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From Cosmology to Complex Biological Systems

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Declaration of Authorship

I, Francesco BAJARDI, declare that this thesis titled *The Topological Invariant Approach: From Cosmology to Complex Biological Systems* and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a PhD at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
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- I have acknowledged all main sources of help.
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"I managed to get a quick PhD
- though when I got it I knew
almost nothing about physics.
But I did learn one big thing:
that no one knows everything,
and you don't have to."

S. Weinberg

Preface

Topological invariants are usually defined as quantities which are preserved under homeomorphism transformations. This means that they are mathematical objects which do not depend on the local form of the spacetime, but only relies on its global structure, the topology. They are largely used in all physics branches, from gravitation up to complex systems, due to their capability of reducing the complexity of the dynamics and leading to exact solutions. This thesis is aimed at describing their applications in different contexts, such as modified theories of gravity, standard electromagnetism and biological systems. Regarding the former, it is well known that Einstein General Relativity is still considered the best accepted theory describing the gravitational interaction, but several shortcomings arise in the so called strong regimes. As a matter of facts, despite its success, General Relativity presents many unsolved issues and puzzles at any scales. Such problems can be partially solved by modified theories of gravity, which aim to extend the Einstein-Hilbert action to a more general one including other geometric terms. These latter can mimick the role of Dark Energy and Dark Matter, providing an effective energy-momentum tensor of the gravitational field. Among all the possible modifications of the starting actions, in this thesis modifications related to topological invariants are considered. Modified theories of gravity, often lead to higher-order field equations which cannot be analytically solved even in cosmological backgrounds. In this framework, reducing the order of the field equations, topological invariants can be particularly useful in order to find out exact solutions, well describing the today observations at the large scales. Moreover, as pointed out in the second part of the thesis, topological invariants can be used to construct gauge-invariant Lagrangians, which allow to fix the high-energy issues arising in the attempt of merging the formalism of Quantum Mechanics with that of General Relativity. In the second part of this work we will focus on a modified theory of gravity including a function of the Gauss–Bonnet topological surface term, showing that suitable field equations allow to find out exact solutions in a cosmological and in a spherically symmetric background.

The starting action is selected by means of the Noether Symmetry approach, a selection criterion aimed at finding theories containing symmetries. Modified $f(\mathcal{G})$ gravity, with $\mathcal{G}$ being the Gauss–Bonnet term, is studied in a D-dimensional spacetime, where higher-order cosmological and black holes solutions are provided. The Gauss–Bonnet scalar is, then, coupled to a dynamical scalar field, in order to make a comparison with the standard scalar-tensor theory of gravity. In all cases, General Relativity can be recovered as a particular limit.

In the third part of the thesis, we consider the Chern–Simons theory in odd dimensions. It is based on the Chern-Simons forms, whose exterior derivatives provide topological surface terms. This property make the theory *quasi*-Gauge invariants, namely invariant under gauge transformations up to a boundary term. We show that from very general and basic theories such as classical and quantum theories of gravity, Chern–Simons theory can lead to far beyond closely related fields to push concepts and applications to complex systems, there including the interactions between biomolecules, such as nucleic acids and proteins. Indeed, after providing cosmological and spherically symmetric solutions in D dimensions, we show that the theory can be also applied to biological systems. While applications to our Universe seems to be a straightforward consequence for a testbed of the theory itself, the use of Chern–Simons theory in understanding complex systems might look unusual and non-conventional.

Biological systems often exhibit complicated topological structures, such as nucleic acids or proteins, since different parts of the same molecule may assume a complicated three-dimensional shape (tertiary structure). When two or more tertiary structures interact, the resulting system fold into a quaternary structure, whose schematization represents one of the most controversial and discussed branch of science, due to the important implications in biology, microbiology, medicine *etc.* As an example, from the spatial configuration assumed by the DNA, it is possible to infer the place in which genomic mutations might occur, as well as the difference among phenotypes.

The link between Chern–Simons theories and the dynamics/interactions of complex biomolecules is the topological nature of the former which can be essential to describe the complicated physico-chemical and biological behavior of the latter, very much relying on their topology. Basically, the main idea is to describe the DNA curvature by using the same formalism adopted for the spacetime, treating the interactions among biological systems as driven by the same general principles that govern the gravitational interaction.

By merging the schematization approach lying behind the Chern–Simons theory with the more conventional ones coming from bioinformatic, it is possible to implement the nowadays knowledge of the biological scenario. In particular, the deterministic aspect of the former can be combined with bioinformatic techniques, which treat the biological issues from a stochastic point of view.

As a final remark, in light of the above mentioned applications, it is worth pointing out that this thesis can be understood as a first step towards the development of the so called "Topological Invariant Approach". More precisely, we aim to show that topological invariants can be considered in the framework of different fields to describe the corresponding dynamics, ranging from cosmology, black holes, up to complex systems. Throughout the history of physics it is possible to identify several approaches, developed to solve specific issues, but which subsequently spread out in different fields, because of their general validity. This is the case *e.g.* of symmetries, which nowadays play a fundamental role in almost all branches of science. Similarly, though topological invariants arose with the purpose of addressing evidences provided by the gravitational interaction, the same structure can be also applied to apparently unrelated fields. In particular, the vision on which topology, geometry and topological invariants are based on, is the key point of the approach. In this way, once addressing a configuration space to the given system, the evolution can be described under the same formalism as the space-time, so that the research for topological and geometrical features of the configuration space can provide information about the related dynamics. Therefore, the link with the gravitational interaction appears natural and straightforward, and the dynamical behavior of galaxies, stars and planets can be addressed to other different models.

To conclude, this thesis is organized as follows: in Chaps. 1 and 2 successes and shortcomings of General Relativity are outlined, and the main classes of modified theories of gravity are discussed. In Chaps. 3-8, different theories of gravity involving the Gauss–Bonnet scalar are studied, as well as scalar-tensor theories and non-local theories. Specifically, in Chap. 3, cosmological aspects such as energy conditions and slow–roll inflation are discussed in the framework of $f(\mathcal{G})$ gravity. The form of the starting action is selected by symmetry considerations, namely using the Noether symmetry approach. The prescription pursued to find out analytic cosmological solutions by Noether's approach is based on Ref. [47], while applications to early stages of the Universe and energy conditions on Ref. [146]. In Chap. 5 we find out exact solutions for $f(\mathcal{G})$ gravity in a spherically symmetric background, following Ref. [155]. In Chap. 6 we compare two classes of non-local integral kernel theories of Gauss–Bonnet gravity, outlining the main results of Ref. [158]. In Chap. 7 the equivalence between metric and affine scalar-tensor theories is discussed, remarking the differences and the common features. In particular, a function of the scalar field $f(\phi)$ is coupled to the scalar curvature (Sec. 7.1), to the torsion scalar (Sec. 7.2) and to the Gauss–Bonnet scalar (Sec. 7.3). For further details see Ref. [162]. The third part is devoted to basic foundations and applications of Chern–Simons theory. After outlining its main aspects in Chap. 9, in Chaps. 10 and 11 Chern–Simons gravity is applied to cosmology and spherical symmetry [171]. Finally, in Chaps 12 and 13 the applications to electromagnetism and biological system is respectively considered [218, 217]. With regards to this latter, in Sec. 13.1.1 the theory is applied to KRAS human gene, in order to study the effect of induced mutations to selected sequences. In Sec. 13.1.2 the same analysis is performed to SARS-COV 2 virus.

Keywords: Topological Invariants; Modified Theories of Gravity; Complex Systems.

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This thesis is the result of three intense years of effort, experiences, satisfactions and, overall, of personal growth.

I was little more than a child when started this experience that taught me very much, among which, as a last resort, some physics notions. I am still little more than a child, but hope that I am tracing the right way which one day will lead myself being the man I would like to be.

I owe a tremendous debt of gratitude to the many people who helped make this journey a reality. I want to acknowledge anyone who pushed me to accomplish more than I could on my own and who supported me during these years, with fruitful suggestions, discussions, criticisms and comments, but also with friendship and true affection.

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Moving to Naples was initially tough, but now the department of physics feels like home, thanks to the availability, kindness and courtesy of colleagues and professors, to which I would like to express my sincere gratitude.

I acknowledge my family for the unconditional love and continuous support during my lifetime, I owe you this and any other goals I will hopefully reach.

It goes without saying that I would have never been able to reach this point without any of the above mentioned people.

Thank you all.

Napoli, May 2021,

Francesco Bajardi.

List of Publications

During my PhD I published 17 papers in different topics, in collaboration with several research groups. Not all of them are part of this thesis, but all of them increased my background and my personal experience, consequently enriching the content of this thesis. The list of publications is outlined below:

- **F. Bajardi**, F. Bascone, S. Capozziello "*DNA Mutations Description via Chern-Simons Current*" Submitted to Scientific Reports (2021)

- **F. Bajardi**, F. Bascone, S. Capozziello "*Renormalizability of alternative theories of gravity: differences between power counting and entropy argument*" Universe **7** (5), 148 (2021) 10.3390/universe7050148

- S. Capozziello, C. Altucci, **F. Bajardi**, A. Basti, N. Beverini, G. Carelli, D. Ciampini, A. Di Virgilio, F. Fuso, U. Giacomelli, E. Maccioni, P. Marsili, A. Ortolan, A. Porzio, A. Simonelli, G. Terreni, R. Velotta "*Constraining Theories of Gravity by GINGER Experiment*" Eur. Phys. J. Plus **136**, 394 (2021) 10.1140/epjp/s13360-021-01373-4

- A. Di Virgilio, C. Altucci, **F. Bajardi**, A. Basti, N. Beverini, S. Capozziello, G. Carelli, D. Ciampini, F. Fuso, U. Giacomelli, E. Maccioni, P. Marsili, A. Ortolan, A. Porzio, A. Simonelli, G. Terreni, R. Velotta "*Sensitivity limit investigation of a Sagnac gyroscope through linear regression analysis*" The European Physical Journal C **81**, 400 (2021) 10.1140/epjc/s10052-021-09199-1

- **F. Bajardi**, S. Capozziello "*Noether Symmetries and Quantum Cosmology in Extended Teleparallel Gravity*" Int. J. Geom. Meth. In Mod. Phys. (2021) 10.1142/S0219887821400028

- **F. Bajardi**, D. Vernieri, S. Capozziello "*Bouncing Cosmology in $f(Q)$ Symmetric Teleparallel Gravity*" Eur. Phys. J. Plus **135**, 912 (2020) 10.1140/epjp/s13360-020-00918-3

- **F. Bajardi**, S. Capozziello, D. Vernieri "*Non-Local Curvature and Gauss-Bonnet Cosmologies by Noether Symmetries*" Eur. Phys. J. Plus **135**, 942 (2020) 10.1140/epjp/s13360-020-00944-1

- **F. Bajardi**, S. Capozziello "*Equivalence of Non-Minimally Coupled Cosmologies by Noether Symmetry*" Int. J. Mod. Phys. D **29**, no.14, 2030015 (2020) 10.1142/S0218271820300153

- M. Singh, C. Zannella, V. Folliero, R. Di Girolamo, **F. Bajardi**, A. Chianese, L. Altucci, A. Damasco, M. R. Del Sorbo, C. Imperatore, M. Rossi, M. Valadan, M. Varra, A. Vergara, G. Franci, M. Galdiero, and C. Altucci "*Combating actions of green 2D-materials on gram positive and negative bacteria and enveloped viruses*", Frontiers in Bioengineering and Biotechnology, **8** 2296-4185 (2020) 10.3389/fbioe.2020.569967

- **F. Bajardi**, S. Capozziello "*$f(\mathcal{G})$ Noether Cosmology*", The European Physical Journal C, **80** 704 (2020) 10.1140/epjc/s10052-020-8258-2

- Z. Urban, **F. Bajardi**, S. Capozziello "*The Noether-Bessel-Hagen Symmetry Approach for Dynamical Systems*", Int. J. Geom. Meth. Mod. Phys. **17**, no.14, 2050215 (2020) 10.1142/S0219887820502151

- F. Bosi, A. Di Virgilio, U. Giacomelli, A. Simonelli, G. Terreni, A. Basti, N. Beverini, G.

Carelli, D. Ciampini, F. Fuso, E. Maccioni, P. Marsilli, F. Stefani, A. Ortolan, A. Porzio, C. Altucci, **F. Bajardi**, S. Capozziello, R. Velotta, "*Sagnac gyroscopes, GINGERINO and GINGER*". Journal of Physics: Conference Series, **1468**, 16th International Conference of Topics in Astroparticle and Underground Physics (TAOP 2019) 9-13 September 2019, Toyama, Japan. (2020) 10.1088/1742-6596/1468/1/012243

- C. Imperatore, M. Valadan, L. Tartaglione, M. Persico, A. Ramunno, M. Menna, M. Casertano, C. Dell'Aversano, M. Singh, M. d'Aulisio Garigliota, **F. Bajardi**, E. Morelli, C. Fattorusso, C. Altucci and M. Varra "*Exploring the photodynamic properties of two antiproliferative benzodiazopyrrole derivatives*", Int. J. Mol. Sci. 21(4), **1246** (2020) 10.3390/ijms21041246

- **F. Bajardi**, S. Capozziello, K. Dialektopoulos. "*Higher dimensional static and spherically symmetric solutions in extended Gauss-Bonnet gravity*", Symmetry **12**, no.3, 372 (2020) 10.3390/sym12030372

- R. Benedetti, **F. Bajardi**, L. Altucci, S. Capozziello, M. Del Sorbo, A. Nebbioso, M. Singh, H. Stunnenberg, M. Valadan, and C. Altucci "*Different Approaches to Unveil Biomolecule Configurations and Their Mutual Interactions*", Analytical Letters **54**, no.1-2, 40-56 (2019) 10.1080/00032719.2020.1716241.

- S. Capozziello and **F. Bajardi**, "*Gravitational waves in modified gravity*," Int. J. Mod. Phys. D **28**, no.05, 1942002 (2019) 10.1142/S0218271819420021

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Notation

We will set $\hbar = c = 8\pi G = 1$ unless otherwise indicated and we will use the following notation:

1. For the indexes:
   - Greek indexes $\{\alpha, \beta, \gamma ... = 0{,}1{,}2{,}3\} \rightarrow$ label the four dimensional curved space-time coordinates
   - Latin indexes $\{a, b, c ... = 0{,}1{,}2{,}3\} \rightarrow$ label the four dimensional flat space-time coordinates
   - Middle indexes $\{i, j, k ... = 1{,}2{,}3\} \rightarrow$ label the spatial coordinates
   - Symmetrization over the indexes will be indicated by the curly bracket, while anti-symmetrization by the square bracket
2. Let A_μ be a generic four-vector, we adopt the following:
   - $D_\nu A_\mu = A_{\mu;\nu}$ is the covariant derivative in terms of the Levi-Civita connection
   - $\partial_\nu A_\mu = A_{\mu,\nu}$ is the standard partial derivative
   - Christoffel connection will be indicated equivalently by $\Gamma^\alpha_{\beta\gamma}$ or $g^{\alpha\sigma}\{\sigma, \beta\gamma\}$
   - $\nabla_\nu A_\mu \rightarrow$ is the covariant derivative in terms of any connection except for the Levi-Civita connection.
3. We use the symbol $\mathscr{L}$ for Lagrangian density, while the Lagrangian will be denoted by $\mathcal{L}$.
4. For the Einstein tensor we use the notation $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$
5. The derivative with respect to the variable will be indicated by the subscript variable or sometimes by the subscript variable in the partial derivative

6. $\Box$ stands for the four-dimensional D'Alembert operator $\Box = g_{\mu\nu}\nabla^{\mu}\nabla^{\nu}$
7. $\mathcal{X}$ represents the generator of a certain symmetry, while $X = \mathcal{X} + \dot{\eta}^i\partial_{\dot{q}^i}$ is the Noether vector

The metric signature adopted is $(+,-,-,-)$.

We will introduce less important symbols during construction.

List of Acronyms

- ADM: *Arnowitt-Deser-Misner*
- CFT: *Conformal Field Theory*
- AdS: *Anti de Sitter*
- DEC: *Dominant Energy Condition*
- FLRW: *Friedmann-Lemaitre-Robertson-Walker*
- GR: *General Relativity*
- GW: *Gravitational Wave*
- IDGs: *Infinite Derivative Theories of Gravity*
- IKGs: *Integral Kernel Theories of Gravity*
- IR: *Infrared Light*
- NEC: *Null Energy Condition*
- PN: *Post-Newtonian*
- QFT: *Quantum Field Theory*
- RBD: *Receptor Binding Domain*
- SEC: *Strong Energy Condition*
- STEGR: *Symmetric Teleparallel Equivalent of General Relativity*
- TEGR: *Teleparallel Equivalent of General Relativity*
- UV: *Ultraviolet Light*
- WDW: *Wheeler-DeWitt*
- WEC: *Weak Energy Condition*

To those who inspired it and will not read it

Part I

INTRODUCTION AND PRELIMINARIES

1

Overview of General Relativity: Successes and Shortcomings

[1] The Hilbert-Einstein action, linear in the Ricci curvature scalar R, gives rise to the field equations of General Relativity (GR), which is the theory of gravity capable of fitting a huge amount of phenomena ranging from gravitational waves (GWs), astrophysical compact objects, black holes up to cosmology. At the astrophysical scales, GR soon obtained a great success after the observations of the light deflection, followed by the Radar Echo Delay and the exact estimation of the precession of the perihelion of Mercury in its orbit around the sun. The above mentioned successes come from the application of the theory to a spherically symmetric space-time of the form

$$ds^2 = P(r,t)^2 dt^2 - Q(r,t)^2 dr^2 - r^2 d\Omega^2, \tag{1.1}$$

with Ω being the two-sphere defined as $d\Omega^2 \equiv d\theta^2 - \sin^2\theta d\phi^2$. Once replacing the interval (1.1) in the Einstein field equations, it turns out that the only solution is

$$ds^2 = \left(1 - \frac{r_S}{r}\right) dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 - r^2(sin^2\theta d\phi^2 + d\theta^2), \tag{1.2}$$

which is static and contains two intrinsic singularities. One of them is an intrinsic divergence occurring for $r = 0$, due to the curvature generated by the compact object, at the

[1] In this Chapter we restore the Newton constant G_N, subsequently set to $1/8\pi$.

center of which any information is missed. The other singularity occurs when the radius is equal to the so called "Schwarzschild radius" r_S, defined as

$$r_S = 2G_N M, \tag{1.3}$$

with G_N being the Newton coupling constant and M the mass of the compact object. It can be shown that this latter singularity is coordinate-dependent, and can be deleted by means of an appropriate transformation (Kruskal-Szekeres coordinates). The plane $r = r_S$ is the "Event Horizon" and can be interpreted as the boundary beyond which events cannot affect an observer. The recent black hole image at the center of M87 galaxy, showed that these theoretical predictions are consistent with experimental observations [1].

The application of GR to homogeneous and isotropic space-times led to better understand the cosmological evolution crossed by the Universe, from the Big Bang to the Dust Matter Dominated Era. Using a cosmological perfect fluid with equation of state $p = \gamma\rho$, the Einstein field equations provide the solution

$$\frac{a}{a_0} = \left(\frac{t}{t_0}\right)^{\frac{2}{3(\gamma+1)}} \qquad \rho(t) = \left[6\pi G_N(1+\gamma^2)t^2\right]^{-1}, \tag{1.4}$$

where a spatially–flat Friedmann–Lemaitre–Robertson–Walker (FLRW) universe of the form

$$ds^2 = dt^2 - a(t)^2[dx_1^2 + dx_2^2 + dx_3^2], \tag{1.5}$$

must be considered to obtain Eq. (1.4). Depending on the value of γ, three different epochs can be identified:

- $\gamma = \frac{1}{3} \rightarrow$ Radiation fluids
- $\gamma = 1 \rightarrow$ Stiff matter fluids
- $\gamma = 0 \rightarrow$ Dust matter fluids

Experimental observations confirm that the evolution of the Universe went through dif-

ferent epochs, predicted by GR cosmology with high precision.
At the astrophysical scales, linearized Einstein field equations show that GR admits the presence of GWs propagating outward from their source at the speed of light. Specifically, considering a small perturbation $h_{\mu\nu}$ of the Minkowski flat metric tensor

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1\,, \tag{1.6}$$

a D'Alembert equation of the form

$$\Box h^{\mu}_{\nu} = -2\left(T^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}T\right)\,, \tag{1.7}$$

[2] can be obtained from the field equations, where T is the trace of the energy–momentum tensor T^{μ}_{ν}. In vacuum the above equation describes propagating waves at the speed of light. Using the TT gauge condition, the general solution reads:

$$\begin{gathered} h_{\mu\nu} = e^{+}_{\mu\nu}h_{+} + e^{\times}_{\mu\nu}h_{\times}\,, \\ e^{+}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad e^{\times}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{gathered} \tag{1.8}$$

According to the standard model, GWs in GR are described by spin-2 massless particles, with spin orientated in the same direction of motion.
For many years, GWs represented only a theoretical solution of field equations. In 2015, the Laser Interferometer Gravitational-Wave Observatory (LIGO) revealed a GW event (GW150914) and opened a new window in astrophysics and cosmology [2].
The GW production occurred during the merging of two black holes with masses of 29 $M_{\odot}$ and 36 $M_{\odot}$. The merging process produced a black hole of 62 $M_{\odot}$. The remaining (3 $M_{\odot}$) mass-energy was released in form of gravitational radiation. The observation gave a

[2] The operator $\Box$ is the D'Alembert operator defined as $D_{\mu}D^{\mu}$, with D_{μ} being the covariant derivative

double result: confirmed the existence of GWs and of stellar mass black holes.
After this first detection, several other events have been observed thanks to the LIGO-VIRGO collaboration, and further detections are expected in the forthcoming years.
When the cosmological constant is considered and dominating, the vacuum solution of the Einstein field equations in a FLRW space-time provides a scale factor of the form

$$a(t) = a_0 e^{\sqrt{\frac{\Lambda}{3}}\,t}, \tag{1.9}$$

denoting an exponentially accelerated universe.
The cosmological constant was introduced to explain the today observed accelerated cosmic expansion, physically interpreted as a form of energy which should represent the 68% of the Universe, called *Dark Energy.* The today accepted formulation of gravity, includes the cosmological constant as a fundamental component in the Einstein field equations, which therefore reads as:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G_N\, T_{\mu\nu}, \tag{1.10}$$

where $R_{\mu\nu}$ is the Ricci tensor, R the Ricci scalar, $g_{\mu\nu}$ the metric tensor and $T_{\mu\nu}$ the energy–momentum tensor of matter fields.
Those mentioned above are only a few part of the results gained by GR during more than one hundred years. In spite of all this, it also provided some results which disagree with experiments. For instance GR is not able to predict the right correlation between mass and radius of compact objects. Another example is given by the speed of the farest stars orbiting around the center of a given galaxy, which is experimentally lower than theoretically expected (see galaxy rotation curve problem [3]). To theoretically fix this issue, the missing matter was addressed to a fluid with zero pressure, called *Dark Matter*. It is supposed to represent the 26.8% of the Universe but has never been observed directly.

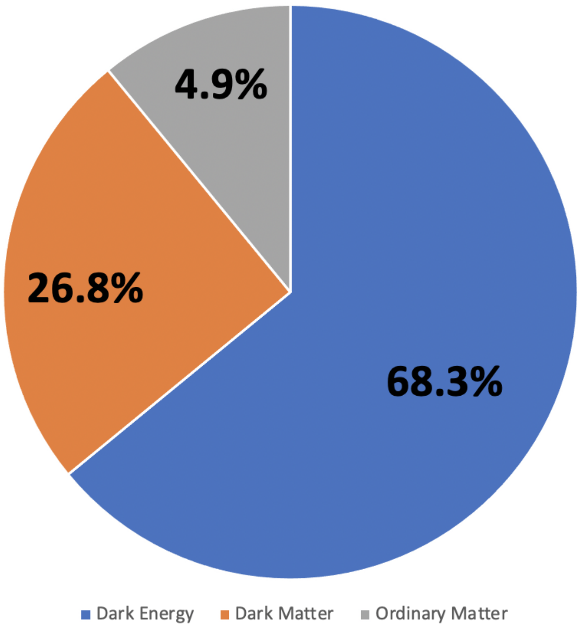

Figure 1: *Energy–matter content in the Universe.*

As the accelerated expansion cannot be predicted by GR without invoking the presence of Dark Energy, the galaxy rotation curve cannot be fitted without Dark Matter; however, even considering the cosmological constant, at the quantum level there is a discrepancy of 120 order of magnitude between the theoretically predicted value and the experimentally calculated one.

On the other hand, the "local" formulation of GR seems completely in disagreement with the intrinsic "non-locality" of quantum mechanics.

Quantum mechanics was the most revolutionary theory of the last century, which opened the doors to a completely new vision of physics at the high energy. The determinism of classical mechanics was replaced by a probabilistic interpretation of small-scale phenomena, which seemed to be the only way to fit all the experimental results. As we gained a theory capable of describing almost all the evidences provided by the quantum world, we lost the capability to exactly predict the time evolution of the system. Soon after, Quantum Field Theory (QFT) arose with the purpose to describe all the fundamental interactions under the same standard. It was soon clear that this prescription could not be applied to the gravitational interaction. Indeed, as quantum mechanics is probabilistic by nature, gravity is in turn described by Einstein's GR, where non-local interactions are not allowed. So far, a theory capable of describing both the large-scale structure and the Ultraviolet (UV) scale results is still missing. Moreover, neither QFT nor GR hold at the Planck scale, where a new physics is probably needed. On the one hand, despite all the experimental confirmations of quantum mechanics, we still miss its deep meaning; on the other hand, although GR is mathematically consistent and well developed, it presents some inconsistencies even at the large-scales. Any attempt to merge the formalism of GR with that of QFT have failed. Even though QFT in curved space-time addressed several evidences provided by the small-scales observations (such as Hawking Radiation, Unruh effect or cosmic inflation) it suffers several shortcomings. Indeed, it turns out that GR can be renormalized up to the second loop level [4], which means that incurable divergences arise once adapting the same scheme as QFT to gravity. In addiction, unlike the other fundamental interactions, GR cannot be treated under a Yang-Mills formalism, due to the lack of a Hilbert space and a probabilistic interpretation of the wave function. For these reasons, a coherent and self-consistent theory of quantum gravity is one of the most studied topic nowadays [5, 6, 7, 8, 9, 10, 11]. In the last few years, the quantum formalism was adapted to cosmology, where the dynamics can be reduced considering a minisuperspace of the variables. It represents a "toy model" which does not claim to be complete, but yields several important results in the understanding of the early-stage of our Universe [12, 13, 14, 15, 16].

2

Modified Theories of Gravity

Before introducing the main classes of modified theories of gravity it is useful to overview the state of art of GR modifications and the reasons why extending gravity. As mentioned above, while the electroweak and the strong interaction are Yang-Mills gauge theories, GR is invariant under diffeomorphism transformations, which involve coordinates instead of fields. Moreover, according to the geometric description of gravity, in view of a possible quantum scheme, the space-time metric should represent both a dynamical field and the background; this is not the case of other interactions, whose treatment is simplified by the assumption that the space-time is supposed to be flat. In 1988 Lasenby, Doran and Gull proposed to deal with the flat tangent space of the Riemannian manifold, treating GR as a gauge theory with respect to the local Lorentz group [17]; in order to pass from the curved to the flat space-time, a mathematical tool called *tetrad fields* is necessary, which in turn becomes the fundamental dynamical field. The formalism adopted is the so called Einstein–Cartan formalism [18, 19, 20, 21], where the connection is generally independent of the metric and the two-form curvature must be found through Cartan's structure equations [18, 22]. This implies that curvature can be used along with torsion to simultaneously label the space-time, so that the theory reduces to standard GR as soon as anti-symmetric part of the connection vanishes. This approach is not aimed at solving all the problems occurring in GR at the small scales regime, since even under an Einstein-Cartan formalism, several shortcomings are still suffered. As an example, neglecting the asymptotic safety scenario [23, 24, 25, 26], the theory can be renormalized only up to the one-loop level. At the large scales, early and late-time Universe acceleration cannot be predicted without introducing Dark Energy, as well as the galaxy rotation curve cannot be fitted without Dark Matter. In this framework, modified theories of gravity arose with the purpose of solving such shortcomings, by taking into account alternatives to the Hilbert–Einstein action. In the first instance, they can be distinguished in two main categories: purely metric theories and metric-affine theories. The former (which will be the main focus of this thesis) admits the metric tensor as the only fundamental filed. The latter disentangles the contribution of the metric from the affine connection, such that no relations between $\Gamma^{\alpha}_{\mu\nu}$ and $g_{\mu\nu}$ occur. This prescription is usually called "Palatini formalism" (see [27, 28, 29, 30] for basic foundations and applications). One of the most famous extensions of GR is the $f(R)$ gravity, which introduces into the action a function of the scalar curvature. Similarly, the $f(T)$ gravity considers a function of the so called *torsion scalar* into the action. However, the Hilbert–Einstein Lagrangian can be extended in several ways, such as introducing the coupling between geometry and scalar fields, higher-than-fourth order terms involving the D'Alembert operator $\Box^n$, or higher-order curvature invariants (as well as $R^{\mu\nu}R_{\mu\nu}$ or $R^{\mu\nu p\sigma}R_{\mu\nu p\sigma}$). All these theories can be treated either with respect to the purely metric or to the Palatini formalism. In this thesis, we assume the affine connection to be linked to the metric tensor, such that the corresponding field equation solutions can be uniquely determined by knowing the space-time line element. Specifically, we will focus on those modified theories of gravity somehow related to topological invariants, such us modified Gauss–Bonnet gravity and Chern–Simons gravity. An exhaustive treatment regarding other modified theories of gravity can be found in [31]. For specific discussions see *e.g.* [32, 33, 34, 35, 36] for $f(R)$ gravity, [37, 38, 39, 40, 41, 42] for $f(T)$ gravity, [43, 44, 45, 46] for scalar-tensor gravity, [47, 48, 49] for actions depending on second-order curvature scalars. Lagrangians of most modified theories of gravity contain unknown functions which cannot be directly constrained by experimental observations. Therefore, it comes natural wondering how to select the shape of the function among all possible choices.

