*

$$|\Delta V| \leqslant C(\Omega)|\Delta P_F|. \tag{3.14}$$

Proof. By definition, we have that

$$\Omega \subseteq \{x \in \mathbb{R}^n \mid F^o(x) \leqslant r^F_{\max}\},$$

that is the Wulff shape $\mathcal{W}_{r^F_{\max}}$ of radius $r^F_{\max}$ centered at the origin, and it results

$$A_\epsilon \subset \mathcal{W}_{r^F_{\max}} \cap \partial T_\epsilon.$$

We can define the anisotropic diameter of Ω as

$$\operatorname{diam}_F(\Omega) = \sup\{F^o(x-y) \mid x, y \in \Omega\}.$$

We recall that there exist $a, b > 0$ such that for every $x \in \mathbb{R}^n$

$$a|x| \leqslant F(x) \leqslant b|x|;$$

$$\frac{1}{b}|x| \leqslant F^o(x) \leqslant \frac{1}{a}|x|.$$

Then, we have that

$$\frac{\operatorname{diam}_F(\Omega)}{2} \leqslant \frac{\operatorname{diam}(\Omega)}{2a} \leqslant \frac{1}{a}\sqrt{2r_{\max}(\Omega)\epsilon} \leqslant \frac{b}{a}\sqrt{2r^F_{\max}(\Omega)\epsilon}. \tag{3.15}$$

We follow now the construction described in [31, Lemma 2.5]. Without loss of generality, we can suppose that the x_n axis lies in the direction of the outer normal to T_ϵ. Let $A'_\epsilon \subset \mathbb{R}^{n-1}$ the projection of A_ϵ onto the subspace $\{x_n = 0\}$. We denote with $g(\cdot) : A'_\epsilon \to \mathbb{R}$ the concave function describing $\partial\Omega \backslash \partial\Omega_\epsilon$. We can observe that $g(0) = r^F_{\max}$. We define then

$$h : A'_\epsilon \to \mathbb{R}$$

$$h(y) := g(y) - \left(r^F_{\max}(\Omega) - \epsilon\right).$$

By construction, we have that $\max h = \epsilon = h(0)$. As in [31],

$$-\Delta V = |\Delta V| = \int_{A'_\epsilon} h(y)\, dy \geqslant \epsilon \frac{\mathcal{L}^{n-1}(A'_\epsilon)}{n}, \tag{3.16}$$

and so, using (3.15), (3.16) and the Sobolev-Poincaré inequality, we obtain

$$|\Delta V| =\leqslant C(n)\frac{a^2}{b^2} 2r^F_{\max}(\Omega)(\kappa_{n-1})^{2/(n-1)} \int_{A'_\epsilon} |Dh|^2\, dy.$$

On the other hand, we have:

$$-\Delta P_F = \int_{A'_\epsilon} \left(\sqrt{1 + |Dg(y)|^2} - 1\right) F(\nu(y))\, dy \geqslant K(\Omega) \int_{A'_\epsilon} |Dg(y)|^2 F(\nu(y))\, dy$$

and so we have the desired result.

□

Lemma 3.11. *Let Ω be a bounded, open convex set of $\mathbb{R}^n$, then*

$$|\Delta M_F| \leqslant p(r^F_{\max}(\Omega))^{p-1}\Delta V + (r^F_{\max}(\Omega))^p \Delta P_F + o(\Delta P_F) + o(\Delta V). \tag{3.17}$$

Proof.

$$\begin{aligned}
&-\Delta M_F = \\
&\int_{A'_\epsilon} (g^p(y) + [F^o(y)]^p)\sqrt{1+|Dg|^2}F(\nu(y))dy - \\
&-\int_{A'_\epsilon} \left[\left(r^F_{\max}(\Omega)-\epsilon\right)^p + [F^o(y)]^p\right] F(\nu(y))dy = \\
&= \int_{A'_\epsilon} \left(g^p(y) - \left(r^F_{\max}(\Omega)-\epsilon\right)^p\right)\sqrt{1+|Dg(y)|^2}\, F(\nu(y))\, dy + \\
&+ \int_{A'_\epsilon} \left[\left(r^F_{\max}(\Omega)-\epsilon\right)^p + (F^o(y))^p\right]\left(\sqrt{1+|Dg(y)|^2}-1\right) F(\nu(y))\, dy = I_1 + I_2.
\end{aligned}$$

We prove now that

$$I_1 \geqslant -2r^F_{\max}(\Omega)\Delta V + o(\Delta V).$$

Considering that $r^F_{\max}(\Omega) = g(0)$ and using the convexity inequality, we have that

$$\begin{aligned}
&\int_{A'_\epsilon} \left(g^p(y) - \left(r^F_{\max}(\Omega)-\epsilon\right)^p\right)\sqrt{1+|Dg(y)|^2}\, F(\nu(y))\, dy \geqslant \\
&\geqslant \int_{A'_\epsilon} h(y)\ p\ g(0)^{p-1}\ F(\nu(y))dy + o(\Delta V) \geqslant p\ g(0)^{p-1} a \int_{A'_\epsilon} h(y)\ dy + o(\Delta V).
\end{aligned}$$

Then, we will prove that

$$I_2 \geqslant -2r^F_{\max}(\Omega)\Delta V + o(\Delta P_F).$$

Firstly, we observe that

$$0 < \int_{A'_\epsilon} (F^o(y))^p \left(\sqrt{1+|Dg(y)|^2}-1\right) F(\nu_{\partial\Omega}(y))\, dy \leqslant \left(\operatorname{diam}_F(A'_\epsilon)\right)^p (-\Delta P_F)$$

and that $\dfrac{o(\Delta P_F)}{\Delta P_F} \to 0$ implies that, for every costant c,

$$c\Delta P_F \geqslant o(\Delta P_F).$$

Using these last two relations, we can deduce that

$$\begin{aligned}
&\int_{A'_\epsilon} (F^o(y))^p \left(\sqrt{1+|Dg(y)|^2}-1\right) F(\nu(y))\, dy \\
&\geqslant -\int_{A'_\epsilon} (F^o(y))^p \left(\sqrt{1+|Dg(y)|^2}-1\right) F(\nu(y))\, dy \geqslant \\
&\geqslant \left(\operatorname{diam}_F(A'_\epsilon)\right)^p (\Delta P_F) \geqslant o(\Delta P_F).
\end{aligned}$$

To conclude, we observe that

$$\begin{aligned}
&\int_{A'_\epsilon} \left(r^F_{\max}(\Omega) - \epsilon\right)^p \left(\sqrt{1+|Dg(y)|^2} - 1\right) F(\nu(y))\, dy = \\
&= \int_{A'_\epsilon} \left[(r^F_{\max}(\Omega))^p - p\epsilon + o(\epsilon)\right] \left(\sqrt{1+|Dg(y)|^2} - 1\right) F(\nu(y))\, dy = \\
&= -(r^F_{\max}(\Omega))^p \Delta P_F + \int_{A'_\epsilon} (p\epsilon + o(\epsilon)) \left(\sqrt{1+|Dg(y)|^2} - 1\right) F(\nu(y))\, dy \\
&\geqslant -(r^F_{\max}(\Omega))^p \Delta P_F + o(\Delta P_F).
\end{aligned}$$

□

Proposition 3.12. *Let Ω be a bounded, open convex set of $\mathbb{R}^n$ such that*

$$E_F(\Omega) > 0, \tag{3.18}$$

then Ω is not a minimizer of $\mathcal{F}(\cdot)$.

Proof. We can write the following:

$$\begin{aligned}
&\frac{\Delta\mathcal{F}(\Omega)}{\mathcal{F}(\Omega)} = \Delta \ln\left[M_F(\Omega)(P_F(\Omega))^{-1}(V(\Omega))^{-\frac{p}{n}}\right] + o(\Delta P_F) + o(\Delta V) = \\
&= \Delta\ln(M_F(\Omega)) - \Delta\ln(P_F(\Omega)) - \frac{p}{n}\Delta\ln(V(\Omega)) + o(\Delta P_F) + o(\Delta V) = \\
&\frac{\Delta M_F(\Omega)}{M_F(\Omega)} - \frac{\Delta P_F(\Omega)}{P_F(\Omega)} - \frac{p}{n}\frac{\Delta V(\Omega)}{V(\Omega)} + o(\Delta P_F) + o(\Delta V).
\end{aligned}$$

Then, using (3.17), we have that

$$\Delta\mathcal{F}(\Omega) = \frac{1}{V(\Omega)^{\frac{p}{n}} P_F(\Omega)} \left(\Delta M_F - \frac{\Delta P_F}{P_F(\Omega)} M_F(\Omega) - \frac{p}{n}\frac{\Delta V}{V(\Omega)} M_F(\Omega)\right) + o(\Delta P_F) + o(\Delta V) = \tag{3.19}$$

$$\begin{aligned}
&= \frac{1}{V(\Omega)^{\frac{p}{n}} P_F(\Omega)} \Bigg[p\left(\left(r^F_{\max}(\Omega)\right)^{p-1} - \frac{W_F(\Omega)}{nV(\Omega)}\right)\Delta V + \\
&\qquad \left((r^F_{\max}(\Omega))^p - \frac{M_F(\Omega)}{P_F(\Omega)}\right)\Delta P_F\Bigg] + o(\Delta P_F) + o(\Delta V) = \\
&= \frac{1}{V(\Omega)^{\frac{p}{n}} P_F(\Omega)} \left[pE_F(\Omega)\Delta V + \left((r^F_{\max}(\Omega))^p - \frac{M_F(\Omega)}{P_F(\Omega)}\right)\Delta P_F\right] + o(\Delta P_F) + o(\Delta V)
\end{aligned}$$

Since (3.18) holds, Ω cannot be a ball centered at the origin. It follows that

$$(r^F_{\max}(\Omega))^p - \frac{M_F(\Omega)}{P_F(\Omega)} > 0.$$

Considering also that $\Delta V < 0$ and $\Delta P_F < 0$, we can conclude that

$$\Delta\mathcal{F} < 0.$$

□

Wulff shapes are the unique minimizers having vanishing Excess

Proposition 3.13. *Let Ω be a bounded, open convex set of $\mathbb{R}^n$ such that*

$$E_F(\Omega) = 0, \tag{3.20}$$

then either Ω is the Wulff shape centered at the origin or it is not a minimizer of $\mathcal{F}(\cdot)$.

Proof. From (3.14), (3.20), (3.19), we obtain the following expression

$$\Delta\mathcal{F}(\Omega) = \frac{1}{V(\Omega)^{\frac{p}{n}} P_F(\Omega)} \left[\left((r^F_{\max}(\Omega))^p - \frac{M_F(\Omega)}{P_F(\Omega)} \right) \Delta P_F \right] + o(\Delta P_F).$$

If

$$(r^F_{\max}(\Omega))^p = \frac{M_F(\Omega)}{P_F(\Omega)},$$

then Ω is a Wulff shape centered at the origin. If $\Delta\mathcal{F} < 0$, then Ω is not a minimizer. Thus, we have proved the desired claim. $\square$

3.2 Study of the orthotropic $\infty-$ Laplace eigenvalue problem of Steklov type

3.2.1 The $p-$orthotropic Laplace eigenvalue with Steklov boundary condition: definitions and notations.

We fix $p > 1$ and an open bounded convex set $\Omega \subseteq \mathbb{R}^n$ and consider the Steklov problem for the orthotropic p-Laplacian operator on Ω, sometimes called pesudo p-Laplacian, as studied in [27], that is

$$\begin{cases} -\widetilde{\Delta}_p u = 0 & \text{on } \Omega \\ \sum_{j=i}^n |u_{x_j}|^{p-2} u_{x_j} \nu_j = \sigma |u|^{p-2} u \rho_p & \text{on } \partial\Omega, \end{cases} \tag{3.21}$$

where u_{x_j} is the partial derivative of u with respect to x_j, $\nu = (\nu_1, \ldots, \nu_n)$ is the outer normal of $\partial\Omega$, $\rho_p(x) = \|\nu_{\partial\Omega}(x)\|_{\ell^{p'}}$, p' is the coniugate exponent of p, and

$$-\widetilde{\Delta}_p u = \operatorname{div}\left(\mathcal{A}_p(\nabla u)\right), \qquad \mathcal{A}_p(\nabla u) = \left(|u_{x_1}|^{p-2} u_{x_1}, \ldots, |u_{x_n}|^{p-2} u_{x_n}\right).$$

We will use the following notation: for any $x \in \mathbb{R}^n$ and $p \geqslant 1$

$$\|x\|^p_{\ell^p} = \sum_{j=1}^n |x_j|^p,$$

while for $p = \infty$ we have

$$\|x\|_{\ell^\infty} = \max_{j=1,\ldots,n} |x_j|.$$

Solutions of (3.21) are to be interpreted in the weak sense; we recall here the definition of weak solution.

Definition 3.1. Let $u \in W^{1,p}(\Omega)$. We say that u is a weak solution of (3.21) if

$$\int_\Omega \langle \mathcal{A}_p(\nabla u), \nabla\varphi \rangle dx = \sigma \int_{\partial\Omega} |u|^{p-2} u \varphi \rho_p \, d\,\mathcal{H}^{n-1} \qquad \forall \varphi \in W^{1,p}(\Omega).$$

It has been shown in [27, Section 4] that the Steklov problem (3.21) admits a non-decreasing sequence of eigenvalues

$$0 = \sigma_{1,p}(\Omega) < \sigma_{2,p}(\Omega) \leqslant \cdots$$

where the first eigenvalue is trivial for any $p > 1$ and corresponds to constant eigenfunctions. We denote the first non-trivial eigenvalue $\sigma_{2,p}(\Omega) =: \Sigma_p^p(\Omega)$. In [27] a variational characterization of Σ_p^p is shown. Indeed, we have that

$$\Sigma_p^p(\Omega) = \min\left\{ \frac{\int_\Omega \|\nabla u\|_{\ell^p}^p \, dx}{\int_{\partial\Omega} |u|^p \rho_p(x) d\mathcal{H}^{n-1}}, \; u \in W^{1,p}(\Omega), \; \int_{\partial\Omega} |u|^{p-2} u \rho_p(x) d\mathcal{H}^{n-1} = 0 \right\}. \tag{3.22}$$

Finally, we observe that (for instance for C^2 functions) we can rewrite the orthotropic p-Laplacian operator in such a way to explicitly see where the second derivatives come into play:

$$\widetilde{\Delta}_p u = \sum_{j=1}^n (p-1)|u_{x_j}|^{p-2} u_{x_j,x_j}. \tag{3.23}$$

3.2.2 Viscosity solutions of the p-orthotropic Steklov problem

In the following we will need to work with viscosity solutions to the Steklov problem (3.21). Let us consider in this section Ω a C^1 open bounded convex subset of $\mathbb{R}^n$. Thus, we denote

$$F_p : (\xi, X) \in \mathbb{R}^n \times \mathbb{R}^{n\times n} \mapsto -\sum_{j=1}^n (p-1)|\xi_j|^{p-2} X_{j,j}$$

and

$$B_p : (\sigma, x, u, \xi) \in \mathbb{R} \times \partial\Omega \times \mathbb{R} \times \mathbb{R}^n \mapsto \sum_{j=1}^n |\xi_j|^{p-2}\xi_j \nu_j(x) - \sigma |u|^{p-2} u \rho_p(x).$$

Following [72], the Steklov problem (3.21) can be *formally* rewritten as

$$\begin{cases} F_p(\nabla u, \nabla^2 u) = 0, & \text{on } \Omega \\ B_p(\sigma, x, u, \nabla u) = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.24}$$

As a consequence, the functions F_p and B_p can be used to define viscosity solutions for the Steklov problem (3.21) (see, for instance, [86]). We give now the following definitions

Definition 3.2. Let u be a lower (upper) semi-continuous function on $\overline{\Omega}$ and $\Phi \in C^2(\bar{\Omega})$. We say that Φ is *touching from below* (*above*) u in $x_0 \in \overline{\Omega}$ if and only if $u(x_0) - \Phi(x_0) = 0$ and $u(x) > \Phi(x)$ ($u(x) < \Phi(x)$) for any $x \neq x_0$ in $\bar{\Omega}$.

Definition 3.3. A lower (upper) semi-continuous function u on $\bar{\Omega}$ is said to be a *viscosity supersolution* (*subsolution*) of (3.24) if for any function $\Phi \in C^2(\bar{\Omega})$ touching from below (above) u in $x_0 \in \bar{\Omega}$ one has

- $F_p(\nabla\Phi(x_0), \nabla^2\Phi(x_0)) \geqslant (\leqslant) 0$ with $x_0 \in \Omega$;
- $\max\{F_p(\nabla\Phi(x_0), \nabla^2\Phi(x_0)), B_p(\sigma, x_0, \Phi(x_0), \nabla\Phi(x_0))\} \geqslant (\leqslant)$, with $x_0 \in \partial\Omega$

Finally, we say that a continuous function u on $\bar{\Omega}$ is a *viscosity solution* if it is both viscosity subsolution and supersolution.

We recall here this result, that is proved in [92, Section 10].

Lemma 3.14. *Let $n \in \mathbb{N}$ and $x, y \in \mathbb{R}^n$. For $p \geqslant 2$ we have*

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle \geqslant 2^{2-p}|x - y|^p.$$

Now we are ready to show the following result, which is the p-orthotropic version of [72, Lemma 2.1].

Proposition 3.15. *Fix $p \geqslant 2$. Let u be a weak solution of the Steklov problem* (3.21) *that is continuous in $\overline{\Omega}$. Then it is a viscosity solution of* (3.24).

Proof. Let us show that u is a viscosity supersolution of (3.24), since for the subsolution the proof is analogous. Let us consider $\Phi \in C^2(\Omega)$ touching from below u in $x_0 \in \Omega$. Let us first consider $x_0 \in \Omega$. We want to show that

$$F_p(\nabla\Phi(x_0), \nabla^2\Phi(x_0)) \geqslant 0.$$

Let us suppose by contradiction that

$$F_p(\nabla\Phi(x_0), \nabla^2\Phi(x_0)) < 0.$$

Since $\Phi \in C^2$, there exists a radius $r > 0$ such that for any $x \in B_r(x_0)$ it holds

$$F_p(\nabla\Phi(x), \nabla^2\Phi(x)) < 0.$$

Consider then

$$m = \inf_{x\in\partial B_r(x_0)} |u(x) - \Phi(x)| = \inf_{x\in\partial B_r(x_0)} (u(x) - \Phi(x))$$

and define $\Psi(x) = \Phi(x) + \frac{m}{2}$. Since Ψ and Φ differ only by a constant, $\nabla\Psi = \nabla\Phi$ and $\nabla^2\Psi = \nabla^2\Phi$. Hence, for any $x \in B_r(x_0)$

$$F_p(\nabla\Psi(x), \nabla^2\Psi(x)) < 0,$$

that is to say

$$-\widetilde{\Delta}_p\Psi(x) < 0.$$

This leads, for any non-negative test function $\varphi \in W_0^{1,p}(B_r(x_0))$ with $\varphi \not\equiv 0$, to

$$\sum_{j=1}^{n} \int_{B_r(x_0)} |\Psi_{x_j}|^{p-2}\Psi_{x_j}\varphi_{x_j}\,dx < 0.$$

Moreover, being u a weak solution of (3.21), we have, for any $\varphi \in W_0^{1,p}(B_r(x_0))$,

$$\sum_{j=1}^{n} \int_{B_r(x_0)} |u_{x_j}|^{p-2}u_{x_j}\varphi_{x_j}\,dx = 0.$$

Thus we get, for any non-negative text function $\varphi \in W_0^{1,p}(B_r(x_0))$,

$$\sum_{j=1}^{n} \int_{B_r(x_0)} (|\Psi_{x_j}|^{p-2}\Psi_{x_j} - |u_{x_j}|^{p-2}u_{x_j})\varphi_{x_j}\,dx < 0.$$

Let us observe that $\Psi(x_0) - u(x_0) = \frac{m}{2} > 0$. On the other hand, since u and Φ are continuous, there exists a radius $r_* > 0$ such that $u(x) - \Phi(x) \geqslant \frac{m}{2}$ for any $x \in B_r(x_0) \backslash B_{r_*}(x_0)$. In particular, for any $x \in B_r(x_0) \backslash B_{r_*}(x_0)$ it holds $\Psi(x) - u(x) \leqslant 0$. Thus, the function $\varphi = (\Psi - u)^+ \chi_{B_r(x_0)}$ can be expressed as

$$\varphi(x) = \begin{cases} (\Psi - u)^+ & x \in B_r(x_0) \\ 0 & x \notin B_{r_*}(x_0), \end{cases}$$

where the two definitions agree in $B_r(x_0) \backslash B_{r_*}(x_0)$. Finally, we can observe that, being Ψ and u both in $W^{1,p}(B_r(x_0))$, $\Psi - u$ is in $W^{1,p}(B_r(x_0))$ and then also its positive part (see [?] and references therein). Since we have shown that $\varphi \in W_0^{1,p}(B_r(x_0))$, we can use it as a test function to achieve

$$\sum_{j=1}^{n} \int_{\{\Psi > u\} \cap B_r(x_0)} (|\Psi_{x_j}|^{p-2} \Psi_{x_j} - |u_{x_j}|^{p-2} u_{x_j})(\Psi_{x_j} - u_{x_j}) dx < 0.$$

Thus, by Lemma 3.14, we obtain

$$\begin{aligned} 0 &\leqslant \sum_{j=1}^{n} \int_{\{\Psi > u\} \cap B_r(x_0)} |\Psi_{x_j} - u_{x_j}|^p dx \\ &\leqslant C(p) \sum_{j=1}^{n} \int_{\{\Psi > u\} \cap B_r(x_0)} (|\Psi_{x_j}|^{p-2} \Psi_{x_j} - |u_{x_j}|^{p-2} u_{x_j})(\Psi_{x_j} - u_{x_j}) dx < 0, \end{aligned}$$

which is absurd.
Now let us consider $x_0 \in \partial\Omega$. As before, let us argue by contradiction, supposing that

$$\max\{F_p(\nabla\Phi(x_0), \nabla^2\Phi(x_0)), B_p(\sigma, x_0, u(x_0), \nabla\Phi(x_0))\} < 0.$$

Thus, since $\Phi \in C^2$ and $u \in C^0$, there exists a radius $r > 0$ such that, for any $x \in B_r(x_0) \cap \Omega$, it holds

$$F_p(\nabla\Phi(x), \nabla^2\Phi(x)) < 0,$$

while, for any $x \in B_r(x_0) \cap \partial\Omega$, it holds

$$\max\{F_p(\nabla\Phi(x), \nabla^2\Phi(x)), B_p(\sigma, x, u(x), \nabla\Phi(x))\} < 0.$$

As before, let us consider

$$m = \inf_{x \in \partial B_r(x_0) \cap \overline{\Omega}} |u(x) - \Phi(x)| = \inf_{x \in \partial B_r(x_0) \cap \overline{\Omega}} (u(x) - \Phi(x))$$

and define $\Psi(x) = \Phi(x) + \frac{m}{2}$. We have that, for any $x \in B_r(x_0) \cap \Omega$, it holds

$$F_p(\nabla\Psi(x), \nabla^2\Psi(x)) < 0,$$

while, for any $x \in B_r(x_0) \cap \partial\Omega$, it holds

$$\max\{F_p(\nabla\Psi(x), \nabla^2\Psi(x)), B_p(\sigma, x, u(x), \nabla\Psi(x))\} < 0.$$

From the fact that $F_p(\nabla\Psi(x), \nabla^2\Psi(x)) < 0$, we achieve

$$-\Delta_p \Psi(x) < 0.$$

Let us consider a non-negative test function $\varphi \in W^{1,p}(B_r(x_0) \cap \Omega)$ such that $\varphi \not\equiv 0$ and $\varphi = 0$ on $\partial B_r(x_0) \cap \Omega$. It holds

$$\sum_{j=1}^{n} \int_{B_r(x_0)\cap\Omega} |\Psi_{x_j}|^{p-2}\Psi_{x_j}\varphi_{x_j}dx < \sum_{j=1}^{n} \int_{B_r(x_0)\cap\partial\Omega} |\Psi_{x_j}|^{p-2}\Psi_{x_j}\varphi\nu^j_{\partial\Omega}d\,\mathcal{H}^{n-1}\,.$$

Now, since $B_p(\sigma, x, u(x), \nabla\Psi(x)) < 0$, we have, for $x \in B_r(x_0) \cap \partial\Omega$,

$$\sum_{j=1}^{n} |\Psi_{x_j}(x)|^{p-2}\Psi_{x_j}(x)\nu^j_{\partial\Omega}(x) < \sigma|u(x)|^{p-2}u(x)\rho_p(x)$$

and consequently

$$\sum_{j=1}^{n} \int_{B_r(x_0)\cap\Omega} |\Psi_{x_j}|^{p-2}\Psi_{x_j}\varphi_{x_j}dx < \sigma \int_{B_r(x_0)\cap\partial\Omega} |u|^{p-2}u\varphi\rho_p d\,\mathcal{H}^{N-1}\,.$$

Moreover, being u a weak solution of (3.21), we have

$$\sum_{j=1}^{N} \int_{B_r(x_0)\cap\Omega} |u_{x_j}|^{p-2}u_{x_j}\varphi_{x_j}dx = \sigma \int_{B_r(x_0)\cap\partial\Omega} |u|^{p-2}u\varphi.\rho_p d\,\mathcal{H}^{N-1}\,.$$

Hence we obtain

$$\sum_{j=1}^{n} \int_{B_r(x_0)\cap\Omega} (|\Psi_{x_j}|^{p-2}\Psi_{x_j} - |u_{x_j}|^{p-2}u_{x_j})\varphi_{x_j}dx < 0.$$

Let us consider $\varphi = (\Psi - u)^+\chi_{B_r(x_0)\cap\bar{\Omega}}$. Arguing as before we have that $\varphi \in W^{1,p}(\Omega \cap B_r(x_0))$ and $\varphi = 0$ on $\partial B_r(x_0) \cap \Omega$, thus we can use it as test function to achieve

$$\begin{aligned} 0 &\leqslant \sum_{j=1}^{n} \int_{\{\Psi>u\}\cap B_r(x_0)\cap\Omega} |\Psi_{x_j} - u_{x_j}|^p dx \\ &\leqslant C(p) \sum_{j=1}^{n} \int_{\Omega} (|\Psi_{x_j}|^{p-2}\Psi_{x_j} - |u_{x_j}|^{p-2}u_{x_j})(\Psi_{x_j} - u_{x_j})dx < 0, \end{aligned}$$

which is absurd. □

Remark 3.16. Concerning the regularity of a weak solution u of $-\widetilde{\Delta}_p u = 0$, let us observe that for $p \geqslant 2$ orthotropic p-harmonic functions are locally Lipschitz in Ω (see [21]) and in particular in dimension 2 they are $C^1(\Omega)$ for any $p > 1$ (see [20]). We will actually work with $p \to +\infty$, hence we can suppose $p > n$. In such case, Morrey's embedding theorem ensures that $u \in C^0(\bar{\Omega})$. We can conclude that for $p > N$, every weak solution of (3.21) is a viscosity solution of (3.24).