One possible remedy, largely considered in the literature (see App. B), is to use a selection criterion aimed at finding actions containing symmetries. It is called *Noether Symmetry Approach* and will be particularly used in the second part of this work to select modified Gauss–Bonnet theories with symmetries. Specifically, as better pointed out in App. B, Noether theorem can be used as an approach to reduce the dynamics and find out exact analytic solutions of the field equations in modified gravity.

2.1 Brief Introduction on Modified Theories of Gravity

Modified theories of gravity, in some context, are capable of fixing GR inconsistencies, at infrared (IR) and UV scales. GR can be modified in several ways, depending on the scale and on the theoretical issues considered. As a matter of fact, GR does not account for the most general classical theory of gravity, but it relies on several assumptions. Most of them are motivated neither by experimental observations nor by strong theoretical reasons, but was introduced with the aim to construct a suitable theory leading to analytic solutions. It is beyond any discussion that the description of the gravitational interaction through the space-time geometry was perhaps the greatest intuition of XX century, and the consequent approval marked a turning point in the physical comprehension of phenomena. However, in order to gain such a predictive power and to obtain analytic results, many hypothesis was adopted; in what follows we analyze the main assumptions lying behind GR.

- Equivalence Principle and Symmetric Connection.

Let us first consider the assumption of symmetric connection, based on the requirement for the validity of the Equivalence Principle. The Weak Equivalence Principle affirms that there is no difference between gravitational field and accelerated systems, so that a free falling reference frame is completely equivalent to a system with no gravitational field. In other words, according to the weak Equivalence Principle, it is always possible to locally link the curved space-time to a flat tangent Minkowski space-time. The equivalence between the gravitational and the intertial mass is then automatically implied, and

nowadays several experiments confirm such equivalence with a precision of 1 part over 10^{14}. Despite this, it is just an assumption motivated by macroscopic observations, which surely holds at large scales, though it is still unclear whether it keeps being valid at Planck scales. In order to treat the gravitational interaction under the same standard as the other fundamental interactions and to construct a coherent theory of quantum gravity, such a precision of 10^{-14} makes a crucial difference, since admits the possibility that at higher scales the ratio m_g/m_I might pull rapidly away from 1.
As usual in GR, the form of the Christoffel connection can be found by imposing the "metricity condition" $D_\alpha g_{\mu\nu} = 0$, by means of which the following identity

$$D_\alpha g_{\mu\nu} + D_\nu g_{\alpha\mu} - D_\mu g_{\nu\alpha} = 0, \tag{2.1}$$

must hold. Considering the definition of the covariant derivative and assuming $\Gamma^\alpha_{[\mu\nu]} = 0$, it turns out that the only possible connection in GR is the Levi–Civita connection, that is:

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha p} \left(\partial_\mu g_{p\nu} + \partial_\nu g_{\mu p} - \partial_p g_{\mu\nu} \right). \tag{2.2}$$

However, once the metricity condition and the hypothesis of symmetric connection are relaxed, the same computation leads to a more general form of $\Gamma^\alpha_{\mu\nu}$, comprehending other non-trivial terms. It reads [50, 51, 52]:

$$\Gamma^\alpha{}_{\mu\nu} = \breve{\Gamma}^\alpha{}_{\mu\nu} + \frac{1}{2} g^{\alpha\lambda} \Big(T_{\mu\lambda\nu} + T_{\nu\lambda\mu} + T_{\lambda\mu\nu} \Big) + \frac{1}{2} g^{\alpha\lambda} \Big(-Q_{\mu\nu\lambda} - Q_{\nu\mu\lambda} + Q_{\lambda\mu\nu} \Big), \tag{2.3}$$

where $\breve{\Gamma}^\alpha{}_{\mu\nu}$ denotes the Levi-Civita connection and $Q_{\beta\mu\nu}$, $T^\alpha{}_{\mu\nu}$ are rank-three tensors defined as:

$$Q_{\beta\mu\nu} = \nabla_\beta g_{\mu\nu} \qquad T^\alpha_{\mu\nu} = 2\Gamma^\alpha_{[\mu\nu]}. \tag{2.4}$$

The formalism in which the metric is disentangled from the connection, so that this latter is no longer "metric compatible", is known as *Einstein-Cartan-Sciama-Kibble* formalism. Standard GR is recovered when $\Gamma^\alpha{}_{\mu\nu} = \breve{\Gamma}^\alpha{}_{\mu\nu}$, namely when both $T^\alpha_{\mu\nu}$ and $Q_{\beta\mu\nu}$ vanish. In order to show that the Equivalence Principle implies the presence of symmetric connection as the only possible connection, let us consider a generic covector A_μ and the rank-two tensor $D_\nu A_\mu$, namely the covariant derivative of A_μ. Requiring that A_μ can be recast as the derivative of a scalar field ϕ, namely $A_\mu = \partial_\mu \phi$, the quantity $D_{[\nu} A_{\mu]}$ reads as:

$$D_{[\nu} A_{\mu]} = -\Gamma^{\alpha}_{[\mu\nu]} \partial_\alpha \phi. \tag{2.5}$$

If the Equivalence Principle holds, than there exist a free falling reference frame where the covariant derivative turns into the standard partial derivative, so that the LHS must vanish. Being a rank-two tensor, the LHS must be null in all reference frames as well as the antisymmetric part of the Christoffel connection $\Gamma^{\alpha}_{[\mu\nu]}$. On the contrary, this latter must be taken into account in the general form of the connection, when the Equivalence Principle is not assumed to hold. Nevertheless, field equations arising from general Christoffel connections require other additional conditions to be solved. In this case the metric is not enough to uniquely determine the dynamics of the system, which needs to be further constrained by other external impositions. Reversing the argument, by choosing a metric compatible connection, the metric tensor turns out to be the only fundamental field needed to solve the field equations. The consequences of such an assumption do not have to be addressed to the theoretical side only, since neglecting the antisymmetric part of the connection is a strict *ansatz* also affecting the physical description of the space-time. When the rank-3 tensor $T^{\alpha}_{\mu\nu}$ is included in the general connection, torsion arises in the geometric description of the space-time. For this reason, $T^{\alpha}_{\mu\nu}$ is usually called *Torsion Tensor*. According to GR, torsion is not admitted as a component of the space-time, which is described by curvature only.

Similarly, by relaxing the metricity principle, according to which the covariant derivative of the metric tensor must vanish, the rank-3 tensor $Q^{\alpha}{}_{\mu\nu}$ arises in the Christoffel connection. As better pointed out in Sec. 2.5, this implies a space-time in which the norm of a vector changes while parallel transported along a closed path.

Therefore, if all the three terms in Eq. (2.3) are considered at the same level, it is possible to construct theories where the space-time is ruled by torsion or non-metricity (or both). The former theory is usually called *Teleparallel Equivalent of General Relativity* (TEGR), while the latter is the *Symmetric Teleparallel Equivalent of General Relativity* (STEGR).

- Second-Order Field Equations

Another assumption which is based neither on strong theoretical foundations nor on experimental observations is that of second-order field equations, according to which the Hilbert–Einstein action is chosen as linearly dependent on the scalar curvature. This choice is historically motivated by the requirement for the equivalence between gravitational and electromagnetic field equations. In principle, there are not strong theoretical motivations which impose the gravitational field equations to be of the second–order with respect to the metric. The only non-trivial four–dimensional action respecting the general covariance and leading to second–order field equations must be linear in the scalar curvature. It is the well known Hilbert–Einstein action, namely:

$$S_{H-E} = \frac{1}{2}\int \sqrt{-g} R \, d^4x. \tag{2.6}$$

By relaxing this hypothesis, several extensions of GR can be developed, with actions also involving higher-order geometric terms, like *e.g.* R^2, $R_{\mu\nu}R^{\mu\nu}$ or $R^{\mu\nu p\sigma}R_{\mu\nu p\sigma}$. More precisely, functions of such second–order curvature invariants can be generally considered, as well as the coupling between geometry and scalar fields. As an example, the action

$$S = \int \sqrt{-g} F(\phi, R, \Box^z R, R^{\mu\nu}R_{\mu\nu}, R^{\mu\nu p\sigma}R_{\mu\nu p\sigma}) \qquad z \in \mathbb{Z}, \tag{2.7}$$

depending on the scalar curvature R, on its D'Alembertian $\Box^z R$, on the higher–order terms $R_{\mu\nu}R^{\mu\nu}$, $R^{\mu\nu p\sigma}R_{\mu\nu p\sigma}$ and on a scalar field ϕ, leads generally to $2z+4$-th order field equations. Moreover, a Klein–Gordon-like equation can be obtained by varying the action with respect to the scalar field.

As pointed out in the previous section, though GR achieved several successes in the large scales, it manifests some shortcomings which call its validity into question. Therefore, it comes natural wondering whether GR could be the low–energy limit of a grand unified theory, which holds from the Planck scale up to the cosmological scale. Modified theories of gravity might represent a good candidate towards a final comprehension of the gravitational field, since most of them is capable of settling several puzzles at small and large scales. In what follows we discuss the most known GR modifications, sketching their main features and the most important results.

2.2 Curvature Extensions

The simplest extension of GR is the so called $f(R)$ gravity, and includes a function of the scalar curvature into the action:

$$S = \int \sqrt{-g} f(R)\, d^4x. \tag{2.8}$$

By varying the action with respect to the metric, one gets:

$$G_{\mu\nu} = \frac{1}{f_R(R)} \left\{ \frac{1}{2} g_{\mu\nu} \left[f(R) - R f_R(R) \right] + f_R(R)_{;\mu;\nu} - g_{\mu\nu} \Box f_R(R) \right\}, \tag{2.9}$$

being $G_{\mu\nu}$ the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ and $f_R(R)$ the first derivative of $f(R)$ with respect to R. Notice that when $f_R = 1$, Einstein field equations are recovered. By comparing Eq. (2.9) with GR field equations in vacuum (1.10), it is possible to interpret the quantity

$$\overline{T}^{GF}_{\mu\nu} = \frac{1}{f_R(R)} \left\{ \frac{1}{2} g_{\mu\nu} \left[f(R) - R f_R(R) \right] + f_R(R)_{;\mu;\nu} - g_{\mu\nu} \Box f_R(R) \right\}, \tag{2.10}$$

as an effective energy–momentum tensor of the gravitational field, so that the $f(R)$ field equations take the form $G_{\mu\nu} = \overline{T}^{GF}_{\mu\nu}$. In this way, the RHS can play the role of effective energy density and pressure, violating the energy conditions and leading to an accelerated expansion at the cosmological scales in the late-time. Therefore, as pointed out in the

the second part of the thesis, Dark Matter and Dark Energy are mimicked by geometric contributions, and no exotic matter is needed to fit the current astrometric data. In [53], for instance, quintessence model is addressed to geometry provided by the modification of the gravitational action.

Among the $f(R)$ extensions, great success was gained by the *Starobinsky* model, firstly treated in [54] and whose action is:

$$S = \frac{1}{2} \int \sqrt{-g} \left(R + \alpha R^2 \right) d^4x. \tag{2.11}$$

The above action was firstly considered in order to explain the evolution of the Universe in its early stage, without invoking the presence of dynamical scalar fields non–minimally coupled to gravity. Moreover, by means of the linearization of the metric tensor

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad h^{\nu}_{\mu} \ll \delta^{\nu}_{\mu}, \tag{2.12}$$

it turns out that the constant α can be understood as the mass of the GW [55]. To be more precise, by means of Eq. (2.12), the action (2.11) yields a D'Alembert equation of the form $(\Box + \alpha)h_{\mu\nu} = 0$, with $\Box \equiv g_{\mu\nu} D^{\mu} D^{\nu}$. As GR field equations are of the second order, $f(R)$ gravity leads generally to fourth–order field equations. However, extensions introduce new degrees of freedom due to which, often, analytic solutions cannot be found. Higher than fourth–order field equations can be obtained by further generalizing the action (2.8), including higher derivatives of the scalar curvature, namely:

$$S = \int \sqrt{-g}\, f(R, \Box R, \Box^2 R, ...\Box^k R)\, d^4x. \tag{2.13}$$

The variational principle (for $k > 0$) yields

$$
\begin{aligned}
G^{\mu\nu} = \frac{1}{\mathcal{M}} \Bigg\{ & \frac{1}{2} g^{\mu\nu}(F - \mathcal{M}R) + (g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\nu} g^{\lambda\sigma}) \mathcal{M}_{;\lambda;\sigma} \\
& + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{i} (g^{\mu\nu} g^{\lambda\sigma} + g^{\mu\lambda} g^{\nu\sigma}) (\Box^{j-1} R)_{;\sigma} \left(\Box^{i-j} \frac{\partial F}{\partial \Box^{i} R} \right)_{;\lambda} \\
& - g^{\mu\nu} g^{\lambda\sigma} \left[(\Box^{j-1} R)_{;\sigma} \left(\Box^{i-j} \frac{\partial F}{\partial \Box^{i} R} \right) \right]_{;\lambda} \Bigg\} ,
\end{aligned} \tag{2.14}
$$

being $\mathcal{M}$ defined as

$$
\mathcal{M} \equiv \sum_{j=0}^{k} \left(\Box^{j} \frac{\partial F}{\partial \Box^{j} R} \right) . \tag{2.15}
$$

The differential equations (2.14), which reduce to those of $f(R)$ gravity when $k = 0$, are of $2k + 4$-th order. These theories are usually considered in order to provide a solution for the renormalizability problem of gravity at IR scales. The coupling constants of higher–order terms in the geometry, have non-negative mass dimensions, which make the corresponding theory power-counting renormalizable. As a matter of facts, when the gravitational action is expanded around a gaussian fixed point, the coupling constant mass dimensions suggests whether the theory can be renormalized. In GR, the coupling constant turns out to have a negative mass dimensions, though the asymptotic safety argument endows a possible convergence at UV scales. Nonetheless, as mentioned at the beginning of this chapter, GR cannot be recast as a gauge invariant theory with respect to unitary groups, like other interactions. On the other hand, when $k < 0$, the D'Alembert operator turns into a propagator and the action is said to be *Non-Local* [56, 57, 58, 59]. Non-Local actions, give rise to renormalizable unitary theories also well fitting the cosmological predictions [60, 61, 62, 63, 64, 65].

2.3 Coupling Gravity to Scalar Fields

In this section we consider an extension of the Hilbert–Einstein action, including the non-minimal coupling between geometry and scalar fields. The first model was introduced by

A. Linde and A. Guth in [66, 67] to address the evidences provided by the cosmological data in the early time, when the inflationary phase was predominant. Inflation is usually thought as generated by a scalar field ϕ, called inflaton, which is supposed to be the responsible for the accelerated expansion of the Universe. According to this vision, inflaton should be made by scalar field driving the cosmic acceleration between 10^{-34} and 10^{-35} seconds after the initial expansion, generating a isotropic and homogeneous universe. The theory can also describe the production of particles after the early-time accelerated expansion (Reheating) [68, 69, 70].

The most general action including the coupling $f(\phi)$, the kinetic term $\omega(\phi)$ and the potential $V(\phi)$ is:

$$S = \int \sqrt{-g} \left\{ f(\phi)\; R + \frac{\omega(\phi)}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi) \right\} d^4x. \tag{2.16}$$

By varying the above action with respect to the metric one gets

$$f(\phi) G_{\mu\nu} = \frac{\omega(\phi)}{4} g_{\mu\nu}\; \phi^{;\alpha} \phi_{,\alpha} - \frac{\omega(\phi)}{2} \phi_{,\mu} \phi_{\nu} - \frac{1}{2} V(\phi) g_{\mu\nu} - g_{\mu\nu} \Box f(\phi) + f(\phi)_{,\mu,\nu}. \tag{2.17}$$

In the limits $\omega(\phi) = \phi^{-1}$, $f(\phi) = \phi$, the so called *Brans-Dicke* gravity is recovered. It is worth noticing that even in this case, by means of the definition

$$\begin{aligned} \overline{T}_{\mu\nu} = -\frac{1}{2f(\phi)} \quad & \left[-\frac{\omega(\phi)}{2} g_{\mu\nu}\; \phi^{;\alpha} \phi_{,\alpha} + \omega(\phi) \phi_{,\mu} \phi_{\nu} \right. \\ & \left. + V(\phi) g_{\mu\nu} + 2 g_{\mu\nu} \Box f(\phi) - 2 f(\phi)_{,\mu,\nu} \right], \end{aligned} \tag{2.18}$$

the field equations can be recast as $G_{\mu\nu} = \overline{T}^{GF}_{\mu\nu}$, with $\overline{T}^{GF}_{\mu\nu}$ being the effective energy–momentum tensor provided by the dynamical scalar field and the geometry. The variation of the action with respect to the scalar field ϕ provides the Klein-Gordon equation

$$2\omega(\phi) \Box \phi - f_{\phi}(\phi) R + V_{\phi}(\phi) + \omega_{\phi}(\phi) \partial^{\mu} \phi \partial_{\mu} \phi = 0. \tag{2.19}$$

Therefore, the scalar field can be intended as an effective space-time dependent Newton's constant, whose value depends on the space-time point considered. Coupling gravity to a scalar field can also describe the evolution of the Universe across different epochs, from the very early stage up to the late time.
Moreover, it can be shown that fourth–order theories provide the same dynamics as second–order theories non–minimally coupled to a scalar field. In particular, the former can be addressed to the latter (and *vice versa*) by means of conformal transformations.

2.4 Gravity as a Theory of Translation Group: Teleparallel Gravity

In the previous sections we discussed some possible modifications of GR, which extend the Hilbert–Einstein action introducing functions of second–order curvature invariants. All these theories, share the common feature of having the same structure as GR, maintaining the general covariance and the invariance under diffeomorphism transformations. They can be classified as *extensions* of GR and the Hilbert–Einstein action can be exactly recovered under some limits. In this section we introduce the main aspects of a theory of gravity which cannot be thought as an extension of GR, but describes the gravitational interaction from a different point of view. At the beginning of this chapter we mentioned the possibility of discarding the Levi–Civita and the non-metricity contributions in the general Christoffel connection (2.3), giving rise to a theory which describes the space-time through torsion instead of curvature, the so called TEGR.
Specifically, by means of the definitions

$$K^{\rho}{}_{\mu\nu} \equiv \frac{1}{2} g^{\rho\lambda}\Big(T_{\mu\lambda\nu} + T_{\nu\lambda\mu} + T_{\lambda\mu\nu}\Big), \tag{2.20}$$

$$S^{p\mu\nu} \equiv K^{\mu\nu p} - g^{p\nu} T^{\sigma\mu}{}_{\sigma} + g^{p\mu} T^{\sigma\nu}{}_{\sigma}\,, \tag{2.21}$$

$$T \equiv T^{p\mu\nu} S_{p\mu\nu}\,, \tag{2.22}$$

it turns out that the actions

$$S_{GR} \equiv \frac{1}{2}\int d^4x\,\sqrt{-g}\,R, \tag{2.23}$$

$$S_{TEGR} \equiv \frac{1}{2}\int d^4x\,\sqrt{-g}\,T\,, \tag{2.24}$$

differ from each other only by a four-divergence. In this case the general connection labeling the geometry is

$$\Gamma^{\alpha}{}_{\mu\nu} = K^{\alpha}{}_{\mu\nu}, \tag{2.25}$$

where $K^{\alpha}{}_{\mu\nu}$ is the *Contorsion Tensor* and $S^{p\mu\nu}$ the *Superpotential*.
The same result can be obtained by considering local translations in a flat space-time, so that the gravitational interaction is understood as a gauge theory of translation group in the locally flat tangent space-time. The flat and the curved space-times can be linked by means of a mathematical tool called *tetrad fields*. They are rank-2 tensor with mixed indexes, belonging to the anholonomic frame (flat space-time) and to the holonomic frame (curved space-time), respectively. They can be easily introduced by considering a coordinate transformation $x^\mu \to \tilde{x}^\mu$, by means of which the metric tensor varies as:

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \partial_\mu \tilde{x}^\alpha \partial_\nu \tilde{x}^\beta g_{\alpha\beta}. \tag{2.26}$$

Imposing the transformed reference frame to be the Minkowski space-time, namely $\tilde{g}_{\mu\nu} = \eta_{ab}$, the above transformation takes the form:

$$\eta_{ab} = \partial_a \tilde{x}^\alpha \partial_b \tilde{x}^\beta g_{\alpha\beta} \tag{2.27}$$

and, equivalently

$$g_{\mu\nu} = \partial_\mu \tilde{x}^a \partial_\nu \tilde{x}^b \eta_{ab}. \tag{2.28}$$

By defining the tetrad fields as

$$e^a_\mu = \partial_\mu \tilde{x}^a, \tag{2.29}$$

the relations below naturally follow:

$$e^a_\mu e^\mu_b = \delta^a_b \; ; \qquad e^a_\mu e^\nu_a = \delta^\nu_\mu. \tag{2.30}$$

Thanks to tetrad fields, any quantity in the Riemannian manifold can be projected into the flat tangent space-time. Tetrad fields can be used to formally recast TEGR as a gauge theory of translation group.
In the reference frame where the spin connection vanishes, it turns out that the torsion tensor (2.4) can be written in terms of tetrad fields as:

$$T^p{}_{\mu\nu} = (\Gamma^p_{\mu\nu} - \Gamma^p_{\nu\mu}) = (e^p_a \partial_\mu e^a_\nu - e^p_a \partial_\nu e^a_\mu). \tag{2.31}$$

The TEGR connection (2.25), therefore, becomes:

$$\Gamma^\alpha_{\mu\nu} = e^\alpha_a \partial_\mu e^a_\nu \tag{2.32}$$

and is called Weitzenbock connection. When the general Christoffel connection is chosen such that the only contribution is provided by the contorsion tensor, the Riemann curvature can be written as:

$$\mathcal{R}^\lambda{}_{p\nu\mu} = \nabla_\nu K^\lambda{}_{p\mu} - \nabla_\mu K^\lambda{}_{p\nu} + K^\lambda{}_{\sigma\nu} K^\sigma{}_{p\mu} - K^\lambda{}_{\sigma\mu} K^\sigma{}_{p\nu}. \tag{2.33}$$

More generally, when also the Levi–Civita connection is included, the total curvature takes the form

$$R^\lambda{}_{p\nu\mu} = \breve{R}^\lambda{}_{p\nu\mu} + \mathcal{R}^\lambda{}_{p\nu\mu}, \tag{2.34}$$

with $\breve{R}^\lambda{}_{p\nu\mu}$ being the standard Riemann tensor written in terms of the Levi–Civita connection $\breve{\Gamma}^p{}_{\mu\nu}$. The teleparallel action can be constructed by means of the definitions (2.20), (2.21), (2.22). When TEGR is thought as a gauge theory of translation group, the determinant of tetrad fields e is comprehended into the action, so that this latter reads as:

$$S = \frac{1}{2}\int e\, T\, d^4x. \tag{2.35}$$

Moreover, from Eq. (2.34) it is possible to show that the standard scalar curvature and the torsion scalar satisfy the relation

$$\breve{R} - \frac{2}{e}\partial_\mu (e\, T^{\nu\mu}{}_{\nu}) = -T, \tag{2.36}$$

namely they only differ for a boundary term. For instance, in a cosmological spatially flat background, the Torsion Scalar T reads as:

$$T = -6\frac{\dot{a}^2}{a^2}. \tag{2.37}$$

Therefore, the teleparallel action can be recast in terms of the Hilbert–Einstein action S_{GR} as:

$$S_{TEGR} = -S_{GR} + 2\int \partial_\mu (e\, T^{\nu\mu}{}_{\nu})\, d^4x. \tag{2.38}$$

This implies that S_{TEGR} and S_{GR} formally yield the same equations of motion. In the first order formalism, where the Lagrangian depends on the fields and on their first derivatives, the Euler–Lagrange equations with respect to the tetrad fields, *i.e.*

$$\frac{\partial \mathscr{L}_{TEGR}}{\partial e^a_p} - \partial_\sigma \frac{\partial \mathscr{L}_{TEGR}}{\partial(\partial_\sigma e^a_p)} = 0, \tag{2.39}$$

yield in vacuum:

$$\frac{4}{e}\partial_\mu \left(e\, S_a{}^{\mu\beta}\right) - 4T^\sigma{}_{\mu a} S_\sigma{}^{\beta\mu} - T e^\beta_a = 0. \tag{2.40}$$

When the contorsion tensor is included as a fundamental part of the Christoffel connection, the space-time turns out to be labeled by torsion. This means that when a vector is parallel transported around a closed path, its final position will be shifted with respect to the initial one. If also curvature is comprehended, the final vector results shifted and rotated.

The boundary term in Eq. (2.36) does not play any role in the dynamics provided by the standard teleparallel action. However, extensions of TEGR consider the boundary term as a fundamental degree of freedom which allows to exactly recover $f(R)$ gravity (see *e.g.* [71, 72]).

In this context, similarly to the extensions of GR, TEGR can be extended in several ways, *e.g.* by including into the action a function of the torsion and of the boundary term $f(T, B)$, non-local terms [73], non-minimally coupled scalar fields [45], second-order torsion invariants [74], or higher-order terms [75].

2.5 Symmetric Teleparallel Equivalent of General Relativity

So far, we discussed modifications of Einstein GR including the anti symmetric part of the Christoffel connection, or extra terms in the gravitational action. In this section we overview the mathematical and physical aspects of theories not respecting the metricity principle. We show that a theory described only by non-metricity, without torsion and curvature, is formally equivalent to GR and TEGR at the level of equations. This theory is the previously defined STEGR, in which the general connection (2.3) is reduced to

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda} \Big(-Q_{\mu\nu\lambda} - Q_{\nu\mu\lambda} + Q_{\lambda\mu\nu} \Big) \equiv L^{\alpha}{}_{\mu\nu}, \tag{2.41}$$

where $Q_{\beta\mu\nu}$ is defined as

$$Q_{\beta\mu\nu} = \nabla_{\beta} g_{\mu\nu} \tag{2.42}$$

and where the rank–three tensor $L^{\alpha}{}_{\mu\nu}$ is called *Disformation Tensor*. By introducing the quantities

$$
\begin{aligned}
&Q \equiv -\frac{1}{4}Q_{\alpha\mu\nu}\left[-2L^{\alpha\mu\nu}+g^{\mu\nu}\left(Q^{\alpha}-\tilde{Q}^{\alpha}\right)-\frac{1}{2}\left(g^{\alpha\mu}Q^{\nu}+g^{\alpha\nu}Q^{\mu}\right)\right],\\
&Q_{\mu} \equiv Q_{\mu}{}^{\lambda}{}_{\lambda}\,,\\
&\tilde{Q}_{\mu} \equiv Q_{\alpha\mu}{}^{\alpha}\,,
\end{aligned}
$$

it turns out that the action

$$
\mathcal{S}_{STEGR} \equiv \frac{1}{2}\int d^4x\,\sqrt{-g}\,Q+\mathcal{S}^{(m)}\,, \tag{2.43}
$$

leads to the same field equations as the GR and TEGR actions. This means that the torsion scalar T, the non–metricity scalar Q and the Ricci scalar R differ from each other for boundary terms. GR and STEGR can be straightforwardly compared by considering the connection

$$
\Gamma^{\alpha}{}_{\mu\nu} = \breve{\Gamma}^{\alpha}{}_{\mu\nu} + L^{\alpha}{}_{\mu\nu}, \tag{2.44}
$$

which comprehends the Levi–Civita connection and the Disformation tensor. Moreover, the GR Ricci scalar $\breve{R}$ and the non–metricity scalar can be linked through the relation

$$
\breve{R} = -Q - \nabla_{\alpha}\left(Q^{\alpha}+\tilde{Q}^{\alpha}\right). \tag{2.45}
$$

Further details about non-metric theories can be found *e.g.* in [76, 77, 78].
Notice that the same procedure as the previous section cannot be applied to STEGR, which cannot be thought as a gauge theory in the locally flat tangent space-time. As the torsion and the curvature are respectively the gauge fields associated to local translation and to Lorentz transformations, no gauge transformation can be associated to non–metricity. This formally yields a fundamental difference among the three theories, although they lead to the same equations of motion.

2.6 The Geometric Trinity of Gravity

Taking into account the results achieved in previous sections, here we introduce the so called *trinity of gravity*. We showed that TEGR and STEGR are totally equivalent to GR up to a boundary term in the starting Lagrangian. They can be obtained from the general Christoffel connection

$$\Gamma^{\rho}{}_{\mu\nu} = \breve{\Gamma}^{\rho}{}_{\mu\nu} + K^{\rho}{}_{\mu\nu} + L^{\rho}{}_{\mu\nu} \,. \tag{2.46}$$

Specifically:

$$\begin{aligned} &\mathrm{GR} \rightarrow L^{\rho}{}_{\mu\nu} = K^{\rho}{}_{\mu\nu} = 0 \,, \\ &\mathrm{TEGR} \rightarrow \breve{\Gamma}^{\rho}{}_{\mu\nu} = L^{\rho}{}_{\mu\nu} = 0 \,, \\ &\mathrm{STEGR} \rightarrow \breve{\Gamma}^{\rho}{}_{\mu\nu} = K^{\rho}{}_{\mu\nu} = 0 \,. \end{aligned} \tag{2.47}$$

It follows that the most general form of the scalar curvature, written in terms of the general connection (2.46) is

$$\begin{aligned} R \;=\; & \breve{R} + \left(K^{\rho}{}_{\nu\mu} + L^{\rho}{}_{\nu\mu}\right)\left(K^{\beta}{}_{\beta\rho} + L^{\beta}{}_{\beta\rho}\right) g^{\nu\mu} + \\ & - \left(K^{\rho}{}_{\beta\mu} + L^{\rho}{}_{\beta\mu}\right)\left(K^{\beta}{}_{\nu\rho} + L^{\beta}{}_{\nu\rho}\right) g^{\nu\mu} + \\ & + \nabla_{\beta}\left[\left(K^{\beta}{}_{\nu\mu} + L^{\beta}{}_{\nu\mu}\right) g^{\nu\mu} - \left(K^{\nu}{}_{\nu\mu} + L^{\nu}{}_{\nu\mu}\right) g^{\nu\mu}\right] , \end{aligned} \tag{2.48}$$

with $\breve{R}$ being the standard Ricci scalar. When the space-time is labeled by the Levi–Civita connection, the dynamics is ruled by the curvature, while torsion and non–metricity vanish everywhere. In this case, the parallel transport of a given vector around a closed path yields a vector with same norm but different orientation. An angular displacement therefore occurs due to the curvature of the manifold, as showed in Fig. **2**.

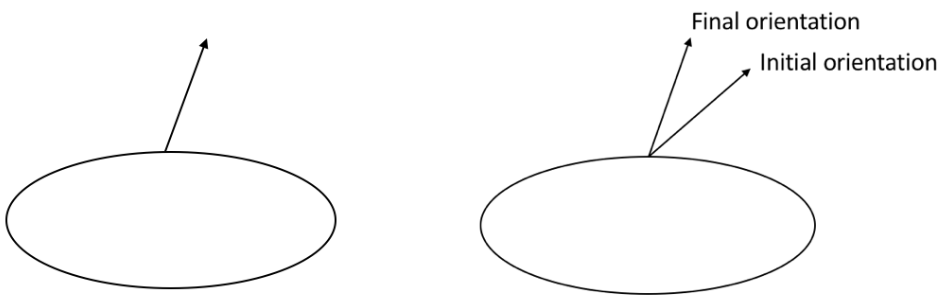

Figure 2: *Comparison between a closed path in the flat space-time and in the curved space-time.*

When only the contorsion tensor occurs in the general connection, the parallel transport yields a shift of the vector after performing a closed path. The result is a radial displacement between the starting and the final point.

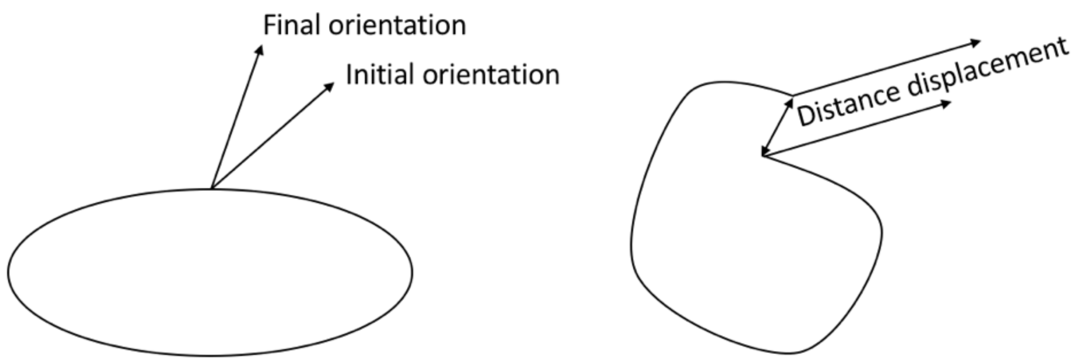

Figure 3: *Comparison between a closed path in the flat space-time and in a space-time with torsion.*

Finally, when both curvature and torsion vanish, non-metricity describes a space-time in which a propagating vector continuously changes its modulus.

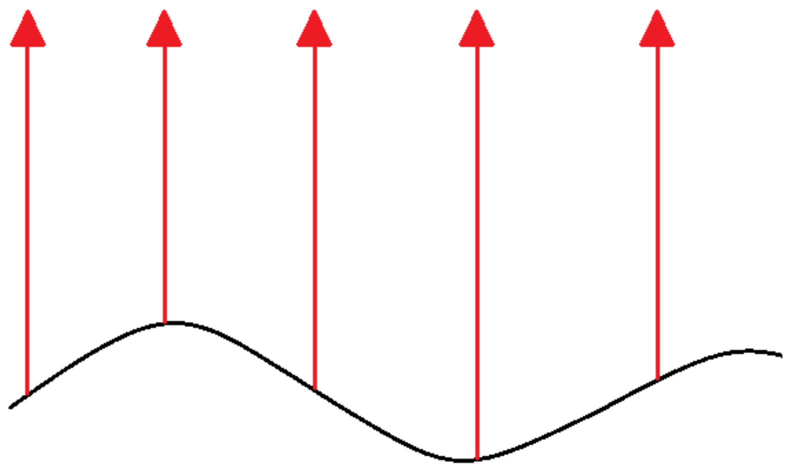

Figure 4: *Space-time with non-metricity: a propagating vector field maintains the same direction but changes its length.*

As pointed out in [79], non-metricity, torsion and curvature can be thought as a geometric *Trinity* of gravity, since all of them leads to totally equivalent theories by different formalism. Despite this, $f(R)$ gravity is quite different with respect to $f(T)$ and $f(Q)$ extensions, where the boundary terms B_Q and B_T play a non-trivial role in the dynamics. In Fig. **5**, a useful scheme of all possible non-extended theories with curvature, torsion and non-metricity is outlined.

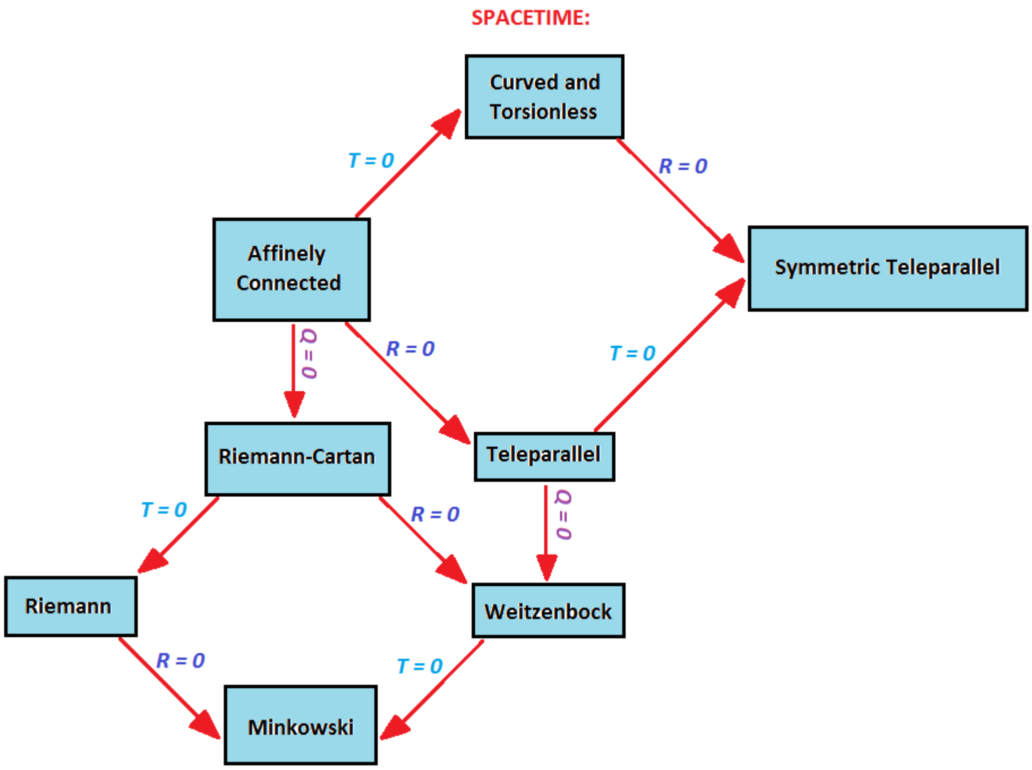

Figure 5: *Classification of alternative theories containing curvature, torsion or non-metricity.*

To conclude this chapter, several extended and alternative theories of gravity arose after GR was formulated, and most of them is capable of providing solutions for GR shortcomings at large scales (see *e.g.* [33, 80, 81, 82, 83, 84] for astrophysical scales, [71, 85, 86, 87] for cosmological scales).

Many other approaches have been developed to address small scales problems, such as *String Theory* [88, 89, 90, 91, 92], *Kaluza-Klein Theory* [93, 94], *Loop Quantum Gravity* [95, 96, 97], *Horava-Lifshitz Gravity* [98, 99, 100, 101], *Non-Local Gravity* [102, 103, 104] *etc.* While GR turns out to be non-renormalizable from the two-loop level, the above mentioned theories are both renormalizable and unitary.

In this thesis, the introduction of modified theories of gravity represents a fundamental step towards a full comprehension of Gauss–Bonnet (treated in Part II) and Chern–Simons theories (treated in Part III). Both indeed are alternatives to GR, which aim to fix the low and high energy problems suffered by Einstein's theory. The former considers a function of a topological surface term into the action, while the latter is an odd-dimensional theory based on gauge-invariant Lagrangians coming from topological invariants.

Part II

MODIFIED GAUSS–BONNET THEORY

3

Introduction to Part II: Gauss–Bonnet Theory

This chapter is devoted to the introduction of the Gauss–Bonnet theory of gravity, mainly investigated throughout the second part of the thesis. In Chap. 2 we introduced modified theories of gravity, outlining extensions and modifications of the gravitational action. We mentioned that the assumption of second-order field equations can be relaxed, by introducing other curvature invariants. GR assumes that the Riemann tensor must be contracted with the metric tensor such that the starting action is

$$S_{H-E} = \frac{1}{2}\int \sqrt{-g}\, g^{\mu p} g^{\nu\sigma} R_{\mu\nu p\sigma}\, d^4x. \tag{3.1}$$

In this way, several other curvature scalars are neglected, which in principle may contribute to the dynamics at the same level as the scalar curvature. For instance, by means of the definitions

$$P \equiv g^{\mu p} R^{\nu\sigma} R_{\mu\nu p\sigma} \qquad Q \equiv R^{\mu\nu p\sigma} R_{\mu\nu p\sigma}, \tag{3.2}$$

the general action

$$S = \int \sqrt{-g}\, f(R, P, Q)\, d^4x, \tag{3.3}$$

can be considered. The corresponding field equations read as:

$$
\begin{aligned}
& f_R(R,P,Q)G_{\mu\nu} = \left[\frac{1}{2}g_{\mu\nu}f(R,P,Q) - Rf_R(R,P,Q)\right] \\
- \quad & \left(g_{\mu\nu}\Box - D_\mu D_\nu\right) f_R(R,P,Q) \\
- \quad & 2\left[f_P(R,P,Q)R^\alpha_\mu R_{\alpha\nu} + f_Q(R,P,Q)R_{p\sigma\alpha\mu}R^{p\sigma\alpha}{}_\nu\right] \\
- \quad & g_{\mu\nu}D_p D_\sigma\left[f_P(R,P,Q)R^{p\sigma}\right] - \Box\left[f_P(R,P,Q)R_{\mu\nu}\right] \\
+ \quad & 2D_\sigma D_p\left[f_P(R,P,Q)R^p_{\{\mu}\delta^\sigma_{\nu\}} + 2f_Q(R,P,Q)R^p{}_{\{\mu\nu\}}{}^\sigma\right],
\end{aligned}
\tag{3.4}
$$

where $\{\}$ stands for the anti-commutator and where f_P and f_Q denote the derivative of f with respect to P and Q, respectively. When $f_P = f_Q = 0$, $f(R)$ field equations in Eq. (2.9) are restored. Among all the possible combinations of R, P and Q, the only action leading to ghost-free modes is:

$$
S = \int \sqrt{-g}\left(R^2 - 4P + Q\right) d^4x. \tag{3.5}
$$

The quantity $\mathcal{G} \equiv R^2 - 4P + Q$ is called *Gauss–Bonnet scalar* and is of particular interest in the context of modified theories of gravity. In four dimensions $\mathcal{G}$ is a topological surface term which, according to the generalized Gauss–Bonnet theorem, integrated over the manifold provides a topological invariant. Specifically, the four–dimensional representation of the Gauss–Bonnet scalar is the Euler density, so that the identity

$$
\int_{\mathcal{M}} \sqrt{-g}\,\mathcal{G}\, d^4x = \chi(\mathcal{M}) \tag{3.6}
$$

holds, being $\chi(\mathcal{M})$ the Euler characteristic. Assuming gravity as a gauge theory of the local Lorentz group on the tangent bundle, the Gauss–Bonnet term can be written as:

$$
\mathcal{G} = \epsilon_{a_1,a_2,a_3....a_n} R^{a_1,a_2} \wedge R^{a_3,a_4} \wedge e^{a_5} \wedge ... \wedge e^{a_n}, \tag{3.7}
$$

being R^{a_i,a_j} the two form curvature, e^k the set of zero forms defining the basis and $\epsilon_{a_1,a_2,a_3....a_n}$ the Levi-Civita symbol. This Gauss–Bonnet term is part of the D-dimensional Lovelock Lagrangian [105, 106] which, in four dimensions, can be expressed as:

$$\mathscr{L}^{(4)} = \epsilon_{abcd} \left[\alpha_2 R^{ab} \wedge R^{cd} + \alpha_1 R^{ab} \wedge e^c \wedge e^d + \alpha_0 e^a \wedge e^a \wedge e^b \wedge e^c \wedge e^d \right] \ , \tag{3.8}$$

where the first term is the Gauss–Bonnet invariant, the second is the Ricci scalar and the third is the cosmological constant.

Despite the impossibility of dealing with the Gauss–Bonnet scalar in four dimensions, considering any function $f(\mathcal{G}) \neq \mathcal{G}$ can be mathematically and physically relevant, also in 4D. In general, $f(R, \mathcal{G})$ gravity is taken into account to recover GR, in a given limit, often assuming a starting action of the form

$$S = \int \sqrt{-g} \left(\frac{1}{2} R + f(\mathcal{G}) \right) d^4x. \tag{3.9}$$

GR is thus safely recovered when $f(\mathcal{G}) = 0$, so that the function plays the role of an effective cosmological constant given by curvature. The above action has been extensively studied. In [107], cosmologically viable models are considered by studying the stability of a late-time de-Sitter solution and the existence of radiation and matter epochs. In [108], possible power-law scaling solutions have been taken into account, by developing the scalar-tensor equivalent of the above theory. In particular, in [109], authors show that density perturbations cause instabilities. In [110], the author shows that the above theory is ruled out as a possible explanation of the late-time acceleration by Solar System tests. In [111], the Gauss–Bonnet term is added to a $f(R)$ five-dimensional Lagrangian and a static spherically symmetric solution is studied. In [112] the authors study the energy bounds for Gauss–Bonnet gravity in an AdS_7 background. Finally in [113], a mimetic version of the above theory is considered and solutions that unify the inflation era together with Dark Energy are provided. In addition, Dark Matter can be described in the framework of this model. In [114], the Newtonian and Post-Newtonian (PN) limit of (3.9) is studied in detail.

For almost half a century, higher dimensional theories of gravity have been studied in many different contexts in the literature [115]. The aforementioned puzzling phenomena in gravity can sometimes be explained by invoking extra dimensions [116, 117, 118, 119]. Braneworlds and other higher dimensional modifications of Einstein's theory, *e.g.* Lovelock theory [120] have been considered as possible extensions in the hunt for a self-consistent theory of gravity.

All of the above researches deal with a theory that safely recovers GR in the background or in some limit. This means that if one switches off the effect of the Gauss–Bonnet contribution, i.e. $f(\mathcal{G}) \to 0$, then the action reduces to the Hilbert–Einstein one and GR is recovered. This happens because GR has to be restored in view of observations and experimental tests.

Observational and theoretical constraints have been obtained also for other forms of $f(R, \mathcal{G})$, but pure $f(\mathcal{G})$ theories are not, in general, considered because GR seems excluded. However, it turns out that GR can be restored as a particular case of $f(\mathcal{G})$ gravity and the further degrees of freedom related to $R_{\mu\nu}$ and $R^{\alpha}{}_{\lambda\mu\nu}$ can be neglected with respect to R. This happens if particular symmetries are adopted like in homogeneous and isotropic cosmology or in other specific cases. The action only containing a function of $\mathcal{G}$ can be helpful in reducing the dynamics and analytically solve the equations of motion. For instance, in some cosmological context, the second order curvature invariants P and Q are comparable to R^2, so that $\sqrt{\mathcal{G}} \sim \sqrt{R^2}$, up to a constant factor. Therefore, $\mathcal{G}$ and R^2 can be considered dynamically equivalent on the solutions (up to a constant factor) if homogeneity and isotropy hold [47]. In a spherically symmetric configuration, as pointed out in Chap. 5, exact solutions can be found in any dimensions.

Moreover, as showed in [55], the action (3.3) yields new polarization spin-2 modes in the GWs, though the linearization of the metric tensor in the $f(\mathcal{G})$ theory yields the same dispersion relation as GR [121]. It means that the $f(\mathcal{G})$ Gauss–Bonnet gravity is the only modified action where no ghost modes occur, but only the spin-2 massless modes of GR and the scalar mode.

Therefore, the general Gauss–Bonnet gravity theory discussed in this part is given by the action

$$S = \int \sqrt{|g|} f(\mathcal{G})\, d^{d+1}x \,, \tag{3.10}$$

where d labels the spatial dimensions. The above action starts being topologically trivial in three dimensions or less, so that $f(\mathcal{G})$ gravity may represent a valuable alternative to Einstein GR to address issues at small and large scales.

By varying Eq. (3.10) with respect to the metric, we get the field equations

$$\begin{aligned}
&-\left(2RR_{\mu\nu} - 4R_{\mu p}R^{p}{}_{\nu} + 2R_{\mu}{}^{p\sigma\tau}R_{\nu p\sigma\tau} - 4R^{\alpha\beta}R_{\mu\alpha\nu\beta}\right) f_{\mathcal{G}}(\mathcal{G}) \\
&+\left(2RD_\mu D_\nu + 4G_{\mu\nu}\Box - 4R^{p}_{\{\nu}D_{\mu\}}D_p + 4g_{\mu\nu}R^{p\sigma}D_pD_\sigma \right.\\
&\left. -4R_{\mu\alpha\nu\beta}D^\alpha D^\beta\right) f_{\mathcal{G}}(\mathcal{G}) + \frac{1}{2}g_{\mu\nu}f(\mathcal{G}) = 0 \,,
\end{aligned} \tag{3.11}$$

where $f_{\mathcal{G}}$ is the derivative of f with respect to $\mathcal{G}$. It is worth considering also the trace of Eq. (3.11), that is

$$\left(\frac{d+1}{2}\right) f(\mathcal{G}) - 2\mathcal{G} f_{\mathcal{G}}(\mathcal{G}) - 2(d-2)\left(R\Box - 2R^{\mu\nu}D_\mu D_\nu\right) f_{\mathcal{G}}(\mathcal{G}) = 0 \,. \tag{3.12}$$

This can be seen as the equation of motion for the new scalar degree of freedom introduced in this theory.

As mentioned above, in four dimensions (*i.e.* $d = 3$), a linear term in $\mathcal{G}$ is trivial because, as a topological invariant, it turns into a surface term and the related integral is null. Despite this, cosmological and spherically symmetric solutions can be valid even in four dimensions, if the Gauss–Bonnet coupling constant diverges when $d = 3$. This idea was recently proposed *e.g.* in Refs. [122, 123], where the authors deal with a four–dimensional theory in which the Gauss–Bonnet term contributes to the dynamics.

Nonetheless, as in four dimensions GR perfectly fits the current observations at Solar

System scales, in more dimensions the Gauss–Bonnet term might provide a description for a higher-dimensional universe.

This second part of the thesis is organized as follows: in Chap. 4 cosmological aspects of modified Gauss–Bonnet gravity are treated and exact solutions are found for selected models. The same function is used in Sec. 4.3 to study the energy conditions and the slow–roll inflation. The application to a spherically symmetric background is investigated in Chap. 5, where black hole solutions in higher dimensions are provided.

In Chap. 6 non-local Gauss–Bonnet theories of gravity are analyzed. Finally, in Chap. 7, a comparison among metric and affine scalar-tensor theories of gravity is pursued. The conclusive remarks of this part are finally relied to Chap. 8

4

$f(\mathcal{G})$ Cosmology

In this chapter we discuss the results provided by $f(\mathcal{G})$ cosmology. To this purpose we deal with the covariant representation of $\mathcal{G}$, which is given by Eq. (3.5). More precisely, we consider a general analytic function of $\mathcal{G}$ and the action

$$S = \int \sqrt{-g} f(\mathcal{G})\, d^{d+1}x + S^m, \tag{4.1}$$

with S^m being the matter action. By varying Eq. (4.1) with respect to the metric, we find the following field equations

$$\begin{aligned} &2RD_\mu D_\nu f_\mathcal{G}(\mathcal{G}) - 2g_{\mu\nu} R\Box f_\mathcal{G}(\mathcal{G}) - 4R^\lambda_\mu D_\lambda D_\nu f_\mathcal{G}(\mathcal{G}) + 4R_{\mu\nu}\Box f_\mathcal{G}(\mathcal{G}) \\ &+4g_{\mu\nu}R^{p\sigma} D_p D_\sigma f_\mathcal{G}(\mathcal{G}) + 4R_{\mu\nu p\sigma} D^p D^\sigma f_\mathcal{G}(\mathcal{G}) + \frac{1}{2} g_{\mu\nu}[f(\mathcal{G}) - \mathcal{G} f_\mathcal{G}(\mathcal{G})] = T_{\mu\nu}, \end{aligned} \tag{4.2}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of matter fields. In [114, 124, 125, 126], one can found the generalization of this action to $f(R, \mathcal{G})$ gravity in 4-dimensions.

In order to obtain the form of the Gauss–Bonnet scalar in cosmology, we have to calculate the $d+1$-dimensional Riemann tensor, Ricci tensor and Ricci scalar in a cosmological metric. We choose a spatially flat FLRW line element of the form

$$ds^2 = dt^2 - a(t)^2 \delta_{ij} dx^i dx^j\,, \tag{4.3}$$

where the indexes i, j label all the spatial dimensions and run from 1 to d. The not-null curvature components are:

$$R_{00} = d\frac{\ddot{a}}{a}\,, \qquad R_{ij} = \Big[(d-1)\dot{a}^2 + a\ddot{a}\Big]\,\delta_{ij}\,,$$
$$R = -2d\frac{\ddot{a}}{a} - d(d-1)\frac{\dot{a}^2}{a^2}\,, \qquad R_{0i0j} = a\ddot{a}\delta_{ij}\,, \qquad R_{ijm\ell} = a^2\dot{a}^2\delta_{im}\delta_{j\ell}\,. \tag{4.4}$$

By properly contracting the above quantities, the D-dimensional Gauss–Bonnet term turns out to be

$$\mathcal{G} = \frac{p(d)\,\Big[(d-3)\dot{a}^4 + 4a\dot{a}^2\ddot{a}\Big]}{a^4} \equiv p(d)\frac{\Big[(d-3)\dot{a}^4 + 4a\dot{a}^2\ddot{a}\Big]}{a^4}, \tag{4.5}$$

with $p(d) = d(d-1)(d-2)$. As we can see, in less than four dimensions (*i.e.* $d < 3$) it vanishes regardless of the value of the scale factor, while in 4-dimensions ($d = 3$) it turns into a topological surface term of the form

$$\mathcal{G} = 24\frac{\dot{a}^2\ddot{a}}{a^3}. \tag{4.6}$$

Dynamics can be derived either starting from field equations (4.2) or from the Euler-Lagrange equations provided by a point-like Lagrangian. Because of our further considerations related to the Noether theorem, let us pursue the latter approach. Lagrangian can be found thanks to the Lagrange multipliers method, with constraint (4.5), as follows:

$$S = \int \left[\sqrt{-g}f(\mathcal{G}) - \lambda\left\{\mathcal{G} - \frac{p(d)\,\Big[(d-3)\dot{a}^4 + 4a\dot{a}^2\ddot{a}\Big]}{a^4}\right\} + \mathcal{L}_m\right] d^{d+1}x\,, \tag{4.7}$$

being $\mathcal{L}_m$ the matter Lagrangian. Considering the cosmological volume element in $d+1$-dimensions, the action can be written as

$$S = 2\pi^2 \int \left[a^d f(\mathcal{G}) - \lambda\left\{\mathcal{G} - \frac{p(d)\,\Big[(d-3)\dot{a}^4 + 4a\dot{a}^2\ddot{a}\Big]}{a^4}\right\} + \mathcal{L}_m\right] d^{d+1}x\,. \tag{4.8}$$

By varying the action with respect to $\mathcal{G}$, we can find the Lagrange multiplier λ:

$$\delta S = \frac{\partial S}{\partial \mathcal{G}} \delta \mathcal{G} = a^d f_{\mathcal{G}}(\mathcal{G}) - \lambda = 0 \qquad \lambda = a^d f_{\mathcal{G}}(\mathcal{G}) \ . \tag{4.9}$$

Replacing the result in Eq. (4.8) and integrating out the second derivative, the Lagrangian finally takes the form:

$$\mathcal{L} = \frac{1}{3} a^{d-4} \Big[(3-d) p(d) \dot{a}^4 f_{\mathcal{G}}(\mathcal{G}) + 3a^4 [f(\mathcal{G}) - \mathcal{G} f_{\mathcal{G}}(\mathcal{G})] - 4a p(d) \dot{a}^3 \dot{\mathcal{G}} f_{\mathcal{G}\mathcal{G}}(\mathcal{G}) \Big] + \mathcal{L}_m . \tag{4.10}$$

The dynamical system is given by the two Euler-Lagrange equations coming from Lagrangian (4.10), with respect to the scale factor a and the Gauss–Bonnet scalar $\mathcal{G}$, respectively. The system is completed by the energy condition $E_{\mathcal{L}} = \left(\dot{a} \partial_{\dot{a}} + \dot{\mathcal{G}} \partial_{\dot{\mathcal{G}}} - 1 \right) \mathcal{L} = 0$, so that the final set of equations of motion consist in the following three partial differential equations:

$$\begin{cases} d a^4 [f(\mathcal{G}) - \mathcal{G} f_{\mathcal{G}}(\mathcal{G})] + 4(d-3) p(d) a \dot{a}^2 [\ddot{a} f_{\mathcal{G}}(\mathcal{G}) + \dot{a} \dot{\mathcal{G}} f_{\mathcal{G}}(\mathcal{G})] \\ +4a^2 p(d) \dot{a} [2 \dot{\mathcal{G}} \ddot{a} f_{\mathcal{G}\mathcal{G}}(\mathcal{G}) + \dot{a} \ddot{\mathcal{G}} f_{\mathcal{G}\mathcal{G}}(\mathcal{G}) + \dot{a} \dot{\mathcal{G}}^2 f_{\mathcal{G}\mathcal{G}\mathcal{G}}(\mathcal{G})] + (d-3)(d-4) p(d) \dot{a}^4 f_{\mathcal{G}}(\mathcal{G}) = 0 \\ \\ \mathcal{G} = p(d) \dfrac{\left[(d-3) \dot{a}^4 + 4a \dot{a}^2 \ddot{a} \right]}{a^4} \\ \\ (3-d) p(d) \dot{a}^4 f_{\mathcal{G}}(\mathcal{G}) - a^4 [f(\mathcal{G}) - \mathcal{G} f_{\mathcal{G}}(\mathcal{G})] - 4a p(d) \dot{a}^3 \dot{\mathcal{G}} f_{\mathcal{G}\mathcal{G}}(\mathcal{G}) = 0. \end{cases} \tag{4.11}$$

It is worth noticing that the equation for $\mathcal{G}$ provides exactly the cosmological constraint on the Gauss–Bonnet scalar (4.5). However it is not possible to solve the above equations without selecting the form of the $f(\mathcal{G})$ function. In order to do this, we adopt the Noether symmetry approach, by means of which one can select reliable models according to the existence of symmetries. The approach is also physically motivated because symmetries correspond to conservation laws, as pointed out in App. B. Specifically, we use Noether's theorem as a selection criterion to find out viable models leading to exact solutions. This allows to reduce the dynamics and analytically solve the equations of motion, as well as to deal with the Hamiltonian formalism and obtain the wave function of the Universe.