3.2.3 The orthotropic ∞-Laplacian: heuristic derivation

We want to study problem 3.21 as $p \to +\infty$. To do this, we need to introduce the orthotropic ∞-Laplacian, as the formal limit as $p \to +\infty$ of $\widetilde{\Delta}_p$. The operator $\widetilde{\Delta}_p$ can be interpreted as the anistropic p-Laplace operator associated to the norm $\mathcal{F}_p(x) = \|x\|_{\ell^p}$, i. e.

$$\widetilde{\Delta}_p u = \operatorname{div}\left(\frac{1}{p}\nabla_x \mathcal{F}_p^p(\nabla u)\right).$$

In the classic case the ∞-Laplacian Δ_∞ was achieved from the p-Laplacian Δ_p by dividing by $(p-2)|\nabla u|^{p-4}$ and then formally taking the limit as $p \to +\infty$ (see [93]). We work in the same fashion, by using $\|\nabla u\|_{\ell^p}$. Before doing this, let us recall the following easy result.

Lemma 3.17. *The functions $\|\cdot\|_{\ell^p}$ uniformly converge to $\|\cdot\|_{\ell^\infty}$ as $p \to +\infty$ and to $\|\cdot\|_{\ell^1}$ as $p \to 1$ in any compact set $K \subseteq \mathbb{R}^n$.*

Proof. Let us recall that for any $x \in \mathbb{R}^n$

$$\|x\|_{\ell^\infty} \leqslant \|x\|_{\ell^p} \leqslant n^{\frac{1}{p}} \|x\|_{\ell^\infty} \tag{3.25}$$

thus we have that for any compact $K \subseteq \mathbb{R}^n$ (setting $M_\infty = \max_{x \in K} \|x\|_{\ell^\infty}$)

$$|\, \|x\|_{\ell^p} - \|x\|_{\ell^\infty} \,| \leqslant (1 - n^{\frac{1}{p}}) \|x\|_{\ell^\infty} \leqslant M_\infty (1 - n^{\frac{1}{p}}).$$

Let us also recall that

$$\|x\|_{\ell^p} \leqslant \|x\|_{\ell^1} \leqslant n^{1-\frac{1}{p}} \|x\|_{\ell^p}$$

thus we have that for any compact $K \subseteq \mathbb{R}^n$ (setting $M_1 = \max_{x \in K} \|x\|_{\ell^1}$)

$$|\, \|x\|_{\ell^p} - \|x\|_{\ell^1} \,| \leqslant (1 - n^{\frac{1}{p}-1}) \|x\|_{\ell^1} \leqslant M_1 (1 - n^{\frac{1}{p}-1}).$$

$\square$

The previous Lemma allows us to work directly with $\|\nabla u\|_{\ell^\infty}$, instead of working with $\|\nabla u\|_{\ell^p}$. Suppose $u \in C^2$ and write

$$\widetilde{\Delta}_p u = (p-1) \sum_{j=1}^n |u_{x_j}|^{p-4} u_{x_j}^2 u_{x_j,x_j},$$

i.e.

$$\frac{\widetilde{\Delta}_p u}{p-1} = \sum_{j=1}^n |u_{x_j}|^{p-4} u_{x_j}^2 u_{x_j,x_j}.$$

Dividing everything by $\|\nabla u\|_{\ell^\infty}^{p-4}$, we achieve

$$\frac{\widetilde{\Delta}_p u}{(p-1) \|\nabla u\|_{\ell^\infty}^{p-4}} = \sum_{j=1}^n \left| \frac{u_{x_j}}{\|\nabla u\|_{\ell^\infty}} \right|^{p-4} u_{x_j}^2 u_{x_j,x_j}. \tag{3.26}$$

If we consider the set

$$I(x) := \{ j \leqslant n : \ |x_j| = \|x\|_{\ell^\infty} \},$$

we can rewrite equation (3.26) as

$$\frac{\widetilde{\Delta}_p u}{(p-1) \|\nabla u\|_{\ell^\infty}^{p-4}} = \sum_{j \in I(\nabla u(x))} u_{x_j}^2 u_{x_j,x_j} + \sum_{j \notin I(\nabla u(x))} \left| \frac{u_{x_j}}{\|\nabla u\|_{\ell^\infty}} \right|^{p-4} u_{x_j}^2 u_{x_j,x_j}.$$

Finally, taking the limit as $p \to +\infty$ and recalling that for any $j \notin I_\infty(\nabla u(x))$ we have $\left| \frac{u_{x_j}}{\|\nabla u\|_{\ell^\infty}} \right| < 1$, we achieve

$$\widetilde{\Delta}_\infty u = \lim_{p \to +\infty} \frac{\widetilde{\Delta}_p u}{(p-1) \|\nabla u\|_{\ell^\infty}^{p-4}} = \sum_{j \in I(\nabla u(x))} u_{x_j}^2 u_{x_j,x_j} = \|\nabla u\|_{\ell^\infty}^2 \sum_{j \in I(\nabla u(x))} u_{x_j,x_j}.$$

The same result holds also if we use $\|\nabla u\|_{\ell^p}$ in place of $\|\nabla u\|_{\ell^\infty}$, since, by uniform convergence, for p big enough and $j \notin I(\nabla u(x))$, we still have $\left| \frac{u_{x_j}}{\|\nabla u\|_{\ell^p}} \right| < 1$.

We stress the fact that the computations above are just heuristics, whose aim is to obtain an expected form of the limit operator; it turns out that such heuristics actually lead to the limit operator of the orthotropic p-Laplacian. Indeed, the orthotropic ∞-Laplacian has been introduced in [15] as

$$\widetilde{\Delta}_\infty u = \sum_{j \in I(\nabla u(x))} u_{x_j}^2 u_{x_j,x_j}.$$

In the same paper the authors prove that this operator is related to the problem of the Absolutely Minimizing Lipschitz Extension with respect to the ℓ^∞ on $\mathbb{R}^n$ (as the ∞-Laplacian is related to the same problem with respect to the ℓ^2 norm, as shown in [7]). In particular, in [15] it is shown that, if $u \in C^2(\Omega) \cap W^{1,\infty}(\Omega)$ is such that for any $D \subset\subset \Omega$ and any $w \in u + W_0^{1,\infty}(\Omega)$ it holds

$$\left\| \|\nabla u\|_{\ell^\infty} \right\|_{L^\infty(D)} \leqslant \left\| \|\nabla w\|_{\ell^\infty} \right\|_{L^\infty(D)},$$

then u solves

$$-\widetilde{\Delta}_\infty u = 0.$$

In the following we will work with a limit problem arising from (3.21) as $p \to +\infty$ that will take into account the operator $\widetilde{\Delta}_\infty$.

3.2.4 Limit eigenvalues

We will study in the following the behaviour of the Steklov eigenvalues as $p \to +\infty$. As we stated before, for any $p > 1$ we have $\sigma_{1,p}(\Omega) = 0$, thus we have $\lim_{p\to+\infty} \sigma_{1,p}(\Omega) = 0$. For this reason we focus on $\Sigma_p(\Omega)$.
To determine $\lim_{p\to+\infty} \Sigma_p(\Omega)$, we need first to fix some notations. We define

$$d_1(x,y) = \|x-y\|_{\ell^1}, \quad x,y \in \mathbb{R}^n.$$

For fixed x_0, the function $x \mapsto d_1(x,x_0)$ is such that $\|\nabla d_1(x,x_0)\|_{\ell^\infty} = 1$ almost everywhere, as observed in [39]. Moreover, let us define the quantity

$$\operatorname{diam}_1(E) = \sup_{x,y\in E} d_1(x,y).$$

Now let us recall the variational characterization of $\Sigma_p^p(\Omega)$ given in equation (3.22) and let us denote

$$\begin{aligned} \mathcal{R}_p[u] &= \frac{\int_\Omega \|\nabla u\|_{\ell^p}^p \, dx}{\int_{\partial\Omega} |u|^p \rho_p(x) d\mathcal{H}^{n-1}}, \\ \mathcal{M}_p[u] &= \int_{\partial\Omega} |u|^{p-2} u \rho_p(x) d\mathcal{H}^{n-1}, \\ \mathcal{U}_p &= \left\{ u \in W^{1,p}(\Omega) : \ \mathcal{M}_p[u] = 0 \right\} \end{aligned}$$

to rewrite $\Sigma_p^p(\Omega) = \min_{u\in\mathcal{U}_p} \mathcal{R}_p[u]$. We consider on $L^p(\Omega)$ the norm

$$\|u\|_{L^p(\Omega)}^p = \fint_\Omega |u|^p dx = \frac{1}{V(\Omega)} \int_\Omega |u|^p dx.$$

On $\partial\Omega$, we define the measure $\mathcal{H}_p = \rho_p \lfloor \mathcal{H}^{n-1}$ and consider for any $p, q \geqslant 1$

$$\|u\|_{L^p(\partial\Omega,\mathcal{H}_q)}^p = \frac{1}{\mathcal{H}^{n-1}(\partial\Omega)} \int_{\partial\Omega} |u|^p d\mathcal{H}_q.$$

Recall that if $q = 2$, then $\rho_2 \equiv 1$ and $\|u\|_{L^p(\partial\Omega,\mathcal{H}_2)} = \|u\|_{L^p(\partial\Omega)}$. From the equivalence of the ℓ^p norms on $\mathbb{R}^n$, that for $p > q$ is given by

$$\|x\|_{\ell^p} \leqslant \|x\|_{\ell^q} \leqslant n^{\frac{1}{q}-\frac{1}{p}} \|x\|_{\ell^p},$$

we have that, for $p > q$,

$$\rho_q(x) \leqslant \rho_p(x) \leqslant n^{\frac{1}{p'}-\frac{1}{q'}} \rho_q(x).$$

Moreover, we have from Lemma 3.17 and this equivalence, the following result.

Lemma 3.18. *For any $p, q_1, q_2 \geqslant 1$ we have $u \in L^p(\partial\Omega, \mathcal{H}_{q_1})$ if and only if $u \in L^p(\partial\Omega, \mathcal{H}_{q_2})$ and, if $1 \leqslant q_1 < q_2 \leqslant +\infty$,*

$$\|u\|_{L^p(\partial\Omega,\mathcal{H}_{q_1})} \leqslant \|u\|_{L^p(\partial\Omega,\mathcal{H}_{q_2})} \leqslant n^{\frac{1}{q_2^*}-\frac{1}{q_1^*}} \|u\|_{L^p(\partial\Omega,\mathcal{H}_{q_1})}(x).$$

Moreover, as $q \to +\infty$, we have $\|u\|_{L^p(\partial\Omega,\mathcal{H}_q)} \to \|u\|_{L^p(\partial\Omega,\mathcal{H}_\infty)}$ and, as $q \to 1$, we have $\|u\|_{L^p(\partial\Omega,\mathcal{H}_q)} \to \|u\|_{L^p(\partial\Omega,\mathcal{H}_1)}$

The latter property is due to the fact that since $\rho_p(x) = \|\nu(x)\|_{\ell^{p'}}$ and $\nu(x) \in \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ is the unit sphere of $\mathbb{R}^n$ with respect to the ℓ^2 norm (that is a compact set), $\rho_p(x) \to \rho_\infty(x)$ uniformly as $p \to \infty$ and $\rho_p(x) \to \rho_1(x)$ uniformly as $p \to 1$.
Now, let us observe that we can recast $\mathcal{R}_p[u]$ as

$$\mathcal{R}_p[u] = \frac{\int_\Omega \|\nabla u\|_{\ell^p}^p \, dx}{\frac{1}{V(\Omega)} \int_{\partial\Omega} |u|^p d\,\mathcal{H}_p}.$$

Moreover, we have the following lower-semicontinuity property.

Lemma 3.19. *Fix $p \geqslant 2$ and let $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$. Then,*

$$\mathcal{R}_p[u] \leqslant \liminf_{n\to+\infty} \mathcal{R}_p[u_n].$$

Now let us denote with $u_{2,p} \in \mathcal{U}_p$ the minimizer of $\mathcal{R}_p$ such that

$$\frac{1}{V(\Omega)} \int_{\partial\Omega} |u_{2,p}|^p d\,\mathcal{H}_p = 1. \tag{3.27}$$

In particular, in such a case,

$$\Sigma_p^p(\Omega) = \fint_\Omega \|\nabla u_{2,p}\|_{\ell^p}^p \, dx. \tag{3.28}$$

We first give the following technical Lemma.

Lemma 3.20. *Let Ω be a bounded open convex subset of $\mathbb{R}^n$ and $u \in W^{1,\infty}(\Omega)$. Then*

$$|u(x) - u(y)| \leqslant \|\|\nabla u\|_{\ell^\infty}\|_{L^\infty} \operatorname{diam}_1(\Omega), \ \forall x, y \in \overline{\Omega}. \tag{3.29}$$

Proof. Let us recall that, by definition of polar norm, $|\langle x, y\rangle| \leqslant \|x\|_{\ell^\infty} \|y\|_{\ell^1}$. Now fix $x, y \in \Omega$ and observe that since Ω is convex $(1-t)x + ty \in \Omega$ for any $t \in [0,1]$. Define the function

$$v(t) = u((1-t)x + ty), \ t \in [0,1]$$

and observe that $v \in W^{1,\infty}([0,1])$. Hence in particular v is absolutely continuous and

$$v(0) - v(1) = \int_0^1 v'(t)dt$$

where v is the weak derivative. We have

$$u(x) - u(y) = \int_0^1 \langle \nabla u((1-t)x + ty), y - x\rangle dt$$

and then

$$\begin{aligned} |u(x) - u(y)| &\leqslant \int_0^1 |\langle \nabla u((1-t)x + ty), y - x\rangle| dt \\ &\leqslant \int_0^1 \|\nabla u((1-t)x + ty)\|_{\ell^\infty} \|x - y\|_{\ell^1} \, dt \\ &\leqslant \| \|\nabla u\|_{\ell^\infty} \|_{L^\infty(\Omega)} \operatorname{diam}_1(\Omega). \end{aligned}$$

Finally, by Morrey's embedding theorem, we know that $u \in C^0(\overline{\Omega})$, thus inequality (3.29) holds also for $x, y \in \partial\Omega$. □

Now we show the following result.

Proposition 3.21. *It holds*

$$\lim_{p \to +\infty} \Sigma_p(\Omega) = \frac{2}{\operatorname{diam}_1(\Omega)} =: \Sigma_\infty(\Omega).$$

Proof. First of all, let us show that $\limsup_{p\to+\infty} \Sigma_p \leqslant \frac{2}{\operatorname{diam}_1(\Omega)}$. To do this, we consider $x_0 \in \Omega$ and we observe that, being Ω an open set, $d_1(x, x_0) > 0$ for any $x \in \partial\Omega$. Indeed, if $d_1(x, x_0) = 0$ for some $x \in \partial\Omega$, being d_1 a distance, we should have $x = x_0$ and then $x_0 \in \Omega \cap \partial\Omega = \emptyset$. In particular, this implies that $\mathcal{M}_p[d_1(\cdot, x_0)] > 0$.
Define the function $w_p(x) = d_1(x, x_0) - c_p$ where $c_p \in \mathbb{R}$ is chosen in such a way that $w_p \in \mathcal{U}_p$. Let us recall that $\|\nabla w_p\|_{\ell^\infty} = 1$ almost everywhere in $\Omega \backslash \{x_0\}$, hence we have, by equation (3.25),

$$\fint_\Omega \|\nabla w_p\|_{\ell^p}^p \, dx \leqslant n.$$

Moreover, we have, from Lemma 3.18,

$$\|w_p\|_{L^p(\partial\Omega, \mathcal{H}_\infty)} \leqslant n^{\frac{1}{p}} \|w_p\|_{L^p(\partial\Omega, \mathcal{H}_p)}.$$

Thus, recalling that $\Sigma_p(\Omega) \leqslant \mathcal{R}[w_p]^{\frac{1}{p}}$, we achieve

$$\begin{aligned} \Sigma_p(\Omega) &\leqslant \frac{\left(\fint_\Omega \|\nabla w_p\|_{\ell^p}^p \, dx\right)^{\frac{1}{p}}}{\left(\frac{1}{V(\Omega)} \int_{\partial\Omega} |w_p|^p \rho_p(x) d\,\mathcal{H}^{n-1}\right)^{\frac{1}{p}}} \\ &= \frac{\left(\fint_\Omega \|\nabla w_p\|_{\ell^p}^p \, dx\right)^{\frac{1}{p}}}{\left(\frac{\mathcal{H}^{n-1}(\partial\Omega)}{V(\Omega)}\right)^{\frac{1}{p}} \|w_p\|_{L^p(\partial\Omega, \mathcal{H}_p)}} \\ &\leqslant \frac{n^{\frac{1}{p}}}{\left(\frac{\mathcal{H}^{n-1}(\partial\Omega)}{V(\Omega)}\right)^{\frac{1}{p}} n^{-\frac{1}{p}} \|w_p\|_{L^p(\partial\Omega, \mathcal{H}_\infty)}}. \end{aligned} \tag{3.30}$$

Now let us observe that, since $\mathcal{M}_p(w_p) = 0$, w_p must change sign on $\partial\Omega$. Since $0 \leqslant d_1(x, x_0) \leqslant \operatorname{diam}_1(\Omega)$, we have $c_p \in [0, \operatorname{diam}_1(\Omega)]$. Up to a subsequence, we can suppose $c_p \to c \in [0, \operatorname{diam}_1(\Omega)]$ as $p \to +\infty$ and, setting $w = d_1(x, x_0) - c$, we have that $w_p \to w$ uniformly. Hence, as $p \to +\infty$,

$$\|w_p\|_{L^p(\partial\Omega, \mathcal{H}_\infty)} \to \sup_{x\in\partial\Omega} |d_1(x, x_0) - c|;$$

taking the lim sup as $p \to +\infty$ in (3.30), we have

$$\limsup_{p\to+\infty} \Sigma_p(\Omega) \leqslant \frac{1}{\sup_{x\in\partial\Omega} |d_1(x, x_0) - c|}. \tag{3.31}$$

Now let us observe that

$$|d_1(x, x_0) - c| \geqslant \inf_{c\in[0,\operatorname{diam}_1(\Omega)]} |d_1(x, x_0) - c| = \frac{d_1(x, x_0)}{2},$$

thus we get

$$\sup_{x\in\partial\Omega} |d_1(x, x_0) - c| \geqslant \frac{\sup_{x\in\partial\Omega} d_1(x, x_0)}{2}.$$

Plugging this relation into equation (3.31), we achieve

$$\limsup_{p\to+\infty} \Sigma_p(\Omega) \leqslant \frac{2}{\sup_{x\in\partial\Omega} d_1(x, x_0)}.$$

Since this inequality holds for any $x_0 \in \Omega$, we can take the infimum as $x_0 \in \Omega$ to obtain

$$\limsup_{p\to+\infty} \Sigma_p(\Omega) \leqslant \frac{2}{\operatorname{diam}_1(\Omega)}.$$

Now let us show that $\liminf_{p\to+\infty} \Sigma_p(\Omega) \geqslant \frac{2}{\operatorname{diam}_1(\Omega)}$. To do this, let us consider $m > n$ and $p > m$. Since $p > 2$, we have

$$\|\nabla u_{2,p}\|_{\ell^p} \geqslant n^{\frac{1}{p}-2} \|\nabla u_{2,p}\|_{\ell^2},$$

and then, by Hölder inequality and definition of $u_{2,p}$,

$$\Sigma_p(\Omega) = \left(\fint_\Omega \|\nabla u_{2,p}\|_{\ell^p}^p \, dx\right)^{\frac{1}{p}} \geqslant n^{\frac{1}{p}-2} \left(\fint_\Omega \|\nabla u_{2,p}\|_{\ell^2}^p \, dx\right)^{\frac{1}{p}} \geqslant n^{\frac{1}{p}-2} \left(\fint_\Omega \|\nabla u_{2,p}\|_{\ell^2}^m \, dx\right)^{\frac{1}{m}}.$$

Since $W^{1,m}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$, we can suppose (up to a subsequence) that there exists a function $u_{2,\infty} \in C^0(\overline{\Omega})$ such that $u_{2,p} \to u_{2,\infty}$ uniformly on $\overline{\Omega}$ and weakly in $W^{1,m}(\Omega)$. Now let us fix any $1 \leqslant q < p$ and observe that, by lower semicontinuity of the functional

$$u \in W^{1,q}(\Omega) \mapsto (\mathcal{H}^{n-1}(\partial\Omega))^{\frac{1}{q}} \mathcal{R}_q[u] \in \mathbb{R}$$

with respect to the weak convergence in $W^{1,q}$ as stated in Lemma 3.19, we have

$$\frac{\left(\fint_\Omega \|\nabla u_{2,\infty}\|_{\ell^\infty}^q \, dx\right)^{\frac{1}{q}}}{\left(\frac{1}{\mathcal{H}^{N-1}(\partial\Omega)} \int_{\partial\Omega} |u_{2,\infty}|^q \rho_\infty d\,\mathcal{H}^{N-1}\right)^{\frac{1}{q}}} \leqslant \liminf_{p\to+\infty} \frac{\left(\fint_\Omega \|\nabla u_{2,p}\|_{\ell^\infty}^q \, dx\right)^{\frac{1}{q}}}{\left(\frac{1}{\mathcal{H}^{N-1}(\partial\Omega)} \int_{\partial\Omega} |u_{2,p}|^q \rho_\infty d\,\mathcal{H}^{N-1}\right)^{\frac{1}{q}}}.$$

By Hölder inequality we get

$$\frac{\left(\fint_\Omega \|\nabla u_{2,\infty}\|_{\ell^\infty}^q \, dx\right)^{\frac{1}{q}}}{\left(\frac{1}{\mathcal{H}^{n-1}(\partial\Omega)} \int_{\partial\Omega} |u_{2,\infty}|^q \rho_\infty d\,\mathcal{H}^{n-1}\right)^{\frac{1}{q}}} \leqslant \liminf_{p\to+\infty} \frac{\left(\fint_\Omega \|\nabla u_{2,p}\|_{\ell^\infty}^p \, dx\right)^{\frac{1}{p}}}{\left(\frac{1}{\mathcal{H}^{n-1}(\partial\Omega)} \int_{\partial\Omega} |u_{2,p}|^q \rho_\infty d\,\mathcal{H}^{n-1}\right)^{\frac{1}{q}}}$$

and then, by using (3.25),

$$\frac{\left(\fint_\Omega \|\nabla u_{2,\infty}\|^q_{\ell^\infty}\,dx\right)^{\frac{1}{q}}}{\left(\frac{1}{\mathcal{H}^{n-1}(\partial\Omega)}\int_{\partial\Omega}|u_{2,\infty}|^q\rho_\infty d\,\mathcal{H}^{n-1}\right)^{\frac{1}{q}}} \leqslant \liminf_{p\to+\infty}\frac{\left(\fint_\Omega \|\nabla u_{2,p}\|^p_{\ell^p}\,dx\right)^{\frac{1}{p}}}{\left(\frac{1}{\mathcal{H}^{n-1}(\partial\Omega)}\int_{\partial\Omega}|u_{2,p}|^q\rho_\infty d\,\mathcal{H}^{n-1}\right)^{\frac{1}{q}}}.$$