4.1 Research For Symmetries and Exact Solutions in d+1 Dimensions

We apply the first prolongation of Noether vector (B.3) to the Lagrangian (4.10), whose generator takes the form:

$$\mathcal{X} = \xi(a, \mathcal{G}, t)\partial_t + \alpha(a, \mathcal{G}, t)\partial_a + \beta(a, \mathcal{G}, t)\partial_{\mathcal{G}} \,. \tag{4.12}$$

In order to find symmetries, we consider the Noether identity (B.2) and set terms with derivative powers of a and $\mathcal{G}$ equal to zero. Noether symmetry approach yields a system of four differential equations plus the constraints on the infinitesimal generators α, β, ξ. It reads:

$$\begin{cases} d\, a^2\alpha(f - \mathcal{G}f_{\mathcal{G}}) - 4p(d)\dot{a}^2\dot{\mathcal{G}}f_{\mathcal{G}\mathcal{G}}\partial_t\alpha - a^3\left[\beta f'_{\mathcal{G}\mathcal{G}} - (f - \mathcal{G}f_{\mathcal{G}})\partial_t\xi\right] = 0 \\ (d-3)\left[f_{\mathcal{G}}\partial_t\alpha + a f_{\mathcal{G}\mathcal{G}}\partial_t\beta\right] = 0 \\ (d-3)\alpha f_{\mathcal{G}\mathcal{G}} + a\beta f_{\mathcal{G}\mathcal{G}\mathcal{G}} + a f_{\mathcal{G}\mathcal{G}}\left(3\partial_a\alpha + \partial_{\mathcal{G}}\beta - 3\partial_t\xi\right) = 0 \\ (d-3)(d-4)\alpha f_{\mathcal{G}} + (d-3)a\beta f_{\mathcal{G}\mathcal{G}} - (d-3)a f_{\mathcal{G}}\left(3\partial_t\xi - 4\partial_a\alpha\right) + 4a^2 f_{\mathcal{G}\mathcal{G}}\partial_a\beta = 0\,, \end{cases} \tag{4.13}$$

with $\xi = \xi(t)\,, \alpha = \alpha(a,t)\,, g = g_0\,.$ Here, we neglect *a priori* the possibility $p(d) = 0$. Only three solutions satisfy the whole system; all of them provides the same dependence of the infinitesimal generators on the variables, namely

$$\alpha = \alpha_0 a\,, \quad \beta = \beta_0\mathcal{G}\,, \quad \xi = \xi_0 t + \xi_1, \tag{4.14}$$

but different values of the constants α_0, β_0, ξ_0. The final solutions with the corresponding infinitesimal generators are:

$$
\begin{aligned}
&1): \ \alpha = \alpha_0 a\,, \quad \beta = \beta_0 \mathcal{G}\,, \quad \xi = \alpha_0 \left(\frac{d}{3}\right) t + \xi_1\,, \qquad f(\mathcal{G}) = f_0 \mathcal{G} \\
&2): \ \alpha = \alpha_0 a\,, \quad \beta = -4\xi_0 \mathcal{G}\,, \quad \xi = \xi_0 t + \xi_1\,, \qquad f(\mathcal{G}) = \frac{4 f_0 \xi_0}{\alpha_0 d + \xi_0} \mathcal{G}^{\frac{\alpha_0 d + \xi_0}{4\xi_0}} \\
&3): \ \alpha = 0\,, \quad \beta = \beta_0 \mathcal{G}\,, \quad \xi = 0\,, \qquad f(\mathcal{G}) = f_0 \mathcal{G} + f_1,
\end{aligned}
\tag{4.15}
$$

where the exponent of the second function must be different from 1. The first and the third solution are non-trivial only in more than 4 dimensions, while the second provides contributions to the equations of motion even for $d = 3$. Without loss of generality, in order to find the dynamics of the scale factor, we choose the function $f(\mathcal{G}) = f_0 \mathcal{G}^k$, where we define

$$
\frac{\alpha_0 d + \xi_0}{4\xi_0} = k \tag{4.16}
$$

and incorporate the coefficient of $\mathcal{G}^k$ into f_0. In this way, the point-like Lagrangian can be written as

$$
\mathcal{L} = -\frac{1}{3} a^{d-4} \mathcal{G}^{k-2} \left[3(k-1) a^4 \mathcal{G}^2 + k(d-3)\mathcal{G} p(d) \dot{a}^4 + 4k(k-1) a p(d) \dot{a}^3 \dot{\mathcal{G}} \right]. \tag{4.17}
$$

The Euler-Lagrange equations (4.11) can now be exactly solved providing the following solutions:

$$
a(t) = a_0 e^{qt}\,, \qquad \mathcal{G}(t) = (d+1) p(d)\, q^4\,, \qquad k = \frac{d+1}{4} \tag{4.18}
$$

$$
a(t) = a_0 t^{-4\frac{(k-1)(4k-1)}{4k-d-1}}\,, \qquad \mathcal{G}(t) = \frac{256 \left[(k-1)\,(4k-1)\right]^3 \left[4 + (d+1)(4k-5)\right] p(d)}{((d+1)t - 4kt)^4}, \tag{4.19}
$$

with q constant. It is worth noticing that the de-Sitter-like expansion only holds in more than 4 dimensions, unlike the power-law solutions which is valid even for $d = 3$. However, the $d = 3$ case deserves a separate discussion. To this purpose, in the next section we will study the four-dimensional $f(\mathcal{G})$ cosmology with matter fields. As a final remark, it is interesting to observe that the function containing a linear Gauss–Bonnet term leads to a vacuum exponential acceleration with several free parameters to be constrained by experimental observations.

4.2 $f(\mathcal{G})$ Cosmology in 4-Dimensions

Let us now specifically discuss the four-dimensional limit of the previously discussed case; in particular, we will derive Noether symmetries coming from the 4-dimensional Lagrangian, as well as the related cosmological solutions in presence of matter. Then, the selected function will be studied in terms of energy conditions and slow–roll inflation. To deal with a Lagrangian description, we introduce the matter Lagrangian through the choice $\mathcal{L}_m = \rho_0 a^{-3w}$, where w represents the ratio between pressure and density $p = w\,\rho$, that is the Equation of State of a perfect fluid. For $w = 0$, we have dust matter, while for $w = \frac{1}{3}$, we have radiation. The case $w = -1$, in turn, corresponds to the cosmological constant. Therefore, being $p(3) = 6$, the Lagrangian (4.10), in 4-dimensions, is

$$\mathcal{L}^{(4)} = a^3[f(\mathcal{G}) - \mathcal{G} f_{\mathcal{G}}(\mathcal{G})] - 8\dot{a}^3 f_{\mathcal{G}\mathcal{G}}(\mathcal{G})\dot{\mathcal{G}} + \rho_0 a^{-3w}\ . \tag{4.20}$$

The Euler-Lagrange equations of the above Lagrangian read as:

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{\mathcal{G}}} = \dfrac{\partial \mathcal{L}}{\partial \mathcal{G}} \quad \to \quad \mathcal{G} = 24\dfrac{\dot{a}^2\ddot{a}}{a^3} \\ \\ \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{a}} = \dfrac{\partial \mathcal{L}}{\partial a} \quad \to \quad a^2[f(\mathcal{G}) - \mathcal{G} f_{\mathcal{G}}(\mathcal{G})] + 16\dot{a}\ddot{a}\dot{\mathcal{G}} f_{\mathcal{G}\mathcal{G}}(\mathcal{G}) + 8\dot{a}^2[f_{\mathcal{G}\mathcal{G}}(\mathcal{G})\ddot{\mathcal{G}} + f_{\mathcal{G}\mathcal{G}\mathcal{G}}(\mathcal{G})\dot{\mathcal{G}}^2] \\ \qquad\qquad\qquad\qquad\quad +3\rho_0 w a^{-3w-1}\ . \end{cases} \tag{4.21}$$

Notice that, by construction, the first equation is the Lagrange multiplier in 4-dimensions. The above set of partial differential equations must be integrated with the energy condition

$$E_{\mathcal{L}} = \dot{a}\frac{\partial \mathcal{L}}{\partial \dot{a}} + \dot{\mathcal{G}}\frac{\partial \mathcal{L}}{\partial \dot{\mathcal{G}}} - \mathcal{L} = 0 \,, \tag{4.22}$$

which gives

$$a^3[f(\mathcal{G}) - \mathcal{G} f_{\mathcal{G}}(\mathcal{G})] + 24\dot{a}^3 f_{\mathcal{G}\mathcal{G}}(\mathcal{G})\dot{\mathcal{G}} + \rho_0 a^{-3w} = 0 \,. \tag{4.23}$$

Applying the Noether condition (B.2) to the Lagrangian (4.20), we get a system of two differential equations. With respect to the D-dimensional case, we have two less equations. This is due to the fact that the second equation of (4.13) vanishes identically for $d = 3$, while the fourth trivially reduces to $\partial_a \beta = 0$. Thus the system takes the form:

$$\begin{cases} 3\alpha a^2[f(\mathcal{G}) - \mathcal{G} f_{\mathcal{G}}(\mathcal{G}) - w\rho_0 a^{-3(w+1)}] - \beta a^3 \mathcal{G} f_{\mathcal{G}\mathcal{G}}(\mathcal{G}) + \partial_t \xi a^3 [f(\mathcal{G}) - \mathcal{G} f_{\mathcal{G}}(\mathcal{G})] = 0 \\ 3\partial_a \alpha f_{\mathcal{G}\mathcal{G}}(\mathcal{G}) + \beta f_{\mathcal{G}\mathcal{G}\mathcal{G}}(\mathcal{G}) - 3\partial_t \xi \; f_{\mathcal{G}\mathcal{G}}(\mathcal{G}) + \partial_{\mathcal{G}} \beta \; f_{\mathcal{G}\mathcal{G}}(\mathcal{G}) = 0 \\ \xi = \xi(t)\,, \quad \alpha = \alpha(a)\,, \quad \beta = \beta(\mathcal{G}) \quad g = g_0. \end{cases} \tag{4.24}$$

The presence of the matter Lagrangian does not cause any changes in the system resolution, so that the function and the infinitesimal generators turn out to be the same as Eq. (4.15) with $d = 3$, namely

$$\begin{cases} \alpha = \alpha_0 a\,, \quad \xi_0 t + \xi_1\,, \quad \beta = -4\xi_0 \mathcal{G}\,, \quad g = g_0 \\ f(\mathcal{G}) = \dfrac{f_0 \mathcal{G}^k}{k}\,, \qquad k \neq 0,\, 1\,. \end{cases} \tag{4.25}$$

By using the above solutions and incorporating the constant k into f_0, we can rewrite the point-like Lagrangian as

$$\mathcal{L} = -f_0(k-1)\mathcal{G}^k a^3 - 8 f_0 k(k-1)\mathcal{G}^{k-2}\dot{a}^3\dot{\mathcal{G}} - \rho_0 a^{-3w} \,. \tag{4.26}$$

Euler-Lagrange equations and energy condition coming from (4.26) lead to the system

$$\begin{cases} \mathcal{G} = 24\dfrac{\dot{a}^2\ddot{a}}{a^3} \\ \\ -8f_0k(k-1)(k-2)\dot{a}^2\mathcal{G}^{k-3}\dot{\mathcal{G}}^2 - 8f_0k(k-1)\dot{a}\mathcal{G}^{k-2}\left(2\ddot{a}\dot{\mathcal{G}} + \dot{a}\ddot{\mathcal{G}}\right) \\ +f_0(k-1)a^2\mathcal{G}^k - \rho_0 w a^{-3w-1} = 0 \\ \\ -24f_0k(k-1)\dot{a}^3\mathcal{G}^{k-2}\dot{\mathcal{G}} + f_0(k-1)a^3\mathcal{G}^k + \rho_0 a^{-3w} = 0. \end{cases} \tag{4.27}$$

There are two kind of solutions of the above system; the first can be obtained neglecting the matter Lagrangian. In this case, when geometric contributions are greater than matter ones, the only solution reads

$$a(t) = a_0 t^{1-4k} \qquad \mathcal{G}(t) = -96k(1-4k)^3 t^{-4} \equiv \mathcal{G}_0 t^{-4}\ , \tag{4.28}$$

which is a power-law expansion and, as expected, it is contained into (4.19). Without neglecting $\mathcal{L}_m$, we find another set of solutions, namely:

$$\begin{cases} a(t) = a_0 t^{1-4k} \qquad \mathcal{G}(t) = -96k(1-4k)^3 t^{-4} \equiv \mathcal{G}_0 t^{-4}\ , \qquad w = 0 \\ \\ a(t) = e^{nt} \quad \mathcal{G}(t) = 24m^4 \quad w = -1. \end{cases} \tag{4.29}$$

The former is the solution for dust matter, while the latter describes a Dark-Energy dominated universe. Moreover, from Eq. (4.28), we can distinguish the cosmological epochs crossed by the Universe even in vacuum, as depending on the geometrical contributions

$$\begin{aligned}
k &= \frac{1}{8} \ \rightarrow\ a(t) \sim t^{\frac{1}{2}} \quad \mathcal{G} = -\frac{3}{2}t^{-4} \ \rightarrow\ \text{Radiation} \\
k &= \frac{1}{6} \ \rightarrow a(t) \sim t^{\frac{1}{3}} \quad \mathcal{G} = -\frac{16}{27}t^{-4} \ \rightarrow\ \text{Stiff matter} \\
k &= \frac{1}{12} \ \rightarrow\ a(t) \sim t^{\frac{2}{3}} \quad \mathcal{G} = -\frac{64}{27}t^{-4} \ \rightarrow\ \text{Dust matter}
\end{aligned} \tag{4.30}$$

Cosmological solutions (4.28) are, therefore, in agreement with the FLRW solutions of GR but are recovered without imposing the Ricci scalar in the gravitational action. It is worth noticing that in all cases the Gauss–Bonnet term turns out to be negative, so that the function $f(\mathcal{G}) = f_0\mathcal{G}^k$ may lead to some problems for fractional values of k. To avoid these kind of singularities, we want to stress that the function $f(\mathcal{G})$ is still a solution of Noether's system even including the modulus of the Gauss–Bonnet term, *i.e.* $f(\mathcal{G}) = f_0|\mathcal{G}|^k$. The same happens in several other modified theories; for example, in $f(R)$ gravity, the Noether approach provides $f(R) \sim R^{3/2}$ [127], whose time power-law solution $a(t) \sim t^p$ leads to a complex function for $p < 0$ and $p > 1/2$. Hence, without loss of generality and in agreement with Noether's approach, we can always require the function into the action to be positive. However, as shown in [128] for $f(R) \sim |R|^{3/2}$, some exact solutions can imply transitions from decelerated/accelerated behaviors, that is dust/Dark Energy behaviors, according to the values of the solution parameters. In the present case, however, we are discussing only exact solutions emerging from Noether's symmetries, where there is no change of concavity in the evolution of the scale factor and then no transitions from decelerated to accelerated universes (and *vice versa*).

4.2.1 Quantum Cosmology and the Wave Function of the Universe

Taking into account the function selected by symmetry considerations, here we use the Wheeler–DeWtt (WDW) equation to find out the wave function of the Universe and the corresponding classical trajectories. Basic foundations, applications and interpretations of the wave functions are discussed in App. C. The results of the previous section allow to suitably develop the quantum cosmology for the minisuperspace $T\mathcal{S} \equiv \{a, \dot{a}, \mathcal{G}, \dot{\mathcal{G}}\}$. Starting from Lagrangian (4.26), we can calculate the related Hamiltonian as a function of momenta:

$$\mathcal{H} = \frac{f_0}{k}\mathcal{G}^k a^3 + \pi_a \left(-\frac{\pi_{\mathcal{G}}}{8f_0}\mathcal{G}^{2-k}\right)^{\frac{1}{3}} , \tag{4.31}$$

where $\pi_a = \frac{\partial\mathcal{L}}{\partial\dot{a}}$ and $\pi_{\mathcal{G}} = \frac{\partial\mathcal{L}}{\partial\dot{\mathcal{G}}}$, according to the Legendre transformations.
In this form, the Hamiltonian cannot be quantized due to the presence of the fractional exponent. Nonetheless, thanks to the Noether symmetries, we can insert a cyclic variable into Eq. (4.31), which allows to fully quantize the theory. Specifically, from Eq. (4.28), it is easy to see that the quantity $\dfrac{\dot{a}}{\mathcal{G}^k}$ is a constant of motion. Therefore, by defining

$$\frac{\dot{a}^3}{\mathcal{G}^{3k}} = \Sigma_0 , \tag{4.32}$$

the momentum $\pi_{\mathcal{G}}$ can be rewritten in terms of Σ_0 as:

$$\pi_{\mathcal{G}} = -8f_0\Sigma_0\mathcal{G}^{4k-2} . \tag{4.33}$$

Replacing this result into Eq. (4.31), the new Hamiltonian written in the tangent minusuperspace $T\mathcal{S}' = \{a, \pi_a, \mathcal{G}\}$, reads

$$\mathcal{H} = \frac{f_0}{k}\mathcal{G}^k a^3 + \pi_a \left(\Sigma_0\mathcal{G}^{3k}\right)^{\frac{1}{3}} . \tag{4.34}$$

Now, thanks to the quantization rules coming from the Arnowitt-Deser-Misner (ADM) formalism (see App. C), it is possible to recast the conjugate momenta as operators:

$$\begin{cases} \pi_{\mathcal{G}} = -i\dfrac{\partial}{\partial \mathcal{G}} \\ \\ \pi_a = -i\dfrac{\partial}{\partial a} \\ \\ \mathcal{H}\psi = 0\ . \end{cases} \tag{4.35}$$

The third equation is the WDW equation and ψ is the wave function of the Universe. Being $\pi_{\mathcal{G}} = -8f_0\Sigma_0\mathcal{G}^{4k-2}$, it follows that the quantity $\pi_{\mathcal{G}}\mathcal{G}^{2-4k}$ is a constant of motion. More precisely, the quantized equation of momentum can be written as:

$$i\frac{\partial}{\partial \mathcal{G}}\psi(a,\mathcal{G}) = 8f_0\Sigma_0\mathcal{G}^{4k-2}\psi(a,\mathcal{G}), \tag{4.36}$$

and together with the WDW equation yields

$$\begin{cases} \pi_{\mathcal{G}}\psi = -i\dfrac{\partial}{\partial \mathcal{G}}\psi \quad \rightarrow \quad \psi(a,\mathcal{G}) = A(a)\ \exp\left\{i\ \dfrac{8f_0\Sigma_0\mathcal{G}^{4k-1}}{1-4k}\right\} \\ \mathcal{H}\psi = 0 \quad \rightarrow \quad \dfrac{f_0}{k}\left(\Sigma_0\right)^{-\frac{1}{3}} a^3 A(a) - i\dfrac{\partial A(a)}{\partial a} = 0\ . \end{cases} \tag{4.37}$$

The latter equation can be solved with respect to $A(a)$ and the result can be replaced in the former one. After some computations, the wave function turns out to be

$$\psi(a,\mathcal{G}) = \psi_0 \exp\left\{i\left[-\frac{f_0}{4k}\left(\Sigma_0\right)^{-\frac{1}{3}} a^4 + \frac{8f_0\Sigma_0\mathcal{G}^{4k-1}}{1-4k}\right]\right\}\ . \tag{4.38}$$

According to the Hartle criterion, an oscillating wave function means correlations among variables and then the possibility to find classical trajectories (i.e. observable universes). In fact, considering the Wentzel-Kramers-Brillouin (WKB) approximation, the wave function can be recast as $\psi(a,\mathcal{G}) \sim e^{iS}$ (where S is the action), so that in the $f(\mathcal{G})$ model the action we can be identified with the quantity:

$$S = -\frac{f_0}{4k}\left(\Sigma_0\right)^{-\frac{1}{3}} a^4 + \frac{8 f_0 \Sigma_0}{1-4k}\mathcal{G}^{4k-1}. \tag{4.39}$$

Hamilton-Jacobi equation with respect to the scale factor provides the third equation of motion in (4.27), namely:

$$\frac{\partial S}{\partial a} = \pi_a \quad \rightarrow \quad \mathcal{G}^k a^3 = 24\mathcal{G}^{k-2}\dot{a}^3\dot{\mathcal{G}}\ . \tag{4.40}$$

The second Hamilton-Jacobi equation $\left(\frac{\partial S}{\partial \mathcal{G}} = \pi_{\mathcal{G}}\right)$ instead, is nothing but the identity $\pi_{\mathcal{G}} = \Sigma_0$. The energy condition is trivially recovered as a linear combination of the two, so that observable universes are selected by Noether's approach.

As reported in App. C, oscillatory behaviors of the wave function of the Universe are related to conserved quantities; therefore, if the number of symmetries is equal to the variables of minisuperspace, the dynamical system is fully integrable and the wave function fully oscillating.

4.2.2 The case $f(\mathcal{G}) = f_0\mathcal{G}^n + f_1\mathcal{G}^k$

As pointed out in the previous section, no exponential solutions occur in vacuum when $f(\mathcal{G}) = f_0\mathcal{G}^k$ cosmology is considered. On the other hand, here we show that a starting action of the form

$$S = \int \sqrt{-g}\left(\mathcal{G}^n + \mathcal{G}^k\right) d^4x, \tag{4.41}$$

leads to a de Sitter-like expansion without cosmological constant. Even though the action (4.41) is not directly a solution of the Noether system, it could be very relevant for several reasons. At the beginning of this chapter we affirmed that in some epochs GR can be recovered from $f(\mathcal{G})$ gravity by the choice $f(\mathcal{G}) = \sqrt{\mathcal{G}}$, when a cosmological background is considered. Therefore the function

$$f(\mathcal{G}) = f_0\mathcal{G}^{\frac{1}{2}} + f_1\mathcal{G}^n \ , \tag{4.42}$$

can be easily compared to the case $f(R,\mathcal{G}) = R + f(\mathcal{G})$, often discussed in literature in view to recover GR in suitable limits [129, 130].
In order not to lose generality, we extend Eq. (4.42) to a function of the form $f(\mathcal{G}) = f_0\mathcal{G}^n + f_1\mathcal{G}^k$, so that Eq. (4.42) is recovered when $n = 1/2$. The Lagrangian is a particular case of that in Eq. (4.20) and reads:

$$\mathcal{L} = -a^3\left[f_0(n-1)\mathcal{G}^n + f_1(k-1)\mathcal{G}^k\right] - 8\left[f_0 n(n-1)\mathcal{G}^{n-2} + f_1 k(k-1)\mathcal{G}^{k-2}\right]\dot{a}^3\dot{\mathcal{G}} \ . \tag{4.43}$$

The Euler-Lagrange equations and the energy condition are:

$$\begin{aligned}
\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\mathcal{G}}} = \frac{\partial\mathcal{L}}{\partial\mathcal{G}} \quad &\to \quad \mathcal{G} = 24\frac{\dot{a}^2\ddot{a}}{a^3}\\
\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{a}} = \frac{\partial\mathcal{L}}{\partial a} \quad &\to \quad 3a^2\left[f_0(n-1)\mathcal{G}^n + f_1(k-1)G^k\right]\\
&\qquad -24\dot{a}\left[f_0 n\left(n^2-3n+2\right)\dot{a}\mathcal{G}^{n-3}\dot{\mathcal{G}}^2 + \right.\\
&\qquad +f_1 k\left(k^2-3k+2\right)\dot{a}\mathcal{G}^{k-3}\dot{\mathcal{G}}^2\\
&\qquad +f_0 n(n-1)\mathcal{G}^{n-2}\left(2\ddot{a}\dot{\mathcal{G}} + \dot{a}\ddot{G}\right) + \\
&\qquad \left. +f_1 k(k-1)\mathcal{G}^{k-2}\left(2\ddot{a}\dot{\mathcal{G}} + \dot{a}\ddot{G}\right)\right] = 0\\
\dot{a}\frac{\partial\mathcal{L}}{\partial\dot{a}} + \dot{\mathcal{G}}\frac{\partial\mathcal{L}}{\partial\dot{\mathcal{G}}} - \mathcal{L} = 0 \quad &\to \quad a^3\left[f_0(n-1)\mathcal{G}^n + f_1(k-1)\mathcal{G}^k\right]\\
&\qquad -24\dot{a}^3\dot{\mathcal{G}}\left[f_0 n(n-1)\mathcal{G}^{n-2} + f_1 k(k-1)\mathcal{G}^{k-2}\right] = 0.
\end{aligned} \tag{4.44}$$

The system admits the following de Sitter solution:

$$a(t) = a_0 e^{mt} \quad \mathcal{G}(t) = 24m^4 \quad \text{with} \ \ m = \left[-24^{n-k}\,\frac{f_0}{f_1}\left(\frac{n-1}{k-1}\right)\right]^{\frac{1}{4(k-n)}} \ . \tag{4.45}$$

This means that Dark Energy [129] and inflation [131] can be easily recovered in this framework.

4.3 Energy Conditions

As it is well known, in GR the energy conditions only rely to the energy-momentum tensor of matter, which automatically satisfies all the four inequalities on the pressure and the energy density. The energy conditions are also required in order to demonstrate several important theorems, such as the no hair theorem [132] or the laws of black hole thermodynamics [133]. Specifically, the energy conditions in natural units read as:

$$\begin{aligned}
&\text{Null Energy Condition (NEC)} \rightarrow \rho + p \geq 0 \\
&\text{Weak Energy Condition (WEC)} \rightarrow \rho \geq 0 \ ; \ \rho + p \geq 0 \\
&\text{Dominant Energy Condition (DEC)} \rightarrow \rho - |p| \geq 0 \\
&\text{Strong Energy Condition (SEC)} \rightarrow \rho + p \geq 0 \ ; \ \rho + 3p \geq 0 \ ,
\end{aligned} \tag{4.46}$$

where ρ is the energy density and p the pressure of the fluid. The last relation implies that gravity must be attractive, while the others require the pressure and the energy to be non-negative. In GR, where the only energy momentum tensor is that of the standard matter, all the energy conditions are identically satisfied.
On the other hand, in extended gravity, the field equations can be written as

$$F G_{\mu\nu} = T_{\mu\nu}^{(Grav\ Field)} + T_{\mu\nu}^{(Matter)}, \tag{4.47}$$

where F is a generic function depending on the particular theory considered and $T_{\mu\nu}^{(Grav\ Field)}$ is the so called energy-momentum tensor of the gravitational field. A complete discussion on this topic can be found in [31, 85, 134].
From a cosmological point of view, energy conditions can be used to test the validity of the theory by means of the cosmographic parameters. The approach is straightforwardly dis-

cussed for some extended theories of gravity, *e.g.* in Refs. [135, 136, 137, 138]. In particular, in Ref. [138], the authors start from an action of the form $S = \int \sqrt{-g}\,(R + f(\mathcal{G}))\; d^4x$, recovering therefore the Einstein tensor in the field equations. By a similar procedure, in Sect. 4.3 we show that the energy conditions for the gravitational field can be written for an action only containing the function $f(\mathcal{G})$, without assuming the GR limit as a requirement when $f(\mathcal{G})$ vanishes. Subsequently we compare the result with $R + f(\mathcal{G})$ gravity. In both cases, the Einstein tensor can be isolated, so that the effective energy density and the effective pressure can be written in terms of geometry. Specifically, we consider the function $f(\mathcal{G}) = f_0\mathcal{G}^k$ in a FLRW metric and write the energy conditions in terms of the cosmographic parameters.
Moreover, the effective energy density and pressure of the gravitational field can be used in order to find out the slow–roll parameters and then to check whether the theory admits a cosmological inflation in the early time.
Inflationary model was introduced by A. Linde and A. Guth [66, 67] to address the evidences provided by the cosmological data. Inflation is usually thought as generated by a scalar field ϕ, called inflaton, which is supposed to be the responsible for the accelerated expansion of the Universe. Soon after, the Starobinsky model [139] showed that the additional geometric contributions occurring in modified theories of gravity can be intended as an effective scalar potential capable of driving the inflation. Therefore, nowadays inflation can be realized in several ways [140, 141, 142, 143, 144].
In Sect. 4.3.1.1 we consider the so called *new inflation* or *slow–roll inflation*, according to which inflation is driven by a scalar field rolling down a potential energy hill. Inflation occurs as soon as the scalar field rolling is slow with respect to the Universe expansion.
By using the energy conditions, we require the slow–roll parameters to be small during inflation. This allows to put some constraint to the $f(\mathcal{G})$ function, selecting those which predict an inflationary universe.