Recalling equations (3.27) and (3.28), we have

$$\begin{aligned}\frac{\left(\fint_\Omega \|\nabla u_{2,\infty}\|^q_{\ell^\infty}\,dx\right)^{\frac{1}{q}}}{\left(\frac{1}{\mathcal{H}^{n-1}(\partial\Omega)}\int_{\partial\Omega}|u_{2,\infty}|^q\rho_\infty d\,\mathcal{H}^{n-1}\right)^{\frac{1}{q}}} &\leqslant \liminf_{p\to+\infty}\frac{\left(\frac{1}{V(\Omega)}\int_{\partial\Omega}|u_{2,p}|^p\rho_p d\,\mathcal{H}^{n-1}\right)^{\frac{1}{p}}}{\left(\frac{1}{\mathcal{H}^{n-1}(\partial\Omega)}\int_{\partial\Omega}|u_{2,p}|^q\rho_\infty d\,\mathcal{H}^{n-1}\right)^{\frac{1}{q}}}\Sigma_p(\Omega)\\ &= \liminf_{p\to+\infty}\frac{\left(\frac{\mathcal{H}^{n-1}(\partial\Omega)}{V(\Omega)}\right)^{\frac{1}{p}}\|u_{2,p}\|_{L^p(\partial\Omega,\mathcal{H}_p)}}{\|u_{2,p}\|_{L^q(\partial\Omega,\mathcal{H}_\infty)}}\Sigma_p(\Omega)\end{aligned}$$

and, by using Lemma 3.18, we achieve

$$\begin{aligned}\frac{\left(\fint_\Omega \|\nabla u_{2,\infty}\|^q_{\ell^\infty}\,dx\right)^{\frac{1}{q}}}{\left(\frac{1}{\mathcal{H}^{n-1}(\partial\Omega)}\int_{\partial\Omega}|u_{2,\infty}|^q\rho_\infty d\,\mathcal{H}^{n-1}\right)^{\frac{1}{q}}} &\leqslant \liminf_{p\to+\infty}\frac{\left(\frac{\mathcal{H}^{n-1}(\partial\Omega)}{V(\Omega)}\right)^{\frac{1}{p}}\|u_{2,p}\|_{L^p(\partial\Omega,\mathcal{H}_\infty)}}{\|u_{2,p}\|_{L^q(\partial\Omega,\mathcal{H}_\infty)}}\Sigma_p(\Omega)\\ &= \frac{\|u_{2,\infty}\|_{L^\infty(\partial\Omega)}}{\|u_{2,\infty}\|_{L^q(\partial\Omega,\mathcal{H}_\infty)}}\liminf_{p\to+\infty}\Sigma_p(\Omega);\end{aligned}$$

finally let us take the limit as $q\to+\infty$ to obtain

$$\frac{\left\|\|\nabla u_{2,\infty}\|_{\ell^\infty}\right\|_{L^\infty(\Omega)}}{\|u_{2,\infty}\|_{L^\infty(\partial\Omega)}} \leqslant \liminf_{p\to+\infty}\Sigma_p(\Omega). \tag{3.32}$$

Now we want to estimate the left-hand side of the previous inequality. To do this, let us recall that, for any $p>1$,

$$\int_{\partial\Omega}|u_{2,p}|^{p-2}u_{2,p}\rho_p d\,\mathcal{H}^{n-1}=0$$

hence, in particular,

$$\|(u_{2,p})_+\|_{L^{p-1}(\partial\Omega,\mathcal{H}_p)} = \|(u_{2,p})_-\|_{L^{p-1}(\partial\Omega,\mathcal{H}_p)}.$$

By using the previous identity we have

$$\begin{aligned}0 &\leqslant \left|\|(u_{2,\infty})_+\|_{L^{p-1}(\partial\Omega,\mathcal{H}_p)} - \|(u_{2,\infty})_-\|_{L^{p-1}(\partial\Omega,\mathcal{H}_p)}\right|\\ &\leqslant \left|\|(u_{2,\infty})_+\|_{L^{p-1}(\partial\Omega,\mathcal{H}_p)} - \|(u_{2,p})_+\|_{L^{p-1}(\partial\Omega,\mathcal{H}_p)}\right|\\ &+ \left|\|(u_{2,p})_-\|_{L^{p-1}(\partial\Omega,\mathcal{H}_p)} - \|(u_{2,\infty})_-\|_{L^{p-1}(\partial\Omega,\mathcal{H}_p)}\right|\\ &\leqslant \|(u_{2,\infty})_+ - (u_{2,p})_+\|_{L^{p-1}(\partial\Omega,\mathcal{H}_p)} + \|(u_{2,\infty})_- - (u_{2,p})_-\|_{L^{p-1}(\partial\Omega,\mathcal{H}_p)}\\ &\leqslant \|(u_{2,\infty})_+ - (u_{2,p})_+\|_{L^{p-1}(\partial\Omega,\mathcal{H}_\infty)} + \|(u_{2,\infty})_- - (u_{2,p})_-\|_{L^{p-1}(\partial\Omega,\mathcal{H}_\infty)}\\ &\leqslant \left(\frac{\mathcal{H}_\infty(\partial\Omega)}{\mathcal{H}^{n-1}(\partial\Omega)}\right)^{\frac{1}{p-1}}\left(\|(u_{2,\infty})_+ - (u_{2,p})_+\|_{L^\infty(\partial\Omega)} + \|(u_{2,\infty})_- - (u_{2,p})_-\|_{L^\infty(\partial\Omega)}\right).\end{aligned} \tag{3.33}$$

Now let us observe that

$$n^{-\frac{1}{p}} \left\| (u_{2,\infty})_\pm \right\|_{L^{p-1}(\partial\Omega,\mathcal{H}_\infty)} \leqslant \left\| (u_{2,\infty})_\pm \right\|_{L^{p-1}(\partial\Omega,\mathcal{H}_p)} \leqslant \left\| (u_{2,\infty})_\pm \right\|_{L^{p-1}(\partial\Omega,\mathcal{H}_\infty)}$$

and $\lim_{p\to+\infty} \left\| (u_{2,\infty})_\pm \right\|_{L^{p-1}(\partial\Omega,\mathcal{H}_\infty)} = \left\| (u_{2,\infty})_\pm \right\|_{L^\infty(\partial\Omega)}$, thus we have

$$\lim_{p\to+\infty} \left\| (u_{2,\infty})_\pm \right\|_{L^{p-1}(\partial\Omega,\mathcal{H}_p)} = \left\| (u_{2,\infty})_\pm \right\|_{L^\infty(\partial\Omega)}.$$

Taking the limit as $p \to +\infty$ in (3.33), by also using the uniform convergence of $u_{2,p}$ towards $u_{2,\infty}$ on $\partial\Omega$, we obtain

$$0 \leqslant \left| \left\| (u_{2,\infty})_+ \right\|_{L^\infty(\partial\Omega)} - \left\| (u_{2,\infty})_- \right\|_{L^\infty(\partial\Omega)} \right| \leqslant 0;$$

thus, being $u_{2,\infty} \in C^0(\overline{\Omega})$,

$$\max_{x\in\partial\Omega} u_{2,\infty}(x) = -\min_{x\in\partial\Omega} u_{2,\infty}(x).$$

Let us consider $x_M, x_m \in \partial\Omega$ respectively a maximum and minimum point of $u_{2,\infty}$ on $\partial\Omega$ and observe that $u_{2,\infty}(x_M) = -u_{2,\infty}(x_m)$. This means that x_M and x_m are both maximum points for $|u_{2,\infty}|$ on $\partial\Omega$ and

$$2 \left\| u_{2,\infty} \right\|_{L^\infty(\partial\Omega)} = u_{2,\infty}(x_M) - u_{2,\infty}(x_m).$$

By (3.29), we obtain

$$\left\| u_{2,\infty} \right\|_{L^\infty(\partial\Omega)} = \frac{u_{2,\infty}(x_M) - u_{2,\infty}(x_m)}{2} \leqslant \frac{\operatorname{diam}_1(\Omega)}{2} \left\| \left\| \nabla u \right\|_{\ell^\infty} \right\|_{L^\infty(\Omega)}.$$

Plugging last inequality in equation (3.32), we obtain

$$\frac{2}{\operatorname{diam}_1(\Omega)} \leqslant \liminf_{p\to+\infty} \Sigma_p(\Omega),$$

concluding the proof. □

By using the function $u_{2,\infty}$ defined in the previous proof, we can also exploit the behaviour of $\Sigma_\infty(\Omega)$ as a minimizer of a Rayleigh quotient.

Proposition 3.22. *It holds*

$$\Sigma_\infty(\Omega) = \min \left\{ \frac{\left\| \left\| \nabla u \right\|_{\ell^\infty} \right\|_{L^\infty}}{\left\| u \right\|_{L^\infty(\partial\Omega)}},\ u \in W^{1,\infty}(\Omega),\ \max_{x\in\partial\Omega} u(x) = -\min_{x\in\partial\Omega} u(x) \neq 0 \right\}.$$

Proof. Let us consider $u \in W^{1,\infty}(\Omega)$ such that

$$u_M := \max_{x\in\partial\Omega} u(x) = -\min_{x\in\partial\Omega} u(x) =: -u_m.$$

Then, being Ω an open bounded convex set, we know that $u \in W^{1,p}(\Omega)$ for any $p \geqslant 1$. Now let us consider $p_n \to +\infty$ as $n \to +\infty$. For each $n \in \mathbb{N}$, let us define c_n such that

$$\int_{\partial\Omega} |u + c_n|^{p_n - 2}(u + c_n)\rho_{p_n}\, d\mathcal{H}^{n-1} = 0. \tag{3.34}$$

Now, since $u_M := -u_m$, we know that u changes sign. Moreover, also $u + c_n$ must change sign for any $n \in \mathbb{N}$. Hence we have that $c_n \in [-u_M, u_M]$. Let us then consider a subsequence (let us call it still c_n) such that $c_n \to c \in [-u_M, u_M]$ and let us define $u_n = u + c_n$: it is easy to check that $u_n \to u$ in $C^0(\overline{\Omega})$. By Equation (3.34) we have

$$\|(u+c_n)_+\|_{L^{p_n-1}(\partial\Omega,\mathcal{H}_{p_n})} = \|(u+c_n)_-\|_{L^{p_n-1}(\partial\Omega,\mathcal{H}_{p_n})}$$

and then, taking the limit as $n \to +\infty$, by uniform convergence, we have

$$u_M + c = \max_{x\in\partial\Omega}(u+c) = -\min_{x\in\partial\Omega}(u+c) = -u_m - c$$

and then, since $u_M = -u_m$, $c = 0$.
Now let us observe that, by definition, $u_n \in \mathcal{U}_p$, thus, by definition of Σ_{p_n}, we achieve (recalling that $\nabla u_n = \nabla u$)

$$\Sigma_{p_n}(\Omega) \leqslant \frac{|\Omega|^{\frac{1}{p_n}} \left(\fint_\Omega \|\nabla u\|_{\ell^{p_n}}^{p_n}\, dx\right)^{\frac{1}{p_n}}}{\mathcal{H}_{p_n}(\partial\Omega)^{\frac{1}{p_n}} \left(\fint_{\partial\Omega} |u_n|^{p_n} \rho_{p_n}(x) d\mathcal{H}^{n-1}\right)^{\frac{1}{p_n}}}.$$

Now, since u_n converges uniformly, we have, by taking the limit as $n \to +\infty$,

$$\Sigma_\infty(\Omega) \leqslant \frac{\|\|\nabla u\|_{\ell^\infty}\|_{L^\infty}}{\|u\|_{L^\infty(\partial\Omega)}}.$$

Since $u \in W^{1,\infty}(\Omega)$ is such that $\max_{x\in\partial\Omega} u(x) = -\min_{x\in\partial\Omega} u(x)$ is arbitrary, then we have

$$\Sigma_\infty(\Omega) \leqslant \inf\left\{\frac{\|\|\nabla u\|_{\ell^\infty}\|_{L^\infty}}{\|u\|_{L^\infty(\partial\Omega)}},\ u \in W^{1,\infty}(\Omega),\ \max_{x\in\partial\Omega} u(x) = -\min_{x\in\partial\Omega} u(x)\right\}.$$

Finally, let us observe that $u_{2,\infty} \in W^{1,\infty}(\Omega)$ and it is such that $\max_{x\in\partial\Omega} u_{2,\infty}(x) = -\min_{x\in\partial\Omega} u_{2,\infty}(x)$ and then

$$\frac{\left\|\|\nabla u_{2,\infty}\|_{\ell^\infty}\right\|_{L^\infty}}{\|u_{2,\infty}\|_{L^\infty(\partial\Omega)}} \geqslant \Sigma_\infty(\Omega).$$

However, we have also

$$\frac{\left\|\|\nabla u_{2,\infty}\|_{\ell^\infty}\right\|_{L^\infty}}{\|u_{2,\infty}\|_{L^\infty(\partial\Omega)}} \leqslant \frac{2}{\mathrm{diam}_\infty(\Omega)} = \Sigma_\infty(\Omega)$$

concluding the proof. □

Remark 3.23. Let us observe that, defining

$$\mathcal{R}_\infty[u] = \frac{\|\|\nabla u\|_{\ell^\infty}\|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\partial\Omega)}}$$

for $u \in W^{1,\infty}(\Omega)$ with $u \not\equiv 0$ on $\partial\Omega$, the function $u_{2,\infty}$ is a minimizer of $\mathcal{R}_\infty$ in

$$\mathcal{U}_\infty = \left\{u \in W^{1,\infty}(\Omega),\ \max_{x\in\partial\Omega} u(x) = -\min_{x\in\partial\Omega} u(x) \neq 0\right\}.$$

Moreover, the previous Proposition also implies that, for any $u \in \mathcal{U}_\infty$, it holds

$$\|u\|_{L^\infty(\partial\Omega)} \leqslant \frac{1}{\Sigma_\infty(\Omega)} \|\|\nabla u\|_{\ell^\infty}\|_{L^\infty(\Omega)},$$

thus $1/\Sigma_\infty(\Omega)$ represents the best constant of a trace-type inequality in $\mathcal{U}_\infty$.

Next step is to characterize $u_{2,\infty}$ as a solution (in the viscosity sense) of a boundary-value problem involving the orthotropic ∞-Laplacian as Ω is regular enough.

Theorem 3.24. *Let Ω be an open set with C^1 boundary. The limit $u_{2,\infty}$ is a viscosity solution of*

$$\begin{cases} -\widetilde{\Delta}_\infty u_{2,\infty} = 0 & \text{on } \Omega \\ \Lambda(x, u, \nabla u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.35}$$

where

$$\Lambda(x,u,\eta) = \begin{cases} \min\left\{ \|\eta\|_{\ell^\infty} - \Sigma_\infty(\Omega)|u| \,,\, \sum_{j\in I(\eta(x))} \eta_{x_j}(x)\nu_j(x)\right\} & \text{if } u > 0 \\ \max\left\{\Sigma_\infty(\Omega)|u| - \|\eta\|_\infty \,, \sum_{j\in I(\eta(x))} \eta_{x_j}(x)\nu_j(x)\right\} & \text{if } u < 0 \\ \sum_{j\in I(\eta(x))} \eta_{x_j}(x)\nu_j(x) & \text{if } u = 0 \end{cases}$$

Proof. First of all, we prove that $-\widetilde{\Delta}_\infty u_{2,\infty} = 0$ in the viscosity sense in Ω. In order to do that, let us take a test function Φ touching u from above in $x_0 \in \Omega$. In the proof of Proposition 3.21, we have shown that the sequence u_{2,p_i} converges uniformly to $u_{2,\infty}$; it follows that $u_{2,p_i} - \Phi$ has a maximum at some point $x_i \in \Omega$ with $x_i \to x_0$. In Proposition 3.15 it is proven that u_{2,p_i} is a viscosity solution of $-\widetilde{\Delta}_{p_i} u_{2,p_i} = 0$, so we obtain that

$$-(p_i - 1)\sum_{j=1}^n |\Phi_{x_j}|^{p_i-4}\Phi_{x_j}^2 \Phi_{x_j x_j} \leqslant 0,$$

that can be rewritten as

$$\begin{aligned} -(p_i-1)\Bigg[&\|\nabla\Phi\|_{\ell^\infty}^{p_i-4} \sum_{j\in I(\nabla\Phi(x_i))} \Phi_{x_j}^2(x_i)\Phi_{x_j,x_j}(x_i) \\ &+ \sum_{j\notin I(\nabla\Phi(x_i))} \left|\Phi_{x_j}(x_i)\right|^{p_i-4} \Phi_{x_j}(x_i)^2 \Phi_{x_j,x_j}(x_i)\Bigg] \leqslant 0. \end{aligned}$$

Dividing by $(p_i - 1)\,\|\nabla\Phi\|_{\ell^\infty}^{p_i-4}$ and passing to the limit, we obtain that $-\widetilde{\Delta}_\infty\Phi(x_0) \leqslant 0$. Working in the same way, if Φ is touching u from below in $x_0 \in \Omega$, we achieve $-\widetilde{\Delta}_\infty\Phi(x_0) \geqslant 0$ and then $-\widetilde{\Delta}_\infty u_{2,\infty} = 0$ in the viscosity sense in Ω.

Now we deal with the boundary conditions. Let us consider $x_0 \in \partial\Omega$ and $u(x_0) > 0$. Let assume that Φ touches u from below in x_0. Since u_{p_i} converges uniformly to $u_{2,\infty}$, we have that $u_{p_i} - \Phi$ admits a minimum in some point $x_i \in \overline{\Omega}$, with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, we already have $-\widetilde{\Delta}_\infty\Phi(x_0) \geqslant 0$. So, we study the case $x_i \in \partial\Omega$ ultimately for any i.

If $\nabla\Phi(x_0) = 0$, then $\frac{\partial\Phi}{\partial\nu}(x_0) = 0$. Let now $\nabla\Phi(x_0) \neq 0$; we have that

$$\sum_{j=1}^n \left|\Phi_{x_j}(x_i)\right|^{p_i-2}\Phi_{x_j}(x_i)\,\nu_{\partial\Omega}^j(x_i) \geqslant \Sigma_{p_i}^{p_i}(\Omega)\,|\Phi(x_i)|^{p_i-2}\Phi(x_i)\rho_{p_i}(x_i),$$

and, dividing by $\|\nabla\Phi(x_i)\|_{\ell^\infty}^{p_i-2}$,

$$\sum_{j=1}^n \left|\frac{\Phi_{x_j}(x_i)}{\|\nabla\Phi(x_i)\|_{\ell^\infty}}\right|^{p_i-2} \Phi_{x_j}(x_i)\,\nu_{\partial\Omega}^j(x_i) \geqslant \Sigma_{p_i}^{p_i/(p_i-1)}(\Omega)\left|\frac{\Sigma_{p_i}^{p_i/(p_i-1)}(\Omega)\Phi(x_i)}{\|\nabla\Phi(x_i)\|_{\ell^\infty}}\right|^{p_i-2}\Phi(x_i)\rho_{p_i}(x_i). \tag{3.36}$$

Passing to the limit in the left-hand side, we have

$$\lim_{i\to+\infty}\sum_{j=1}^{n}\left|\frac{\Phi_{x_j}(x_i)}{\|\nabla\Phi(x_i)\|_{\ell^\infty}}\right|^{p_i-2}\Phi_{x_j}(x_i)\,\nu^j_{\partial\Omega}(x_i)=\sum_{j\in I(\nabla\Phi(x_0))}\Phi_{x_j}(x_0)\nu^j_{\partial\Omega}(x_0).$$

From this we can deduce that the limit superior of the right-hand side in (3.36) is finite. Since

$$\frac{\Sigma_{p_i}^{p_i/(p_i-1)}(\Omega)\Phi(x_i)}{\|\nabla\Phi(x_i)\|_{\ell^\infty}}\to\frac{\Sigma_\infty(\Omega)|\Phi(x_0)|}{\|\nabla\Phi(x_0)\|_{\ell^\infty}},$$

to have a finite limit on the right-hand side of (3.36), we need

$$\frac{\Sigma_\infty(\Omega)|\Phi(x_0)|}{\|\nabla\Phi(x_0)\|_{\ell^\infty}}\leqslant 1.$$

From this last condition we have

$$\|\nabla\Phi(x_0)\|_{\ell^\infty}\geqslant\Sigma_\infty(\Omega)|\Phi(x_0)|\geqslant 0,$$

and then, taking the limit in equation (3.36),

$$\sum_{j\in I(\nabla\Phi(x_0))}\Phi_{x_j}(x_0)\nu^j_{\partial\Omega}(x_0)\geqslant 0.$$

Hence, if Φ is touching u from below in x_0, we have

$$\max\left\{\min\left\{\sum_{j\in I(\nabla\Phi(x_0))}\Phi_{x_j}(x_0)\nu^j_{\partial\Omega}(x_0),\|\nabla\Phi(x_0)\|_{\ell^\infty}-\Sigma_\infty(\Omega)|\Phi(x_0)|\right\},-\widetilde{\Delta}\Phi(x_0)\right\}\geqslant 0. \tag{3.37}$$

Now assume that Φ is touching u from above in x_0. Since u_{2,p_i} converges uniformly to $u_{2,\infty}$, we have that $u_{2,p_i}-\Phi$ admits a maximum in some point $x_i\in\overline{\Omega}$, with $x_i\to x_0$. If $x_i\in\Omega$ for infinitely many i, arguing as before, we obtain $-\widetilde{\Delta}\Phi_{2,\infty}(x_0)\leqslant 0$. If $x_i\in\partial\Omega$ ultimately for any i, then

$$\sum_{j=1}^{N}\left|\Phi_{x_j}(x_i)\right|^{p_i-2}\Phi_{x_j}(x_i)\,\nu^j_{\partial\Omega}(x_i)\leqslant\Sigma_{p_i}^{p_i}(\Omega)\,|\Phi(x_i)|^{p_i-2}\Phi(x_i)\rho_{p_i}(x_i).$$

If $\nabla\Phi(x_0)=0$, then $\frac{\partial\Phi}{\partial\nu}(x_0)=0$; otherwise we obtain

$$\sum_{j=1}^{n}\left|\frac{\Phi_{x_j}(x_i)}{\|\nabla\Phi(x_i)\|_{\ell^\infty}}\right|^{p_i-2}\Phi_{x_j}(x_i)\,\nu^j_{\partial\Omega}(x_i)\leqslant\Sigma_{p_i}^{p_i/(p_i-1)}(\Omega)\left|\frac{\Sigma_{p_i}^{p_i/(p_i-1)}(\Omega)\Phi(x_i)}{\|\nabla\Phi(x_i)\|_{\ell^\infty}}\right|^{p_i-2}\Phi(x_i)\rho_{p_i}(x_i).$$

From this last inequality, if $\Sigma_\infty(\Omega)|\Phi(x_0)|<\|\nabla\Phi(x_0)\|_{\ell^\infty}$, then, taking the limit,

$$\sum_{j\in I(\nabla\Phi(x_0))}\Phi_{x_j}(x_0)\nu^j_{\partial\Omega}(x_0)\leqslant 0.$$

Hence,

$$\min\left\{\min\left\{\sum_{j\in I(\nabla\Phi(x_0))}\Phi_{x_j}(x_0)\nu^j_{\partial\Omega}(x_0),\|\nabla\Phi(x_0)\|_{\ell^\infty}-\Sigma_\infty(\Omega)|\Phi(x_0)|\right\},-\widetilde{\Delta}\Phi(x_0)\right\}\leqslant 0. \tag{3.38}$$

Now let us suppose $u(x_0) < 0$ and assume that Φ is touching u from above in x_0. Since u_{2,p_i} converges uniformly to $u_{2,\infty}$, we have that $u_{2,p_i} - \Phi$ admits a maximum at some point $x_i \in \overline{\Omega}$, with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, arguing as before, we obtain that $-\widetilde{\Delta}\Phi_{2,\infty}(x_0) \leqslant 0$. If $x_i \in \partial\Omega$ ultimately for any i, then

$$\sum_{j=1}^{n} \left|\Phi_{x_j}(x_i)\right|^{p_i-2} \Phi_{x_j}(x_i)\, \nu^j_{\partial\Omega}(x_i) \leqslant \Sigma^{p_i}_{p_i}(\Omega)\, |\Phi(x_i)|^{p_i-2}\Phi(x_i)\rho_{p_i}(x_i).$$

If $\nabla\Phi(x_0) = 0$, then $\frac{\partial\Phi}{\partial\nu}(x_0) = 0$; otherwise we obtain

$$\sum_{j=1}^{n} \left|\frac{\Phi_{x_j}(x_i)}{\|\nabla\Phi(x_i)\|_{\ell^\infty}}\right|^{p_i-2} \Phi_{x_j}(x_i)\, \nu^j_{\partial\Omega}(x_i) \leqslant \Sigma^{p_i/(p_i-1)}_{p_i}(\Omega) \left|\frac{\Sigma^{p_i/(p_i-1)}_{p_i}(\Omega)\Phi(x_i)}{\|\nabla\Phi(x_i)\|_{\ell^\infty}}\right|^{p_i-2} \Phi(x_i)\rho_{p_i}(x_i). \tag{3.39}$$