4.3.1 $f(\mathcal{G})$ Cosmology

Once we introduced the main features of the energy conditions in extended theories of gravity, in this section we write the energy conditions for the $f(\mathcal{G})$ cosmology, constraining the theory by the cosmographic parameters. Let us begin by considering the $f(\mathcal{G})$ action

$$S = \int \sqrt{-g}\, f(\mathcal{G})\, d^4x + S^{(m)}\ , \tag{4.48}$$

with $S^{(m)}$ being the matter action. The corresponding field equations (4.2) can be rewritten by isolating Einstein tensor $G_{\mu\nu}$ in all terms in which it appears, recasting Eq. (4.2) as:

$$\begin{aligned} G_{\mu\nu}(2R-4\Box)f_{\mathcal{G}}(\mathcal{G}) &= T_{\mu\nu} - \Big[\Big(R^2 - 4R_{\mu p}R^{p}{}_{\nu} + 2R_{\mu}{}^{p\sigma\tau}R_{\nu p\sigma\tau} \\ - & \ 4R^{\alpha\beta}R_{\mu\alpha\nu\beta}\Big) f_{\mathcal{G}}(\mathcal{G}) - \Big(2RD_{\mu}D_{\nu} - 4R^{p}_{\{\nu}D_{\mu\}}D_p + 4g_{\mu\nu}R^{p\sigma}D_pD_{\sigma} \\ - & \ 4R_{\mu\alpha\nu\beta}D^{\alpha}D^{\beta}\Big) f_{\mathcal{G}}(\mathcal{G}) - \frac{1}{2}g_{\mu\nu}f(\mathcal{G})\Big]\ . \end{aligned} \tag{4.49}$$

Notice that the term in the square bracket of the RHS can be intended as the effective energy-momentum tensor of the gravitational field, introduced in Eq. (4.47), while $(2R - 4\Box)f_{\mathcal{G}}(\mathcal{G})$ accounts for the function F. Therefore, the components of the Einstein tensor can be interpreted as the analogue of energy density and pressure, namely

$$G^0_0 = \frac{1}{(2R-4\Box)f_{\mathcal{G}}(\mathcal{G})}\left(\rho^{GravField} + \rho_0\right) \tag{4.50}$$

$$G^i_j = -\frac{\delta^i_j}{(2R-4\Box)f_{\mathcal{G}}(\mathcal{G})}\left(p^{GravField} + p_0\right), \tag{4.51}$$

where ρ_0 and p_0 are the matter density and pressure respectively. In a spatially flat cosmological universe of the form

$$ds^2 = dt^2 - a(t)^2 d\mathbf{x}^2, \tag{4.52}$$

the field equations of $f(\mathcal{G}) = f_0\mathcal{G}^k$ gravity can be written as:

$$\begin{aligned} G^0_0 = \frac{1}{(2R-4\Box)f_{\mathcal{G}}(\mathcal{G})} \quad & \Big\{\rho_0 + \frac{f_0}{2}\mathcal{G}^{k-3}\Big[-24k(k-1)H^2\left(\mathcal{G}\ddot{\mathcal{G}} + (k-2)\dot{\mathcal{G}}^2\right) \\ & -72k\mathcal{G}^2H^4 - 48k\mathcal{G}H^2\left(2\mathcal{G}(\dot{H}+H^2) + (k-1)H\dot{\mathcal{G}}\right) + \mathcal{G}^3\Big]\Big\}\,, \end{aligned} \tag{4.53}$$

$$\begin{aligned} G^1_1 = -\frac{1}{(2R-4\Box)f_{\mathcal{G}}(\mathcal{G})} \quad & \Big\{p_0 + \frac{f_0}{2}\mathcal{G}^{k-3}\Big[16k(k-1)(\dot{H}+H^2)\left(\mathcal{G}\ddot{\mathcal{G}} + (k-2)\dot{\mathcal{G}}^2\right) \\ & +24k\mathcal{G}^2H^4 + 16k\mathcal{G}(\dot{H}+H^2)\left(3\mathcal{G}(\dot{H}+H^2) + 2(k-1)H\dot{\mathcal{G}}\right) \\ & +24k\mathcal{G}H^2\left(4\mathcal{G}(\dot{H}+H^2) + (k-1)H\dot{G}\right) - \mathcal{G}^3\Big]\Big\}\,, \end{aligned} \tag{4.54}$$

with H being the Hubble constant $H \equiv \dot{a}/a$. The second term in the RHS of Eq. (4.53) is the cosmological density of gravitational field $\rho^{(GF)}$, as the same term in Eq. (4.54) is the pressure. Specifically:

$$\begin{aligned} \rho^{(GF)} = \frac{f_0}{2}\mathcal{G}^{k-3} \quad & \Big[-24k(k-1)H^2\left(\mathcal{G}\ddot{\mathcal{G}} + (k-2)\dot{\mathcal{G}}^2\right) \\ & -72k\mathcal{G}^2H^4 - 48k\mathcal{G}H^2\left(2\mathcal{G}(\dot{H}+H^2) + (k-1)H\dot{\mathcal{G}}\right) + \mathcal{G}^3\Big]\,, \end{aligned} \tag{4.55}$$

$$\begin{aligned} p^{(GF)} = \frac{f_0}{2}\mathcal{G}^{k-3} \quad & \Big[16k(k-1)(\dot{H}+H^2)\left(\mathcal{G}\ddot{\mathcal{G}} + (k-2)\dot{\mathcal{G}}^2\right) + 24k\mathcal{G}^2H^4 \\ & +16k\mathcal{G}(\dot{H}+H^2)\left(3\mathcal{G}(\dot{H}+H^2) + 2(k-1)H\dot{\mathcal{G}}\right) \\ & +24k\mathcal{G}H^2\left(4\mathcal{G}(\dot{H}+H^2) + (k-1)H\dot{\mathcal{G}}\right) - \mathcal{G}^3\Big]\,, \end{aligned} \tag{4.56}$$

so that each energy condition can be split in two contributions, namely

$$\begin{aligned}
&\text{NEC} \rightarrow \rho_0 + \rho^{(GF)} + p_0 + p^{(GF)} \geq 0 \\
&\text{WEC} \rightarrow \rho_0 + \rho^{(GF)} \geq 0 \; ; \; \rho_0 + \rho^{(GF)} + p_0 + p^{(GF)} \geq 0 \\
&\text{DEC} \rightarrow \rho_0 + \rho^{(GF)} - |p_0| - \left|p^{(GF)}\right| \geq 0 \\
&\text{SEC} \rightarrow \rho_0 + \rho^{(GF)} + p_0 + p^{(GF)} \geq 0 \; ; \; \rho_0 + \rho^{(GF)} + 3p_0 + 3p^{(GF)} \geq 0 \; .
\end{aligned} \tag{4.57}$$

Starting from the above inequalities, in what follows we focus on a particular subcase, requiring that the standard matter and the gravitational field must respect the energy conditions separately. Assuming that the ordinary matter automatically satisfies all the energy conditions, the only constraints capable of providing the validity range of the parameter k are:

$$\begin{aligned}
&\text{NEC} \rightarrow \rho^{(GF)} + p^{(GF)} \geq 0 \\
&\text{WEC} \rightarrow \rho^{(GF)} \geq 0 \; ; \; \rho^{(GF)} + p^{(GF)} \geq 0 \\
&\text{DEC} \rightarrow \rho^{(GF)} - |p^{(GF)}| \geq 0 \\
&\text{SEC} \rightarrow \rho^{(GF)} + p^{(GF)} \geq 0 \; ; \; \rho^{(GF)} + 3p^{(GF)} \geq 0 \; .
\end{aligned} \tag{4.58}$$

Replacing Eqs. (4.55) and (4.56) into the system (4.58), the energy conditions yield

$$\begin{aligned}
NEC \rightarrow \quad & 4f_0 k \mathcal{G}^{k-3} \Big[(k-1)\mathcal{G}\left(2\ddot{\mathcal{G}}\dot{H} - H^2\ddot{\mathcal{G}} + 4H\dot{\mathcal{G}}\dot{H} + H^3\dot{\mathcal{G}}\right) \\
& -(k-1)(k-2)\dot{\mathcal{G}}^2\left(H^2 - 2\dot{H}\right) + 6\mathcal{G}^2\dot{H}\left(\dot{H} + 2H^2\right)\Big] \geq 0
\end{aligned}$$

$$WEC \; \begin{cases}
4f_0 k \mathcal{G}^{k-3} \Big[(k-1)\mathcal{G}\left(2\ddot{\mathcal{G}}\dot{H} - H^2\ddot{\mathcal{G}} + 4H\dot{\mathcal{G}}\dot{H} + H^3\dot{\mathcal{G}}\right) \\
-(k-1)(k-2)\dot{\mathcal{G}}^2\left(H^2 - 2\dot{H}\right) + 6\mathcal{G}^2\dot{H}\left(\dot{H} + 2H^2\right)\Big] \geq 0 \\
\qquad\qquad\qquad \text{and} \\
\frac{f_0}{2}\mathcal{G}^{k-3}\Big[-24k(k-1)H^2\left(\mathcal{G}\ddot{\mathcal{G}} + (k-2)\dot{\mathcal{G}}^2\right) - 72k\mathcal{G}^2H^4 \\
-48k\mathcal{G}H^2\left(2\mathcal{G}(\dot{H} + H^2) + (k-1)H\dot{\mathcal{G}}\right) + \mathcal{G}^3\Big] \geq 0
\end{cases}$$

$$
DEC \begin{cases}
4f_0 k\mathcal{G}^{k-3}\Big[(k-1)\mathcal{G}\left(2\ddot{\mathcal{G}}\dot{H}-H^2\ddot{\mathcal{G}}+4H\dot{\mathcal{G}}\dot{H}+H^3\dot{\mathcal{G}}\right) \\
-(k-1)(k-2)\dot{G}^2\left(H^2-2\dot{H}\right)+6\mathcal{G}^2\dot{H}\left(\dot{H}+2H^2\right)\Big]\geq 0 \quad \text{if } p<0 \\
\text{and} \\
f_0\mathcal{G}^{k-3}\Big[-4k(k-1)(k-2)\dot{\mathcal{G}}^2\left(2\dot{H}+5H^2\right) \\
-24k\mathcal{G}^2\left(6H^2\dot{H}+\dot{H}^2+7H^4\right) \\
-4k(k-1)\mathcal{G}\left(2\ddot{\mathcal{G}}\dot{H}+5H^2\ddot{\mathcal{G}}+4H\dot{\mathcal{G}}\dot{H}+13H^3\dot{\mathcal{G}}\right)+\mathcal{G}^3\Big]\geq 0 \quad \text{if } p\geq 0
\end{cases}
$$

$$
SEC \begin{cases}
4f_0 k\mathcal{G}^{k-3}\Big[(k-1)\mathcal{G}\left(2\ddot{\mathcal{G}}\dot{H}-H^2\ddot{\mathcal{G}}+4H\dot{\mathcal{G}}\dot{H}+H^3\dot{\mathcal{G}}\right) \\
-(k-1)(k-2)\dot{G}^2\left(H^2-2\dot{H}\right)+6\mathcal{G}^2\dot{H}\left(\dot{H}+2H^2\right)\Big]\geq 0 \\
\text{and} \\
f_0\mathcal{G}^{k-3}\Big[12k(k-1)(k-2)\dot{\mathcal{G}}^2\left(2\dot{H}+H^2\right) \\
+24k\mathcal{G}^2\left(10H^2\dot{H}+3\dot{H}^2+7H^4\right) \\
+12k(k-1)\mathcal{G}\left(2\ddot{\mathcal{G}}\dot{H}+H^2\ddot{\mathcal{G}}+4H\dot{\mathcal{G}}\dot{H}+5H^3\dot{\mathcal{G}}\right)-\mathcal{G}^3\Big]\geq 0\,.
\end{cases}
$$

With the aim to study the validity and the violation of the energy conditions, we introduce the cosmographic parameters j (jerk), q (deceleration) and s (snap), defined as:

$$q=-1-\frac{\dot{H}}{H^2} \quad j=1+\frac{\ddot{H}+3\dot{H}H}{H^3} \quad s=1+\frac{\dddot{H}+3\ddot{H}H+3\dot{H}^2+6H^2\dot{H}+H\ddot{H}}{H^4}\,; \tag{4.59}$$

in particular, we can use the experimental values of q, j, s in order to study the behavior of the energy conditions as a function of k. First, notice that the cosmological expression of the Gauss–Bonnet scalar $\mathcal{G}=24H^2(H^2+\dot{H})$ can be recast in terms of the cosmographic parameters as:

$$\mathcal{G}=-24qH^4 \qquad \dot{\mathcal{G}}=24(3q+2q^2+j)H^5 \qquad \ddot{\mathcal{G}}=24(s-12q-15q^2-2q^3-6j-6jq)H^6\,. \tag{4.60}$$

Therefore, the pressure and the energy density become:

$$
\begin{aligned}
\rho^{GF} &= \frac{f_0 2^{3k-1}3^k\left(-H^4 q\right)^k}{q^3}\Big\{2j(k-1)kq[k(2q+3)-q-4] \\
&+j^2k(k-1)(k-2)+q[k^3q(2q+3)^2-k^2(10q^3+25q^2+21q+s) \\
&+k(6q^3+9q^2+15q+s)+q^2]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
p^{GF} &= \frac{f_0 2^{3k-1}3^{k-1}\left(-H^4 q\right)^k}{q^2}\Big\{2j^2k(k-1)(k-2)+jk(k-1)[4(2k-1)q^2 \\
&+4(3k-4)q+3]+2k^3q^2(2q+3)^2-k^2q(20q^3+50q^2+36q+2s-9) \\
&+2kq(6q^3+10q^2+15q+s-6)-3q^2]\Big\},
\end{aligned}
$$

respectively. Though the value of the term $\left(-qH^4\right)^k$ depends on the specific k considered, it is a total factor that multiplies all the rest of the inequality. Moreover, the numerical value of the deceleration parameter is usually negative, so that such total term can be neglected in the computation of the energy conditions. First we choose $f_0 > 0$, in order to recover the GR coupling as soon as $k = 1/2$. Considering Eqs. (4.59) and (4.60), and using the experimental values of j, k, s provided in [145] ($q = -0.81$, $j = 2.16$, $s = -0.22$), the energy conditions are satisfied for:

$$
\begin{aligned}
&\text{NEC} \to k \le 0 \vee 1.457 \le k \le 2.977 \\
&\text{WEC} \to k \le 0 \\
&\text{DEC} \to k \le 0 \\
&\text{SEC} \to 1.457 \le k \le 2.977.
\end{aligned} \tag{4.61}
$$

It is worth remarking that the above result holds regardless of the value of the Hubble constant H, since it multiplies all the energy conditions as a total factor with an even power. As we can see, no values of k simultaneously satisfy all the energy conditions when

$f_0 > 0$. Considering $f_0 < 0$, the energy conditions provide

$$\begin{aligned}
&\text{NEC} \rightarrow 0 \leq k \leq 1.457 \vee k \geq 2.977 \\
&\text{WEC} \rightarrow 0.093 \leq k \leq 1.457 \vee k \geq 2.977 \\
&\text{DEC} \rightarrow 0.113 \leq k \leq 1.457 \vee k \geq 2.977 \\
&\text{SEC} \rightarrow 0 \leq k \leq 0.189,
\end{aligned} \tag{4.62}$$

and are simultaneously satisfied for k comprehended in the range

$$0.113 < k < 0.189. \tag{4.63}$$

By considering the vacuum solution of the field equations in $f(\mathcal{G})$ cosmology (4.28), we notice that also the scale factor is constrained by the energy conditions. According to the values of k selected when $f_0 < 0$, the scale factor (4.28) respects the energy conditions if

$$a(t) = a_0 t^n \text{ with } 0.24 \leq n \leq 0.55. \tag{4.64}$$

Nevertheless, the energy conditions validity for other values of k and n might occur in a different epoch, where the behavior of the gravitational interaction was different than the current one. In this case, the function labeling the theory might assume the form given by Eq. (4.41), *i.e.*

$$f(\mathcal{G}) = f_0 \mathcal{G}^{k_1} + f_1 \mathcal{G}^{k_2}. \tag{4.65}$$

For weakly time-depending coupling constants, the possibility that at some epochs the contribution of f_1 (f_0) was predominant with respect to that of f_0 (f_1), can be taken into account.

4.3.1.1 Slow–Roll Inflation

When the Big Bang model was largely affirmed as the best candidate to describe the initial phase of our Universe, it was soon evident that this prescription was not in agreement with standard GR. The nowadays observed spatial flatness of the Universe cannot be explained by Einstein's gravity, as well as the reason why photons of the CMB coming from opposite directions today have the same temperature, while the size of causally connected regions at the last scattering is at most one degree. When the Hilbert–Einstein action is considered, these problems can be solved only by imposing *ad hoc* initial conditions. The nowadays accepted vision of the inflation considers a minimal coupling between geometry and a scalar field ϕ, so that the starting action is:

$$S = \frac{1}{2}\int \sqrt{-g}\left[R + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right] d^4x. \tag{4.66}$$

In this way, the field equations together with the Klein-Gordon equation, can be written in a cosmological background as:

$$\begin{cases} \dfrac{\dot{a}^2}{a^2} = \dfrac{1}{6}\left[\dfrac{\dot{\phi}^2}{2} + V(\phi)\right] - \dfrac{1}{a^2} \\ \dfrac{\ddot{a}}{a} = -\dfrac{1}{6}\left[\dot{\phi}^2 - V(\phi)\right] \\ \ddot{\phi} + 3\dfrac{\dot{a}}{a}\dot{\phi} + \partial_\phi V(\phi) = 0. \end{cases} \tag{4.67}$$

Slow-roll inflation, mentioned at the beginning of this section, can be also realized in modified theories of gravity, imposing the magnitude of the slow–roll parameters ε and η to be small, namely

$$|\varepsilon| \equiv \left|-\frac{\dot{H}}{H^2}\right| \ll 1 \quad |\eta| \equiv \left|-\frac{\ddot{H}}{2H\dot{H}}\right| \ll 1. \tag{4.68}$$

The first condition comes from the requirement for an accelerated expansion of the Universe through the relation

$$\frac{\ddot{a}}{a} = H^2\left(1 + \frac{\dot{H}}{H^2}\right) = H^2\left(1 - \varepsilon\right) > 0. \tag{4.69}$$

With regards to the second condition, inflation occurs when the potential energy is predominant with respect to the other terms. Then, the potential should have a minimum at the end of the inflation. In the slow–roll approximation, where the scalar field is expected to roll slowly, the field equations together with the Klein–Gordon equation yield the condition

$$-\frac{\ddot{\phi}}{H\dot{\phi}} = -\frac{\ddot{H}}{2H\dot{H}} \ll 1. \tag{4.70}$$

Inflation ends when the potential starts decreasing towards its minimum, so that the velocity of the scalar field cannot be neglected if compared to the Universe expansion. Therefore, the scalar field approaches the minimum of the potential and starts oscillating around the equilibrium position. Such an oscillations yields particles creation, which dissipate the energy of the scalar field raising the temperature of the Universe. This thermalization is called *reheating*, a phenomenon which has dropped during the inflationary expansion. This superluminal expansion of the early Universe is able to solve both the flatness problem and the temperature of photons coming from CMB. All the cosmic structures began to grow after the end of the inflationary and radiation epochs. Mathematically speaking, the end of inflation occurs when the slow-roll parameters become of order unity.

Similarly, in modified theories of gravity, the extra geometric terms can play the role of kinetic energy and potential of some time–depending scalar field, naturally providing inflation without any minimal coupling.

Here we use the previously written energy conditions, in order to find out the expression of the slow–roll parameters ε and η in terms of k. To this purpose, we rewrite the first two components of the field equations (4.49) as:

$$\begin{aligned} H^2 &= \frac{1}{3(2R-4\Box)f_{\mathcal{G}}(\mathcal{G})}\left\{\frac{f_0}{2}\mathcal{G}^{k-3}\left[-24k(k-1)H^2\left(\mathcal{G}\ddot{\mathcal{G}}+(k-2)\dot{\mathcal{G}}^2\right)\right.\right. \\ &\quad \left.\left. -72k\mathcal{G}^2H^4-48k\mathcal{G}H^2\left(2\mathcal{G}(\dot{H}+H^2)+(k-1)H\dot{\mathcal{G}}\right)+\mathcal{G}^3\right]\right\}, \end{aligned}$$

$$
\begin{aligned}
\dot{H} \;=\; & -\frac{3}{2}H^2 - \frac{1}{2(2R-4\Box)f_{\mathcal{G}}(\mathcal{G})}\left\{\frac{f_0}{2}\mathcal{G}^{k-3}\left[16k(k-1)(\dot{H}+H^2)\left(\mathcal{G}\ddot{\mathcal{G}}+(k-2)\dot{\mathcal{G}}^2\right)\right.\right. \\
& +24k\mathcal{G}^2H^4 + 16k\mathcal{G}(\dot{H}+H^2)\left(3\mathcal{G}(\dot{H}+H^2)+2(k-1)H\dot{\mathcal{G}}\right) \\
& \left.\left.+24k\mathcal{G}H^2\left(4\mathcal{G}(\dot{H}+H^2)+(k-1)H\dot{G}\right)-\mathcal{G}^3\right]\right\}.
\end{aligned}
$$

Replacing the vacuum solution of $f(\mathcal{G})$ gravity (4.28) in the above equations and considering the explicit expression of the Gauss–Bonnet term

$$\mathcal{G} = 24H^2(H^2+\dot{H}), \tag{4.71}$$

the magnitude of the slow–roll parameters turns out to be

$$|\varepsilon| = |\eta| = \left|\frac{1}{1-4k}\right|. \tag{4.72}$$

The conditions for the inflation

$$|\varepsilon| \ll 1 \qquad |\eta| \ll 1, \tag{4.73}$$

provide the constraint

$$k \ll 0 \vee k \gg \frac{1}{2}. \tag{4.74}$$

This means that cosmological inflation in $f(\mathcal{G})$ gravity occurs only when k is strictly negative or when it is much higher than $1/2$. Interestingly notice that the value $k = 1/2$ is the limit in which $f(\mathcal{G}) = \mathcal{G}^k$ gravity behaves like Einstein GR in a cosmological spatially flat background. As before, we also notice that the deceleration parameters can be written in terms of k as:

$$q = -1 + \frac{1}{4k-1}, \tag{4.75}$$

so that when $k \ll 0 \vee k \gg 1/2$, q turns out to be negative as we expect. The negative value of q corresponds to an accelerated expansion of the Universe, in agreement with the result provided by Eq. (4.74). Slow–roll conditions are not sufficient to establish whether the theory is able to fit the cosmological data, and a further study is necessary. Nevertheless, it is a first step aimed at verifying whether the theory might be a good candidate for the inflationary model.

4.3.2 Energy Conditions in $R + f(\mathcal{G})$ Cosmology

With the aim to compare $f(\mathcal{G})$ with $R + f(\mathcal{G})$ gravity, we study the energy conditions for this latter theory. Therefore, we consider a starting action of the form

$$S = \int \sqrt{-g}\left(\frac{R}{2} + f_0\mathcal{G}^k\right) d^4x \tag{4.76}$$

and study the energy conditions by using cosmographic parameters. For general k, the field equations can be written as

$$\begin{aligned} G_{\mu\nu} &= T_{\mu\nu} + \frac{f_0}{2}\mathcal{G}^k g_{\mu\nu} - kf_0\Big(2RR_{\mu\nu} - 4R_{\mu p}R^p{}_\nu + 2R_\mu{}^{p\sigma\tau}R_{\nu p\sigma\tau} \\ &\quad -4R^{\alpha\beta}R_{\mu\alpha\nu\beta}\Big)\mathcal{G}^{k-1} + kf_0\Big(2RD_\mu D_\nu + 4G_{\mu\nu}\Box - 4R^p_{\{\nu}D_{\mu\}}D_p \\ &\quad +4g_{\mu\nu}R^{p\sigma}D_pD_\sigma - 4R_{\mu\alpha\nu\beta}D^\alpha D^\beta\Big)\mathcal{G}^{k-1}\,, \end{aligned} \tag{4.77}$$

so that the RHS can be intended as an effective energy-momentum tensor, which vanishes as soon as $f_0 = 0$. In a cosmological spatially flat space-time, the not null components of the field equations read

$$G^0_0 = \rho_0 + \frac{f_0}{2}\mathcal{G}^{k-2}\Big\{24kH^2\Big[(k-1)H\dot{\mathcal{G}} - \mathcal{G}(\dot{H} + H^2)\Big] + 1\Big\} \tag{4.78}$$

$$\begin{aligned} G^1_1 &= -\Big\{p_0 - \frac{f_0}{2}\mathcal{G}^{k-3}\Big[-24H^2k\mathcal{G}^2\left(H^2+\dot{H}\right) + \mathcal{G}^3 + \\ &\quad +8H(k-1)k\Big(2\mathcal{G}\left(H^2+\dot{H}\right)\dot{\mathcal{G}} + H\left(\mathcal{G}\ddot{\mathcal{G}} + (k-2)\dot{\mathcal{G}}^2\right)\Big)\Big]\Big\}\,. \end{aligned} \tag{4.79}$$

As before, the second terms in the RHS of the above equations can be intended as effective energy densities and pressures of the gravitational field. In this way, Eqs. (4.78) and (4.79) can be written in terms of the cosmographic parameters (4.59) as:

$$\rho^{GF} = \frac{2^{3k-1}3^k f_0(-H^4 q)^k}{q^2}(k-1)[jk - q^2 + kq(3+2q)] \tag{4.80}$$

$$\begin{aligned} p^{GF} = & \frac{2^{3k-1}3^{k-1} f_0(-H^4 q)^k}{q^3}(k-1)[j^2 k(k-2) + 2jkq(-3+3k+2kq) \\ & +3q^3 + k^2 q^2(3+2q)^2 - kq(6q+3q^2+2q^3+s)], \end{aligned} \tag{4.81}$$

where we used the form of $\mathcal{G}$ given by Eq. (4.60). As we did in the previous section, we assume that the density and the pressure of matter must satisfy the energy conditions separately, and use the same values of the cosmographic parameters. In this way, when $f_0 > 0$ the energy conditions are satisfied for:

$$\begin{aligned} &\text{NEC} \rightarrow k \leq 0 \vee 1 \leq k \leq 4.371 \\ &\text{WEC} \rightarrow k \leq 0 \vee 1 \leq k \leq 4.371 \\ &\text{DEC} \rightarrow -1.866 \leq k \leq 0 \vee 1.573 \leq k \leq 4.371 \\ &\text{SEC} \rightarrow k \leq -0.313 \vee 1 \leq k \leq 3.129\,. \end{aligned} \tag{4.82}$$

The above inequalities admit a common solution, that is

$$-1.866 \leq k \leq -0.313 \vee 1.573 \leq k \leq 3.129. \tag{4.83}$$

Moreover, as showed in Ref. [146], vacuum solutions of the field equations yield an exponential scale factor when $k \neq 1/2$, namely

$$a(t) = a_0 \exp\left\{\left[\frac{24^k f_0(1-k)}{3}\right]^{\frac{1}{2-4k}}\right\}. \tag{4.84}$$

In the first range of Eq. (4.83), that is $-1.866 \leq k \leq -0.313$, the argument of the exponential function is positive and the scale factor describes an exponentially accelerated universe. In the second range, when $1.573 \leq k \leq 3.129$, the argument turns out to be negative, leading to a bouncing cosmological model. When $k = 1/2$ another solution occurs, that is

$$a(t)_{\pm} = a_0 t^{\frac{4f_0^2+3\pm\sqrt{3}\sqrt{16f_0^2+3}}{2\left(3-2f_0^2\right)}}. \tag{4.85}$$

In this case, the scale factor $a(t)_+$ describes an accelerating universe when

$$-\sqrt{\frac{3}{2}} < f_0 < \sqrt{\frac{3}{2}}. \tag{4.86}$$

On the contrary, when $a_-(t)$ is considered, the power of t is always negative regardless of the value of f_0. Since $k = 1/2$ violates all the inequalities (4.82), when $f_0 > 0$ and $k = 1/2$ a power-law universe acceleration occurs and energy conditions are violated. Interestingly notice that Eq. (4.86) provides the same range as Eq. (4.92). This means that only an energy condition violation may lead to an inflationary universe.
Assuming now a negative coupling constant, namely $f_0 < 0$, the energy conditions yield the constraints

$$\begin{aligned}
&\text{NEC} \rightarrow k \leq 0 \vee 1 \leq k \leq 4.371 \\
&\text{WEC} \rightarrow 0.630 < k < 1 \\
&\text{DEC} \rightarrow \nexists k \in \mathbb{R} \\
&\text{SEC} \rightarrow 0 \leq k \leq 1 \vee k \geq 4.371
\end{aligned} \tag{4.87}$$

and cannot be simultaneously satisfied for any real value of k. However, it is worth noticing that all those k comprehended in the range $1 < k < 4.371$ violate the SEC, which means that the geometric contributions in $R + f(\mathcal{G})$ gravity can act as a repulsive source of gravitational field.

4.3.2.1 Slow–Roll Inflation

Here we show that inflation can be realized also by including the scalar curvature into the action. Therefore, we start from the action (4.76), namely

$$S=\int\sqrt{-g}\left(\frac{R}{2}+f_0\mathcal{G}^k\right)d^4x, \tag{4.88}$$

in order to write the corresponding field equations in a cosmological spatially flat background. With the aim to constrain the value of the coupling constant f_0 admitting slow-roll inflation, we consider the vacuum field equations of $R+f_0\sqrt{\mathcal{G}}$ gravity, namely

$$H^2 = \frac{f_0}{6}\mathcal{G}^{-\frac{3}{2}}\left\{12H^2\left[-\frac{1}{2}H\dot{\mathcal{G}}-\mathcal{G}(\dot{H}+H^2)\right]+1\right\} \tag{4.89}$$

$$\begin{aligned}\dot{H} &= -\frac{3}{2}H^2+\frac{f_0}{4}\mathcal{G}^{-\frac{5}{2}}\left[-12H^2\mathcal{G}^2\left(H^2+\dot{H}\right)+\mathcal{G}^3+\right.\\ &\left.-2H\left(2\mathcal{G}\left(H^2+\dot{H}\right)\dot{\mathcal{G}}+H\left(\mathcal{G}\ddot{\mathcal{G}}-\frac{3}{2}\dot{\mathcal{G}}^2\right)\right)\right],\end{aligned} \tag{4.90}$$

whose only power-law solution is

$$a_\pm(t)=a_0\,t^{\frac{4f_0^2+3\pm\sqrt{3}\sqrt{16f_0^2+3}}{2\left(3-2f_0^2\right)}}. \tag{4.91}$$

Replacing a_+[1] in the field equations (4.89) and considering the conditions for the inflation in Eq. (4.68), it turns out that slow-roll inflation is admitted by $R+f_0\sqrt{\mathcal{G}}$ theory of gravity when

$$f_0\sim\pm\sqrt{\frac{3}{2}}. \tag{4.92}$$

In particular, the more f_0 approaches the values $\pm\sqrt{3/2}$, the faster the scalar field rolls down the potential hill. These values are in agreement with the range provided by the

[1]No coupling constants are selected by $a_-(t)$

energy conditions in Eq. (4.86).

5

Modified Gauss–Bonnet Black Holes

Here we adopt Noether symmetry approach for $f(\mathcal{G})$ gravity in a $d+1$ dimensional spherically symmetric background, with the aim to find out suitable black holes solutions. Let us consider now a static and spherically symmetric *ansatz* for the metric, that reads

$$ds^2 = P(r)^2 dt^2 - Q(r)^2 dr^2 - r^2 d\Omega^2_{d-1}\,, \tag{5.1}$$

where $d\Omega^2_{d-1} = \sum_{j=1}^{d-1} d\theta_j^2 + \sin^2\theta_j d\phi^2$ is the metric element of the $(d-1)$-sphere, for a space-time labeled by coordinates $x^\mu = (t, r, \theta_1, \theta_2, ..., \theta_{d-2}, \phi)$. Before proceeding, an important comment is necessary here; we assume that the metric (5.1) is not dynamical, which means that Birkhoff's theorem should be valid for these models. This is not proven and we take it for granted in theories such as (3.10). However, there are a lot of references in the literature claiming to have found cases where a generalization of Birkhoff's theorem could exist [147, 148, 149, 150, 151, 152].

Starting from the metric (5.1), the not-null components of the D-dimensional Riemann tensor, Ricci tensor and Ricci scalar are:

$$
\begin{aligned}
R_{0101} &= P\left(\frac{P'Q'}{Q} - P''\right) \quad R_{0i0j} = -\frac{rP'P}{Q^2}\delta_{ij}\Big|_{i,j\geq 2} \\
R_{1i1j} &= -\frac{rQ'}{Q}\delta_{ij}\Big|_{i,j\geq 2} \quad R_{ikj\ell} = -r^2\left(1-\frac{1}{Q^2}\right)\delta_{ij}\delta k\ell\Big|_{i,j,k,\ell\geq 2} \\
R_{00} &= -\frac{P\left[rP'Q' - (d-1)QP' - rQP''\right]}{rQ^3} \quad R_{11} = \frac{(d-1)PQ' + r(P'Q' - QP'')}{rPQ} \\
R_{ij} &= d-2-\frac{(d-2)P + rP'}{PQ^2} + \frac{rQ'}{Q^3}\delta_{ij}\Big|_{i,j\geq 2} \\
R &= \frac{2r\left\{Q\left[(d-1)P' + rP''\right] - rP'Q'\right\} + (1-d)P\left[(d-2)Q^3 + (2-d)Q + 2rQ'\right]}{r^2PQ^3},
\end{aligned}
\tag{5.2}
$$

so that the Gauss–Bonnet term in arbitrary $(d+1)$ dimensions takes the form

$$
\begin{aligned}
\mathcal{G}^{(d+1)} = \frac{(d-2)(d-1)}{r^4PQ^5} \quad & \Big\{(d-3)P\left(Q^2-1\right)\left[(d-4)Q^3 - (d-4)Q + 4rQ'\right] \\
& -4r\Big[(d-3)Q^3P' + rQ^3P'' - (d-3)QP' - rQP'' \\
& -rQ^2P'Q' + 3rP'Q'\Big]\Big\} \,,
\end{aligned}
\tag{5.3}
$$

where the prime denotes the derivative with respect to the radial coordinate and we set for simplicity $\theta_j = \pi/2$. Note that for $d \leq 2$ (*i.e* in less than four dimensions), the above scalar vanishes identically, while for $d = 3$, it becomes a topological surface term, as it happens in the cosmological case.