Now, if we pass to the limit superior on the right hand side, arguing as before and recalling this time that $\Phi(x_0) < 0$, we obtain a finite quantity; this implies

$$\frac{\Sigma_\infty(\Omega)|\Phi(x_0)|}{\nabla\Phi(x_0)} \leqslant 1.$$

Moreover, taking the limit in (3.39), since $\Phi(x_0) < 0$,

$$\sum_{j\in I(\nabla\Phi(x_0))} \Phi_{x_j}(x_0)\nu^j_{\partial\Omega}(x_0) \leqslant 0.$$

Therefore,

$$\min\left\{\max\left\{\sum_{j\in I(\nabla\Phi(x_0))} \Phi_{x_j}(x_0)\nu^j_{\partial\Omega}(x_0), -\|\nabla\Phi(x_0)\|_{\ell^\infty} + \Sigma_\infty(\Omega)|\Phi(x_0)|\right\}, -\widetilde{\Delta}\Phi(x_0)\right\} \leqslant 0. \tag{3.40}$$

Now assume that Φ is touching u from below in x_0. Since u_{2,p_i} converges uniformly to $u_{2,\infty}$, we have that $u_{2,p_i} - \Phi$ admits a minimum at some point $x_i \in \overline{\Omega}$, with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, arguing as before, we obtain that $-\widetilde{\Delta}\Phi_{2,\infty}(x_0) \geqslant 0$. If $x_i \in \partial\Omega$ ultimately for any i, then

$$\sum_{j=1}^{n} \left|\Phi_{x_j}(x_i)\right|^{p_i-2} \Phi_{x_j}(x_i)\, \nu^j_{\partial\Omega}(x_i) \geqslant \Sigma^{p_i}_{p_i}(\Omega)\, |\Phi(x_i)|^{p_i-2}\Phi(x_i)\rho_{p_i}(x_i).$$

If $\nabla\Phi(x_0) = 0$, then $\frac{\partial\Phi}{\partial\nu}(x_0) = 0$; otherwise we obtain

$$\sum_{j=1}^{n} \left|\frac{\Phi_{x_j}(x_i)}{\|\nabla\Phi(x_i)\|_{\ell^\infty}}\right|^{p_i-2} \Phi_{x_j}(x_i)\, \nu^j_{\partial\Omega}(x_i) \geqslant \Sigma^{p_i/(p_i-1)}_{p_i}(\Omega) \left|\frac{\Sigma^{p_i/(p_i-1)}_{p_i}(\Omega)\Phi(x_i)}{\|\nabla\Phi(x_i)\|_{\ell^\infty}}\right|^{p_i-2} \Phi(x_i)\rho_{p_i}(x_i). \tag{3.41}$$

If $\Sigma_\infty(\Omega)|\Phi(x_0)| < \|\nabla\Phi(x_0)\|_{\ell^\infty}$, then, taking the limit in equation (3.41), we achieve

$$\sum_{j\in I(\nabla\Phi(x_0))} \Phi_{x_j}(x_0)\nu^j_{\partial\Omega}(x_0) \geqslant 0$$

and consequently

$$\max\left\{\max\left\{\sum_{j\in I(\nabla\Phi(x_0))}\Phi_{x_j}(x_0)\nu^j_{\partial\Omega}(x_0), -\|\nabla\Phi(x_0)\|_{\ell^\infty}+\Sigma_\infty(\Omega)|\Phi(x_0)|\right\}, -\widetilde{\Delta}\Phi(x_0)\right\}\geqslant 0. \tag{3.42}$$

Now let us suppose that $u(x_0)=0$ and assume that Φ is touching u from below in x_0. Since u_{2,p_i} converges uniformly to $u_{2,\infty}$, we have that $u_{2,p_i}-\Phi$ admits a minimum at some point $x_i\in\overline{\Omega}$, with $x_i\to x_0$. If $x_i\in\Omega$ for infinitely many i, arguing as before, we obtain that $-\widetilde{\Delta}\Phi_{2,\infty}(x_0)\geqslant 0$. If $x_i\in\partial\Omega$ ultimately for any i, then

$$\sum_{j=1}^n\left|\Phi_{x_j}(x_i)\right|^{p_i-2}\Phi_{x_j}(x_i)\,\nu^j_{\partial\Omega}(x_i)\geqslant\Sigma^{p_i}_{p_i}(\Omega)\,|\Phi(x_i)|^{p_i-2}\Phi(x_i)\rho_{p_i}(x_i).$$

If $\nabla\Phi(x_0)=0$, then $\frac{\partial\Phi}{\partial\nu}(x_0)=0$; otherwise we obtain

$$\sum_{j=1}^n\left|\frac{\Phi_{x_j}(x_i)}{\|\nabla\Phi(x_i)\|_{\ell^\infty}}\right|^{p_i-2}\Phi_{x_j}(x_i)\,\nu^j_{\partial\Omega}(x_i)\geqslant\Sigma^{p_i/(p_i-1)}_{p_i}(\Omega)\left|\frac{\Sigma^{p_i/(p_i-1)}_{p_i}(\Omega)\Phi(x_i)}{\|\nabla\Phi(x_i)\|_{\ell^\infty}}\right|^{p_i-2}\Phi(x_i)\rho_{p_i}(x_i).$$

Since $0=\Sigma_\infty(\Omega)|\Phi(x_0)|<\|\nabla\Phi(x_0)\|_{\ell^\infty}$, we obtain

$$\sum_{j\in I(\nabla\Phi(x_0))}\Phi_{x_j}(x_0)\nu^j_{\partial\Omega}(x_0)\geqslant 0,$$

hence

$$\max\left\{\sum_{j\in I(\nabla\Phi(x_0))}\Phi_{x_j}(x_0)\nu^j_{\partial\Omega}(x_0), -\widetilde{\Delta}\Phi(x_0)\right\}\geqslant 0. \tag{3.43}$$

Finally, assume that Φ is touching u from above in x_0. Since u_{2,p_i} converges uniformly to $u_{2,\infty}$, we have that $u_{2,p_i}-\Phi$ admits a maximum at some point $x_i\in\overline{\Omega}$, with $x_i\to x_0$. If $x_i\in\Omega$ for infinitely many i, arguing as before, we obtain that $-\widetilde{\Delta}\Phi_{2,\infty}(x_0)\leqslant 0$. If $x_i\in\partial\Omega$ ultimately for any i, then

$$\sum_{j=1}^n\left|\Phi_{x_j}(x_i)\right|^{p_i-2}\Phi_{x_j}(x_i)\,\nu^j_{\partial\Omega}(x_i)\leqslant\Sigma^{p_i}_{p_i}(\Omega)\,|\Phi(x_i)|^{p_i-2}\Phi(x_i)\rho_{p_i}(x_i).$$

If $\nabla\Phi(x_0)=0$, then $\frac{\partial\Phi}{\partial\nu}(x_0)=0$; otherwise we obtain

$$\sum_{j=1}^n\left|\frac{\Phi_{x_j}(x_i)}{\|\nabla\Phi(x_i)\|_{\ell^\infty}}\right|^{p_i-2}\Phi_{x_j}(x_i)\,\nu^j_{\partial\Omega}(x_i)\leqslant\Sigma^{p_i/(p_i-1)}_{p_i}(\Omega)\left|\frac{\Sigma^{p_i/(p_i-1)}_{p_i}(\Omega)\Phi(x_i)}{\|\nabla\Phi(x_i)\|_{\ell^\infty}}\right|^{p_i-2}\Phi(x_i)\rho_{p_i}(x_i).$$

Since $0=\Sigma_\infty(\Omega)|\Phi(x_0)|<\|\nabla\Phi(x_0)\|_{\ell^\infty}$, we obtain

$$\sum_{j\in I(\nabla\Phi(x_0))}\Phi_{x_j}(x_0)\nu^j_{\partial\Omega}(x_0)\leqslant 0,$$

hence

$$\min\left\{\sum_{j\in I(\nabla\Phi(x_0))}\Phi_{x_j}(x_0)\nu^j_{\partial\Omega}(x_0), -\widetilde{\Delta}\Phi(x_0)\right\}\leqslant 0. \tag{3.44}$$

The Theorem follows from (3.37)-(3.38)-(3.40)-(3.42)-(3.43)-(3.44).

□

3.2.5 Brock-Weinstock and Weinstock type inequalities for the orthotropic ∞-Laplacian

Let us denote, for $p \in [1, \infty]$,

$$\mathcal{W}_p := \{x \in \mathbb{R}^n \mid \|x\|_{\ell^p} \leqslant 1\}$$

and, for any bounded convex set $\Omega \subseteq \mathbb{R}^n$,

$$\mathcal{P}_p(\Omega) := \int_{\partial\Omega} \rho_p(x) d\mathcal{H}^{n-1}(x), \quad \mathcal{M}_p(\Omega) := \int_{\partial\Omega} |x|^p \rho_p(x) d\mathcal{H}^{n-1}(x),$$

that are, respectively, the anisotropic perimeter and the boundary p-momentum with respect to the ℓ^p norm on $\mathbb{R}^n$.

We are interested in Brock-Weinstock and Weinstock type inequalities. Let us define the scaling invariant shape operator

$$\mathcal{F}_p(\Omega) := \frac{\mathcal{M}_p(\Omega)}{\mathcal{P}_p(\Omega) V(\Omega)^{\frac{p}{n}}}.$$

Then, for any $p \in (1, \infty)$ and for any open bounded convex set $\Omega \subseteq \mathbb{R}^n$, it holds

$$\mathcal{F}_p(\Omega) \geqslant \mathcal{F}_p(\mathcal{W}_p). \tag{3.45}$$

Moreover, let us observe that, by definition of $\mathcal{W}_p$ and by using the relation $nV(\mathcal{W}_p) = \mathcal{P}_p(\mathcal{W}_p)$,

$$\frac{nV(\mathcal{W}_p)}{\mathcal{M}_p(\mathcal{W}_p)} = \frac{\mathcal{P}_p(\mathcal{W}_p)}{\mathcal{P}_p(\mathcal{W}_p)} = 1. \tag{3.46}$$

In the general case of the orthotropic p-Laplacian, the following Brock-Weinstock type inequality (restricted to bounded convex open sets) has been proven in [27]

$$\Sigma_p^p(\Omega) V(\Omega)^{\frac{p-1}{n}} \leqslant V(\mathcal{W}_p)^{\frac{p-1}{n}}. \tag{3.47}$$

As a first step, we want to improve the previous inequality, to include in some way the perimeter.

Theorem 3.25. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded convex set and $p > 1$. Consider $q \geqslant 0$ and $r \in [0, n]$ such that $\frac{p}{n} = q + \frac{r}{n}$. Then, we have*

$$\Sigma_p^p(\Omega)\, \mathcal{P}_p(\Omega)^{\frac{r-1}{n-1}} V(\Omega)^q \leqslant \mathcal{P}_p(\mathcal{W}_p)^{\frac{r-1}{n-1}} V(\mathcal{W}_p)^q. \tag{3.48}$$

Proof. Let us first recall that by [27, Lemma 7.1], we can use the functions x_i with $i = 1, \dots, n$ as test functions in the Rayleigh quotient $\mathcal{R}_p$ for Σ_p^p (up to a rigid movement of Ω), with

$$\mathcal{R}_p[x_i] = \frac{V(\Omega)}{\int_{\partial\Omega} |x_i|^p \rho_p(x) d\mathcal{H}^{n-1}(x)},$$

hence, for any $i = 1, \dots, N$, we have

$$\Sigma_p^p(\Omega) \int_{\partial\Omega} |x_i|^p \rho_p(x) d\mathcal{H}^{n-1}(x) \leqslant V(\Omega).$$

Summing over i we have

$$\Sigma_p^p(\Omega) \leqslant \frac{nV(\Omega)}{\mathcal{M}_p(\Omega)}. \tag{3.49}$$

Now let us write inequality (3.45) explicitly to achieve

$$\frac{\mathcal{M}_p(\Omega)}{\mathcal{P}_p(\Omega)V(\Omega)^{\frac{p}{n}}} \geqslant \frac{\mathcal{M}_p(\mathcal{W}_p)}{\mathcal{P}_p(\mathcal{W}_p)V(\mathcal{W}_p)^{\frac{p}{n}}}$$

and then

$$\mathcal{M}_p(\Omega) \geqslant \frac{\mathcal{M}_p(\mathcal{W}_p)\,\mathcal{P}_p(\Omega)V(\Omega)^{\frac{p}{n}}}{\mathcal{P}_p(\mathcal{W}_p)V(\mathcal{W}_p)^{\frac{p}{n}}}.$$

Using this inequality in equation (3.49) we get

$$\Sigma_p^p(\Omega) \leqslant \frac{nV(\Omega)\,\mathcal{P}_p(\mathcal{W}_p)V(\mathcal{W}_p)^{\frac{p}{n}}}{\mathcal{M}_p(\mathcal{W}_p)\,\mathcal{P}_p(\Omega)V(\Omega)^{\frac{p}{n}}},$$

that can be recast as

$$\Sigma_p^p(\Omega) \leqslant \frac{n\,\mathcal{P}_p(\mathcal{W}_p)V(\mathcal{W}_p)^{\frac{p}{N}}}{\mathcal{M}_p(\mathcal{W}_p)\,\mathcal{P}_p(\Omega)^{\frac{r-1}{n-1}}V^q(\Omega)}\left(\frac{V(\Omega)^{1-\frac{1}{n}}}{\mathcal{P}_p(\Omega)}\right)^{\frac{n-r}{n-1}}.$$

Let us recall the anisotropic standard isoperimetric inequality (see [2, Proposition 2.3]):

$$\frac{V(\Omega)^{1-\frac{1}{n}}}{\mathcal{P}_p(\Omega)} \leqslant \frac{V(\mathcal{W}_p)^{1-\frac{1}{n}}}{\mathcal{P}_p(\mathcal{W}_p)}.$$

Thus, since $r < n$ and then $\frac{n-r}{n-1} > 0$, we have

$$\Sigma_p^p(\Omega) \leqslant \frac{n\,\mathcal{P}_p(\mathcal{W}_p)V(\mathcal{W}_p)^{\frac{p}{n}}}{\mathcal{M}_p(\mathcal{W}_p)\,\mathcal{P}_p(\Omega)^{\frac{r-1}{n-1}}V(\Omega)^q}\left(\frac{V(\mathcal{W}_p)^{1-\frac{1}{n}}}{\mathcal{P}_p(\mathcal{W}_p)}\right)^{\frac{n-r}{n-1}}$$

and then, recalling that $p/n = q + r/n$, we finally get

$$\Sigma_p^p(\Omega)\,\mathcal{P}_p(\Omega)^{\frac{r-1}{n-1}}V(\Omega)^q \leqslant \frac{nV(\mathcal{W}_p)}{\mathcal{M}_p(\mathcal{W}_p)}\,\mathcal{P}_p(\mathcal{W}_p)^{\frac{r-1}{n-1}}V(\mathcal{W}_p)^q.$$

Equality (3.46) concludes the proof.

□

Remark 3.26. Let us observe that Theorem 3.25 includes inequality (3.47). Indeed, for any bounded convex set Ω and any $p > 1$ we can fix $r = 1$ and then $q = \frac{p-1}{n}$ in inequality (3.48) to obtain the desired result.

Moreover, let us observe that, in general, inequality (3.48) implies inequality (3.47). Indeed, since the left-hand side of equation (3.48) is scaling invariant, we can always suppose $\mathcal{P}_p(\Omega) = \mathcal{P}_p(\mathcal{W}_p)$. Thus, the aforementioned equation becomes

$$\Sigma_p^p(\Omega)V^q(\Omega) \leqslant V^q(\mathcal{W}_p).$$

Multiplying both sides by $V(\Omega)^{\frac{p-r}{n}}$ we have

$$\Sigma_p^p(\Omega)V^{\frac{p-1}{n}}(\Omega) \leqslant V(\mathcal{W}_p)^qV^{\frac{p-r}{n}}(\Omega) \leqslant V^{\frac{p-1}{n}}(\mathcal{W}_p),$$

where the last inequality follows from the anisotropic isoperimetric inequality.

As in [27], we are not able to detect equality cases. However, let us stress out that equality could not hold even for $\mathcal{W}_p$ if $\Sigma_p^p(\mathcal{W}_p) < 1$. Let us recall that in general it is known that $\Sigma_p^p(\mathcal{W}_p) \leqslant 1$, but determining if it is actually equal to 1 or not is still an open problem, except that for $p = 2$.

An improvement that involves only the perimeter can be shown if $p \leqslant n$. Indeed, we have the following Corollary.

Corollary 3.27. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded convex set and $p \in (1, n]$. Then, we have*

$$\Sigma_p^p(\Omega)\, \mathcal{P}_p(\Omega)^{\frac{p-1}{n-1}} \leqslant \mathcal{P}_p(\mathcal{W}_p)^{\frac{p-1}{n-1}}.$$

Proof. Just observe that if $p \in (1, n]$, we can choose $r = p$ and $q = 0$ in equation (3.48). □

Remark 3.28. If the conjecture by Brasco and Franzina in [27] reveals to be true, i. e. the fact that $\Sigma_p^p(\mathcal{W}_p) = 1$, last result implies the Weinstock inequality for the orthotropic p-Laplacian as $p \in (1, n]$.

In any case, we can recast equation (3.47) as

$$\Sigma_p(\Omega) V^{\frac{p-1}{np}}(\Omega) \leqslant V^{\frac{p-1}{np}}(\mathcal{W}_p)$$

and then take the limit as $p \to +\infty$ to obtain

$$\Sigma_\infty(\Omega) V(\Omega)^{\frac{1}{n}} \leqslant V(\mathcal{W}_\infty)^{\frac{1}{n}}, \tag{3.50}$$

that cannot be rewritten in a full scaling-invariant form since $\Sigma_\infty(\mathcal{W}_\infty) = 1/n$. Moreover, being $V(\mathcal{W}_\infty) = 2^n$, equation (3.50) can be rewritten as

$$\Sigma_\infty(\Omega) V(\Omega)^{\frac{1}{N}} \leqslant 2.$$

However, we can improve such inequality by means of an anisotropic isodiametric inequality.

Corollary 3.29. *For any bounded convex open set $\Omega \subset \mathbb{R}^n$ it holds*

$$\Sigma_\infty(\Omega) V(\Omega)^{1/n} \leqslant \Sigma_\infty(\mathcal{W}_1) V(\mathcal{W}_1)^{1/n}. \tag{3.51}$$

Equality holds if and only if Ω is equivalent to $\mathcal{W}_1$ up to translations and scalings.

Proof. Let us observe that, by [111, Proposition 2.1], we have

$$\frac{2}{\operatorname{diam}_1(\Omega)} V^{1/n}(\Omega) \leqslant V^{1/n}(\mathcal{W}_1)$$

Recalling that $\Sigma_\infty(\Omega) = \frac{2}{\operatorname{diam}_1(\Omega)}$ and $\Sigma_\infty(\mathcal{W}_1) = \frac{2}{\operatorname{diam}_1(\mathcal{W}_1)} = 1$ we conclude the proof. Equality cases follow from [111, Proposition 2.1]. □

Remark 3.30. Let us observe that inequality (3.51) implies inequality (3.50), since $V(\mathcal{W}_1) = 2^{\frac{n}{2}}$.

On the other hand, a Weinstock-type inequality in the planar case follows from the Rosenthal-Szasz inequality in Radon planes (see [9]). To give this result we need to introduce the concept of width in our case. Fix $n = 2$ and consider any bounded open convex set Ω. For each direction v there exists two supporting lines r_1, r_2 for Ω that are orthogonal to v in the Euclidean sense. We call width of Ω in the direction v the distance $\omega(v) = d_1(r_1, r_2)$. With this in mind, we can give the following result.

Corollary 3.31. *For any open bounded convex set $\Omega \subseteq \mathbb{R}^2$ it holds*

$$\Sigma_\infty(\Omega) P_\infty(\Omega) \leqslant \Sigma_\infty(\mathcal{W}_1) P_\infty(\mathcal{W}_1). \tag{3.52}$$

Equality holds if and only if Ω is of constant width, i.e. if and only if $\omega(v) \equiv \operatorname{diam}_1(\Omega)$.

Proof. Let us recall that the Rosenthal-Szasz inequality for Radon planes [9, Theorem 1.1], specified to the plane $(\mathbb{R}^2, \|\cdot\|_{\ell^1})$, is given by

$$\frac{2P_\infty(\Omega)}{\mathrm{diam}_1(\Omega)} \leqslant P_\infty(\mathcal{W}_1).$$

Recalling that $\Sigma_\infty(\mathcal{W}_1) = 1$ and $\Sigma_\infty(\Omega) = \frac{2}{\mathrm{diam}_1(\Omega)}$ we conclude the proof. Equality cases follow from equality cases of the Rosenthal-Szasz inequality in [9, Theorem 1.1]. $\square$

Chapter 4

Some results about the Robin type boundary conditions in the linear and non linear case

In this Chapter we focus our attention on varius problem involving a Robin boundary condition type. In Section 4.1 we prove two bounds for the first Robin eigenvalue of the Finsler Laplacian with negative boundary parameter in the planar case. In the constant area problem, we show that the Wulff shape is the maximizer only for values which are close to 0 of the boundary parameter and, in the fixed perimeter case, that the Wulff shape maximizes the first eigenvalue for all values of the parameter.

In Section 4.2 we prove that in the planar case the anisotropic maximum curvature is minimized by the ball, among simply connected sets with fixed area. In the linear case this result, proved in [102], plays a role in the study of the asymptotic for the Robin eigenvalue with negative parameter.

In Section 4.3 we study, in dimension $n \geqslant 2$, the eigenvalue problem and the torsional rigidity for the p-Laplacian on convex sets with holes, with external Robin boundary conditions and internal Neumann boundary conditions. We prove that the annulus maximizes the first eigenvalue and minimizes the torsional rigidity when the measure and the external perimeter are fixed.

4.1 Anisotropic Robin Laplace eigenvalue problem in the plane

4.1.1 Definition of the Robin problem in the anisotropic case

Let Ω be a bounded subset of $\mathbb{R}^2$ of class C^2. We consider the anisotropic eigenvalue problem with Robin boundary conditions. We fix a Finsler norm F, a negative number α and we study the following problem:

$$\lambda_{1,F}(\alpha,\Omega) = \min_{\substack{u\in W^{1,2}(\Omega)\\ u\neq 0}} J(u), \tag{4.1}$$

where

$$J(u)=\frac{\displaystyle\int_\Omega (F(\nabla u))^2\ dx+\alpha\int_{\partial\Omega}|u|^2F(\nu)\ d\mathcal{H}^1}{\displaystyle\int_\Omega |u|^2\ dx}, \tag{4.2}$$

and $\nu_{\partial\Omega}$ is the outer normal to $\partial\Omega$. Using a constant as test function, we obtain the following inequality

$$\lambda_{1,F}(\alpha,\Omega)\leqslant\alpha\frac{P_F(\Omega)}{|\Omega|}\leqslant 0, \tag{4.3}$$

where $P_F(\Omega)$ is the anisotropic perimeter of Ω as defined in (1.8). The minimizers u of problem (4.1) satisfy the following eigenvalue problem

$$\begin{cases}-\mathrm{div}\,(F(\nabla u)F_\xi(\nabla u))=\lambda_{1,F}(\alpha,\Omega)u & \text{in } \Omega\\ \langle F(\nabla u)F_\xi(\nabla u),\nu_{\partial\Omega}\rangle+\alpha F(\nu_{\partial\Omega})u=0 & \text{on } \partial\Omega,\end{cases} \tag{4.4}$$

that is, in the weak sense

$$\int_\Omega F(\nabla u)\,\langle D_\xi F(\nabla u),D\varphi\rangle\ dx+\alpha\int_{\partial\Omega}u\varphi F(\nu_{\partial\Omega})\ d\mathcal{H}^1=\lambda_{1,F}(\alpha,\Omega)\int_\Omega u\varphi\ dx, \tag{4.5}$$

for all $\varphi\in W^{1,2}(\Omega)$. The following proposition is proved in [52].

Proposition 4.1. *There exists a function $u\in C^{1,\alpha}(\Omega)\cap C(\overline{\Omega})$ which realizes the minimum in (4.1) and satisfies the anisotropic Robin Problem (4.4). Moreover, $\lambda_{1,F}(\alpha,\Omega)$ is the first eigenvalue of the Robin problem and the first eigenfunctions are positive (or negative) in Ω.*

4.1.2 Isoperimetric estimates with a volume constraint

We are interested to find an estimate for $\lambda_1(\alpha,\Omega)$ when is given a volume constraint.

Theorem 4.2. *For bounded planar domains of class C^2 and fixed area, there exists a negative number α_*, depending only on the area, such that the following inequality holds $\forall\alpha\in[\alpha_*,0]$:*

$$\lambda_{1,F}(\alpha,\Omega)\leqslant\lambda_{1,F}(\alpha,\mathcal{W}_\Omega^\#),$$

where $\mathcal{W}_\Omega^\#$ is the Wulff shape of the same area as Ω.

In order to prove Theorem 4.2 we adapt in the anisotropic case the proof contained in [64]. This proof makes use of the classical method of parallel coordinates, developed for the Euclidean case in [110] and for the Riemanian case in [117].

We assume that $\partial\Omega$ is composed by a finite union of C^2 Jordan curves $\Gamma_0,\dots,\Gamma_N$, where Γ_0 is the outer boundary of Ω, i.e. Ω lies in the interior Ω_0 of Γ_0. We observe that, if $N=0$, then Ω is simply connected and $\Omega=\Omega_0$. We denote by

$$L_0^F:=P_F(\Omega_0)=\int_{\partial\Omega_0}F(\nu)\ d\mathcal{H}^1$$

the outer anisotropic perimeter. Therefore, by the anisotropic isoperimetric inequality (see Theorem 1.15), we have

$$(L_0^F)^2\geqslant 4\kappa A_0, \tag{4.6}$$

where $A_0 = V(\Omega)$ denotes the area of Ω (not of Ω_0).

We now introduce the *anisotropic parallel coordinate method* based at the outer boundary Γ_0. Let $\rho_F : \Omega_0 \to (0, \infty)$ be the anisotropic distance function from the outer boundary Γ_0, that is $\rho_F(x) = d_F(x, \Gamma_0)$. Let

$$A_F(t) = V(\{x \in \Omega \mid 0 < \rho_f(x) < t\})$$

denote the area of $\Omega_t = \Omega \backslash \tilde{\Omega}_t$ and let us consider the following quantity

$$L_F(t) = \int_{\rho_F^{-1}(t) \cap \Omega} F(\nu(x))\, d\mathcal{H}^1(x).$$

Remark 4.3. By Lemma 1.16, we obtain that, for almost every $t \in [0, r_F(\Omega_0)]$,

$$A_F'(t) = L_F(t). \tag{4.7}$$

Step 1: use of the anisotropic parallel coordinates.

Let $\phi : [0, r_F(\Omega)] \to \mathbb{R}$ be a smooth function and consider the test function $u = \phi \circ A_F \circ \rho_F$, which is Lipschitz in Ω. Using the anisotropic parallel coordinates, the coarea formula and the fact that $F(D\rho_F) = 1$, we obtain the following relations:

$$\begin{aligned}
||u||^2_{L^2(\Omega)} &= \int_\Omega u^2(x)\, dx = \int_\Omega (\phi \circ A_F \circ \rho_F(x))^2\, dx = \\
= \int_0^{r_F(\Omega)} &\left(\int_{\{\rho_F(x) = t\}} (\phi \circ A_F \circ \rho_F(x))^2 \frac{1}{|\nabla \rho_F(x)|}\, d\mathcal{H}^1(x) \right) dt \\
&= \int_0^{r_F(\Omega)} \phi(A_F(t))^2\, P_F(\{\rho_F(x) < t\})\, dt = \\
&= \int_0^{r_F(\Omega)} \phi(A_F(t))^2\, A_F'(t)\, dt;
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega \left(F^2\left(\nabla u(x)\right)\right) dx = \int_\Omega F^2\left(\phi'\left(A_F \circ \rho_F\left(x\right)\right) A_F'\left(\rho_F\left(x\right)\right) \nabla \rho_F\left(x\right)\right) dx = \\
&= \int_\Omega \left(\phi'\left(A_F \circ \rho_F\left(x\right)\right)\right)^2 \left(A_F'\left(\rho_F\left(x\right)\right)\right)^2 dx = \int_0^{r_F(\Omega)} \left(\phi'\left(A_F\left(t\right)\right)\right)^2 \left(A_F'\left(t\right)\right)^3 dt;
\end{aligned}$$

$$\begin{aligned}
&\int_{\partial\Omega} |u(x)|^2 F(\nu(x))\, d\mathcal{H}^1(x) = \int_{\partial\Omega} \left(\phi \circ A_F \circ \rho_F\left(x\right)\right)^2 F(\nu(x))\, d\mathcal{H}^1(x) = \\
&= \left(\phi \circ A_F\left(0\right)\right)^2 P_F(\Omega) \geqslant \phi^2(0)\, L_0.
\end{aligned}$$

Therefore, we have that

$$\lambda(\Omega) \leqslant \frac{\int_0^{r_F(\Omega)} \left(\phi'\left(A_F\left(t\right)\right)\right)^2 \left(A_F'\left(t\right)\right)^3 dt + \alpha\, \phi^2(0)\, L_0^F}{\int_0^{r_F(\Omega)} \phi(A_F(t))^2\, A_F'(t)\, dt}. \tag{4.8}$$

Step 2: from domains to annuli.

We adapt in the anisotropic case the idea contained in [110]. We consider the following change of variables:

$$R(t) := \frac{\sqrt{\left(L_0^F\right)^2 - 4\kappa A_F(t)}}{2\kappa} \tag{4.9}$$

on the interval $[r_1, r_2]$, where

$$r_1 := R\left(r_F\left(\Omega\right)\right) = \frac{\sqrt{\left(L_0^F\right)^2 - 4\kappa A_0}}{2\kappa}, \qquad r_2 := R(0) = \frac{L_0^F}{2\kappa}. \tag{4.10}$$

Remark 4.4. Thanks to (4.6), the transformation (4.9) is well defined on the set $[0, r_F(\Omega)]$.

We introduce now the function

$$\psi(r) := \phi\left(\frac{\left(L_0^F\right)^2}{4\kappa} - \kappa r^2\right)$$

and we obtain the following expressions:

$$\int_\Omega u^2(x)\, dx = 2\kappa \int_{r_1}^{r_2} \left(\psi(r)\right)^2 r\, dr;$$

$$\int_\Omega \left(F^2\left(Du(x)\right)\right) dx = 2\kappa \int_{r_1}^{r_2} \left(\psi'(r)\right)^2 \left(R'(r)\right)^2 r\, dr;$$

$$\int_\Omega |u(x)|^2 F(\nu_{\partial\Omega}(x))\, dx \geqslant L_0^F\ \psi(r_2)^2.$$

Remark 4.5. The radii in (4.10) are such that the F-annulus $A^F_{r_1,r_2} := \mathcal{W}_{r_2} \backslash \overline{\mathcal{W}}_{r_1}$ has the same area A_0 as the original domain Ω. We observe that the transformation (4.9) maps the internal part of $\partial\Omega_t$ into the Wulff shape of radius $R(t)$; so Γ_0 is mapped into the Wulff shape of equal anisotropic perimeter. Moreover, Ω_t is mapped in the anisotropic annulus of area $A_F(t)$.

Proposition 4.6. *Let Ω be a bounded planar domain of class C^2, then*

$$|R'(t)| \leqslant 1,$$

where R is defined in (4.9)*.*

Proof. From (4.7) follows that, for almost every $t \in [0, r_F(\Omega)]$ we have

$$R'(t) = -\frac{L_F(t)}{\sqrt{\left(L_0^F\right)^2 - 4\kappa A_F(t)}}. \tag{4.11}$$

Using the Steiner formula we obtain for almost every $t \in [0, r_F(\Omega)]$

$$L_F(t) \leqslant L_0^F - 2\kappa t;$$

$$A_F(t) = \int_0^t L_F(v)\, dv \leqslant L_0^F t - \kappa t^2.$$

Therefore,

$$L_F(t)^2 \leqslant \left(L_0^F\right)^2 - 4\kappa A_F(t),$$

and putting this in (4.11) the thesis follows.

$\square$

We obtain this upper bound

$$\lambda_{1,F}(\alpha,\Omega) \leqslant \inf_{\psi\neq 0} \frac{\int_{r_1}^{r_2} \psi'(r)^2 r\, dr + \alpha\, r_2\, \psi(r_2)^2}{\int_{r_1}^{r_2} \psi(r)^2 r\, dr} := \mu_F(\alpha, A^F_{r_1,r_2}), \tag{4.12}$$

so the infimum is attained for the first eigenfunction of the Laplacian in $A^F_{r_1,r_2}$, with anisotropic Robin boundary condition on $\partial\mathcal{W}_2$ and anisotropic Neumann boundary conditions on $\partial\mathcal{W}_1$. Therefore we have proved the following proposition.

Proposition 4.7. *Let $\alpha \leqslant 0$. For any bounded planar domain Ω of class C^2,*

$$\lambda_{1,F}(\alpha;\Omega) \leqslant \mu(\alpha, A^F_{r_1,r_2}),$$

where $A^F_{r_1,r_2}$ is the anisotropic annulus of the same area as Ω with radii (4.10).

Step 3: from annuli to disks.

Let $\mathcal{W}_{r_1,r_2}$ be the Wulff shape of the same area as the anisotropic annulus $A^F_{r_1,r_2}$, which has the same area A_0 as Ω. So, we have that

$$r_3 = \sqrt{\frac{A_0}{\kappa}}, \tag{4.13}$$

where r_3 is the radius of $\mathcal{W}_{r_1,r_2}$. In [64] we find the following asymptotics as $\alpha \to +\infty$:

$$\lambda_{1,F}(\alpha, \mathcal{W}_{r_1,r_2}) = 2\alpha\frac{r_3}{r_3^2} + O(\alpha^2) \quad \text{(Robin Wulff)}; \tag{4.14}$$

$$\mu_F(\alpha; A^F_{r_1,r_2}) = 2\alpha\frac{r_2}{r_3^2} + O(\alpha^2) \quad \text{(Neumann-Robin annulus)}. \tag{4.15}$$

Using them we can prove that, for $\alpha < 0$ small enough,

$$\mu(\alpha, A^F_{r_1,r_2}) \leqslant \lambda_{1,F}(\alpha, \mathcal{W}_{r_1,r_2}), \tag{4.16}$$

where $\mathcal{W}_{r_1,r_2}$ is the Wulff shape of the same area as the anisotropic annulus $A^F_{r_1,r_2}$. Thus, we have proved the following theorem.

Proposition 4.8. *For any bounded domain Ω of class C^2, there exists a negative number $\alpha_0 = \alpha_0(A_0, L_0^F)$ such that*

$$\lambda_{1,F}(\alpha,\Omega) \leqslant \lambda_{1,F}(\alpha, \mathcal{W}^*_\Omega)$$

holds $\forall \alpha \in [\alpha_0, 0]$, where $\mathcal{W}^*_\Omega$ is the Wulff shape of the same area as Ω.

Remark 4.9. Using the above asymptotics we can show that

$$\frac{d}{d\alpha}\lambda_{1,F}(\alpha,\Omega)|_{\alpha=0} = \frac{P_F(\Omega)}{V(\Omega)}.$$

Step 4: uniform behaviour and conclusion.

In order to complete the proof of the Theorem 4.2, it remains only to show the following fact.

Proposition 4.10. *The constant α_0 of Proposition 4.8 is independent of L_0^F.*

Following [64], we need to show that the neighbourhood of zero, in which (4.16) holds, does not degenerate in both cases when $r_1 \to 0$ and $r_2 \to +\infty$. So, we are going to prove that α_0 remains bounded away from 0 uniformly in this two instances. We fix $\epsilon > 0$ and we consider

$$r_1 = \sqrt{(2\epsilon r_3 + \epsilon^2)}, \qquad r_2 = r_3 + \epsilon,$$

where r_3 is fixed and equal to $\sqrt{A_0/\kappa}$. In an analogous way to the one reported in [64], it can be proved that there exists $\alpha^* < 0$ such that the curve $\Gamma_A : \alpha \longmapsto \mu_F(\alpha, A^F_{r_1,r_2})$ stays below the curve $\Gamma_B : \alpha \longmapsto \lambda_{1,F}(\alpha, \mathcal{W}_{r_3})$ for all $\epsilon > 0$ and $\forall \alpha \in (\alpha^*, 0)$. Because of the simplicity of the eigenvalues, both the curves are analytic. Moreover, taking into account the asymptotics (4.14) and (4.15) we have that

$$\frac{d}{d\alpha}\mu_F(\alpha, \mathcal{W}_{r_1,r_2}) \leqslant \frac{d}{d\alpha}\lambda_{1,F}(\alpha, A^F_{r_1,r_2}).$$

Remark 4.11. We prove that the curves Γ_A are concave in α. Let $\epsilon > 0$ and let ψ be the first eigenfunction $\mu_F(\alpha+\epsilon, A^F_{r_1,r_2})$ of the Laplacian in the anisotropic annulus. We can choose ψ normalised to 1, so we have

$$\mu_F(\alpha+\epsilon, A^F_{r_1,r_2}) = \int_{r_1}^{r_2} \psi'(r)^2 r\, dr + (\alpha+\epsilon)\, r_2\, \psi(r_2)^2. \tag{4.17}$$

Let φ be the first eigenfunction $\mu_F(\alpha, A^F_{r_1,r_2})$ normalized to 1:

$$\mu_F(\alpha, A^F_{r_1,r_2}) = \int_{r_1}^{r_2} \phi'(r)^2 r\, dr + \alpha\, r_2\, \phi(r_2)^2. \tag{4.18}$$

Now, putting ϕ as a test function in the variational formula of $\mu_F(\alpha+\epsilon, A^F_{r_1,r_2})$ we obtain

$$\mu_F(\alpha+\epsilon, A^F_{r_1,r_2}) \leqslant \int_{r_1}^{r_2} \phi'(r)^2 r\, dr + (\alpha+\epsilon)\, r_2\, \phi(r_2)^2 = \mu_F(\alpha, A^F_{r_1,r_2}) + \epsilon\, r_2\, \phi(r_2)^2.$$

In order to prove our claim, we need only to show that

$$\frac{d}{d\alpha}\mu_F(\alpha, A^F_{r_1,r_2}) = \; r_2\, \phi(r_2)^2.$$

We prove the following more general result.

Lemma 4.12. *Let Ω be a bounded subset of $\mathbb{R}^2$ and let u_α an eigenfunction related to the eigenvalue $\lambda_{1,F}(\alpha,\Omega)$, defined in (4.1), such that $\|u_\alpha\|_{L^2(\Omega)} = 1$. Then*

$$\lambda'_{1,F}(\alpha,\Omega) := \frac{d\lambda_{1,F}(\alpha,\Omega)}{d\alpha} = \int_{\partial\Omega} u_\alpha^2 F(\nu) d\mathcal{H}^1. \tag{4.19}$$

Proof. From the variational characterization (4.1) and using the fact that $\|u_\alpha\|_{L^2(\Omega)} = 1$ we have

$$\lambda_{1,F}(\alpha,\Omega) = \int_\Omega F^2(\nabla u_\alpha)\, dx + \alpha \int_{\partial\Omega} u_\alpha^2 F(\nu)\, d\mathcal{H}^1. \tag{4.20}$$

Deriving both sides of (4.20) with respect to α, we obtain

$$\lambda'_{1,F}(\alpha,\Omega) = 2\int_\Omega F(\nabla u_\alpha) D_\xi F(\nabla u_\alpha)\nabla u'_\alpha\, dx + \int_{\partial\Omega} u_\alpha^2 F(\nu)\, d\mathcal{H}^1 + 2\alpha \int_{\partial\Omega} u_\alpha u'_\alpha F(\nu)\, d\mathcal{H}^1. \tag{4.21}$$

Using the weak formulation (4.5) of the problem in the equation (4.21), remembering that u'_α is the derivative with respect to α and it is in the set of the test functions by standard elliptic regularity theory, we obtain

$$\lambda'_{1,F}(\alpha,\Omega) = 2\lambda_{1,F}(\alpha,\Omega)\int_\Omega u_\alpha u'_\alpha\ dx + \int_{\partial\Omega} u_\alpha^2 F(\nu)\ d\mathcal{H}^1, \tag{4.22}$$

and, having in mind that, from the condition $\|u_\alpha\|_{L^2(\Omega)} = 1$,

$$\int_\Omega u_\alpha u'_\alpha\ dx = 0$$

we get, from (4.22), the equation (4.19). □

Therefore, since the Γ_A are concave in α and their derivative with respect to α are increasing with ϵ, we have that the tangent to the curve corresponding to a specific anisotropic annulus intersects Γ_B at one and only one point , α_1, to the left of zero. Thanks to the concavity we can say that, for larger value of ϵ, any Γ_A that intersects Γ_B must do so to the left of α_1.

As far as the case when ϵ is small, we follow closely the proof presented in [64]. We study the intersection points of the two curves Γ_A and Γ_B, comparing the following two equations; the first equation is the equation of the Wulff shape

$$kI_1(kr_3) + \alpha I_0(kr_3) = 0; \tag{4.23}$$

the second equation is the one of the Neumann-Robin anisotropic annulus

$$\begin{aligned} &K_1(k\sqrt{2\epsilon r_3+\epsilon^2})\left[kI_1\left(k\left(r_3+\epsilon\right)\right)+\alpha I_0\left(k\left(r_3+\epsilon\right)\right)\right] - \\ &I_1(k\sqrt{2\epsilon r_3+\epsilon^2})\left[kK_1\left(k\left(r_3+\epsilon\right)\right)-\alpha K_0\left(k\left(r_3+\epsilon\right)\right)\right] = 0. \end{aligned}$$

We denote here with I_ν and K_ν the modified Bessel functions (for their properties we refer to [1]). The solution in α of the intersection is given by

$$\alpha = -k\frac{I_1(kr_3)}{I_0(kr_3)}.$$

The proof that there are no intersections between Γ_A and Γ_B for α close to zero is the same as the one presented in [64]. In this way we have proved Proposition 4.10.

4.1.3 Isoperimetric estimates with a perimeter constraint

Using the method of parallel coordinates we are able to prove also the following theorem.

Theorem 4.13. *Let $\alpha \leqslant 0$ and let $\Omega \subseteq \mathbb{R}^2$ a bounded domain of class C^2. Then*

$$\lambda_{1,F}(\alpha,\Omega) \leqslant \lambda_{1,F}(\alpha,\mathcal{W}_\Omega^*),$$

where $\mathcal{W}_\Omega^*$ is the Wulff shape with the same perimeter as Ω.

The crucial step in order to prove this theorem is given by the following Proposition.

Proposition 4.14. *Let $\alpha < 0$. For any $0 < r_1 < r_2$ we have*

$$\mu_F(\alpha, A^F_{r_1,r_2}) \leqslant \lambda_{1,F}(\alpha,\mathcal{W}_{r_2}).$$

Proof. By symmetry, $\lambda_{1,F}(\alpha, \mathcal{W}_{r_2})$ is the smallest eigenvalue of the following one-dimensional problem

$$\begin{cases} -r^{-(d-1)} & [r^{d-1}\phi'(r)]' = \lambda_{1,F}(\alpha, \mathcal{W}_{r_2})\, \phi(r), r \in [0, r_2] \\ & \phi'(0) = 0 \\ & \phi'(r_2) + \alpha\phi(r_2) = 0. \end{cases} \tag{4.24}$$

We can choose the associated function ϕ_1 to be positive and normalised to 1 and this eigenfunction can be used as a test function. Integrating by parts, we obtain

$$\mu_F(\alpha, A^F_{r_1,r_2}) \leqslant \lambda_{1,F}(\alpha, \mathcal{W}_{r_2}) - r_1\phi(r_1)\phi'(r_1). \tag{4.25}$$

Since ϕ_1 satisfies (4.24), we have for all $r \in [0, r_2]$

$$\left[r\phi_1(r)\phi_1'(r)\right]' = -\lambda_{1,F}(\alpha, \mathcal{W}_{r_2})r\phi_1(r)^2 + r\phi_1'(r)^2 \geqslant 0.$$

and the inequality is due to (4.3). From the above inequality the function $g(r) := r\phi(r)\phi'(r)$ is non-decreasing and using (4.25), we obtain the desired result. □

Remark 4.15. The following monotonicity result holds true. Let be $\mathcal{W}_R$ be a Wulff shape of radius R. If $\alpha < 0$, then

$$R \mapsto \lambda_{1,F}(\alpha, \mathcal{W}_R)$$

is strictly increasing. The above result is proved for the disks in [6] and for the annuli in [124].

Proof of Theorem 4.13. Firstly, we observe that the measure of $\mathcal{W}_{r_2}$ is greater than the measure of $A^F_{r_1,r_2}$ and the perimeter of $\mathcal{W}_{r_2}$, which is equal to L_0 is less than the anisotropic perimeter of $A^F_{r_1,r_2}$. Using Theorem 4.7 and Proposition 4.14 we obtain the thesis for simply connected domains, i .e. when $L_0 = P_F(\Omega)$. Concerning the general case, when there are multiple connected domains, thanks to Remark 4.15, we have that

$$\lambda_{1,F}(\alpha, \mathcal{W}_{r_2}) \leqslant \lambda_{1,F}(\alpha, \mathcal{W}_{r_3}),$$

where $r_3 = P_F(\Omega)/2\kappa$ for all $\alpha \leqslant 0$. □

4.2 Study of an anisotropic inequality for the anisotropic maximum curvature

Theorem 4.16. *Let $\Omega \subseteq \mathbb{R}^2$ such that $\gamma := \partial\Omega$ is a smooth Jordan curve. Then,*

$$k^F_{\max}(\gamma) \geqslant \sqrt{\frac{\kappa}{A(\Omega)}} \tag{4.26}$$

and there is equality if and only if Ω coincides with a Wulff shape.

Proof. Step 1: Uniqueness Using a stanrdard argument we will prove that, if inequality (5.2) is proved, then the equality holds only for Wulff shapes. Let assume that (5.2) is true and, by contradiction, that the equality holds for a curve γ that is not the boundary of a Wulff shape. Thus, there exists a point $x \in \gamma$ such that $k^F_{\partial K}(x) \leqslant k^F_{\max}(\gamma)$, since the Wulff shapes are the only sets with constant anisotropic curvature (see Remark 1.19). By a small local deformation around x, we can construct a smooth Jordan curve γ' such that the following two conditions hold

- $k^F_{\max}(\gamma') = k^F_{\max}(\gamma)$,
- the area A' enclosed by γ' is strictly smaller than the area A enclosed by γ.