In order to calculate the point-like Lagrangian of the theory with respect to the line element (5.1), we introduce a Lagrange multiplier as [74, 153, 154]

$$
\mathcal{S} = \int d^{d+1}x\, r^{d-1}PQ\left[f(\mathcal{G}) - \lambda\left(\mathcal{G} - \tilde{\mathcal{G}}\right)\right] \,, \tag{5.4}
$$

with $\tilde{\mathcal{G}}$ being the Gauss–Bonnet term in spherical symmetry (5.3) and λ the Lagrange multiplier given by varying the action with respect to $\mathcal{G}$, i.e. $\lambda = \partial_{\mathcal{G}} f$. Substituting $\tilde{\mathcal{G}}$ and integrating out the second derivatives, we obtain

$$\begin{aligned}\mathscr{L}(r,P,Q,\mathcal{G}) =& r^{d-1}PQ\left[f-\mathcal{G}f_{\mathcal{G}}\right] \\ &+\frac{(d-1)(d-2)r^{d-5}(Q^2-1)}{Q^4}\Big\{(d-3)Pf_{\mathcal{G}}\Big[(d-4)Q(Q^2-1)+4rQ'\Big] \\ &+4r^2QP'\mathcal{G}'f_{\mathcal{G}\mathcal{G}}\Big\}\,. \end{aligned} \tag{5.5}$$

This is the point-like canonical Lagrangian of our theory in a static and spherically symmetric space-time. Its configuration space is $\mathcal{S} = \{P, Q, \mathcal{G}\}$, and the tangent space $\mathcal{TS} = \{P, P', Q, Q', \mathcal{G}, \mathcal{G}'\}$.

5.1 Research for Symmetries

The generator of the point transformations (B.9), in our case, is given by

$$\mathcal{X} = \xi(r,\mathcal{G},P,Q)\partial_r + \eta^{\mathcal{G}}(r,\mathcal{G},P,Q)\partial_{\mathcal{G}} + \eta^{P}(r,\mathcal{G},P,Q)\partial_P + \eta^{Q}(r,\mathcal{G},P,Q)\partial_Q\,, \tag{5.6}$$

where ξ and η^i, with $i = \{\mathcal{G}, P, Q\}$, are components of the vector X. By applying the Noether theorem in Eq. (B.7), we obtain a system of twelve equations, which are not all independent. The resolution of the system provides only two different non-trivial solutions for the $f(\mathcal{G})$ function, that we outline below (see [155] for details).

- **Case 1:** In more than three dimensions, we have $f(\mathcal{G}) = f_0\mathcal{G}^k$ with $k \neq 1$. The symmetry generator for this model is given by

$$\mathcal{X} = c_1 r\partial_r - 4c_1\mathcal{G}\partial_{\mathcal{G}} + (4k-d)c_1 P\partial_P \tag{5.7}$$

and $g = c_2$, with c_1 and c_2 being constants. The invariant quantity (B.19), related to the above symmetry (5.7), is

$$
\begin{aligned}
I =& \frac{1}{Q^3} c_1 f_0 r^{d-4} \mathcal{G}^{k-2} \Bigg[(1-k) r^4 \mathcal{G}^2 P Q^4 + \\
& - 4k(k-1)(d-2)(d-1) r \left(Q^2 - 1\right) \left(r P' + (d-4k) P\right) \mathcal{G}' + \\
& - (d-2)(d-1) k \mathcal{G} \left(Q^2 - 1\right) \left[16(k-1) r P' - (d-4)(d-3) P \left(Q^2 - 1\right) \right] \Bigg] - c_2 \,.
\end{aligned}
\tag{5.8}
$$

It is interesting to point out that the form of $f(\mathcal{G})$ selected by symmetries is a power-law functions, as well as Chap. 4.

- **Case 2:** In five dimensions ($d = 4$) there is also the possibility to have a linear model of the form $f(\mathcal{G}) = f_0 \mathcal{G}$. Its Noether symmetry reads

$$
\mathcal{X} = c_1 r \partial_r + c_2 \partial_P \,, \tag{5.9}
$$

and $g = c_3 - \frac{8 f_0 c_2 (3Q^2 - 1)}{Q^3}$, with c_1, c_2 and c_3 being constants. The preserved quantity related to the generator (5.9) is

$$
I = -\frac{8 f_0}{Q^3} \left(\frac{6r \left(Q^2 - 3\right) \left(c_1 r P' - c_2\right) Q'}{Q} + c_2 \left(3Q^2 - 5\right) \right) - c_3 \,. \tag{5.10}
$$

5.1.1 Spherically Symmetric solutions

From the Noether theorem (B.7), we can build the following Lagrange system

$$
\frac{dt}{\xi} = \frac{dq^i}{\eta^i} = \frac{d\dot{q}^i}{\eta^{[1],i}} \,, \tag{5.11}
$$

which, with the help of Eq. (5.7), yields the zero and first order invariants

$$W^{[0],\mathcal{G}}(r,\mathcal{G}) = \frac{dr}{c_1 r} - \frac{d\mathcal{G}}{-4c_1\mathcal{G}} = \mathcal{G}r^4\,, \tag{5.12}$$

$$W^{[0],P}(r,P) = \frac{dr}{c_1 r} - \frac{dP}{(4k-d)c_1 P} = Pr^{d-4k}\,, \tag{5.13}$$

$$W^{[1],\mathcal{G}}(r,\mathcal{G}) = \frac{dr}{c_1 r} - \frac{d\mathcal{G}}{-4c_1\mathcal{G}} - \frac{d\mathcal{G}'}{-5c_1\mathcal{G}'} = \mathcal{G}'r^5\,, \tag{5.14}$$

$$W^{[1],P}(r,P) = \frac{dr}{c_1 r} - \frac{dP}{(4k-d)c_1 P} - \frac{dP'}{(4k-d-1)c_1 P'} = P'r^{1+d-4k}\,, \tag{5.15}$$

by means of which the equations of motion can be reduced from the second to the first order, providing analytic solutions. The Lagrangian (5.5) for the **Case 1**, i.e. $f = f_0\mathcal{G}^k$, becomes

$$\mathscr{L} = \frac{f_0 r^{d-5}\mathcal{G}^{k-2}}{Q^4}\Big\{\mathcal{G}PQ\Big[(d-4)(d-3)(d-2)(d-1)k\left(Q^4-2Q^2+1\right) - \mathcal{G}(k-1)Q^4r^4\Big] + 4(d-1)(d-2)k\left(Q^2-1\right)r\Big[(d-3)\mathcal{G}PQ' + (k-1)rQ\mathcal{G}'P'\Big]\Big\} \tag{5.16}$$

and the associated Euler-Lagrange equations $\dfrac{\partial \mathscr{L}}{\partial q^i} = \dfrac{d}{dr}\dfrac{\partial \mathscr{L}}{\partial q'^i}$ are

$$\begin{aligned}
P :& (k-1)r^4Q^5 + \Big\{4(d-3)(d-2)(d-1)k(k-1)rQ(Q^2-1)\mathcal{G}'\mathcal{G}^{-2} \\
&+ 4(d-2)(d-1)(k-2)(k-1)kr^2Q(Q^2-1)\mathcal{G}'^2\mathcal{G}^{-3} \\
&+ 8(d-2)(d-1)(k-1)kr^2Q^2Q'\mathcal{G}'\mathcal{G}^{-2} \\
&- 12(d-2)(d-1)(k-1)kr^2(Q^2-1)Q'\mathcal{G}'\mathcal{G}^{-2} \\
&- (d-3)(d-2)(d-1)k(Q^2-1)\Big[(d-4)Q(Q^2-1)+4rQ'\Big]\mathcal{G}^{-1} \\
&+ 4(d-2)(d-1)(k-1)kr^2Q(Q^2-1)\mathcal{G}''\mathcal{G}^{-2} = 0,
\end{aligned} \tag{5.17}$$

$$\begin{aligned}
Q :& (k-1)r^4\mathcal{G}^2PQ^4 \\
&- (d-3)(d-2)(d-1)k\mathcal{G}(Q^2-1)\Big[(d-4)P(Q^2-1)-4rP'\Big] \\
&+ 4(d-2)(d-1)(k-1)kr\mathcal{G}'\Big[(d-5)P(Q^2-1)+r(Q^2-3)P'\Big] = 0\,,
\end{aligned} \tag{5.18}$$

$$\mathcal{G} : \mathcal{G}(r) = \tilde{\mathcal{G}}(r), \tag{5.19}$$

for P, Q and $\mathcal{G}$, respectively.

Solving Eq. (5.19) with respect to $\mathcal{G}(r)$, we recover the spherically symmetric expression of the Gauss–Bonnet scalar $\mathcal{G}(r) = \tilde{\mathcal{G}}$, given by Eq. (5.3). Moreover, Eq. (5.17) and Eq. (5.18) can be further simplified by setting $Q(r) = 1/P(r)$, so that the system of differential equations ends up with one equation of the form

$$\begin{aligned}
(d-1)(d-2)\tilde{\mathcal{G}}^k\Bigg\{&(d-3)(P^2-1)\left[(d-4)(P^2-1)-4(k-2)rPP'\right] - \\
&- 4(k-1)r^2\left[(P^2-1)PP''+(3P^2-1)P'^2\right]\Bigg\} = 0\,.
\end{aligned} \tag{5.20}$$

Notice that the *ansatz* $Q(r) = 1/P(r)$ is needed to find the dynamics of D-dimensional $f(\mathcal{G})$ gravity in spherical symmetry. Moreover, it allows to recover the Schwarzschild solution as a particular limit.

Obviously, for $d = 1, 2$ Eq. (5.20) is automatically satisfied. The rest of the equation accepts three solutions which read

$$P(r)^2 = 1 + e^{-2k_2}\sqrt{k_1 - 4r}\, r^{\frac{3}{2}-\frac{d}{2}} \text{ and } \mathcal{G}(r) = 0 \quad k \neq 1 \quad d \geq 3, \tag{5.21}$$

$$P(r)^2 = P_0^2\left(1 - \frac{k_3}{r^{\frac{d}{2}-2}}\right) \text{ and } \mathcal{G}(r) = 0 \quad k = 1 \quad d \geq 4\,, \tag{5.22}$$

$$P(r)^2 = 1 \pm r^{2-\frac{d}{2}}\sqrt{\frac{4k_1 d}{120\binom{d+1}{d-4}}} \pm r^2\sqrt{\frac{\mathcal{G}_0(d-3)}{120\binom{d+1}{d-4}}} \text{ and } \mathcal{G}(r) = \mathcal{G}_0, \quad f(\mathcal{G}) = f_0 \mathcal{G}^{\frac{d+1}{4}} \quad d \geq 4\,, \tag{5.23}$$

with $k_1, k_2, k_3, P_0, \mathcal{G}_0$ constants. These are general black hole solutions for the theory (3.10) with $f(\mathcal{G}) = f_0\mathcal{G}^k$; in particular, the first one is valid in arbitrary d dimensions with $d \geq 3$, while the others hold in more than four dimensions. Solution (5.23), which is the (A)dS equivalent of $f(\mathcal{G})$ gravity, holds for any $k = \dfrac{d+1}{4}$, in agreement with the trace equation (3.12). In any case, the asymptotic flatness is always recovered in more than five dimensions, as well as the presence of horizons. Specifically, solutions (5.22) and (5.21) admit as horizon $r_S \sim (GM)^{\frac{2}{d-4}}$.

Let us now see some more specific solutions of the system (5.17)-(5.19), analyzing the boundary cases $d = 3$ and $d = 4$. In $d = 3$ we have the following solution for any $P(r)$

$$Q(r) = \frac{1}{3}\left(A(r) - e^{q_0}P'(r) + \frac{e^{2q_0}}{A(r)}P'(r)^2\right)\,, \tag{5.24}$$

with

$$A(r) = \left(\frac{27e^{q_0}}{2}P'(r) - e^{3q_0}P'(r)^3 + \frac{3e^{q_0}}{2}P'(r)\sqrt{81 - 12e^{2q_0}P'(r)^2}\right)^{1/3}.$$

q_0 is an integration constant and the Gauss–Bonnet term vanishes in this case. As an example, by introducing the relation $Q(r) = P(r)^n$, we find that the field equations are satisfied by any $P(r)$ solving the equation

$$k(P^{2n} - 3)P'^2 - P(P^{2n} - 1)P'' = 0. \tag{5.25}$$

The limit $n = -1$ provides back solution (5.21); an interesting analytic solution of Eq. (5.25) occurs for $n = -1/3$, where the components of the interval are:

$$P(r)^2 = -2c_1 \left[(r + c_2) \left(\frac{6r}{M(r)} + 1 \right) \right] + \frac{3}{8} \left[\frac{M(r)^2 + 9}{M(r)} + 3 \right],$$

with

$$\begin{aligned} M(r) \;&=\; \sqrt[3]{128c_1^2 r^2 + 16\left(16c_1c_2 - 9\right)c_1 r + } \\ &\overline{+64\sqrt{c_1^3\left(c_2 + r\right)^3\left(4c_1 r + 4c_2c_1 - 1\right)} +} \\ &\overline{+128c_2^2c_1^2 - 144c_2c_1 + 27}. \end{aligned} \tag{5.26}$$

Moreover, in $d = 4$ we only get the following solutions for constant $\mathcal{G}$

$$P(r)^2 = -\frac{1}{2} \exp\left[tanh^{-1}\left(\sqrt{\frac{\mathcal{G}_0}{30}} \frac{r^2}{2} \right) \right] \sqrt{4 - \frac{\mathcal{G}_0 r^4}{30}} \text{ and } Q(r)^{-2} = 1 + \frac{\sqrt{\mathcal{G}_0} r^2}{2\sqrt{30}} \text{ for } k = 5/4\,, \tag{5.27}$$

$$P(r)^2 = 1 = Q(r)^2 \text{ , for } \mathcal{G}_0 = 0 \text{ and } \forall k\,. \tag{5.28}$$

If we Taylor expand $P(r)^2$ in Eq. (5.27), we find that $P(r)^2 = Q(r)^{-2}$, which is an AdS-like solution, where $\frac{\sqrt{\mathcal{G}_0}}{2\sqrt{30}}$ can be considered as the bulk cosmological constant. To conclude, in Table **I** we outline the spherically symmetric solutions provided in this chapter.

Table I: *Exact static and spherically symmetric solutions in $f(G)$ gravity, with $f(\mathcal{G}) = f_0\mathcal{G}^k$, in arbitrary $d+1$ dimensions.*

$\mathbf{P(r)^2}$	$\mathbf{Q(r)^2}$	$\mathbf{d}$	$\mathbf{k}$
$1 + e^{-2c_2}\sqrt{c_1 - 4r}r^{\frac{3}{2}-\frac{d}{2}}$	$1/P(r)^2$	$d \geq 3$	$k > 0, \neq 1$
$P_0^2\left(1 - \frac{k_3}{r^{\frac{d}{2}-2}}\right)$	$1/P(r)^2$	$d > 3$	$k = 1$
$1 \pm r^{2-\frac{d}{2}}\sqrt{\frac{4k_1 d}{120\binom{d+1}{d-4}}} \pm r^2\sqrt{\frac{\mathcal{G}_0(d-3)}{120\binom{d+1}{d-4}}}$	$1/P(r)^2$	$d > 3$	$k = \frac{d+1}{4}$
$\forall P(r)$	$\frac{1}{3}\left(A(r) - e^{q_0}P'(r) + \frac{e^{2q_0}}{A(r)}P'(r)^2\right)$	$d = 3$	$k > 0$
$-\frac{1}{2}\exp\left[tanh^{-1}\left(\sqrt{\frac{\mathcal{G}_0}{30}}\frac{r^2}{2}\right)\right]\sqrt{4 - \frac{\mathcal{G}_0 r^4}{30}}$	$1 + \frac{\sqrt{\mathcal{G}_0}r^2}{2\sqrt{30}}$	$d = 4$	$k = 5/4$
1	1	$d = 4$	$\forall k$

6

Non–Local Theories

Before considering non-local functions of the Gauss–Bonnet term, let us review the basic aspects of non-local theories of gravity. First of all, they can be divided into two main categories: Infinite Derivative Theories of Gravity (IDGs) and Integral Kernel Theories of Gravity (IKGs). The former are used to be exponential functions of the D'Alembert operator [1] and to overcome UV shortcomings by means of a short-range non-locality. The latter mainly involve the inverse of the D'Alembert operator [2] $\Box^{-1}$ and, by means of long-range non-locality, they are capable of fixing, in principle, the IR problems of GR. The models treated in this paper involve functions of the operator $\Box^{-1}$, which will be applied to the Ricci scalar R and the Gauss–Bonnet invariant $\mathcal{G}$. For this reason, we outline only the properties of IKGs.

In general, the local corrections come from an expansion around the value $s = 0$ of a Schwinger proper time, so that they are valid for small times only, providing UV corrections. On the other hand, IR corrections are represented by the expansion around $s \to \infty$, where the proper time integration becomes divergent. This problem can be solved by considering a non-perturbative approach to calculate the Schwinger proper time integral which allows to capture both the effects of local UV contributions ($s = 0$) and of non-local IR corrections ($s \to \infty$).

[1] In general, they can be also trascendental functions of differential operators.

[2] They can involve integral kernels of differential operators

The corresponding quantum effective action in curved space-time reads [156]:

$$W_0 = -\int d^4x\sqrt{-g}\left[V(x) + V(x)(\Box - V)^{-1}V(x)\right] + \frac{1}{6}\,\boldsymbol{\Sigma}\,, \tag{6.1}$$

where $V(x)$ is the potential and $\boldsymbol{\Sigma}$ a surface term defined as

$$\begin{aligned}\boldsymbol{\Sigma} = \int d^4x\sqrt{-g}\Big\{R - R_{\mu\nu}\,\Box^{-1}G^{\mu\nu}2^{-1}R\left(\Box^{-1}R^{\mu\nu}\right)\Box^{-1}R_{\mu\nu} \\ - R^{\mu\nu}\Big(\Box^{-1}R_{\mu\nu}\Big)\Box^{-1}R\Big(\Box^{-1}R^{\alpha\beta}\Big)\Big(D_\alpha\,\Box^{-1}R\Big)D_\beta\,\Box^{-1}R \\ - 2\left(D^\mu\,\Box^{-1}R^{\nu\alpha}\right)\Big(D_\nu\,\Box^{-1}R_{\mu\alpha}\Big)\Box^{-1}R \\ - 2\left(\Box^{-1}R^{\mu\nu}\right)\Big(D_\mu\,\Box^{-1}R^{\alpha\beta}\Big)D_\nu\,\Box^{-1}R_{\alpha\beta} + O\Big[R^{\,4}_{\mu\nu}\Big]\Big\}\,.\end{aligned} \tag{6.2}$$

The integral operator $\Box^{-1}$ is the responsible for quantum corrections to GR. A simple action containing such an operator was proposed by Deser and Woodard in Ref. [61], where they presented a non-local modified effective theory of gravity capable of explaining the current late-time cosmic acceleration as a mechanism driven by the integral kernel of some differential operator; the corresponding action reads:

$$\mathcal{S} = \int d^4x\sqrt{-g}\,\left[\frac{R}{2} + F\Big(\Box^{-1}R\Big)\right] + \mathcal{S}^{(m)}\,, \tag{6.3}$$

where $F\Big(\Box^{-1}R\Big)$ is an arbitrary function of $\Box^{-1}R$. The field equations associated to the effective theory (6.3) are

$$G_{\mu\nu} + \Delta G_{\mu\nu} = T^{(m)}_{\mu\nu}\,, \tag{6.4}$$

where

$$\begin{aligned}\Delta G_{\mu\nu} = \Big(G_{\mu\nu} + g_{\mu\nu}\,\Box - D_\mu D_\nu\Big)\Big\{F + \Box^{-1}\Big[R\,F_{\Box^{-1}R}\Big]\Big\} \\ + \left[\delta_\mu^{\;(\rho}\,\delta_\nu^{\;\sigma)} - \frac{1}{2}\,g_{\mu\nu}g^{\rho\sigma}\right]\partial_\rho\Big(\Box^{-1}R\Big)\,\partial_\sigma\Big(\Box^{-1}\Big[R\,F_{\Box^{-1}R}\Big]\Big)\,,\end{aligned} \tag{6.5}$$

with the definitions $F \equiv F\Big(\Box^{-1}R\Big)$ and $F_{\Box^{-1}R} \equiv \dfrac{\partial F}{\partial\Big(\Box^{-1}R\Big)}$. It is straightforward to show that the intrinsic nature of operator $\Box^{-1}$ is able to predict naturally the late-time cosmological expansion of the Universe. In this regards, considering a power-law form of the scale factor given by the standard cosmological model

$$a(t) \sim t^q \qquad \Longrightarrow \qquad R(t) \sim 6q(1-2q)\, t^{-2}\,, \tag{6.6}$$

it is possible to approximately evaluate the quantity $\Big(\Box^{-1}R\Big)(t_0)$ at the present time. Indicating with $t_0 \sim 10^{10}\, y$ the current time and with $t_{eq} \sim 10^5\, y$ the time when the CMB radiation originated, the non-local causal effects acting within the interval $[t_{eq}, t_0]$ are:

$$\begin{aligned}\Big(\Box^{-1}R\Big)(t_0) &= \int_{t_{eq}}^{t_0} dt' \frac{1}{a^3(t')} \int_{t_{eq}}^{t'} dt'' a^3(t'') R(t'') = \\ &= \frac{6q(2q-1)}{(3q-1)} \left\{ \log\left(\frac{t_0}{t_{eq}}\right) - \frac{1}{3q-1} + \frac{1}{3q-1}\left(\frac{t_{eq}}{t_0}\right)^{3q-1} \right\}.\end{aligned} \tag{6.7}$$

Taking into account a standard matter dominated universe with $q = 2/3$, we have

$$\Big(\Box^{-1}R\Big)(t_0)\Big|_{q=\frac{2}{3}} \sim 14.0\ . \tag{6.8}$$

The above result suggests that the non-local term leads to the order required by the current cosmic acceleration and avoids the fine tuning of parameters. Furthermore, these corrections occur only at late-times, since during the radiation dominated era the non-local effects are null while, after the onset of matter dominance, the logarithmic dependence make them negligible.

6.1 Non-local Gauss–Bonnet Cosmology

With the above considerations in mind, let us see if the form of the non-local action containing the operator $\Box^{-1}$ applied to the Gauss–Bonnet scalar $\mathcal{G}$ can be selected by the Noether symmetry approach. Let us start by considering the following non-local Gauss–Bonnet action in vacuum

$$S = \int \sqrt{-g} f(\mathcal{G}, \Box^{-1} h(\mathcal{G})) d^4x \,. \tag{6.9}$$

Starting from this, we want to derive the cosmological point-like Lagrangian and then search for Noether symmetries according to the lines sketched in App. B. The first issue is to define a suitable *localization* of the non-local field. We can define:

$$\Box^{-1} h(\mathcal{G}) := \phi \qquad \rightarrow \qquad h(\mathcal{G}) = \Box\phi \,, \tag{6.10}$$

where ϕ is an auxiliary scalar field. Let us now focus on a spatially-flat FLRW cosmological background, with metric $ds^2 = dt^2 - a(t)^2 d\mathbf{x}^2$. In this perspective, we must consider also the dependence of the scalar field on the cosmic time, that is $\phi = \phi(t)$. According to Eq. (4.6) the Gauss–Bonnet scalar can be expressed as a function of the scale factor, so that the cosmological expression of $\mathcal{G}$ can be used as a constraint in the Lagrange multipliers method. Therefore, considering also the localization (6.10), as in [60, 157], we can define the further scalar field $\epsilon(t)$ such that action (6.9) can be written as

$$S = \int \sqrt{-g} \left\{ f(\mathcal{G}, \phi) + \epsilon(t)(\Box\phi - h(\mathcal{G})) \right\} d^4x \,, \tag{6.11}$$

where ϕ and $\mathcal{G}$ have to be treated as separated fields. Using the Lagrange Multipliers method, action (6.11) becomes

$$S = \int \left\{ a^3 f(\mathcal{G}, \phi) + a^3 \epsilon(t) \left[\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - h(\mathcal{G}) \right] - \lambda \left(\mathcal{G} - 24 \frac{\dot{a}^2 \ddot{a}}{a^3} \right) \right\} dt \,. \tag{6.12}$$

The variation of the action with respect to the Gauss–Bonnet term allows to find the Lagrange Multiplier λ:

$$\frac{\delta S}{\delta \mathcal{G}} = \int \left\{a^3 f_{\mathcal{G}}(\mathcal{G},\phi) - a^3\epsilon(t)h_{\mathcal{G}}(\mathcal{G}) - \lambda\right\} dt = 0 \quad \rightarrow \quad \lambda = a^3\left[f_{\mathcal{G}}(\mathcal{G},\phi) - \epsilon(t)h_{\mathcal{G}}(\mathcal{G})\right] . \tag{6.13}$$

Furthermore, from Eq. (6.11), it is easy to verify that once varying the action with respect to $\epsilon(t)$, one recovers the definition $\Box\phi - h(\mathcal{G}) = 0$. From the variation with respect to the scalar field ϕ, we get:

$$\begin{aligned}\frac{\delta S}{\delta \phi} &= \int \sqrt{-g}\left\{f_\phi(\mathcal{G},\phi) - \frac{\delta}{\delta\phi}\left[\epsilon(t)(\Box\phi - h(\mathcal{G}))\right]\right\} d^4x \\ &= \int \sqrt{-g}\left\{f_\phi(\mathcal{G},\phi) - \frac{\delta}{\delta\phi}\left[\epsilon(t)\Box(\phi - \Box^{-1}h(\mathcal{G}))\right]\right\} d^4x \, .\end{aligned} \tag{6.14}$$

Using the divergence theorem, the last term of Eq. (6.14) can be written as:

$$\begin{aligned}&\int \sqrt{-g}\left\{\frac{\delta}{\delta\phi}\left[\epsilon(t)\Box(\phi - \Box^{-1}h(\mathcal{G}))\right]\right\} d^4x = \frac{\delta}{\delta\phi}\int \sqrt{-g}\Box\epsilon(t)(\phi - \Box^{-1}h(\mathcal{G}))d^4x \\ &= \int \sqrt{-g}\,\Box\epsilon(t)\frac{\delta}{\delta\phi}(\phi - \Box^{-1}h(\mathcal{G}))d^4x = \int \sqrt{-g}\,\Box\epsilon(t)d^4x \, ,\end{aligned} \tag{6.15}$$

so the variation with respect to the scalar field ϕ provides the following Klein-Gordon equation:

$$\frac{\delta S}{\delta\phi} = 0 \quad \rightarrow \quad \Box\epsilon(t) = f_\phi(\mathcal{G},\phi) \, . \tag{6.16}$$

After introducing the Lagrange multipliers and integrating out the higher derivatives, the point-like Lagrangian can be written as:

$$\begin{aligned}\mathcal{L} &= a^3\left[f(\mathcal{G},\phi) - \mathcal{G}f_{\mathcal{G}}(\mathcal{G},\phi) - \epsilon h(\mathcal{G}) + \epsilon\mathcal{G}h_{\mathcal{G}}(\mathcal{G})\right] - a^3\dot{\phi}\dot{\epsilon} - 8\dot{a}^3\dot{\mathcal{G}}f_{\mathcal{G}\mathcal{G}}(\mathcal{G},\phi) \\ &+ 8\dot{a}^3\dot{\epsilon}h_{\mathcal{G}}(\mathcal{G}) + 8\epsilon\dot{a}^3\dot{\mathcal{G}}h_{\mathcal{G}\mathcal{G}}(\mathcal{G}) - 8\dot{a}^3\dot{\phi}f_{\mathcal{G}\phi}(\mathcal{G},\phi) \, .\end{aligned} \tag{6.17}$$