In this way we have a contradiction, since

$$k^F_{\max}(\gamma') < \sqrt{\kappa/A'}.$$

Step 2: The inequality holds for convex curves. Let us assume that γ is a convex Jordan curve. Using inequality (1.39), we obtain

$$\frac{\kappa}{A(\Omega)} P_F(\Omega) \leqslant \int_{\partial\Omega} (k^F_{\partial\Omega}(x)^2 F(\nu_{\partial\Omega}(x))\, d\mathcal{H}^1(x) \leqslant \left(k^F_{\max}(\partial\Omega)\right)^2 P_F(\Omega) \tag{4.27}$$

and so inequality (5.2) follows straightforward.
Step 3: The inequality holds for general curves. Using the anisotropic curvature flow, the case of the general curves will be reduced to the case of the convex curves, in the same spirit of [102]. We set $A_0 := A(\Omega)$ and we prove that $k^F_{\max}(\gamma) \geqslant \sqrt{A_0/\kappa} := C$ for every admissible γ. By contradiction, there exists a smooth Jordan curve $\overline{\gamma}$ (not convex) such that

$$k^F_{\max}(\overline{\gamma}) < C. \tag{4.28}$$

Let $u(\cdot, t)$, with $t \in [0, T]$, be the family of curves evolving by anisotropic curvature flow with $u(\cdot, 0) = \overline{\gamma}(\cdot)$; so that at time $t = T$ the area enclosed by $u(\cdot, T)$ is 0. We consider the family

$$U(\cdot, t) := f(t)u(\cdot, t),$$

where f is a non-negative function chosen in such a way that every curve of the family $U(\cdot, t)$ encloses constant area. Therefore,

$$f(t) = \sqrt{\frac{A_0}{A(t)}},$$

where $A(t)$ is the area enclosed by $u_t(\cdot) := u(\cdot, t)$. Moreover, we observe that

$$k^F_U = \left(\frac{1}{f}\right) k^F_u. \tag{4.29}$$

Recalling that we denote by $'$ the derivative with respect to θ, using (4.29) and (1.47), we obtain

$$\begin{aligned}
(\partial_t - \psi\partial_{ss}) \frac{\left(k^F_U\right)^2}{2} &= (\partial_t - \psi\partial_{ss}) \left[\frac{A(t)}{2A_0} \frac{\left(k^F_u\right)^2}{2}\right] = \\
&= A'(t) \frac{\left(k^F_u\right)^2}{2A_0} + \frac{A(t)}{A_0} (\partial_t - \psi\partial_{ss}) \left(k^F_u\right)^2 \leqslant \\
&\leqslant A'(t) \frac{\left(k^F_u\right)^2}{2A_0} + \frac{A(t)}{A_0} \left[\left(3k_u h\phi' + h' k_u \phi\right) \partial_s (k^F_u)^2 + (k^F_u)^4\right] = \\
&= A'(t) \frac{\left(k^F_u\right)^2}{2A_0} + \frac{A(t)}{A_0} \left(k^F_u\right)^4 + \frac{A(t)}{A_0} \left[\left(3k_u h\phi' + h' k_u \phi\right) \partial_s (k^F_u)^2\right] = \\
&= \frac{A'(t)}{A(t)} (k^F_U)^2 + \frac{A_0}{A(t)} (k^F_U)^4 + \frac{A(t)}{A_0} \left[\left(3k_u h\phi' + h' k_u \phi\right) \partial_s (k^F_u)^2\right].
\end{aligned} \tag{4.30}$$

At this point let us introduce some useful notations; we set $k_u^F(\theta,t) := k_{u_t}^F(\theta)$ and $k_U^F(\theta,t) := k_{U_t}^F(\theta)$. Now, by (4.28), there exists $M \in (0,C)$ such that $k_{\overline{\gamma}}^F(\theta) < M$ for every $\theta \in \mathcal{S}^1$ and we want to show that for every $\theta \in \mathcal{S}^1$ and for every t

$$k_u^F(\theta,t) < M < C. \tag{4.31}$$

In order to prove (4.31), we proceed again by contradiction, assuming that there exists $t^* \in (0,T)$ for which it is possible to find a θ^* such that $k_U^F(\theta^*,t^*) = M$. This means that θ^* is a maximum for $k_U^F(\cdot,t^*)$ and, as a consequence, it is a maximum also for $k_u^F(\cdot,t^*)$. So, taking into account that at a maximal point $\partial_s(k_u^F)$ vanishes and $\left(k_u^F\right)_{ss}(\theta^*,t^*)$ is non-positive, from (4.30) we obtain that

$$(\partial_t - \psi\partial_{ss})\,\frac{\left(k_U^F(\theta^*,t^*)\right)^2}{2} \leqslant \frac{M^2}{A'(t^*)}\left(\frac{A'(t^*)}{2} + A_0M^2\right). \tag{4.32}$$

Using then (1.48), we have that

$$\begin{aligned} A'(t^*) = -\int_{u_{t^*}} F(\nu_{u_{t^*}}(s,t^*))k_{u_{t^*}}^F(s,t^*)ds &= -\int_{\partial\Omega_{t^*}} F(\nu_{u_{t^*}}(x))k_{u_{t^*}}^F(x)\, d\mathcal{H}^1(x) \\ &\leqslant -aD\int_{u_{t^*}} k_{u_{t^*}}(x)d\mathcal{H}^1(x) = -2\pi aD, \end{aligned} \tag{4.33}$$

wher Ω_{t^*} is the set enclosed by u_{t^*}. In the last inequality we have used the following facts: that, for every unit vector v, $F(v) \geqslant a$, the fact that the anisotropic curvature is controlled from above by the classical curvature since F is elliptic (see Remark 2.7), and finally the Gauss-Bonnet theorem. As a consequence,

$$(\partial_t - \psi\partial_{ss})\,\frac{\left(k_U^F(\theta^*,t^*)\right)^2}{2} \leqslant -\frac{A_0M^2}{A(t^*)}\left(\frac{\pi aD}{A_0} - M^2\right) < 0, \tag{4.34}$$

since we can assume, using a suitable scaling, that A_0 is such that $\frac{\pi aD}{A_0} = C$. Now, having $\partial_{ss}\left(k_U^F(\theta^*,t^*)\right)^2/2 < 0$, from (4.34), we have that

$$\partial_t\left(k_U^F(\theta^*,t^*)\right)^2 < 0,$$

and so

$$\partial_t\left(k_U^F(\theta^*,t^*)\right) < 0. \tag{4.35}$$

It follows that $k_U^F(\theta^*,t^*-\epsilon) > M$, for $\epsilon > 0$ small enough, which contradicts the choice of t^*. In this way we have proved (4.31).
Now, for the properties of the anisotropic curvature flow (see Section 1.3.4 and the reference therein), we know that for some $\tau > 0$ the curve $U(\cdot,\tau)$ is convex and therefore, thanks to Step 2, we have that for some $\theta \in [0,2\pi]$

$$k^F(\theta,\tau) \geqslant C, \tag{4.36}$$

that contradicts (4.31), concluding the proof.

□

4.3 Robin-Neuman boundary conditions for the $p-$Laplace eigenvalue problem and the torsion problem in the linear case

4.3.1 Definition and properties of the problems

Throughout this Section, we denote by Ω a set such that $\Omega = \Omega_0 \backslash \overline{\Theta}$, where $\Omega_0 \subseteq \mathbb{R}^n$ is an open bounded and convex set and $\Theta \subset\subset \Omega_0$ is a finite union of sets, each of one homeomorphic to a ball of $\mathbb{R}^n$ and with Lipschitz boundary. We define $\Gamma_0 := \partial\Omega_0$ and $\Gamma_1 := \partial\Theta$.

Eigenvalue problem

Let $1 < p < +\infty$, we deal with the following p-Laplacian eigenvalue problem:

$$\begin{cases} -\Delta_p u = \lambda_p^{NR}(\alpha, \Omega)|u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \alpha|u|^{p-2}u = 0 & \text{on } \Gamma_0 \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1. \end{cases} \tag{4.37}$$

We denote as usual by $\partial u/\partial \nu$ the outer normal derivative of u on the boundary and by $\alpha \in \mathbb{R}\backslash\{0\}$ the Robin boundary parameter, observing that the case $\alpha = +\infty$ gives asimptotically the Dirichlet boundary condition. Now we give the definition of eigenvalue and eigenfunction of problem (4.37).

Definition 4.1. The real number λ is an eigenvalue of (4.37) if and only if there exists a function $u \in W^{1,p}(\Omega)$, not identically zero, such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi \, dx + \alpha \int_{\Gamma_0} |u|^{p-2}u\varphi \, d\mathcal{H}^{n-1} = \lambda \int_\Omega |u|^{p-2}u\varphi \, dx$$

for every $\varphi \in W^{1,p}(\Omega)$. The function u is called eigenfunction associated to λ.

In order to compute the first eigenvalue we use the variational characterization, that is

$$\lambda_p^{NR}(\alpha, \Omega) = \min_{\substack{w \in W^{1,p}(\Omega) \\ w \not\equiv 0}} J_0[\alpha, w] \tag{4.38}$$

where

$$J_0[\alpha, w] := \frac{\displaystyle\int_\Omega |\nabla w|^p \, dx + \alpha \int_{\Gamma_0} |w|^p \, d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |w|^p \, dx}.$$

We observe that Ω_0 is convex and hence it has Lipschitz boundary; this ensures the existence of minimizers of the analyzed problems.

Proposition 4.17. *Let $\alpha \in \mathbb{R}\backslash\{0\}$. There exists a minimizer $u \in W^{1,p}(\Omega)$ of* (4.38)*, which is a weak solution to* (4.37)*.*

Proof. First we consider the case $\alpha > 0$. Let $u_k \in W^{1,p}(\Omega)$ be a minimizing sequence of (4.38) such that $||u_k||_{L^p(\Omega)} = 1$. Then, being u_k bounded in $W^{1,p}(\Omega)$, there exist a subsequence, still denoted by u_k, and a function $u \in W^{1,p}(\Omega)$ with $||u||_{L^p(\Omega)} = 1$, such that $u_k \to u$ strongly in $L^p(\Omega)$ and almost everywhere and $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(\Omega)$. As a consequence, u_k converges strongly to u in $L^p(\partial\Omega)$ and so almost everywhere on $\partial\Omega$ to u. Then, by weak lower semicontinuity:

$$\lim_{k\to+\infty} J_0[\alpha, u_k] \geqslant J_0[\alpha, u].$$

We consider now the case $\alpha < 0$. Let $u_k \in W^{1,p}(\Omega)$ be a minimizing sequence of (4.38) such that $||u_k||_{L^p(\Omega)} = 1$. Now, since α is negative, we have the equi-boundness of the functional $J_0[\alpha, \cdot]$, i.e. there exists a constant $C < 0$ such that $J_0[\alpha, u_k] \leqslant C$ for every $k \in \mathbb{N}$. As a consequence

$$||\nabla u_k||^p_{L^p(\Omega)} - C||u_k||^p_{L^p(\Omega)} \leqslant -\alpha$$

and so

$$||u||^p_{W^{1,p}(\Omega)} \leqslant L,$$

where $L := -\alpha/\min\{1, -C\}$. Then, there exist a subsequence, still denoted by u_k, and a function $u \in W^{1,p}(\Omega)$ such that $u_k \to u$ strongly in $L^p(\Omega)$ and $Du_k \rightharpoonup Du$ weakly in $L^p(\Omega)$. So u_k converges strongly to u in $L^p(\partial\Omega)$, and so

$$J_0[\alpha, u] \leqslant \liminf_{k\to\infty} J_0[\alpha, u_k] = \inf_{\substack{v\in W^{1,p}(\Omega)\\ v\not\equiv 0}} J_0[\alpha, v].$$

Finally, u is strictly positive in Ω by the Harnack inequality (see [125]).

□

Now we state some basic properties on the sign and the monotonicity of the first eigenvalue.

Proposition 4.18. *If $\alpha > 0$, then $\lambda_p^{NR}(\alpha, \Omega)$ is positive and if $\alpha < 0$, then $\lambda_p^{NR}(\alpha, \Omega)$ is negative. Moreover, for all $\alpha \in \mathbb{R}\backslash\{0\}$, $\lambda_p^{NR}(\alpha, \Omega)$ is simple, that is all the associated eigenfunctions are scalar multiple of each other and can be taken to be positive.*

Proof. Let $\alpha > 0$, then trivially $\lambda_p^{NR}(\Omega) \geqslant 0$. We prove that $\lambda_p^{NR}(\Omega) > 0$ by contradiction, assuming that $\lambda_p^{NR}(\Omega) = 0$. Thus, we consider a non-negative minimizer u such that $||u||_{L^p(\Omega)} = 1$ and

$$0 = \lambda_p^{NR}(\Omega, \alpha) = \int_\Omega |\nabla u|^p \, dx + \alpha \int_{\Gamma_0} |u|^p \, d\mathcal{H}^{n-1}.$$

So, u has to be constant in Ω and consequently u is 0 in Ω, which contradicts the fact that the norm of u is unitary.

If $\alpha < 0$, choosing the constant as test function in (4.38), we obtain

$$\lambda_p^{NR}(\alpha, \Omega) \leqslant \alpha \frac{P(\Omega_0)}{|\Omega|} < 0.$$

Let $u \in W^{1,p}(\Omega)$ be a function that achieves the infimum in (4.38). First of all we observe that

$$J_0[\alpha, u] = J_0[\alpha, |u|],$$

and this fact implies that any eigenfunction must have constant sign on Ω and so we can assume that $u \geqslant 0$. In order to prove the simplicity of the eigenvalue, we proceed as in [14, 52]. We give here a sketch of the proof. Let u, w be positive minimizers of the functional $J_0[\alpha, \cdot]$, such that $||u||_{L^p(\Omega)} = ||w||_{L^p(\Omega)} = 1$. We define $\eta_t = (tu^p + (1-t)w^p)^{1/p}$, with $t \in [0, 1]$ and we have that $||\eta_t||_{L^p(\Omega)} = 1$. It holds that

$$J_0[\alpha, u] = \lambda_p^{NR}(\alpha, \Omega) = J_0[\alpha, w]. \tag{4.39}$$

Moreover by convexity the following inequality holds true:

$$
\begin{aligned}
|\nabla \eta_t|^p &= \eta_t^p \left| \frac{t u^p \frac{\nabla u}{u} + (1-t) w^p \frac{\nabla w}{w}}{t u^p + (1-t) w^p} \right|^p \\
&\leqslant \eta_t^p \left[\frac{t u^p}{t u^p + (1-t) w^p} \left| \frac{\nabla u}{u} \right|^p + \frac{(1-t) w^p}{t u^p + (1-t) w^p} \left| \frac{\nabla w}{w} \right|^p \right] = t |\nabla u|^p + (1-t) |D w|^p.
\end{aligned} \tag{4.40}
$$

Using now (4.39), we obtain

$$
\lambda_p^{NR}(\alpha, \Omega) \leqslant J_0[\alpha, \eta_t] \leqslant t J_0[\alpha, u] + (1-t) J_0[\alpha, w] = \lambda_p^{NR}(\alpha, \Omega),
$$

and then η_t is a minimizer for $J_0[\alpha, \cdot]$. So inequality (4.40) holds as equality, and therefore $\frac{\nabla u}{u} = \frac{\nabla w}{w}$. This implies that $\nabla(\log u - \log w) = 0$, that is $\log \frac{u}{w} = const$. We conclude passing to the exponentials. $\square$

Proposition 4.19. *The map $\alpha \to \lambda_p^{NR}(\alpha, \Omega)$ is Lipschitz continuous and non-decreasing with respect to $\alpha \in \mathbb{R}$. Moreover $\lambda_p^{NR}(\alpha, \Omega)$ is concave in α.*

Proof. Let $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\alpha_1 < \alpha_2$ and let $w \in W^{1,p}(\Omega)$ be not identically 0. We observe that

$$
\int_\Omega |\nabla w|^p \, dx + \alpha_1 \int_{\Gamma_0} |w|^p \, d\mathcal{H}^{n-1} \leqslant \int_\Omega |\nabla w|^p \, dx + \alpha_2 \int_{\Gamma_0} |w|^p \, d\mathcal{H}^{n-1}.
$$

Now, passing to the infimum on w and taking into account the variatiational characterization, we obtain $\lambda_p^{NR}(\alpha_1, \Omega) \leqslant \lambda_p^{NR}(\alpha_2, \Omega)$.

We prove that $\lambda_p^{NR}(\beta, \Omega)$ is concave in α. Indeed, for fixed $\alpha_0 \in \mathbb{R}$, we have to show that

$$
\lambda_p^{NR}(\alpha, \Omega) \leqslant \lambda_p^{NR}(\alpha_0, \Omega) + \left(\lambda_p^{NR}\right)' (\alpha_0, \Omega) \, (\alpha - \alpha_0) \,, \tag{4.41}
$$

for every $\alpha \in \mathbb{R}$. Let w_0 be the eigenfunction associated to $\lambda_p^{NR}(\alpha_0, \Omega)$ and normalized such that $\int_\Omega w_0^p \, dx = 1$. Hence, we have

$$
\lambda_p^{NR}(\alpha, \Omega) \leqslant \int_\Omega |\nabla w_0|^p \, dx + \alpha \int_{\Gamma_0} |w_0|^p \, d\mathcal{H}^{n-1}. \tag{4.42}
$$

Now, summing and subtracting to the right hand side of (4.42) the quantity $\alpha_0 \int_{\Gamma_0} |w_0|^p d\mathcal{H}^{n-1}$, taking into account that

$$
\lambda_p^{NR}(\alpha_0, \Omega) = \int_\Omega |\nabla w_0|^p \, dx + \alpha_0 \int_{\Gamma_0} |w_0|^p \, d\mathcal{H}^{n-1},
$$

and the fact that

$$
\left(\lambda_p^{NR}\right)' (\alpha_0, \Omega) = \int_{\Gamma_0} |w_0|^p \, d\mathcal{H}^{n-1},
$$

we obtain the desired result (4.41). $\square$

Now we state a result relative to the eigenfunctions of problem (4.37) on the annulus.

Proposition 4.20. *Let r_1, r_2 be two nonnegative real number such that $r_2 > r_1$, and let u be the minimizer of problem (4.38) on the annulus A_{r_1, r_2}. Then u is strictly positive and radially symmetric, in the sense that $u(x) =: \psi(|x|)$. Moreover, if $\alpha > 0$, then $\psi'(r) < 0$ and if $\alpha < 0$, then $\psi'(r) > 0$.*

Proof. The first claim follows from the simplicity of $\lambda_p^{NR}(\alpha, A_{r_1,r_2})$ and from the rotational invariance of problem (4.37). For the second claim, we consider the problem (4.37) with the boundary parameter $\alpha > 0$. The associated radial problem is:

$$\begin{cases} -\dfrac{1}{r^{n-1}}\left(|\psi'(r)|^{p-2}\psi'(r)r^{n-1}\right)' = \lambda_p^{NR}(\alpha, A_{r_1,r_2})\psi^{p-1}(r) & \text{if } r \in (r_1, r_2), \\ \psi'(r_1)|\psi'(r_1)|^{p-2} = 0, \\ |\psi'(r_2)|^{p-2}\psi'(r_2) + \alpha\psi^{p-1}(r_2) = 0. \end{cases}$$

We observe that for every $r \in (r_1, r_2)$

$$-\frac{1}{r^{n-1}}\left(|\psi'(r)|^{p-2}\psi'(r)r^{n-1}\right)' = \lambda_p^{NR}(\alpha, A_{r_1,r_2})\psi^{p-1}(r) > 0, \tag{4.43}$$

and, as a consequence,

$$\left(|\psi'(r)|^{p-2}\psi'(r)r^{n-1}\right)' < 0.$$

Taking into account the boundary conditions $\psi'(r_1) = 0$, it follows that $\psi'(r) < 0$, since

$$|\psi'(r)|^{p-2}\psi'(r)r^{n-1} < 0.$$

If $\beta < 0$, by Remark 4.18, $\lambda_p^{NR}(\alpha, A_{r_1,r_2}) < 0$ and consequently the left side of the equation (4.43) is negative, and hence $\psi'(r) > 0$.

□

Torsional rigidity

Let $\alpha > 0$, we consider the torsional rigidity for the $p-$Laplacian. More precisely, we are interested in

$$\frac{1}{T_p^{NR}(\alpha, \Omega)} = \min_{\substack{w \in W^{1,p}(\Omega) \\ w \not\equiv 0}} K_0[\alpha, w], \tag{4.44}$$

where

$$K_0[\alpha, w] := \frac{\displaystyle\int_\Omega |\nabla w|^p \, dx + \alpha \int_{\Gamma_0} |w|^p \, d\mathcal{H}^{n-1}}{\left|\displaystyle\int_\Omega w \, dx\right|^p}.$$

Problem (4.44), up to a suitable normalization, leads to

$$\begin{cases} -\Delta_p u = 1 & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \alpha |u|^{p-2}u = 0 & \text{on } \Gamma_0 \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1. \end{cases} \tag{4.45}$$

In the following, we state some results for the torsional rigidity, analogously to the ones stated in the previous section for the eigenvalue problems. The proofs can be easily adapted.

Proposition 4.21. *Let $\alpha > 0$, then the following properties hold.*

- *There exists a positive minimizer $u \in W^{1,p}(\Omega)$ of (4.44) which is a weak solution to (4.45) in Ω.*

- *Let r_1, r_2 be two nonnegative real numbers such that $r_2 > r_1$, and ψ be the minimizer of (4.44) on the annulus A_{r_1,r_2}. Then ψ is strictly positive, radially symmetric and strictly decreasing.*
- *The map $\alpha \mapsto \dfrac{1}{T_p^{NR}(\alpha,\Omega)}$ is positive, Lipschitz continuous, non-increasing and concave with respect to α.*

4.3.2 Main results

In this section we state and prove the main results. In the first theorem, we study the problem (4.38), in the second one the problem (4.44). We consider a set Ω defined as at the beginning of this Section.

Theorem 4.22. *Let $\alpha \in \mathbb{R}\backslash\{0\}$ and let Ω be such that $\Omega = \Omega_0\backslash\overline{\Theta}$, where $\Omega_0 \subseteq \mathbb{R}^n$ is an open bounded and convex set and $\Theta \subset\subset \Omega_0$ is a finite union of sets, each of one homeomorphic to a ball of $\mathbb{R}^n$ and with Lipschitz boundary. Let $A = A_{r_1,r_2}$ be the annulus having the same measure of Ω and such that $P(B_{r_2}) = P(\Omega_0)$. Then,*

$$\lambda_p^{NR}(\alpha,\Omega) \leqslant \lambda_p^{NR}(\alpha,A).$$

Proof. We divide the proof in two cases, distinguishing the sign of the Robin boundary parameter. **Case 1: $\alpha > 0$.** We start by considering problem (4.38) with positive value of the Robin parameter. The solution v to (4.38) is a radial function by Proposition 4.20 and we denote by v_m and v_M the minimum and the maximum of v on A. We construct the following test function defined in Ω_0:

$$u(x) := \begin{cases} G(d_e(x)) & \text{if } d_e(x) < r_2 - r_1 \\ v_M & \text{if } d_e(x) \geqslant r_2 - r_1, \end{cases} \tag{4.46}$$

where G is defined as

$$G^{-1}(t) = \int_{v_m}^{t} \frac{1}{g(\tau)}\, d\tau,$$

with $g(t) = |\nabla v|_{v=t}$, defined for $v_m \leqslant t < v_M$, and $d_e(\cdot)$ denotes the distance from $\partial\Omega_0$. We observe that $v(x) = G(r_2 - |x|)$ and u satisfy the following properties: $u \in W^{1,p}(\Omega_0)$ and

$$\begin{aligned} |\nabla u|_{u=t} &= |\nabla v|_{v=t}, \\ u_m &:= \min_{\Omega_0} u = v_m = G(0), \\ u_M &:= \max_{\Omega_0} u \leqslant v_M. \end{aligned}$$

We need now to define the following sets:

$$\begin{aligned} E_{0,t} :=& \{x \in \Omega_0 \ : \ u(x) > t\}, \\ A_t :=& \{x \in A \ : \ v(x) > t\}, \\ A_{0,t} :=& A_t \cup \overline{B}_{r_1}. \end{aligned} \tag{4.47}$$

For simplicity of notation, we will denote by A_0 the set $A_{0,0}$, i.e. the ball B_{r_2}. Since $E_{0,t}$ and $A_{0,t}$ are convex sets, inequalities (1.16) and (1.14) imply

$$-\frac{d}{dt}P(E_{0,t}) \geqslant n(n-1)\frac{W_2(E_{0,t})}{g(t)} \geqslant n(n-1)n^{-\frac{n-2}{n-1}}\omega_n^{\frac{1}{n-1}}\frac{(P(E_{0,t}))^{\frac{n-2}{n-1}}}{g(t)},$$

for $u_m < t < u_M$. Moreover, it holds

$$-\frac{d}{dt}P(A_{0,t}) = n(n-1)n^{-\frac{n-2}{n-1}}\omega_n^{\frac{1}{n-1}}\frac{(P(A_{0,t}))^{\frac{n-2}{n-1}}}{g(t)},$$

for $v_m < t < v_M$. Since, by hypothesis, $P(\Omega_0) = P(B_{r_2})$, using a comparison type theorem, we obtain

$$P(E_{0,t}) \leqslant P(A_{0,t}),$$

for $v_m \leqslant t < u_M$. Let us also observe that

$$\mathcal{H}^{n-1}(\partial E_{0,t} \cap \Omega) \leqslant P(E_{0,t}) \leqslant P(A_{0,t}). \tag{4.48}$$

Using now the coarea formula and (4.48):

$$\begin{aligned}\int_\Omega |Du|^p\, dx &= \int_{u_m}^{u_M} g(t)^{p-1}\, \mathcal{H}^{n-1}\left(\partial E_{0,t} \cap \Omega\right) dt \\ &\leqslant \int_{u_m}^{u_M} g(t)^{p-1} P(E_{0,t})\, dt \leqslant \int_{v_m}^{v_M} g(t)^{p-1} P(A_{0,t})\, dt = \int_A |\nabla v|^p\, dx.\end{aligned} \tag{4.49}$$

Since, by construction, $u(x) = u_m = v_m$ on Γ_0, then

$$\int_{\Gamma_0} u^p\, d\mathcal{H}^{n-1} = u_m^p P(\Omega_0) = v_m^p P(A_0) = \int_{\partial A_0} v^p\, d\mathcal{H}^{n-1}. \tag{4.50}$$

Now, we define $\mu(t) = V(E_{0,t} \cap \Omega)$ and $\eta(t) = V(A_t)$ and using again coarea formula, we obtain, for $v_m \leqslant t < u_M$,

$$\begin{aligned}\mu'(t) = -\int_{\{u=t\}\cap\Omega} \frac{1}{|\nabla u(x)|}\, d\mathcal{H}^{n-1} &= -\frac{\mathcal{H}^{n-1}\left(\partial E_{0,t} \cap \Omega\right)}{g(t)} \geqslant -\frac{P(E_{0,t})}{g(t)} \\ &\geqslant -\frac{P(A_{0,t})}{g(t)} = -\int_{\{v=t\}} \frac{1}{|\nabla v(x)|}\, d\mathcal{H}^{n-1} = \eta'(t).\end{aligned}$$

This inequality holds true also if $0 < t < v_M$. Since $\mu(0) = \eta(0)$ (indeed $V(\Omega) = V(A)$), by integrating from 0 to t, we have:

$$\mu(t) \geqslant \eta(t), \tag{4.51}$$

for $0 \leqslant t < v_M$. If we consider the eigenvalue problem (4.38), we have

$$\int_\Omega u^p\, dx = \int_{v_m}^{v_M} pt^{p-1}\mu(t)dt \geqslant \int_{v_m}^{v_M} pt^{p-1}\eta(t)\, dt = \int_A v^p\, dx. \tag{4.52}$$