The corresponding Euler–Lagrange equations and the energy condition are, respectively:

$$
\begin{cases}
a:\ 8\dot{a}\Big[2\ddot{a}\left(-\dot{\mathcal{G}}f_{\mathcal{G}\mathcal{G}}(\mathcal{G},\phi)-\dot{\phi}f_{\mathcal{G}\phi}(\mathcal{G},\phi)+\epsilon\dot{\mathcal{G}}h_{\mathcal{G}\mathcal{G}}(\mathcal{G})\right)+\dot{a}\left(-\ddot{\mathcal{G}}f_{\mathcal{G}\mathcal{G}}(\mathcal{G},\phi)\right.\\
-2\dot{\mathcal{G}}\dot{\phi}f_{\mathcal{G}\mathcal{G}\phi}(\mathcal{G},\phi)-\dot{\mathcal{G}}^2f_{\mathcal{G}\mathcal{G}\mathcal{G}}(\mathcal{G},\phi)-\ddot{\phi}f_{\mathcal{G}\phi}(\mathcal{G},\phi)-\dot{\phi}^2f_{\mathcal{G}\phi\phi}(\mathcal{G},\phi)+\ddot{\epsilon}h_{\mathcal{G}}(\mathcal{G})\\
\left.+\epsilon\ddot{\mathcal{G}}h_{\mathcal{G}\mathcal{G}}(\mathcal{G})+\epsilon\dot{\mathcal{G}}^2h_{\mathcal{G}\mathcal{G}\mathcal{G}}(\mathcal{G})\right)+2\dot{\epsilon}\left(\ddot{a}h_{\mathcal{G}}(\mathcal{G})+\dot{a}\dot{\mathcal{G}}h_{\mathcal{G}\mathcal{G}}(\mathcal{G})\right)\Big]\\
+a^2\Big[\mathcal{G}f_{\mathcal{G}}(\mathcal{G},\phi)-f(\mathcal{G},\phi)+\dot{\epsilon}\dot{\phi}+\epsilon\left(h(\mathcal{G})-\mathcal{G}h_{\mathcal{G}}(\mathcal{G})\right)\Big]=0\\
\\
\phi:\ \Box\epsilon=f_{\phi}(\mathcal{G},\phi)\\
\\
\mathcal{G}:\ \mathcal{G}=24\dfrac{\dot{a}^2\ddot{a}}{a^3}\\
\\
\epsilon:\ \Box\phi=h(\mathcal{G})\\
\\
EC:\ a^3\left(f(\mathcal{G},\phi)-\epsilon h(\mathcal{G})+\epsilon\mathcal{G}h_{\mathcal{G}}(\mathcal{G})+\dot{\phi}\dot{\epsilon}-\mathcal{G}f_{\mathcal{G}}(\mathcal{G},\phi)\right)\\
-24\dot{a}^3\left(\dot{\epsilon}h_{\mathcal{G}}(\mathcal{G})+\epsilon\dot{\mathcal{G}}h_{\mathcal{G}\mathcal{G}}(\mathcal{G})-\dot{\phi}f_{\mathcal{G}\phi}(\mathcal{G},\phi)-\dot{\mathcal{G}}f_{\mathcal{G}\mathcal{G}}(\mathcal{G},\phi)\right)=0\,.
\end{cases}
$$

Once the forms of the functions $h(\mathcal{G})$ and $f(\mathcal{G},\phi)$ are specified, the above system can provide exact cosmological solutions.
Also here, the Noether theorem can be applied to the Lagrangian (6.17). In such a case the minisuperspace is defined on the configuration space $\mathcal{S}\equiv\{a,\phi,\mathcal{G},\epsilon\}$, so that the symmetry generator takes the explicit form:

$$
\begin{aligned}
\mathcal{X} &= \xi(t,a,\phi,\mathcal{G},\epsilon)\partial_t+\alpha(t,a,\phi,\mathcal{G},\epsilon)\partial_a+\beta(t,a,\phi,\mathcal{G},\epsilon)\partial_\phi\\
&+ \gamma(t,a,\phi,\mathcal{G},\epsilon)\partial_{\mathcal{G}}+\delta(t,a,\phi,\mathcal{G},\epsilon)\partial_\epsilon\,.
\end{aligned}
\tag{6.18}
$$

The system of differential equations coming from the above generator is made of 37 equations but, after deleting all the linear combinations, it reduces to a system of five equations plus the conditions on the generator coefficients [158]. The system admits five different solutions: in two of them the non-local function $f(\mathcal{G},\phi)$ is given by a sum of a function of ϕ and a function of $\mathcal{G}$, *i.e.* $f(\mathcal{G},\phi) = f_1(\mathcal{G}) + f_2(\phi)$. In the other three, solutions are products between the two functions, namely $f(\mathcal{G},\phi) = g_1(\mathcal{G})g_2(\phi)$. The entire set of solutions with the corresponding generators read:

$$
\begin{cases}
I: & \mathcal{X} = (\xi_0 t + \xi_1)\partial_t + \alpha_0 a\partial_a + (\beta_0\phi + \beta_1)\partial_\phi - 4\xi_0\mathcal{G}\partial_\mathcal{G} + \delta_0\epsilon\partial_\epsilon\,, \\
& h(\mathcal{G}) = h_0\mathcal{G}^{\frac{1}{2}+\frac{n}{k}}\,, \quad f(\mathcal{G},\phi) = f_0\mathcal{G}^n + f_1\mathcal{G} + f_2\left(\beta_0\phi + \beta_1\right)^k; \\
& \\
II: & \mathcal{X} = (\xi_0 t + \xi_1)\partial_t + \alpha_0 a\partial_a + (\beta_0\phi + \beta_1)\partial_\phi - 4\xi_0\mathcal{G}\partial_\mathcal{G} + (\delta_0\epsilon + \delta_1)\partial_\epsilon\,, \\
& h(\mathcal{G}) = h_0\mathcal{G}\,, \quad f(\mathcal{G},\phi) = f_0\mathcal{G}^n + f_1\mathcal{G} + f_2(\beta_0\phi + \beta_1)^{2n}; \\
& \\
III: & \mathcal{X} = (\xi_0 t + \xi_1)\partial_t + \alpha_0 a\partial_a + (\beta_0\phi + \beta_1)\partial_\phi - 4\xi_0\mathcal{G}\partial_\mathcal{G} + \delta_0\epsilon\partial_\epsilon\,, \\
& h(\mathcal{G}) = h_0\mathcal{G}^z\,, \quad f(\mathcal{G},\phi) = f_0\mathcal{G}^n(\beta_0\phi + \beta_1)^k; \\
& \\
IV: & \mathcal{X} = (\xi_0 t + \xi_1)\partial_t + \alpha_0 a\partial_a + (\beta_0\phi + \beta_1)\partial_\phi - 4\xi_0\mathcal{G}\partial_\mathcal{G} + (\delta_0\epsilon + \delta_1)\partial_\epsilon\,, \\
& h(\mathcal{G}) = h_0\mathcal{G}\,, \quad f(\mathcal{G},\phi) = f_0\mathcal{G}^n(\beta_0\phi + \beta_1)^k; \\
& \\
V: & \mathcal{X} = (\xi_0 t + \xi_1)\partial_t + \alpha_0 a\partial_a + \beta_1\partial_\phi - 4\xi_0\mathcal{G}\partial_\mathcal{G} + \delta_0\epsilon\partial_\epsilon\,, \\
& h(\mathcal{G}) = h_0\sqrt{\mathcal{G}}\,, \quad f(\mathcal{G},\phi) = f_0\mathcal{G}^n e^{k\phi}, \quad k \equiv \frac{\delta_0 + 4n\xi_0}{\beta_1}\,,
\end{cases}
\tag{6.19}
$$

where ξ_0, ξ_1, α_0, β_0, β_1, δ_0, h_0, f_0, f_1, f_2, n, k are integration constants.

It may seem that the theory is over determined by the large amount of free parameters. However, after solving the equations of motion, the functions will be further constrained to those in agreement with the cosmological solutions. Specifically, it turns out that not all the functions contained in the system (6.19) admit cosmological solutions for the scale factor. As a matter of fact, while the second and the fourth do not admit any cosmological solution, the first and the third can be analytically solved by setting $\beta_1 = 0$. The fifth admits solutions only after constraining the mutual dependence among the parameters.

Let us start by analyzing the Lagrangians corresponding to the functions I and III. They read, respectively:

$$\begin{aligned}\mathcal{L}_I &= a^3\left[f_0(1-n)\mathcal{G}^n - h_0\left(\frac{n}{k}-\frac{1}{2}\right)\epsilon\mathcal{G}^{\frac{n}{k}+\frac{1}{2}} + \dot{\epsilon}\dot{\phi}\right] - 8h_0\left(\frac{n}{k}+\frac{1}{2}\right)\dot{a}^3\dot{\epsilon}\mathcal{G}^{\frac{n}{k}-\frac{1}{2}}\\ &\quad +8f_0n(n-1)\dot{a}^3\dot{\mathcal{G}}\mathcal{G}^{n-2} - 8h_0\left(\frac{n^2}{k^2}-\frac{1}{4}\right)\epsilon\dot{a}^3\dot{\mathcal{G}}\mathcal{G}^{\frac{n}{k}-\frac{3}{2}} + 8f_1k\dot{a}^3\dot{\phi}\phi^{k-1} \qquad (6.20)\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_{III} &= a^3\left[f_0(1-n)\mathcal{G}^n\phi^k + h_0(z-1)\epsilon\mathcal{G}^z\right] + 8h_0z\dot{a}^3\dot{\epsilon}\mathcal{G}^{z-1} + 8h_0z(z-1)\epsilon\dot{a}^3\dot{\mathcal{G}}\mathcal{G}^{z-2}\\ &\quad -8f_0n(n-1)\dot{a}^3\dot{\mathcal{G}}\mathcal{G}^{n-2}\phi^k - 8f_0kn\dot{a}^3\dot{\phi}\mathcal{G}^{n-1}\phi^{k-1} - a^3\dot{\epsilon}\dot{\phi}\,. \qquad (6.21)\end{aligned}$$

From the former Lagrangian, the Euler–Lagrange equations and the energy condition provide a solution given by a time power-law of $a(t)$, namely:

$$a(t)\sim t^{\frac{2}{3}(2n+2kz-k)}\,,\quad \mathcal{G}(t)\sim t^{-4}\,,\quad \phi(t)\sim t^{2-4z}\,,\quad \epsilon(t)\sim t^{2k(1-2z)}\,, \qquad (6.22)$$

$$f(\mathcal{G},\Box^{-1}h(\mathcal{G})) = f_2\mathcal{G}^n(\Box^{-1}\mathcal{G}^z)^k\,. \qquad (6.23)$$

In this case, though exponential solutions do not occur in vacuum, the parameters are not fixed by the equations of motion, so that they might be constrained by observations. On the other hand, the Lagrangian (6.20) gives exact de Sitter-like solutions of the form:

$$a(t)\sim e^{qt}\,,\quad \mathcal{G}(t)\sim \text{Const}\,,\quad \phi(t)\sim t\,,\quad \epsilon(t)\sim t\,,\quad k=1,\ n=\frac{1}{2}\,, \qquad (6.24)$$

$$f(\mathcal{G},\Box^{-1}h(\mathcal{G})) = f_0\sqrt{\mathcal{G}} + f_1\mathcal{G} + f_2\Box^{-1}\mathcal{G} + f_3\,. \qquad (6.25)$$

Therefore, concerning this latter case, the only solution such that the Euler–Lagrange equations, the energy condition, and the Noether system are satisfied, constrains all the

free parameters occurring in the second function. Since $\mathcal{G}$ is a topological invariant, the linear term in $\mathcal{G}$ does not contribute to dynamics so the relevant terms are the square root and the linear non-local terms in $\mathcal{G}$. From a cosmological point of view, this action is equivalent to action (6.3), so the same considerations in [61] hold. In other words, a Dark Energy-like behavior, due to non-local terms, can be achieved both in R and $\mathcal{G}$ descriptions of cosmological dynamics.
Finally, the point-like Lagrangian corresponding to the last solution is:

$$\begin{aligned}\mathcal{L}_V &= 2\dot{a}^3\left[2\mathcal{G}^{-\frac{1}{2}}\dot{\epsilon}-\epsilon\mathcal{G}^{-\frac{3}{2}}\dot{\mathcal{G}}-4f_0n(n-1)\mathcal{G}^{n-2}\dot{\mathcal{G}}e^{k\phi}-4f_0kn\mathcal{G}^{n-1}e^{k\phi}\dot{\phi}\right]\\ &\quad -\frac{1}{2}\mathcal{G}^2a^3\left[2\dot{\epsilon}\dot{\phi}+\epsilon\sqrt{\mathcal{G}}+2f_0(n-1)\mathcal{G}^ne^{k\phi}\right]\end{aligned} \tag{6.26}$$

and the Euler–Lagrange equations. (6.18) can be analytically solved providing two different forms of the scale factor; the first reads as:

$$a(t)\sim e^{q\,t}\,,\quad \phi(t)\sim t\,,\quad \epsilon(t)\sim e^{\sqrt{\frac{8}{3}}kq\,t}\,, \tag{6.27}$$

$$f(\mathcal{G},\Box^{-1}\sqrt{\mathcal{G}})=f_0\mathcal{G}^{\frac{12\sqrt{6}}{4k-\sqrt{6}}}e^{k\phi}\,,\qquad h(\mathcal{G})=\sqrt{\mathcal{G}}\,. \tag{6.28}$$

By comparing the function in Eq. (6.28) with the fifth of Eq. (6.19), we notice that a relation between the free parameters n and k occurs, namely:

$$n=\frac{12\sqrt{6}}{4k-\sqrt{6}}\,. \tag{6.29}$$

Furthermore, we also find power law solutions, namely:

$$a(t)\sim t^q\,,\quad \phi(t)\sim\ln[(1-3q)t]\,,\quad \mathcal{G}(t)\sim\frac{1}{t^4}\,,\quad \epsilon(t)\sim t^{2-4n+\frac{2k\sqrt{6q^3(q-1)}}{3q-1}}\,. \tag{6.30}$$

The second solution, coming from the Lagrangian (6.26), introduces a relation among the parameters n, q and k enlarging the possibility to compare these cosmological behaviors with observational data.

6.1.1 General Relativity plus Non-local Gauss–Bonnet Cosmology

To conclude this discussion, let us treat the case of GR corrected with non-local Gauss–Bonnet terms, considered *e.g.* in [157, 159]. The action is:

$$S = \int \sqrt{-g}\left[\frac{R}{2} + f(\mathcal{G}, \Box^{-1}h(\mathcal{G})\right] d^4x\,, \tag{6.31}$$

that is Eq. (6.9) with the addition of the Hilbert–Einstein term. Let us make use of the Lagrange multipliers method to find the cosmological Lagrangian and, therefore, to apply the Noether symmetry approach. The only difference with respect to the case given by Eq. (6.17) is due to the cosmological form of R, so that the Lagrangian reads as:

$$\begin{aligned}\mathcal{L} \;=\;& a^3\left[f(\mathcal{G},\phi) - \mathcal{G}f_{\mathcal{G}}(\mathcal{G},\phi) - \epsilon h(\mathcal{G}) + \epsilon\mathcal{G}h_{\mathcal{G}}(\mathcal{G})\right] - a^3\dot{\phi}\dot{\epsilon} - 8\dot{a}^3\dot{\mathcal{G}}f_{\mathcal{G}\mathcal{G}}(\mathcal{G},\phi)\\ &+8\dot{a}^3\dot{\epsilon}h_{\mathcal{G}}(\mathcal{G}) + 8\epsilon\dot{a}^3\dot{\mathcal{G}}h_{\mathcal{G}\mathcal{G}}(\mathcal{G}) - 8\dot{a}^3\dot{\phi}f_{\mathcal{G}\phi}(\mathcal{G},\phi) + 3a\dot{a}^2\,,\end{aligned} \tag{6.32}$$

and the only different Euler–Lagrange equation is that related to the scale factor which, in this case, takes the form:

$$\begin{aligned}&8\dot{a}\Big[2\ddot{a}\left(-\dot{\mathcal{G}}f_{\mathcal{G}\mathcal{G}}(\mathcal{G},\phi) - \dot{\phi}f_{\mathcal{G}\phi}(\mathcal{G},\phi) + \epsilon\dot{\mathcal{G}}h_{\mathcal{G}\mathcal{G}}(\mathcal{G})\right)\\ &+\dot{a}\left(-\ddot{\mathcal{G}}f_{\mathcal{G}\mathcal{G}}(\mathcal{G},\phi) - 2\dot{\mathcal{G}}\dot{\phi}f_{\mathcal{G}\mathcal{G}\phi}(\mathcal{G},\phi) - \dot{\mathcal{G}}^2 f_{\mathcal{G}\mathcal{G}\mathcal{G}}(\mathcal{G},\phi) - \ddot{\phi}f_{\mathcal{G}\phi}(\mathcal{G},\phi)\right.\\ &\left.-\dot{\phi}^2 f_{\mathcal{G}\phi\phi}(\mathcal{G},\phi) + \ddot{\epsilon}h_{\mathcal{G}}(\mathcal{G}) + \epsilon\ddot{\mathcal{G}}h_{\mathcal{G}\mathcal{G}}(\mathcal{G}) + \epsilon\dot{\mathcal{G}}^2 h_{\mathcal{G}\mathcal{G}\mathcal{G}}(\mathcal{G})\right)\\ &+2\dot{\epsilon}\left(\ddot{a}h_{\mathcal{G}}(\mathcal{G}) + \dot{a}\dot{\mathcal{G}}h_{\mathcal{G}\mathcal{G}}(\mathcal{G})\right)\Big] + 3(\dot{a}^2 + 2a\ddot{a})\\ &+a^2\Big[\mathcal{G}f_{\mathcal{G}}(\mathcal{G},\phi) - f(\mathcal{G},\phi) + \dot{\epsilon}\dot{\phi} + \epsilon\left(h(\mathcal{G}) - \mathcal{G}h_{\mathcal{G}}(\mathcal{G})\right)\Big] = 0\,.\end{aligned} \tag{6.33}$$

The minisuperspace dimension is the same as the previous case, since the scalar curvature does not introduce any new dynamical variables. By replacing $R(t)$ with its cosmological expression, the Noether system turns out to be the same as the previous section, except for the addition of a further condition on the Noether vector, that is:

$$\alpha - a\partial_t\xi + 2a\partial_a\alpha = 0\,. \tag{6.34}$$

This new link between α and ξ, provided by Eq. (6.34), yields an important implication for the solutions of Noether's system, since it uniquely fixes the value of n and the relation between n and k. We obtain five different generators (with corresponding functions) of the form:

$$\begin{cases} I: \quad \mathcal{X} = (3\alpha_0 t + \xi_1)\partial_t + \alpha_0 a\partial_a + (\beta_0\phi + \beta_1)\partial_\phi - 12\alpha_0\mathcal{G}\partial_\mathcal{G} + \delta_0\epsilon\partial_\epsilon\,, \\ \qquad h(\mathcal{G}) = h_0\mathcal{G}^{\frac{1}{2}+\frac{1}{2k}}\,, \quad f(\mathcal{G},\phi) = f_0\mathcal{G}^{\frac{1}{2}} + f_1\mathcal{G} + f_2\left(\beta_0\phi + \beta_1\right)^k; \\ \\ II: \quad \mathcal{X} = (3\alpha_0 t + \xi_1)\partial_t + \alpha_0 a\partial_a + (\beta_0\phi + \beta_1)\partial_\phi - 12\alpha_0\mathcal{G}\partial_\mathcal{G} + (\delta_0\epsilon + \delta_1)\partial_\epsilon\,, \\ \qquad h(\mathcal{G}) = h_0\mathcal{G}\,, \quad f(\mathcal{G},\phi) = f_0\mathcal{G}^{\frac{1}{2}} + f_1\mathcal{G} + f_2(\beta_0\phi + \beta_1); \\ \\ III: \quad \mathcal{X} = (3\alpha_0 t + \xi_1)\partial_t + \alpha_0 a\partial_a + (\beta_0\phi + \beta_1)\partial_\phi - 12\alpha_0\mathcal{G}\partial_\mathcal{G} + \delta_0\epsilon\partial_\epsilon\,, \\ \qquad h(\mathcal{G}) = h_0\mathcal{G}^{\frac{1-2n}{2k}}\,, \quad f(\mathcal{G},\phi) = f_0\mathcal{G}^n(\beta_0\phi + \beta_1)^k; \\ \\ IV: \quad \mathcal{X} = (3\alpha_0 t + \xi_1)\partial_t + \alpha_0 a\partial_a + (\beta_0\phi + \beta_1)\partial_\phi - 12\alpha_0\mathcal{G}\partial_\mathcal{G} + (\delta_0\epsilon + \delta_1)\partial_\epsilon\,, \\ \qquad h(\mathcal{G}) = h_0\mathcal{G}\,, \quad f(\mathcal{G},\phi) = f_0\mathcal{G}^n(\beta_0\phi + \beta_1)^{1-2n}; \\ \\ V: \quad \mathcal{X} = (3\alpha_0 t + \xi_1)\partial_t + \alpha_0 a\partial_a + \beta_1\partial_\phi - 12\alpha_0\mathcal{G}\partial_\mathcal{G}\,, \\ \qquad h(\mathcal{G}) = h_0\sqrt{\mathcal{G}}\,, \quad f(\mathcal{G},\phi) = f_0\mathcal{G}^n e^{k\phi}. \end{cases} \tag{6.35}$$

By comparing Eqs. (6.19) with Eqs. (6.35), we notice that, with regards to the first two solutions, the introduction of R leads to the further constraint $n = 1/2$. Moreover, in the third and in the fourth cases, the further relations $z = \frac{1-2n}{2k}$ and $k = 1 - 2n$

occur respectively. According to these considerations, it is clear that action (6.31) is fully consistent with (6.9) and then it is not necessary introduce by hand the Hilbert–Einstein term to recover GR in this context.

7

Metric, Affine and Topological Theories Non–Minimally Coupled to a Scalar Field

Despite the need of extending/modifying GR, self-interaction potentials, couplings and kinetic terms give rise to infinite choices which can lead to a frustrating indetermination in fitting observations and addressing conceptual problems. One can adjust models and parameters to match single datasets and phenomena but a theory in agreement with the whole phenomenology seems far to be achieved. In other words, any single theory loses its general predictive power and cannot be used to reproduce a self-consistent cosmic history, starting from UV to IR scales. Therefore, some selection criteria, based on physical requirements, are needed to discriminate among the plethora of modified scalar-tensor gravities. These criteria can be based *e.g.* on symmetries, conservation laws and on general physical motivations.

Here, we want to consider scalar fields non-minimally coupled with different geometric invariants, in particular the Ricci scalar R, the torsion scalar T, and the Gauss–Bonnet scalar $\mathcal{G}$. The aim is to demonstrate that all these non-minimally coupled invariants can give rise to similar cosmological dynamics once we know how to transform each-other. Furthermore, these scalar-tensor theories can be dealt with under the standard of Noether symmetry approach which allows to fix couplings, potentials and kinetic terms

requiring the existence of symmetries and related conserved quantities. In other words, the purpose is to compare, by the Noether symmetries, the dynamics of three different actions, pointing out the equivalence of the three representations of gravity.
Below, we will discuss non-minimal coupling with R, T, and $\mathcal{G}$ scalars showing that analogue (or identical) features emerge if couplings, kinetic terms, and potentials are selected by Noether symmetries. The final result of this study is that all the above mentioned theories provides the same dynamics and the same solutions. This is a further proof of the equivalence between the Gauss–Bonnet models and $f(R)$ gravity, there including GR as a particular limit.
A general scalar-tensor action, written in terms of the scalar curvature, reads as:

$$S=\int\sqrt{-g}\left\{F(\phi)\;R+\omega(\phi)g^{\mu\nu}\phi_{;\mu}\phi_{;\nu}-V(\phi)\right\}d^4x, \tag{7.1}$$

where $F(\phi)$ is the coupling, $\omega(\phi)$ the coefficient of the kinetic term and $V(\phi)$ the potential. The variation of the action with respect to the metric tensor $g_{\mu\nu}$, provides the field equations [160]

$$\begin{aligned}&R_{\mu\nu}F(\phi)-\frac{1}{2}g_{\mu\nu}\left[RF(\phi)+\omega(\phi)\phi_{;\alpha}\phi^{;\alpha}-V(\phi)\right]\\&-F(\phi)_{;\mu;\nu}+g_{\mu\nu}\Box F(\phi)+\omega(\phi)\phi_{;\mu}\phi_{;\nu}=0\,,\end{aligned} \tag{7.2}$$

that clearly reduce to the Einstein equations when $F(\phi)=$ const. and $V(\phi)=\omega(\phi)=0$. Action (7.1) is the paradigm for a very large class of theories. For example, $f(R)$ gravity in metric formalism can be easily recovered from (7.1) if $\omega(\phi)$ is set to zero and (see [31, 161] for details)

$$V(\phi)=\frac{1}{2}\frac{f(R)-Rf_R}{[f_R]^2}\,. \tag{7.3}$$

In general, it turns out that a second–order theory of gravity non-minimally coupled to a scalar field can always be recast as a fourth–order theory. Cosmological solutions considered in this chapter, therefore, can be addressed to $f(R)$, $f(T)$ and $f(\mathcal{G})$ cosmology, respectively.

7.1 Non-Minimally Coupled Curvature Scalar

Let us start our analysis considering the action (7.1), where the form of unknown functions will be fixed by Noether symmetries. After selecting these functions, we will find out analytic cosmological solutions thanks to the reduction of dynamics.
From the variation of the action with respect to the scalar field ϕ, we get the Klein-Gordon equation

$$\omega_\phi(\phi)\phi^{;\alpha}\phi_{;\alpha} + 2\omega(\phi)\Box\phi - RF_\phi(\phi) + V_\phi(\phi) = 0, \tag{7.4}$$

where the subscript ϕ denotes the derivative with respect to ϕ. Together with Eq. (7.2), it completes the set of equations of motion.
The action can be simplified by focusing on a spatially-flat FLRW metric and integrating over the 3-D surface term, so we get

$$S = 2\pi^2 \int a^3 \left[RF(\phi) + \omega(\phi)\dot{\phi}^2 - V(\phi) \right] dt. \tag{7.5}$$

Finally, replacing the cosmological expression of the Ricci scalar into the action and integrating out second order derivatives, we get the cosmological point-like Lagrangian

$$\mathcal{L} = 6F(\phi)a\dot{a}^2 + 6F_\phi(\phi)a^2\dot{a}\dot{\phi} + a^3\omega(\phi)\dot{\phi}^2 - a^3V(\phi). \tag{7.6}$$

As standard when the Lagrangian approach is considered, field equations and Klein-Gordon equation result in the Euler-Lagrange equations along with the energy condition

$E_{\mathcal{L}} = 0$, that is the 00 equation. Therefore the dynamical system turns out to be:

$$\begin{cases} a : 2F\dot{a}^2 + 4a\left(F_\phi \dot{a} + F\ddot{a}\right) + a^2\left(V - \omega\dot{\phi}^2 + 2F_{\phi\phi}\dot{\phi}^2 + 2F_\phi\ddot{\phi}\right) = 0 \\ \phi : 6\dot{a}^2 F_\phi + 6a\omega\dot{a}\dot{\phi} + a\left[6F_\phi\ddot{a} + a\left(V_\phi + \dot{\phi}^2 w_\phi + 2w\ddot{\phi}\right)\right] = 0 \\ E_{\mathcal{L}} = 0 : 6F\dot{a}^2 + 6aF_\phi\dot{\phi}\dot{a} + a^2\left(V + w\dot{\phi}^2\right) = 0 \end{cases} \tag{7.7}$$

and can be solved after the three functions of ϕ are selected through Noether symmetries. The approach can be developed in the two-dimensional minisuperspace $\mathcal{S} = \{a, \phi\}$ whose corresponding symmetry generator is

$$\mathcal{X} = \xi(a,\phi,t)\partial_t + \alpha(a,\phi,t)\partial_a + \beta(a,\phi,t)\partial_\phi\,. \tag{7.8}$$

After equating to zero terms containing same time derivatives of variables, the application of Noether's identity (B.2) to Lagrangian (7.6) provides a system of 10 differential equations. Nevertheless, by imposing *a priori* the condition $\xi = \xi(t)$ (holding for Lagrangians in canonical forms) and neglecting redundant equations, the system reduces to 4 differential equations plus the condition on the infinitesimal generators. Such a system is clearly over determined and cannot provide any explicit form without imposing some constraint. Since we want to investigate functions with physical meaning for cosmology, we replace into the system both power-law and exponential potentials, so that it provides the following solutions [162]

$$\begin{cases} \mathcal{X} = (\xi_0 t + \xi_1)\,\partial_t - \dfrac{\xi_0}{3} a \left(\dfrac{k+c}{k-c}\right)\partial_a + \dfrac{2\xi_0}{k-c}\phi\partial_\phi \\ F(\phi) = F_0\phi^k \quad \omega(\phi) = \omega_0\phi^{k-2} \quad V(\phi) = V_0\phi^c \quad k \neq c \\ \\ \mathcal{X} = \dfrac{k\alpha_0}{3} a^{-\frac{1}{5}}\partial_a + \alpha_0 a^{-\frac{6}{5}}\partial_\phi \\ F(\phi) = F_0\phi^k \quad \omega(\phi) = \omega_0\phi^{k-2} \quad V(\phi) = V_0\phi^k \quad k = \pm\sqrt{\dfrac{2\omega_0}{3F_0}} \\ \\ \mathcal{X} = (\xi_0 t + \xi_1)\,\partial_t - \dfrac{\xi_0}{3} a \left(\dfrac{k+c}{k-c}\right)\partial_a + \dfrac{2\xi_0}{k-c}\partial_\phi \\ F(\phi) = F_0 e^{k\phi} \quad \omega(\phi) = \omega_0 e^{k\phi} \quad V(\phi) = V_0 e^{c\phi} \quad k \neq c \\ \\ \mathcal{X} = -\dfrac{k\alpha_0}{3} a^{-\frac{1}{5}} e^{-\frac{k\omega_0\phi}{F_0k^2+\omega_0}}\partial_a + \alpha_0 a^{-\frac{6}{5}} e^{-\frac{k\omega_0\phi}{F_0k^2+\omega_0}}\partial_\phi \\ F(\phi) = F_0 e^{k\phi} \quad \omega(\phi) = \omega_0 e^{k\phi} \quad V(\phi) = V_0 e^{k\phi} \quad k = \pm\sqrt{\dfrac{2\omega_0}{3F_0}}. \end{cases} \tag{7.9}$$

Furthermore, there is one further solution for constant coefficient of the kinetic term ($\omega(\phi) = 1$), namely

$$\begin{aligned} &\mathcal{X} = -\frac{2(s+1)}{2s+3}\beta_0 a^{s+1}\phi^{\frac{2s^2+4s}{2s+3}}\partial_a + \beta_0 a^s \phi^{\frac{2s^2+6s+3}{2s+3}}\partial_\phi\,, \\ &F(\phi) = \ell(s)\phi^2\,, \quad V(\phi) = V_0\phi^{\frac{6(s+1)}{2s+3}} \quad \ell(s) = \frac{(2s+3)^2}{48(s+1)(s+2)}, \end{aligned} \tag{7.10}$$

with $\alpha_0, \beta_0, \xi_0, k, c, s, \omega_0, F_0, V_0$ real constants. We neglect trivial solutions, such as constant couplings or vanishing potentials. Inserting the above functions into the dynamics, the latter is reduced and the equations of motion can be analytically solved.