Using (4.49)-(4.50)-(4.52), we achieve

$$\begin{aligned}\lambda_p^{NR}(\alpha, \Omega) &\leqslant \frac{\displaystyle\int_\Omega |\nabla u|^p\, dx + \alpha \int_{\Gamma_0} u^p\, d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega u^p\, dx} \\ &\leqslant \frac{\displaystyle\int_A |\nabla v|^p\, dx + \alpha \int_{\partial A_0} v^p\, d\mathcal{H}^{n-1}}{\displaystyle\int_A v^p\, dx} = \lambda_p^{RN}(\alpha, A).\end{aligned}$$

Case 2: $\alpha < 0$**.** We consider now the problem (4.38) with negative Robin external boundary parameter. By Proposition 4.18 the first p-Laplacian eigenvalue is negative. We observe that v is a radial function. We construct now the following test function defined in Ω_0:

$$u(x) := \begin{cases} G(d_e(x)) & \text{if } d_e(x) < r_2 - r_1 \\ v_m & \text{if } d_e(x) \geqslant r_2 - r_1, \end{cases}$$

where G is defined as

$$G^{-1}(t) = \int_t^{v_M} \frac{1}{g(\tau)}\, d\tau,$$

with $g(t) = |\nabla v|_{v=t}$, defined for $v_m < t \leqslant v_M$ with $v_m := \min_A v$ and $v_M := \max_A v$. We observe that u satisfies the following properties: $u \in W^{1,p}(\Omega_0)$ and

$$\begin{aligned} |\nabla u|_{u=t} &= |\nabla v|_{v=t}, \\ u_m &:= \min_{\Omega} u \geqslant v_m, \\ u_M &:= \max_{\Omega} u = v_M = G(0). \end{aligned}$$

We need now to define the following sets:

$$\begin{aligned} \tilde{E}_{0,t} =& \{x \in \Omega_0 \ : \ u(x) < t\}, \\ \tilde{A}_t =& \{x \in A \ : \ v(x) < t\}; \\ \tilde{A}_{0,t} =& \tilde{A}_t \cup \overline{B}_{r_1}. \end{aligned}$$

For simplicity of notation, we will denote by $\tilde{A}_0$ the set $\tilde{A}_{0,0}$, i.e. the ball B_{r_2}. Since $\tilde{E}_{0,t}$ and $\tilde{A}_{0,t}$ are now convex sets, by inequalities (1.17) and (1.14), we obtain

$$\frac{d}{dt} P(\tilde{E}_{0,t}) \geqslant n(n-1) \frac{W_2(\tilde{E}_{0,t})}{g(t)} \geqslant n(n-1) n^{-\frac{n-2}{n-1}} \omega_n^{\frac{1}{n-1}} \frac{\left(P(\tilde{E}_{0,t})\right)^{\frac{n-2}{n-1}}}{g(t)}.$$

Moreover, it holds

$$\frac{d}{dt} P(\tilde{A}_{0,t}) = n(n-1) n^{-\frac{n-2}{n-1}} \omega_n^{\frac{1}{n-1}} \frac{\left(P(\tilde{A}_{0,t})\right)^{\frac{n-2}{n-1}}}{g(t)}.$$

Since, by hypothesis, $P(\Omega_0) = P(B_{r_2})$, using a comparison type theorem, we obtain

$$P(\tilde{E}_{0,t}) \leqslant P(\tilde{A}_{0,t}),$$

for $u_m \leqslant t < v_M$. Moreover, we have

$$\mathcal{H}^{n-1}(\partial \tilde{E}_{0,t} \cap \Omega) \leqslant P(\tilde{E}_{0,t}) \leqslant P(\tilde{A}_{0,t}). \tag{4.53}$$

Using the coarea formula and (4.53),

$$\begin{aligned} \int_\Omega |\nabla u|^p \, dx &= \int_{u_m}^{u_M} g(t)^{p-1}\, \mathcal{H}^{n-1}(\partial \tilde{E}_{0,t} \cap \Omega)\, dt \\ &\leqslant \int_{u_m}^{u_M} g(t)^{p-1} P(\tilde{E}_{0,t})\, dt \leqslant \int_{v_m}^{v_M} g(t)^{p-1} P(\tilde{A}_{0,t})\, dt = \int_A |Dv|^p \, dx. \end{aligned} \tag{4.54}$$

Since, by construction, $u(x) = u_M = v_M$ on Γ_0, it holds

$$\int_{\Gamma_0} u^p \, d\mathcal{H}^{n-1} = u_M^p P(\Omega_0) = v_M^p P(A_0) = \int_{\partial A_0} v^p \, d\mathcal{H}^{n-1}. \tag{4.55}$$

We define now $\tilde{\mu}(t) = V(\tilde{E}_{0,t} \cap \Omega)$ and $\tilde{\eta}(t) = V(\tilde{A}_t)$ and using coarea formula, we obtain, for $u_m \leqslant t < v_M$,

$$\begin{aligned}\tilde{\mu}'(t) = \int_{\{u=t\}\cap\Omega} \frac{1}{|\nabla u(x)|} \, d\mathcal{H}^{n-1} = \frac{\mathcal{H}^{n-1}(\partial \tilde{E}_{0,t} \cap \Omega)}{g(t)} \leqslant \frac{P(\tilde{E}_{0,t})}{g(t)} \\ \leqslant \frac{P(\tilde{A}_{0,t})}{g(t)} = \int_{\{v=t\}} \frac{1}{|\nabla v(x)|} \, d\mathcal{H}^{n-1} = \tilde{\eta}'(t).\end{aligned}$$

Hence $\mu'(t) \leqslant \eta'(t)$ for $v_m \leqslant t \leqslant v_M$. Then, by integrating from t and v_M:

$$|\Omega| - \tilde{\mu}(t) \leqslant |A| - \tilde{\eta}(t),$$

for $v_m \leqslant t < v_M$ and consequently $\tilde{\mu}(t) \geqslant \tilde{\eta}(t)$.

Let us consider the eigenvalue problem (4.38). We have that

$$\int_\Omega u^p \, dx = u_M^p |\Omega| - \int_{u_m}^{u_M} pt^{p-1} \tilde{\mu}(t) dt \leqslant v_M^p |A| - \int_{v_m}^{v_M} pt^{p-1} \tilde{\eta}(t) \, dt = \int_A v^p \, dx. \tag{4.56}$$

By (4.54)-(4.55)-(4.56), we have

$$\begin{aligned}\lambda_p^{NR}(\alpha, \Omega) \leqslant \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx + \alpha \int_{\Gamma_0} u^p \, d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega u^p \, dx} \leqslant \\ \leqslant \frac{\displaystyle\int_A |\nabla v|^p \, dx + \alpha \int_{\partial A_0} v^p \, d\mathcal{H}^{n-1}}{\displaystyle\int_A v^p \, dx} = \lambda_p^{NR}(\alpha, A).\end{aligned}$$

□

Theorem 4.23. *Let $\alpha > 0$ and let Ω be such that $\Omega = \Omega_0 \backslash \overline{\Theta}$, where $\Omega_0 \subseteq \mathbb{R}^n$ is an open bounded and convex set and $\Theta \subset\subset \Omega_0$ is a finite union of sets, each of one homeomorphic to a ball of $\mathbb{R}^n$ and with Lipschitz boundary. Let $A = A_{r_1,r_2}$ be the annulus having the same measure of Ω and such that $P(B_{r_2}) = P(\Omega_0)$. Then,*

$$T_p^{NR}(\alpha, \Omega) \geqslant T_p^{NR}(\alpha, A).$$

Proof. Let v be the function that achieves the minimum in (4.44) on the annulus A. We consider the test function as in (4.46) and the superlevel sets as in (4.47). By (4.51) we have

$$\int_\Omega u \, dx = \int_0^{v_M} \mu(t) dt \geqslant \int_0^{v_M} \eta(t) \, dt = \int_A v \, dx. \tag{4.57}$$

In this way, using (4.49)-(4.50)-(4.57), we conclude

$$
\begin{aligned}
\frac{1}{T_p^{NR}(\alpha,\Omega)} &\leqslant \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx + \alpha \int_{\Gamma_0} u^p \, d\mathcal{H}^{n-1}}{\displaystyle\left|\int_\Omega u \, dx\right|^p} \\
&\leqslant \frac{\displaystyle\int_A |\nabla v|^p \, dx + \alpha \int_{\partial A_0} v^p \, d\mathcal{H}^{n-1}}{\displaystyle\left|\int_A v \, dx\right|^p} = \frac{1}{T_p^{NR}(\alpha,A)}.
\end{aligned}
$$

□

We conclude with some remarks.

Remark 4.24. In [5] the authors prove that the annulus maximizes the first eigenvalue of the p-Laplacian with Neumann condition on internal boundary and Dirichlet condition on external boundary, among sets of $\mathbb{R}^n$ with holes and having a sphere as outer boundary. We explicitly observe that our result includes this case, since

$$
\lim_{\alpha\to+\infty} \lambda_p^{NR}(\alpha,\Omega) = \lambda_p^{ND}(\Omega),
$$

where with $\lambda_p^{ND}(\Omega)$ we denote the first eigenvalue of the p-Laplacian endowed with Dirichlet condition on external boundary and Neumann condition on internal boundary.

Remark 4.25. Let us remark that in the case $p = 2$, we know explicitly the expression of the solution of the problems considered in this section on the annulus $A = A_{r_1,r_2}$.

We denote by J_ν and Y_ν, respectively, the Bessel functions of the first and second kind of order ν (for their definition and properties we refer to [1]). The function that achieves the minimum in $\lambda = \lambda_p^{NR}(\alpha, A)$ is

$$
v(r) = Y_{\frac{n}{2}-2}(\sqrt{\lambda}r_2) r^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(\sqrt{\lambda}r) - J_{\frac{n}{2}-2}(\sqrt{\lambda}r_2) r^{1-\frac{n}{2}} Y_{\frac{n}{2}-1}(\sqrt{\lambda}r),
$$

with the condition

$$
\begin{aligned}
&Y_{\frac{n}{2}-2}(\sqrt{\lambda}r_1)[r_2^{1-\frac{n}{2}} J_{\frac{n}{2}-2}(\sqrt{\lambda}r_2)\sqrt{\lambda} + \alpha r_2^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(\sqrt{\lambda}r_2)] - \\
&J_{\frac{n}{2}-2}(\sqrt{\lambda}r_1)[r_2^{1-\frac{n}{2}} Y_{\frac{n}{2}-2}(\sqrt{\lambda}r_2)\sqrt{\lambda} + \alpha r_2^{1-\frac{n}{2}} Y_{\frac{n}{2}-1}(\sqrt{\lambda}r_2)] = 0.
\end{aligned}
$$

The function that achieves the minimum $1/T = 1/T_p^{NR}(\alpha, A)$ is

$$
v(r) = \frac{1}{2Tn} r^2 + c_1 \frac{(1-n)}{r^n} + c_2,
$$

with

$$
\begin{cases}
c_1 = \frac{1}{\alpha T}\left(\frac{r_2}{n} - \frac{r_1^n}{n r_2^{n-1}} + \frac{\alpha r_2^2}{2n} + \frac{(n-1)\beta}{n}\left(\frac{r_1}{r_2}\right)^n\right) \\
c_2 = -\frac{1}{nT} r_1^n.
\end{cases}
$$

Chapter 5

A reverse type quantitative isoperimetric inequality

Let $\Omega \subseteq \mathbb{R}^n$ an open set with finite Lebesgue measure and $\lambda_1(\Omega)$ the first Dirichlet Laplacian eigenvalue. In this Chapter we work with the following class of admissible sets

$$\mathcal{C}_n := \{\Omega \subseteq \mathbb{R}^n \mid \Omega \text{ convex}, \ V(\Omega) = 1\}$$

and we will prove that there exists a constant $c > 0$, depending only on the dimension n, such that, for every $\Omega \in \mathcal{C}_n$, we have

$$\lambda_1(\Omega) - \lambda_1(B) \geqslant c\,(P(\Omega) - P(B))^2 \,. \tag{5.1}$$

In the following we will denote by B a ball of volume 1.

5.1 Main result

We state now the main result of this Section.

Theorem 5.1. *Let $n \geqslant 2$; there exists a constant $c > 0$, depending only on n, such that, for every $\Omega \in \mathcal{C}_n$, it holds*

$$\lambda_1(\Omega) - \lambda_1(B) \geqslant c\,(P(\Omega) - P(B))^2 \,. \tag{5.2}$$

In order to prove this result, we need to recall the sharp quantitative version of the Faber-Krahn inequality proved in [26]. We recall that the result is sharp, since the power 2 cannot be replaced by any smaller power and that is verified using a suitable family of ellipsoids.

Theorem 5.2 (Quantitative Faber-Krahn)**.** *Let $n \geqslant 2$; there exists a constant $\bar{C} > 0$, depending only on n, such that, for every open set Ω with $V(\Omega) = 1$, it holds*

$$\lambda_1(\Omega) - \lambda_1(B) \geqslant \bar{C}\mathcal{A}_F(\Omega)^2 \tag{5.3}$$

and the exponent 2 is sharp.

In order to prove our main Theorem 5.1, we will prove the following Proposition, that combined with (5.3) will give the desired result. Let us define the following asymmetry functional, known as spherical deviation (see [67]):

$$\widetilde{\mathcal{A}}_{\mathcal{H}}(\Omega) = \inf_{x \in \mathbb{R}^n} \Big\{ d_{\mathcal{H}}(\Omega, B_R(x)) \ : \ B_R(x) \text{ is a ball s.t. } |B_R(x)| = 1 \Big\}. \tag{5.4}$$

Proposition 5.3. *Let $n \geqslant 2$. There exist two constants $C > 0$ and $\delta_0 > 0$, depending only such that, for every $\Omega \in \mathcal{C}_n$ with $\widetilde{\mathcal{A}}_{\mathcal{H}}(\Omega)) < \delta_0$, it holds*

$$\mathcal{A}_F(\Omega) \geqslant C\,(P(\Omega) - P(B))\,. \tag{5.5}$$

Remark 5.4. It is clear that inequality (5.21) cannot be true when Ω is a long and flat domain of fixed volume, since the asymmetry functional is such that $\mathcal{A}_F(\Omega) \in [0,2)$. Let us assume that Ω contains the origin. In this case, we proceed in the following way. First of all, let us recall the result contained in [56] and reported in Chapter 1 (Lemma 1.9) that holds for $\Omega \subseteq \mathbb{R}^n$ convex set:

$$d_{\mathcal{H}}(\Omega, B) \leqslant c\,(\mathrm{diam}(\Omega) + \mathrm{diam}(B))\,\mathcal{A}_F(\Omega)^{1/n}. \tag{5.6}$$

If we are in dimension $n = 2$, since

$$P(\Omega) = \int_0^{2\pi} h(\theta) d\theta,$$

where h is the support function associated to Ω as defined in Definition 1.3, we have

$$d_{\mathcal{H}}(\Omega, B) = ||h_\Omega - \frac{1}{\sqrt{\pi}}||_\infty, \geqslant \frac{1}{2\pi}\int_0^{2\pi} |h_\Omega(\theta) - \frac{1}{\sqrt{\pi}}| \geqslant \frac{1}{2\pi}\Delta P(\Omega). \tag{5.7}$$

So, combining (5.7) and (5.2), we obtain, being C a constant depending only by the dimension,

$$\frac{\Delta\lambda_1(\Omega)}{(\Delta P(\Omega))^2} \geqslant C\frac{\mathcal{A}_F^2(\Omega)}{2\pi\left(\mathrm{diam}(\Omega) + \frac{2}{\sqrt{\pi}}\right)^4 \mathcal{A}_F(\Omega)} \to 0, \tag{5.8}$$

that goes to 0 when the diameter of Ω $\mathrm{diam}(\Omega)$ goes to infinity. Let us consider now the case $n > 2$. We have

$$d_{\mathcal{H}}(\Omega, B) = ||h_\Omega - h_B||_\infty \geqslant ||h_\Omega||_\infty - \frac{1}{\omega_n^{1/n}}$$

where the quantity in the right hand side is positive since we have fixed $V(\Omega) = 1$, and, consequently,

$$\frac{1}{n\omega_n}\int_{\mathbb{S}^{n-1}} h_\Omega(x)\, d\mathcal{H}^{n-1}(x) \leqslant d_{\mathcal{H}}(\Omega, B) + \frac{1}{\omega_n^{1/n}}. \tag{5.9}$$

On the other hand, we have also that

$$d_{\mathcal{H}}(\Omega, B) \geqslant \frac{1}{n\omega_n}\int_{\mathbb{S}^{n-1}} |h_\Omega(x) - h_B(x)|\, d\mathcal{H}^{n-1}(x) \tag{5.10}$$

$$\geqslant \omega(\Omega) - \frac{1}{n\omega_n}\int_{\mathbb{S}^{n-1}} h_\Omega(x)\, d\mathcal{H}^{n-1}(x) - \frac{1}{\omega_n^{1/n}}, \tag{5.11}$$

where $\omega(\Omega)$ is the mean width of Ω as defined in (1.11) in Chapter 1. Consequently, using (5.9) and recalling (1.10),

$$d_{\mathcal{H}}(\Omega, B) \geqslant c_n \frac{W_{n-1}(\Omega) - W_{n-1}(B)}{2}, \tag{5.12}$$

where $W_{n-1}(\Omega)$ is the $(n-1)$-th quermassintegral of Ω, as defined in (1.11). We recall that for $W_{n-1}(\Omega)$ the following Alexandrov-Fenchel inequality (see (1.13) in Chapter 1 with $j = n-1$ and $i = 1$) holds:

$$W_{n-1}(\Omega) \geqslant \frac{n^{\frac{n}{n-1}}}{\omega_n^{1/(n-1)}} P(\Omega)^{1/(n-1)} \tag{5.13}$$

and there is the equality on balls. So, combing (5.12) with (5.13), we obtain

$$d_{\mathcal{H}}(\Omega, B) \geqslant c_n \left(P(\Omega)^{1/(n-1)} - P(B)^{1/(n-1)} \right) \geqslant \bar{c}_n P(\Omega)^{\frac{-n}{n-1}} \Delta P(\Omega), \tag{5.14}$$

where $c_n, \bar{c}_n$ are positive constants depending only on the dimension n. Eventually, using (5.3), (5.6) and (5.14), we obtain, being $C = C(n) > 0$,

$$\frac{\Delta\lambda_1(\Omega)}{(\Delta P(\Omega))^2} \geqslant C \frac{\mathcal{A}_F^2(\Omega)}{P(\Omega)^{\frac{n}{n-1}} \mathcal{A}_F(\Omega)^{1/n} \left(\operatorname{diam}(\Omega) + \frac{2}{\omega_n^{1/n}} \right)^2} \to 0, \tag{5.15}$$

as $\operatorname{diam}(\Omega)$ goes to infinity.

Remark 5.5. With this method of proof, using the sharp inequality proved in [26], we are not able to prove the conjecture contained in [66], that is our starting point, stating that

$$\lambda_1(\Omega) - \lambda_1(B) \geqslant \beta \left(P(\Omega) - P(B) \right)^{3/2}, \tag{5.16}$$

where β is a constant that can be explicitly computed. Our leading idea is indeed to combine inequality (5.3) with an inequality of the form

$$\mathcal{A}_{\mathcal{F}}(\Omega) \geqslant C \left(P(\Omega) - P(B) \right)^{\delta} \tag{5.17}$$

and this last inequality is a sort of 'reverse' quantitative isoperimetric inequality, recalling that the quantitative isoperimetric inequality proved in [69] has the form

$$(P(\Omega) - P(B)) \geqslant C \mathcal{A}_F(\Omega)^2, \tag{5.18}$$

where C is a positive constant, for every Borel set Ω of unit measure. On the contrary to (5.18), in inequality (5.17) the terms of the difference of the perimeter is used as an asymmetry functional and it is situated in the left hand side of the inequality. Proceeding in this way, the target power in (5.17) to prove conjecture (5.16) would be 3/4, but unfortunately the best power is $\delta = 1$ and the "bad" sets in this case are the polygons. As we can see from the computations below we have that

$$\mathcal{A}_{\mathcal{F}}(\mathcal{P}_k^*) \sim \left(P(\mathcal{P}_k^*) - P(B) \right),$$

when $\mathcal{P}_k^*$ is a regular k-gon of area 1. Indeed, the following relations hold:

$$\mathcal{A}_F(\mathcal{P}_k^*) \simeq \epsilon^2, \qquad P(\mathcal{P}_k^*) - P(B) \simeq \epsilon^2.$$

Being B the ball of area 1, if we set R the radius of B, we have that $R = 1/\sqrt{\pi}$. Let us denote by a the apothem of $\mathcal{P}_k^*$, i.e. the segment from the center of the polygon that is perpendicular to one of its sides. Setting $\epsilon = \pi/k$ and Taylor expanding, we obtain that

$$a = \frac{\left(1 - \sin^2(\pi/k)\right)^{1/4}}{\sqrt{k \sin(\pi/k)}} \simeq \frac{\left(1 - \epsilon^2\right)^{1/4}}{\sqrt{\pi}} \simeq \frac{1 - \epsilon^2/4}{\sqrt{\pi}}. \tag{5.19}$$

Now the area A_γ of the circular segment of angle 2γ, that is the region of B which is cut off from the rest by one edge of the polygon,

$$A_\gamma = \frac{1}{2\pi}(\gamma - \sin(\gamma)) \simeq \frac{\sqrt{2}}{\pi}\epsilon^3,$$

since $\gamma = \arccos(a\sqrt{\pi}) \simeq \sqrt{2}\epsilon$. Thus,

$$\mathcal{A}_F(\mathcal{P}_k^*) \simeq 2\sqrt{2\pi}\epsilon^2. \tag{5.20}$$

Using (5.19) we obtain

$$P(\mathcal{P}_k^*) - P(B) = \frac{2}{a} - 2\sqrt{\pi} \simeq \frac{\sqrt{\pi}}{2}\epsilon^2.$$

On the other hand, sets that are smooth without edges do not create problems. If we consider for example the family of ellipses

$$E_\epsilon = \left\{(x,y) \mid x^2 + \frac{y^2}{(1+\epsilon)^2} = 1\right\}$$

we have that (see the computations in [25])

$$\mathcal{A}_{\mathcal{F}}(E_\epsilon) \sim \left(P(E_\epsilon) - P(B)\right)^{1/2},$$

being

$$P(E_\epsilon) - P(B) \simeq \epsilon^2, \qquad \mathcal{A}_{\mathcal{F}}(E_\epsilon) = O(\epsilon).$$

5.1.1 Intermediate result : A geometrical inequality between the asymmetry and the difference of perimeters

In this section we will prove the following proposition.

Proposition 5.6. *Let $n \geqslant 2$. There exist two constants $C > 0$ and $\delta_0 > 0$, depending only on the dimension n, such that, for every $\Omega \in \mathcal{C}_n$ with $\widetilde{\mathcal{A}}_{\mathcal{H}}(\Omega) < \delta_0$, it holds*

$$\mathcal{A}_F(\Omega) \geqslant C\left(P(\Omega) - P(B)\right). \tag{5.21}$$

Let us fix a system of coordinates and $O = (0, \cdots, 0)$ as the origin. Without loss of generality, we can assume that $O \in \Omega$ and that the ball of volume 1 centered at the origin realizes the minimum in (5.4). So, from now on $B = B(O)$ and, consequently, $\widetilde{\mathcal{A}}_{\mathcal{H}}(\Omega) = d_{\mathcal{H}}(\Omega, B)$.

Proof in the planar case

First of all, let us consider the case $n = 2$. The proof of Proposition 5.6 is divided in two main steps: we prove the inequality for a polygonal class of sets that is dense in the class of convex sets with small Hausdorff distance with respect to B and then we can conclude that the inequality is true for every Ω such that $d_{\mathcal{H}}(\Omega, B) < \delta_0$ by a density argument.

We will use the classical polar coordinates representation of convex sets as follows:

$$\Omega = \left\{(r,\theta) \in [0,\infty) \times [0,2\pi) \mid r < \frac{1}{u(\theta)}\right\}, \tag{5.22}$$

where u is a positive and $2\pi-$periodic function, often called the gauge function of Ω, for a reference see e.g [118, 90]. It is well known that Ω is convex if and only if $u'' + u \geqslant 0$. We can write the volume, the perimeter and the asymmetry of Ω in terms of u. In particular, we have that :

$$V(\Omega) = \frac{1}{2}\int_0^{2\pi} \frac{1}{u^2(\theta)}\, d\theta;$$
$$P(\Omega) = \int_0^{2\pi} \frac{\sqrt{u^2 + u'^2}}{u^2} d\theta.$$

We can also compute explicitly the asymmetry, setting $r(\theta) := 1/u(\theta)$, we have

$$\mathcal{A}_F(\Omega) = \int_0^{2\pi} |\int_r^1 r(\theta)\, dr|\, d\theta = \int_0^{2\pi} \frac{|1-r^2(\theta)|}{2} d\theta = \frac{1}{2}\int_0^{2\pi} \frac{|1-u^2(\theta)|}{u^2(\theta)}\, d\theta. \tag{5.23}$$

Let us consider now a set $\Omega \in \mathcal{C}_2$ such that $d_{\mathcal{H}}(\Omega, B) < \delta_0$. Since $d_{\mathcal{H}}(\Omega, B) < \delta_0$, we have that $u(\theta) \simeq 1$. After a rationalization of the denominator and after taking care of adjusting the constant in (5.23), that for convenience we still call C, we have that inequality (5.21) follows from

$$\int_0^{2\pi} |1-u|(1+u)\, d\theta = \int_0^{2\pi} |1-u^2(\theta)|\, d\theta \geqslant C \int_0^{2\pi} \left(u'^2(\theta) + u^2(\theta) - u^4(\theta)\right)\, d\theta. \tag{5.24}$$

Then, if we prove the following

$$\int_0^{2\pi} |1-u(\theta)|\, d\theta \geqslant C \int_0^{2\pi} \left(u'(\theta)\right)^2 d\theta, \tag{5.25}$$

with a still different constant $C > 0$, we obtain (5.24). So, we will prove (5.24) for a class of polygons that are dense in the family of convex sets with small Hausdorff distance from B (see proof of Theorem 3.1.5 in [118]). We give in the following the construction of this class.

Definition 5.1. Let $\Omega \in \mathcal{C}_2$ and such that Ω contains the origin O. We call $\mathcal{P}(\Omega, \bar{\theta})$ the family of the the polygons constructed from Ω in the following way. We consider the radii from the origin, each one of which forming with the adjacent one an angle of amplitudine $\theta_0 \leqslant \bar{\theta}$. We say that $\mathcal{P} \in \mathcal{P}(\Omega, \bar{\theta})$, if $\mathcal{P}$ is the polygon whose boundary is obtained by joining all the consecutive points given by the intersection of the above radii with $\partial\Omega$.

The result we are going to prove is the following.

Lemma 5.7. *There exist $C > 0$, $\delta_0 > 0$ and $\bar{\theta} > 0$ constants such that, for every $\Omega \in \mathcal{C}_2$ with $d_{\mathcal{H}}(\Omega, B) < \delta_0$, if $\mathcal{P} \in \mathcal{P}(\Omega, \bar{\theta})$ is a polygon associated to Ω as in Definition 5.1 and u is the gauge function associated to $\mathcal{P}$, then*

$$\int_0^{2\pi} |1-u(\theta)|\, d\theta \geqslant C \int_0^{2\pi} \left(u'(\theta)\right)^2 d\theta. \tag{5.26}$$

Proof. We will analyse all the possible cases. In the following we are assuming that the segment $\overline{AB} \subset \partial\mathcal{P}$ is one side of the polygon and we call θ_0 the angle $A\hat{O}B$. By definition we have that $\theta_0 \leqslant \bar{\theta}$. We denote by P the point of intersection between the segment $\overline{AB}$ and the ray that forms with the segment $\overline{OA}$ an angle of θ.
Case 1: external tangent. We set $\alpha := O\hat{A}B$, we assume that $\overline{OB}$ has length equal to 1 and that $\alpha + \theta_0 = \pi/2$ and that the segment $\overline{AB}$ lies on the $y-$axis. Moreover we denote $h := |\overline{OA}| - 1$.
We have that

$$h + 1 = \frac{1}{\cos(\theta_0)}; \tag{5.27}$$

Let $\theta \in (0, \theta_0)$. Having $u(\theta) = 1/r(\theta)$, using the sine theorem to the triangle OAP we obtain

$$u(\theta) = \cos(\theta_0 - \theta) \tag{5.28}$$

and, consequently,

$$u'(\theta) = -\sin(\theta_0 - \theta). \tag{5.29}$$

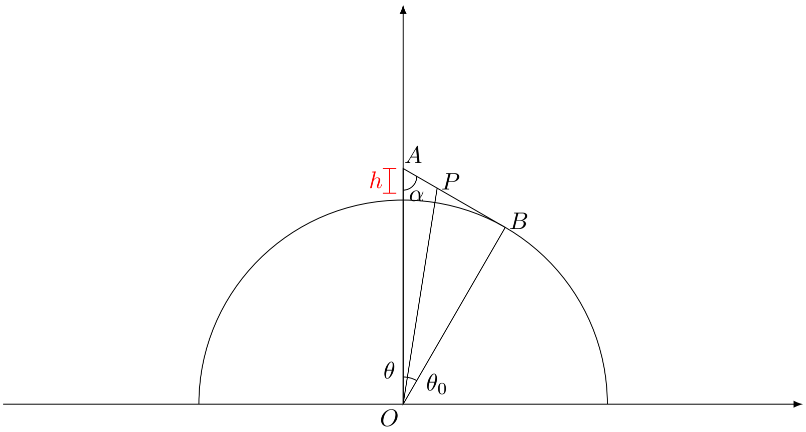

Figure 5.1: External tangent case

Taylor expanding up to the first order with respect to θ_0, we have that

$$\int_0^{\theta_0} \left(u'(\theta)\right)^2 d\theta = \int_0^{\theta_0} \sin^2(\theta_0 - \theta)\, d\theta = \left[\frac{\theta_0}{2} - \frac{\sin(2\theta_0)}{4}\right] \approx \frac{2\theta_0^3}{3} \tag{5.30}$$

and, using (5.27),

$$\int_0^{\theta_0} |1 - u(\theta)|\, d\theta = \int_0^{\theta_0} (1 - u(\theta))\, d\theta = \int_0^{\theta_0} (1 - \cos(\theta_0 - \theta))\, d\theta = \tag{5.31}$$

$$= (\theta_0 - \sin(\theta_0)) \approx \frac{\theta_0^3}{6}. \tag{5.32}$$

Comparing these last two results, we can say that there exists a constant C such that

$$\int_0^{\theta_0} |1 - u(\theta)|\, d\theta \geqslant C \int_0^{\theta_0} \left(u'(\theta)\right)^2 d\theta.$$

Case 2: external intersecting. We are now considering the case when $\alpha + \theta_0 < \pi/2$, the segment $\overline{OB}$ has length 1 and the side $\overline{AB}$ is tangent to the ball $B(O)$ in the point B. We set $z = \pi/2 - \alpha$. In this case we have that

$$h + 1 = \frac{\sin(\alpha + \theta_0)}{\sin(\alpha)}. \tag{5.33}$$

and, using the sine theorem to the triangle OAP,

$$u(\theta) = \frac{\sin(\alpha + \theta)}{\sin(\alpha)(h + 1)}. \tag{5.34}$$

From (5.34) it follows that

$$u'(\theta) = \frac{\cos(\alpha + \theta)}{\sin(\alpha)(h + 1)}. \tag{5.35}$$

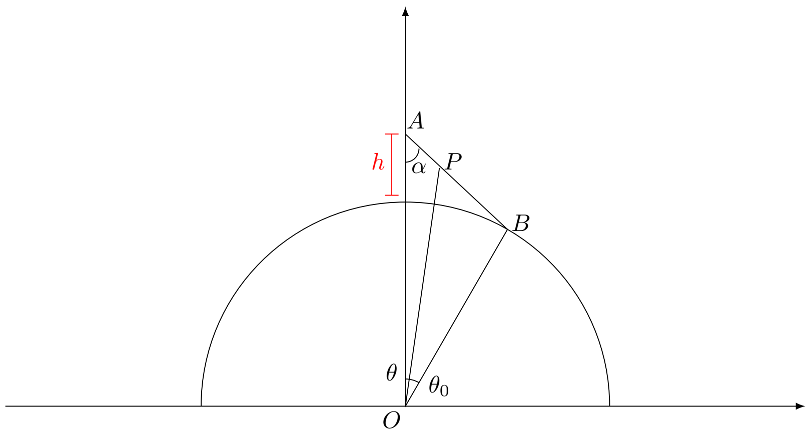

Figure 5.2: External secant case

Still denoting by $I = 1/(\sin(\alpha)(h+1))$ and Taylor expanding up to the first order with respect to θ_0 and z, we have that

$$\int_0^{\theta_0} \left(u'(\theta)\right)^2 d\theta = I^2 \left(\frac{\theta_0}{2} + \frac{\sin(2z - 2\theta_0)}{4} - \frac{\sin(2z)}{4}\right) \simeq \frac{I^2}{3}\left[3z^2\theta_0 - 3z\theta_0^2 + \theta_0^3\right]. \tag{5.36}$$

and, using (5.27),

$$\begin{aligned} &\int_0^{\theta_0} |1 - u(\theta)|\, d\theta = \int_0^{\theta_0} (1 - u(\theta))\, d\theta = I \int_0^{\theta_0} \left(1 - \frac{\cos(\theta_0 - \theta)}{(h+1)\sin(\alpha)}\right) d\theta = \\ =& I\left[\sin(\alpha)(h+1)\theta_0 + \cos(\alpha + \theta_0) - \cos(\alpha)\right] = \\ &= I\left[\theta_0 \cos(\theta_0 - z) + sen(z)\left(\cos(\theta_0) - 1\right) - cos(z)\sin(\theta_0)\right] \simeq \frac{3I}{2}\left[3z\theta_0^2 - 2\theta_0^3\right]. \end{aligned}$$

Comparing these last two results, since we can choose θ_0 such that $z > (2/3)\theta_0$, there exists a costant C such that

$$\int_0^{\theta_0} |1 - u(\theta)|\, d\theta \geqslant C \int_0^{\theta_0} \left(u'(\theta)\right)^2 d\theta.$$

Case 3: External not intersecting. We are now considering the case when $\alpha + \theta_0 \leqslant \pi/2$, the segment $\overline{OB}$ has length strictly greater than 1 and the side $\overline{AB}$ is external to the ball $B(O)$. We set $z = \pi/2 - \alpha$. We consider the line r intersecting the segment $\overline{OA}$, that is parallel to the line passing through the points A and B and that touches the ball. We call h the length of the segment $\overline{AQ}$, where Q is the point of intersection between the segment $\overline{AO}$ and the line r and we call h' the length of the segment $\overline{QQ'}$, where Q' is the the point of intersection between the segment $\overline{AO}$ and the ball $B(O)$. Let us first assume that $\alpha + \theta_0 = \pi/2$. We have that

$$u(\theta) = \frac{\cos(\theta_0 - \theta)}{\sin(\alpha)(h + h' + 1)}. \tag{5.37}$$

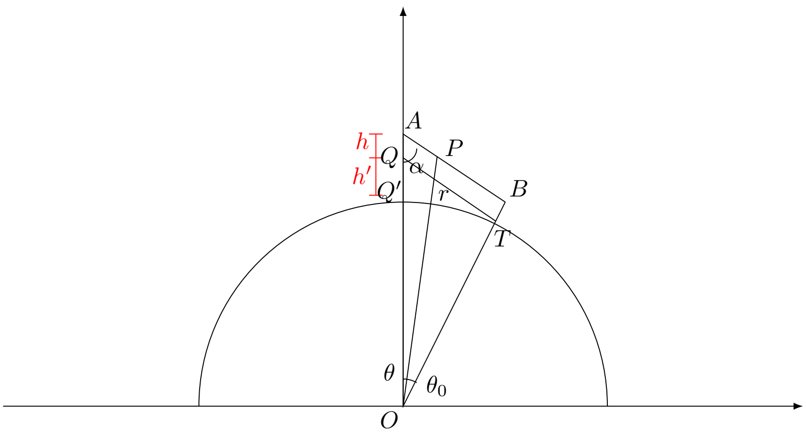

Figure 5.3: External not intersecting case

So, setting $I = 1/\left(\sin(\alpha)(h+h'+1)\right)$, we obtain that

$$\int_0^{\theta_0} \left(u'(\theta)\right)^2 d\theta = I^2 \int_0^{\theta_0} \sin^2(\theta-\theta_0)\, d\theta = I^2 \left(\frac{\theta_0}{2} - \frac{\sin(2\theta_0)}{4}\right) \approx I^2 \frac{4}{3}\theta_0^3. \tag{5.38}$$

Let us denote by A_T the area of the trapeze $ABTQ$ and let us compute A_T:

$$A_T = \frac{\sin(\theta_0)(1+2h+h')}{2} h' \cos(\theta_0) \approx \frac{h'}{2}\sin(2\theta_0) \approx \frac{h'}{2}(2\theta_0 - \frac{4\theta_0^3}{3}). \tag{5.39}$$

So, since

$$\int_0^{\theta_0} (1-u(\theta))\, d\theta \geqslant A_T,$$

we can conclude that there exists a costant $C>0$ such that

$$\int_0^{\theta_0} |1-u(\theta)|\, d\theta \geqslant C \int_0^{\theta_0} \left(u'(\theta)\right)^2 d\theta.$$

Now let us assume that $\alpha + \theta_0 > \pi/2$ and let us set $z = \pi/2 - \alpha$. We have that

$$u(\theta) = \frac{\sin(\theta+\alpha)}{\sin(\alpha)(h+h'+1)} \tag{5.40}$$

and that

$$\frac{1}{I^2}\int_0^{\theta_0} \left(u'(\theta)\right)^2 d\theta = \int_0^{\theta_0} \sin^2(\theta+\alpha)\, d\theta = \frac{\theta_0}{2} - \frac{\sin(2\theta_0+2\alpha)}{4} + \frac{\sin(2\alpha)}{4}$$
$$= \frac{\theta_0}{2} + \frac{\sin(2z)}{4}(1-\cos(2\theta_0)) + \frac{cos(2z)\sin(2\theta_0)}{4} \approx \theta_0 + \theta_0^2 z - z^2\theta_0 - \frac{2}{3}\theta_0^3.$$

If we compute A_T:

$$A_T = \frac{\sin(\theta_0)}{2}[\sin^2(\alpha) sin^2(\beta) + \sin^3(\alpha)\sin(\beta)(1+h+h')]$$

and we can conclude.

Case 3: Internal not touching. Let us assume that the segments $\overline{AO}$ and $\overline{BO}$ are both less than 1 and we call $h := 1 - |\overline{AO}|$ and β the angle $O\hat{B}A$

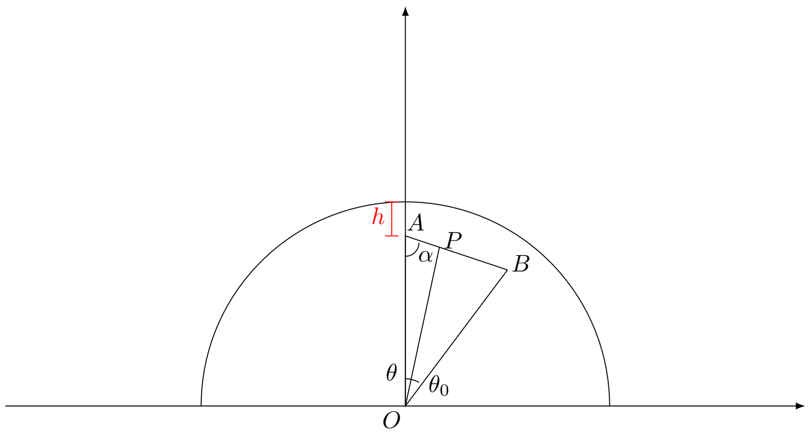

Figure 5.4: Internal not intersecting case

In this case we have that

$$u(\theta) = \frac{\sin(\alpha+\theta)}{(1-h)\sin(\alpha)}.$$

Setting $I = 1/(1-h)\sin(\alpha)$ and $z = \pi/2 - \alpha$,

$$\frac{1}{I^2}\int_0^{\theta_0} \big(u'(\theta)\big)^2 \, d\theta = \frac{\theta_0^2}{2} + \frac{\sin(2z - 2\theta_0)}{4} - \frac{\sin(2z)}{4} \approx \frac{\theta_0^3}{3} - z\theta_0^2 + z^2\theta_0.$$

Let Q be the point that lays in the circumference and in the semi-line that contains the segment $\overline{OA}$ and Q' a point that lays in the semi-line containing $\overline{OB}$, such that the segment $\overline{QQ'}$ is parallel to $\overline{AB}$. Calling A_T the area of trapeze $ABQQ'$, we have

$$A_T = \frac{h(2-h)\sin(\alpha)}{2\sin(\beta)}\sin(\theta_0).$$

We can conclude, since

$$\int_0^{\theta_0} |1 - u(\theta)| \, d\theta \geqslant A_T.$$

Case 4: Internal touching. Let us assume that the side $\overline{AB}$ is now contained in the ball and that the segment $\overline{OB}$ has length less than 1. In this case, setting $I = 1/\sin(\alpha)$ and $z = \pi/2 - \alpha$, we have, when $z \to 0$ that

$$\frac{1}{I^2}\int_0^{\theta_0} \big(u'(\theta)\big)^2 \, d\theta = \int_0^{\theta_0} \cos^2(\theta+\alpha)\, d\theta = \frac{\theta_0}{2} + \frac{\sin(2z - 2\theta_0)}{4} - \frac{\sin(2z)}{4}$$
$$\approx \frac{\theta_0^3}{3} + z^2\theta_0 - z\theta_0^2$$

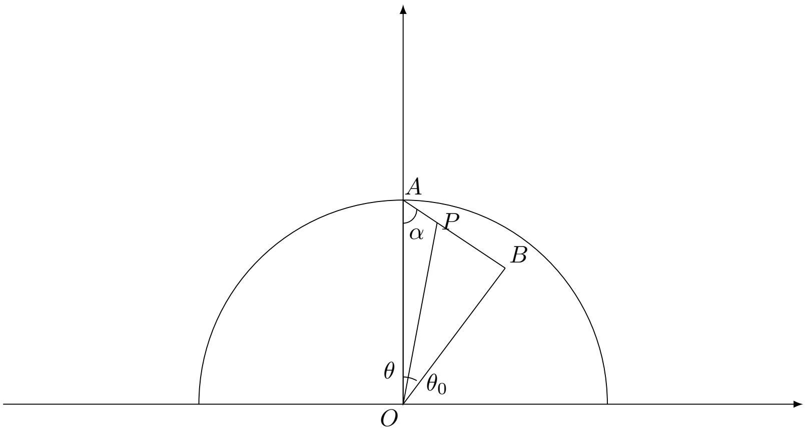

Figure 5.5: Internal secant case

and

$$\frac{1}{I}\int_0^{\theta_0}(u(\theta)-1)\,d\theta = -\sin(z-\theta_0)+\sin(z)-\theta_0\cos(z) \approx \frac{z\theta_0^2}{2}-\frac{\theta_0^3}{6},$$

and we can conclude, since we can show after some computations of Euclidean geometry that z goes to 0 with the same speed as θ_0. If we are in the case that z does not tend to 0, as $\theta_0 \to 0$, we Taylor expand in θ_0 and we easily obtain the desired result.

$\square$

Generalisation in dimension n

Let us assume now that the boundary of $\Omega \in \mathcal{C}_n$ can be parametrized in the following way:

$$\partial\Omega = \{y \in \mathbb{R}^n \mid y = r(x);\ x \in \mathbb{S}^{n-1}\}. \tag{5.41}$$

We will use the following notation: for simplicity we will denote by ∇_τ the tangential gradient $\nabla^{\mathcal{S}^{n-1}}$ as defined in Definition 1.2.

We have the following results (see for the exact computations [67, 68]):

$$\mathcal{A}_F(\Omega) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} |r^n(x)-1|\,d\mathcal{H}^{n-1}(x)$$

and

$$P(\Omega) = \int_{\mathcal{S}^{n-1}} \sqrt{r^{2(n-1)}(x)+r^{2(n-2)}(x)|\nabla_\tau r|^2}\,d\mathcal{H}^{n-1}.$$

If we assume that $d_{\mathcal{H}}(\Omega, B)$ is small, we have that r is sufficiently near to 1, and, consequently,

$$\mathcal{A}_F(\Omega) \leqslant \frac{n+1}{n}\int_{\mathbb{S}^{n-1}} |r(x)-1|\,d\mathcal{H}^{n-1}. \tag{5.42}$$

So, proving Proposition 5.6, is equivalent to prove that there exists $C > 0$ such that

$$\int_{\mathbb{S}^{n-1}} |r(x)-1|\,d\mathcal{H}^{n-1} \geqslant C\int_{\mathbb{S}^{n-1}} \left[|\nabla_\tau r|^2 + r^2 - \frac{1}{r^{2(n-2)}}\right] d\mathcal{H}^{n-1}. \tag{5.43}$$

We note now that

$$r^2 - \frac{1}{r^{2(n-2)}} = \frac{r^{2n-2}-1}{r^{2(n-2)}} = (r-1)\left[\frac{r^{2n-3}+\cdots+1}{r^{2(n-2)}}\right]$$

and the term $\left[\left(r^{2n-3}+\cdots+1\right)/r^{2(n-2)}\right]$ is bounded if $r \in [1-\delta_0, 1+\delta_0]$, so that it can absorbed by the constant C. For this reason, it remains to prove the following inequality

$$\int_{\mathbb{S}^{n-1}} |r(x)-1|\, d\mathcal{H}^{n-1} \geqslant C \int_{\mathbb{S}^{n-1}} |\nabla_\tau r|^2\, d\mathcal{H}^{n-1}.$$

We are going to prove the following result.

Proposition 5.8. *Let $n \geqslant 2$. There exist $C > 0$, $\delta_0 > 0$ costants, depending only on n, such that, for every $\Omega \in \mathcal{C}_n$ with $d_{\mathcal{H}}(\Omega, B) < \delta_0$, then, if r is the function that describes the boundary of Ω as in* (5.41)*, it holds*

$$\int_{\mathbb{S}^{n-1}} |r(x)-1|\, d\mathcal{H}^{n-1}(x) \geqslant C \int_{\mathbb{S}^{n-1}} |\nabla_\tau r(x)|^2\, d\mathcal{H}^{n-1}(x). \tag{5.44}$$

Proof. We consider a system of coordinates $(O, \vec{e_1}, \cdots, \vec{e_n})$ and we shall prove the desired inequality in a neighbourhood ω of the point $x_0 := (0, \cdots, 0, 1)$.
Let $x \in \omega$, there exists $c > 0$ such that

$$|\nabla_\tau r(x)|^2 \leqslant c \sum_{i=1}^{n-1} |\nabla_i u(x)|^2, \tag{5.45}$$

where $\nabla_i u(x)$ is the tangential gradient of u in the circle $S_i := \mathbb{S}^{n-1} \cap (O\vec{x}\ \vec{e_i})$. Inequality (5.45) is a consequence of the fact that the tangent τ_i at C_i in x are almost orthogonal.
We prove now the following inequality: there exists a constant $C > 0$ such that

$$\int_\omega |r(x)-1|\, d\mathcal{H}^{n-1}(x) \geqslant C \int_\omega |\nabla_{n-1} r|^2\, d\mathcal{H}^{n-1}(x). \tag{5.46}$$

In order to do that we consider the spherical coordinates:

$$\begin{cases} x_1 = \cos(\theta_1) \\ x_2 = sin(\theta_1)\cos(\theta_2) \\ \vdots \\ x_{n-1} = \sin(\theta_1)\ldots\sin(\theta_{n-2})\cos(\theta_{n-1}) \\ x_n = \sin(\theta_1)\ldots\sin(\theta_{n-2})\sin(\theta_{n-1}) \end{cases}$$

with $\theta_1 \ldots, \theta_{n-2} \in [0, \pi]$ and $\theta_{n-1} \in [0, 2\pi)$. In this coordinate system the point $x_0 = (0, \cdots, 0, 1)$ corresponds to

$$\begin{aligned} \theta_1 &= \pi/2, \\ \theta_2 &= \pi/2, \\ &\vdots \\ \theta_{n-1} &= \pi/2. \end{aligned}$$

We fix now $\theta_1, \ldots, \theta_{n-2}$ and we let θ_{n-1} vary in a small interval $[\pi/2 - \delta, \pi/2 + \delta]$. We write on the the piece of circle ω' described by this parametrization the bidimensional inequality that we have proved in the previous Section

$$\int_{\omega'} |r(x) - 1| \, d\mathcal{H}^1 \geqslant C \int_{\omega'} |\nabla_{n-1} r|^2 \, d\mathcal{H}^1. \tag{5.47}$$

Now we multiply both sides of (5.47) by the Jacobian $\big(\sin^{n-2}(\theta_1) \sin^{n-3}(\theta_2) \ldots \sin(\theta_{n-2})\big)$ and we integrate in $\theta_1, \theta_2, \ldots, \theta_{n-2}$ in a small neighbourhood, proving in this way inequality (5.46). In a similar way, with a suitable choice of spherical coordinates, we prove that for $i = 1, \ldots, n-2$, also holds

$$\int_{\omega} |r(x) - 1| \, d\mathcal{H}^{n-1}(x) \geqslant C \int_{\omega} |\nabla_i r|^2 \, d\mathcal{H}^{n-1}(x). \tag{5.48}$$

Summing and using (5.45), we can conclude that

$$\int_{\omega} |r(x) - 1| \, d\mathcal{H}^{n-1}(x) \geqslant C \int_{\omega} |\nabla_\tau r|^2 \, d\mathcal{H}^{n-1}(x). \tag{5.49}$$

$\square$

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