Starting from the two main sets of functions selected above, we are going to obtain exact cosmological solutions. It is worth noticing that the choice of exponential potential also leads to exponential coupling and exponential kinetic term, like in string-dilaton

cosmology [163, 164, 165]. This means that the string-dilaton Lagrangian can be naturally obtained from Noether symmetries [166]. From this point of view, solutions occurring in Eq. (7.9) can be considered more general than those provided in [164], since both the solutions outlined there by the authors are contained in the last two of Eq. (7.9). In particular, the exponential potential of string-dilaton cosmology is recovered for $k = -2$ and arbitrary c, while the constant potential is recovered for for $k = -2$ and $c = 0$.
With these considerations in mind, let us solve the Euler-Lagrange equations (7.7) for those cases corresponding to the first and the third solution of Eq. (7.9). In the former case, the Lagrangian (7.6) is:

$$\mathcal{L} = 6F_0\phi^k a\dot{a}^2 + 6F_0 k\phi^{k-1}a^2\dot{a}\dot{\phi} + a^3\omega_0\phi^{k-2}\dot{\phi}^2 - a^3V_0\phi^c \tag{7.11}$$

and the corresponding equations of motion can be analytically solved with the constraint $k = c = 2$, providing a de Sitter-like expansion of the form:

$$a(t) = a_0 e^{qt}\,, \quad \phi(t) = \phi_0 \exp\left\{\frac{1}{2}\left[-3q \pm \sqrt{\frac{-48F_0q^2 - 4V_0 + 9q^2\omega_0}{\omega_0}}\right]t\right\}\,, \quad q, a_0, \phi_0 \in \mathbb{R}. \tag{7.12}$$

Considering the third case of (7.9), the point-like Lagrangian (7.6) can be written as

$$\mathcal{L} = 6F_0 e^{k\phi}a\dot{a}^2 + 6F_0 k e^{k\phi}a^2\dot{a}\dot{\phi} + a^3\omega_0 e^{k\phi}\dot{\phi}^2 - a^3V_0 e^{c\phi} \tag{7.13}$$

and, even in this case, the equations of motion set the value of the parameter k and c, introducing the further constraint $k = c$. Therefore, discarding the solutions with minimal coupling, we find

$$a(t) = a_0 e^{qt}\,, \quad \phi(t) = \frac{3F_0kq \pm \sqrt{(3F_0kq)^2 - 6F_0\omega_0q^2 - V_0\omega_0}}{\omega_0}\,, \quad q \in \mathbb{R}. \tag{7.14}$$

The values of the constants F_0, ω_0, V_0 can be fixed according to cosmological observations [167]. In summary, deSitter-like expansions are provided by symmetries, and will be compared with those of the next sections in order to point out the equivalence among three (apparently) different scalar-tensor theories.

7.2 Non-Minimally Coupled Torsion Scalar

In this section we develop similar considerations for non-minimally coupled TEGR. We will show that dynamics and solutions, derived from Noether symmetries, are equivalent to those obtained in Sec. 7.1. In this sense, symmetries can be a criterion capable of comparing theories coming from different representations of gravity.
Let us consider the teleparallel equivalent of action (7.1), *i.e.*:

$$S = \int e \left[T F(\phi) + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha} - V(\phi) \right] d^4x, \tag{7.15}$$

whose Klein-Gordon equation reads as

$$\omega_\phi(\phi)\phi^{;\alpha}\phi_{;\alpha} + 2\omega(\phi)\Box\phi - TF_\phi(\phi) + V_\phi(\phi) = 0. \tag{7.16}$$

Here e takes the place of $\sqrt{-g}$ and stands for the determinant of tetrad fields. Unlike the previous case, the cosmological expression of torsion does not contain second derivatives which must be integrated out; therefore, the point-like Lagrangian can be easily found only by replacing the relation (2.37) into the action and integrating the three-dimensional surface:

$$\mathcal{L} = -6aF(\phi)\dot{a}^2 + a^3\omega(\phi)\dot{\phi}^2 - a^3V(\phi). \tag{7.17}$$

Note that this Lagrangian is already canonical and the equations of motion are simplified with respect to Eqs. (7.7). They are:

$$\begin{cases} a: \ -2F\dot{a}^2 + a^2(V - \omega\dot{\phi}^2) - 4a(F_\phi \dot{a}\dot{\phi} + F\ddot{a}) = 0 \\ \phi: \ 6\dot{a}^2 F_\phi + 6a\omega\dot{a}\dot{\phi} + a^2(V_\phi + \omega_\phi \dot{\phi}^2 + 2\omega\ddot{\phi}) = 0 \\ E_{\mathcal{L}} = 0: \ -6aF\dot{a}^2 + a^3(V + \omega\dot{\phi}^2) = 0. \end{cases} \tag{7.18}$$

The minisuperspace considered is two-dimensional as in the previous case ($\mathcal{S} = \{a, \phi\}$) and the generator of the symmetry is in turn

$$\mathcal{X} = \xi(t)\partial_t + \alpha(a, \phi, t)\partial_a + \beta(a, \phi, t)\partial_\phi. \tag{7.19}$$

Notice that, being the Lagrangian in a canonical form, the condition $\xi = \xi(t)$ immediately holds. The application of the extended Noether vector to the point-like Lagrangian (7.17) provides a system of 12 equations, which can be reduced to four equations with the constraints on the infinitesimal generators, that is [162]

$$\begin{cases} 6F\partial_\phi \alpha - 2\omega a^2 \partial_a \beta = 0 \\ \alpha F + \beta a F_\phi + aF\partial_t \xi - 2aF\partial_a \alpha = 0 \\ 3\alpha V + \beta a V_\phi - aV\partial_t \xi = 0 \\ 3\alpha\omega + \beta a\omega_\phi - a\omega\partial_t \xi + 2a\omega\partial_\phi \beta = 0 \\ \alpha = \alpha(a, \phi) \quad \beta = \beta(a, \phi) \quad \xi = \xi(t). \end{cases} \tag{7.20}$$

After some manipulations, the system can be recast in two differential equations containing the three functions $F(\phi), \omega(\phi), V(\phi)$ and two unknown infinitesimal generators. It is therefore clear that the system cannot provide a unique solution, and an initial choice must be adopted in order to fix the related dynamics. Therefore, we replace in (7.20) power-law and exponential potentials, which are of cosmological interest. The assumption is not too much strict, since only the form of the potential is needed in order to exactly solve the system. Solutions containing power-law and exponential potentials are:

$$\begin{cases} \mathcal{X} = (\xi_0 t + \xi_1)\,\partial_t - \dfrac{\xi_0}{3} a \left(\dfrac{k+c}{k-c}\right) \partial_a + \dfrac{2\xi_0}{k-c}\phi\partial_\phi \\ F(\phi) = F_0\phi^k \quad \omega(\phi) = \omega_0\phi^{k-2} \quad V(\phi) = V_0\phi^c \quad k \neq c \\ \\ \mathcal{X} = (\xi_0 t + \xi_1)\,\partial_t - \dfrac{\xi_0}{3} a \left(\dfrac{k+c}{k-c}\right) \partial_a + \dfrac{2\xi_0}{k-c}\partial_\phi \\ F(\phi) = F_0 e^{k\phi} \quad \omega(\phi) = \omega_0 e^{k\phi} \quad V(\phi) = V_0 e^{c\phi} \quad k \neq c \\ \\ \mathcal{X} = -\dfrac{k}{3} a\beta(\phi)\partial_a + \beta(\phi)\partial_\phi \\ F(\phi) = F_0 e^{k\phi} \quad \omega(\phi) = \omega_0 e^{k\phi} \quad V(\phi) = V_0 e^{k\phi}. \end{cases} \tag{7.21}$$

Another viable choice is $\omega(\phi) = Const.$, which allows to recover Brans-Dicke gravity as a limit. In this case, the system (7.20) yields

$$\begin{cases} \mathcal{X} = -\dfrac{2\beta_0}{2s+3} a^{s+1}\phi^{-\frac{2s}{2s+3}}\partial_a + \beta_0 a^s \phi^{\frac{3}{2s+3}}\partial_\phi \\ F(\phi) = \dfrac{(2s+3)^2}{48}\phi^2 \quad V(\phi) = V_0\phi^{\frac{6}{2s+3}} \\ \\ \mathcal{X} = -\dfrac{2}{3} a^{\frac{1}{4}}(c_2 + 2c_3\phi)\partial_a + a^{-\frac{3}{4}}(c_1 + c_2\phi + c_3\phi^2)\partial_\phi \\ F(\phi) = \dfrac{3}{64c_3}(c_1 + c_2\phi + c_3\phi^2) \qquad V(\phi) = V_0(c_1 + c_2\phi + c_3\phi^2)^2. \end{cases} \tag{7.22}$$

Also in this case, the exponential solutions of Noether system allow us to find out the teleparallel equivalent of string-dilaton cosmology, namely the string-dilaton action with torsion instead of curvature.

Let us now solve the Euler-Lagrange equations (7.18) for two different set of couplings, potentials and kinetic terms. We choose the most general solutions among those in (7.21), namely the first and the second. In the former case the Lagrangian (7.17) turns out to be:

$$\mathcal{L} = -6F_0 a\phi^k \dot{a}^2 + \omega_0 a^3\phi^{k-2}\dot{\phi}^2 - V_0 a^3\phi^c\,. \tag{7.23}$$

Assuming the condition $k = c$ and the de Sitter-like expansion for the scale factor, we have:

$$a(t) = a_0 e^{qt}\,, \quad \phi(t) = \phi_0 \exp\left\{\pm\sqrt{\frac{6F_0q^2 - V_0}{\omega_0}}\,t\right\}\,, \quad q = \sqrt{\frac{V_0\omega_0}{6F_0\omega_0 - 4k^2F_0^2}}\,. \tag{7.24}$$

By taking into account the second set of functions, the Lagrangian takes the form

$$\mathcal{L} = -6F_0 a e^{k\phi}\dot{a}^2 + \omega_0 a^3 e^{k\phi}\dot{\phi}^2 - V_0 a^3 e^{c\phi}, \tag{7.25}$$

leading to the exponential solutions constrained by the relation $k = c$:

$$a(t) = a_0 e^{qt}\,, \quad \phi(t) = \pm\sqrt{\frac{6F_0q^2 - V_0}{\omega_0}}\,t\,, \quad q = \sqrt{\frac{V_0\omega_0}{6F_0\omega_0 - 4k^2F_0^2}}\,. \tag{7.26}$$

It is worth stressing the difference between the scalar field coupled to the curvature scalar and to the torsion scalars. The Noether approach performed in Sec. 7.1 allows to find exact expressions for the scalar field and for the scale factor, but the analytic relations between the free parameters cannot be obtained analytically. In the case treated here, instead, such a relation can be analytically found, so that an exact solution of Euler-Lagrange equations (7.18) occurs. This is due to the cosmological expression of T which, not containing second derivatives, leads immediately to a canonical Lagrangian.
Similar results occur considering the Gauss–Bonnet topological term non minimally coupled to a scalar field, as we are going to discuss in the forthcoming section.

7.3 Non-Minimally Coupled Gauss–Bonnet Scalar

Here, we will consider functions of $\mathcal{G}$ non-minimally coupled to a scalar field, in order to discuss solutions analogue to the above non-minimally coupled curvature and torsion cases. Let us start by considering the action

$$S=\int\sqrt{-g}\,[\mathcal{G}^nF(\phi)+\omega(\phi)\phi_{;\alpha}\phi^{;\alpha}-V(\phi)]\,d^4x\,,\quad\text{with}\quad n\in\mathbb{R},\tag{7.27}$$

whose Klein-Gordon equation and field equations read, respectively

$$\omega_\phi(\phi)\phi^{;\alpha}\phi_{;\alpha}+2\omega(\phi)\Box\phi-\mathcal{G}^nF_\phi(\phi)+V_\phi(\phi)=0,\tag{7.28}$$

$$\begin{aligned}&\frac{1}{2}g_{\mu\nu}\mathcal{G}^nF(\phi)-2nF(\phi)\left(RR_{\mu\nu}-2R_{\mu\alpha}R^\alpha{}_\nu+R_\mu{}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma}-2R^{\alpha\beta}R_{\mu\alpha\nu\beta}\right)\mathcal{G}^{n-1}\\&+nF(\phi)\left[2RD_\mu D_\nu+4G_{\mu\nu}\Box-4(R^\rho{}_\nu D_\mu+R^\rho{}_\mu D_\nu)D_\rho+4g_{\mu\nu}R^{\rho\sigma}D_\rho D_\sigma\right.\\&\left.-4R_{\mu\alpha\nu\beta}D^\alpha D^\beta\right]\mathcal{G}^{n-1}-\frac{1}{2}g_{\mu\nu}\omega(\phi)\phi_{;\alpha}\phi^{;\alpha}-F(\phi)_{;\mu;\nu}+g_{\mu\nu}\Box F(\phi)\\&+\omega(\phi)\phi_{;\mu}\phi_{;\nu}+\frac{1}{2}g_{\mu\nu}V(\phi)=0.\end{aligned}\tag{7.29}$$

Note that, with respect to Secs. 7.1 and 7.2, we introduced into the action a new degree of freedom, hence the minisuperspace in no longer two-dimensional, but it contains one more variable, that is $\mathcal{S}=\{a,\phi,\mathcal{G}\}$. This is linked to the term $\mathcal{G}^n$, which cannot be treated at the same level as R and T due to the power n. By replacing the cosmological expression of $\mathcal{G}$ into the action, we obtain second order derivatives which cannot be eliminated through a simple integration. In order to find out the point-like Lagrangian, we have to define a further Lagrange multiplier λ which must be introduced into the action through the constraint (4.6), *i.e.*

$$\mathcal{G}=24\frac{\dot{a}^2\ddot{a}}{a^3}.\tag{7.30}$$

After integrating the surface term, the action turns out to be:

$$S=2\pi^2\int a^3\left\{\left[F(\phi)\mathcal{G}^n+\omega(\phi)\dot{\phi}^2-V(\phi)\right]-\lambda\left(\mathcal{G}-24\frac{\dot{a}^2\ddot{a}}{a^3}\right)\right\}dt.\tag{7.31}$$

The Lagrange multiplier can be found by varying the action with respect to the Gauss–Bonnet invariant. It is:

$$\frac{\delta S}{\delta \mathcal{G}} = 0 \quad \rightarrow \quad \lambda = a^3 n \mathcal{G}^{n-1} F(\phi). \tag{7.32}$$

Replacing now the result into the action and integrating out the second derivatives, the point-like Lagrangian becomes

$$\mathcal{L} = (1-n)a^3\mathcal{G}^n F(\phi) - 8n\dot{a}^3\dot{\phi}F_\phi(\phi)\mathcal{G}^{n-1} + a^3\omega(\phi)\dot{\phi}^2 - a^3V(\phi) - 8n(n-1)G^{n-2}\dot{a}^3\dot{\mathcal{G}}F(\phi). \tag{7.33}$$

Clearly the Gauss–Bonnet contribution does not disappear for $n = 1$, since the the surface term can be addressed to the scalar field, providing a non-trivial dynamics.
In this case we have three Euler–Lagrange equations and the energy condition; the further equation is the one for $\mathcal{G}$, which provides the cosmological expression of the Gauss–Bonnet surface term by construction. The equations of motion therefore read

$$\begin{cases} a:\ 3a^2\left[(n-1)F\mathcal{G}^n + V - \omega\dot{\phi}^2\right] - 24nG[t]^{n-3}\dot{a}\left\{(n-1)F\left[2\mathcal{G}\dot{\mathcal{G}}\ddot{a} + (n-2)\dot{a}\dot{\mathcal{G}}^2 + \dot{a}\mathcal{G}\ddot{\mathcal{G}}\right]\right. \\ \left. +2\mathcal{G}^2F_\phi\dot{\phi}\ddot{a} + \dot{a}\mathcal{G}\left[\mathcal{G}\dot{\phi}^2F_{\phi\phi} + 2F_\phi(n-1)\dot{\mathcal{G}}\dot{\phi} + \mathcal{G}F_\phi\ddot{\phi}\right]\right\} = 0 \\ \phi:\ 6a^2\omega\dot{a}\dot{\phi} - 24n\mathcal{G}^{n-1}\dot{a}^2F_\phi\ddot{a} + a^3\left[(n-1)\mathcal{G}^nF_\phi + V_\phi + \omega_\phi\dot{\phi}^2 + 2\omega\ddot{\phi}\right] = 0 \\ \mathcal{G}:\ a^3\mathcal{G} - 24\dot{a}^2\ddot{a} = 0 \\ E_{\mathcal{L}} = 0:\ -8n\mathcal{G}^{n-2}\dot{a}^3\left[2(n-1)F\dot{\mathcal{G}} + 3\mathcal{G}F_\phi\dot{\phi}\right] + a^3\left[(n-1)F\mathcal{G}^n + V + \omega\dot{\phi}^2\right] = 0. \end{cases} \tag{7.34}$$

The generator of the symmetry in the three-dimensional minisuperspace contains one further infinitesimal generator related to $\mathcal{G}$:

$$\mathcal{X} = \xi(a,\phi,\mathcal{G},t)\partial_t + \alpha(a,\phi,\mathcal{G},t)\partial_a + \beta(a,\phi,\mathcal{G},t)\partial_\phi + \gamma(t,a,\phi,\mathcal{G})\partial_{\mathcal{G}}, \tag{7.35}$$

so that the application of $X^{[1]}$ to Lagrangian (7.33) provides the following system of 4 differential equations [162]

$$\begin{cases}(n-1)\left(\gamma F_\phi + F\partial_\phi\gamma\right) + \beta\mathcal{G}F_{\phi\phi} + F_\phi\mathcal{G}\left(-3\partial_t\xi + \partial_\phi\beta + 3\partial_a\alpha\right) = 0\\ \beta\mathcal{G}F_\phi + F\left[(n-2)\gamma + \mathcal{G}\left(-3\partial_t\xi + \partial_{\mathcal{G}}\gamma + 3\partial_a\alpha\right)\right] = 0\\ 3\alpha\omega + a\left[\beta\omega_\phi - \omega(\partial_t\xi - 2\partial_\phi\beta)\right] = 0\\ 3\alpha\mathcal{G}\left[(n-1)F\mathcal{G}^n + V\right] + a(n-1)F\mathcal{G}^n\left(n\gamma + \mathcal{G}\partial_t\xi\right) + a\mathcal{G}\beta\left[(n-1)\mathcal{G}^nF_\phi + V_\phi\right] + a\mathcal{G}V\partial_t\xi = 0\\ \alpha = \alpha(a) \quad \beta = \beta(\phi) \quad \gamma = \gamma(a,\phi,\mathcal{G}) \quad \xi = \xi(t).\end{cases} \tag{7.36}$$

As in the previous cases, the system is overdetermined and admits an infinite class of solutions depending on the form of the unknown coupling, namely

$$\begin{aligned}&\alpha = \alpha_0 a \quad \beta = -\frac{(3\alpha_0 + \xi_0 - 4n\xi_0)F(\phi)}{F_\phi(\phi)} && \gamma = -4\xi_0\mathcal{G} \quad \xi = \xi_0 t + \xi_1\\ &\omega = \frac{F(\phi)^{\frac{-3\alpha_0+(8n-3)\xi_0}{3\alpha_0+\xi_0-4n\xi_0}}F_\phi(\phi)^2}{(3\alpha_0+\xi_0-4n\xi_0)^2} && V = V_0F(\phi)^{\frac{3\alpha_0+\xi_0}{3\alpha_0+\xi_0-4n\xi_0}}.\end{aligned} \tag{7.37}$$

Therefore, by choosing exponential and power-law couplings, Eq. (7.37) can be split in two different solutions:

$$\begin{aligned}&\alpha = \alpha_0 a \quad \beta = -\frac{3\alpha_0+\xi_0-4n\xi_0}{k} \quad \gamma = -4\xi_0\mathcal{G} \quad \xi = \xi_0 t + \xi_1\\ &\omega = \frac{k^2}{(3\alpha_0+\xi_0-4n\xi_0)^2}e^{\frac{k(3\alpha_0-\xi_0)\phi}{3\alpha_0+\xi_0-4n\xi_0}} \qquad V = V_0F_0^{\frac{3\alpha_0+\xi_0}{3\alpha_0+\xi_0-4n\xi_0}}e^{\frac{k(3\alpha_0+\xi_0)\phi}{3\alpha_0+\xi_0-4n\xi_0}}\\ &F(\phi) = F_0e^{k\phi}\end{aligned} \tag{7.38}$$

$$\begin{aligned}&\alpha = \alpha_0 a \quad \beta = -\left(\frac{3\alpha_0+\xi_0-4n\xi_0}{k}\right)\phi \quad \gamma = -4\xi_0\mathcal{G} \quad \xi = \xi_0 t + \xi_1\\ &\omega = \frac{k^2}{(3\alpha_0+\xi_0-4n\xi_0)^2}\phi^{\frac{k(3\alpha_0-\xi_0)}{3\alpha_0+\xi_0-4n\xi_0}-2} \qquad V = V_0F_0^{\frac{3\alpha_0+\xi_0}{3\alpha_0+\xi_0-4n\xi_0}}\phi^{\frac{k(3\alpha_0+\xi_0)}{3\alpha_0+\xi_0-4n\xi_0}}\\ &F(\phi) = F_0\phi^k.\end{aligned} \tag{7.39}$$

The above hold as long as $n \neq 1$; otherwise we obtain the following:

$$\begin{aligned}
&\alpha = 0 \quad \beta = \frac{3\xi_0}{k}\phi \quad \xi = \xi_0 t + \xi_1 \\
&F(\phi) = -\frac{1}{3V_0^2}\phi^k \quad V(\phi) = V_0\phi^{-\frac{k}{3}} \quad \omega(\phi) = \omega_0\phi^{\frac{k}{3}-2}
\end{aligned} \tag{7.40}$$

$$\begin{aligned}
&\alpha = 0 \quad \beta = \frac{3\xi_0}{k}\phi \quad \xi = \xi_0 t + \xi_1 \\
&F(\phi) = \frac{1}{k}e^{k\phi} \quad V(\phi) = V_0 e^{-\frac{k}{3}\phi} \quad \omega(\phi) = \omega_0 e^{\frac{k}{3}\phi},
\end{aligned} \tag{7.41}$$

where, as before, $\xi_0, \alpha_0, \beta_0, \gamma_0, k, n$ are real constants. The $n = 1$ limit is topologically trivial only when $k = 1$, where the contribution of the geometry in the corresponding action turns into a topological surface term. For this reason, there is no interest in investigating cosmological solutions occurring for $k = n = 1$ and, in what follows, we will only focus on the $n = 1/2$ case, which represents the Gauss–Bonnet equivalent to GR in the cosmological framework.

Now we derive cosmological solutions for the Noether symmetry (7.37). We will solve the Euler-Lagrange equations (7.34) for $n = 1/2$ in order to compare the results with the above curvature and torsion cases. For $f(\mathcal{G}) = \mathcal{G}^{1/2}$, Noether's solutions (7.39) can be written as:

$$\begin{aligned}
&\alpha = \frac{\ell}{6}(z+k)a \quad \beta = -\ell \quad \gamma = -2\ell(z-k)\mathcal{G} \quad \xi = \frac{\ell}{2}(z-k)t + \xi_1 \\
&\omega(\phi) = \frac{1}{\ell^2}e^{k\phi} \qquad V(\phi) = \tilde{V}_0 e^{z\phi} \qquad F(\phi) = F_0 e^{k\phi}
\end{aligned} \tag{7.42}$$

$$\begin{aligned}
&\alpha = \frac{\ell}{6}(z+k)a \quad \beta = -\ell\phi \quad \gamma = -2\ell(z-k)\mathcal{G} \quad \xi = \frac{\ell}{2}(z-k)t + \xi_1 \\
&\omega(\phi) = \frac{1}{\ell^2}\phi^{k-2} \qquad V(\phi) = \tilde{V}_0\phi^z \qquad F(\phi) = F_0\phi^k,
\end{aligned} \tag{7.43}$$

where we have defined

$$\ell \equiv \frac{3\alpha_0 - \xi_0}{k} \qquad V_0 F_0^{1+\frac{2\xi_0}{k\ell}} \equiv \tilde{V}_0 \qquad z \equiv k\left(\frac{3\alpha_0 + \xi_0}{3\alpha_0 - \xi_0}\right). \tag{7.44}$$

Let us start by analyzing the action containing a power-law coupling, namely:

$$S = \int \sqrt{-g}\left[F_0\sqrt{\mathcal{G}}\phi^k + \frac{1}{\ell^2}\dot{\phi}^2\phi^{k-2} + \tilde{V}_0\phi^z\right] d^4x, \tag{7.45}$$

corresponding to the solution of Eq. (7.39). After solving the system (7.34), we obtain a de Sitter-like solution which fixes the values of k and z to $k = z = \dfrac{\sqrt{6}}{2\ell^2 F_0}$. It reads as:

$$a(t) = a_0 e^{-\frac{\sqrt{-6\tilde{V}_0}}{2\ell F_0}t}, \quad \phi(t) = \phi_0 e^{\ell\sqrt{-\tilde{V}_0}t}, \quad \mathcal{G}(t) = \frac{54\tilde{V}_0^2}{\ell^4 F_0^4}. \tag{7.46}$$

This means that, by merging the result provided by the Euler-Lagrange equations with those coming from the Noether approach, the only generator associated to this case is:

$$\mathcal{X} = \frac{1}{\sqrt{6}\ell F_0} a\partial_a - \ell\phi\partial_\phi, \tag{7.47}$$

which describes an internal gauge symmetry. Let us now analyze the second solution with exponential coupling, potential and kinetic term; the corresponding action takes the form:

$$S = \int \sqrt{-g}\left[F_0\sqrt{\mathcal{G}}e^{k\phi} + \frac{1}{\ell^2}\dot{\phi}^2 e^{k\phi} + \tilde{V}_0 e^{z\phi}\right] d^4x. \tag{7.48}$$

By replacing Eq. (7.42) into the equations of motion (7.34), it turns out that these latter can be analytically solved by imposing the constraint $k = z$, so that the scale factor and the scalar field behave like

$$\begin{aligned} &a(t) = a_0 \exp\left\{\frac{k\ell}{3}\sqrt{\tilde{V}_0}(1+\sqrt{2})\,t\right\}, \qquad \phi(t) = -\ell\sqrt{\tilde{V}_0}t\,, \\ &\mathcal{G}(t) = \frac{8k^4\ell^4}{27}\tilde{V}_0^2(1+\sqrt{2})^4\,, \qquad k = \frac{3}{\ell^2}\sqrt{\frac{1}{F_0\sqrt{21-12\sqrt{2}}}}. \end{aligned} \tag{7.49}$$

As final remark, it is worth noticing that the non-minimal couplings with the invariants R, T, and $\mathcal{G}$ all admit de Sitter solutions which can be easily compared each-other. It is important to point out that the Gauss–Bonnet topological invariant can be defined also in the case of teleparallel gravity [74, 168] so that the above representations of gravity can be made totally equivalent also at this level. This is due to the fact that in a cosmological context, the explicit expression of the Gauss–Bonnet term turns out to be the same as that of the teleparallel equivalent.

7.4 Equivalence of Hamiltonian Dynamics

We want to show now that for internal symmetries, namely for $\xi(t) = 0$, the Noether approach provides transformation laws allowing to introduce cyclic variables into the point-like cosmological Lagrangian. In order to get internal symmetries, we have to set the infinitesimal generator related to the time variation equal to zero. In the case of Ricci scalar coupled to the scalar field, the only solution containing symmetries which, after setting $\xi(t) = 0$, does not lead to trivial results, is that written in Eq. (7.10). With regards to the teleparallel equivalent, the only compatible solutions are listed in (7.22). For the Gauss–Bonnet term, both solutions can be equivalently considered. It is worth noticing that the generator in Eq. (7.10), the first in Eq. (7.22) and that in Eq. (7.43), under appropriate conditions, are equivalent. For this reason, only the Hamiltonian dynamics provided by the following generators will be investigated:

$$R :\to \begin{cases} \mathcal{X} = -\dfrac{2(s+1)}{2s+3}\beta_0 a^{s+1}\phi^{\frac{2s^2+4s}{2s+3}}\partial_a + \beta_0 a^s \phi^{\frac{2s^2+6s+3}{2s+3}}\partial_\phi \\ F(\phi) = \dfrac{(2s+3)^2}{48(s+1)(s+2)}\phi^2 \quad V(\phi) = V_0\phi^{\frac{6(s+1)}{2s+3}}, \end{cases} \tag{7.50}$$

$$T :\to \begin{cases} \mathcal{X} = -\dfrac{2\beta_0}{2s+3}a^{s+1}\phi^{-\frac{2s}{2s+3}}\partial_a + \beta_0 a^s \phi^{\frac{3}{2s+3}}\partial_\phi \\ F(\phi) = \dfrac{(2s+3)^2}{48}\phi^2 \quad V(\phi) = V_0\phi^{\frac{6}{2s+3}}, \end{cases} \tag{7.51}$$

$$\mathcal{G} :\rightarrow \begin{cases} \mathcal{X} = \dfrac{k\ell}{3} a\partial_a - \ell\phi\partial_\phi \\ F(\phi) = F_0\phi^k \qquad V = V_0\phi^k \qquad \omega(\phi) = \dfrac{1}{\ell^2}\phi^{k-2}\,. \end{cases} \tag{7.52}$$

In the last solution, to obtain the condition $\xi = 0$, we set $k = z$, $\xi_1 = 0$. With the aim to compare the three solutions, we set the coefficient of the kinetic term in the Gauss–Bonnet case (7.52) to be constant, as naturally provided by the Noether approach in the other two cases (7.50) and (7.51). Therefore, by setting $s = 0$ and $k = 2$, (7.50), (7.51) and (7.52) become

$$R :\rightarrow \begin{cases} \mathcal{X} = -\dfrac{2}{3}\beta_0 a\partial_a + \beta_0\phi\partial_\phi \\ F(\phi) = \dfrac{3}{32}\phi^2 \quad V(\phi) = V_0\phi^2, \end{cases} \tag{7.53}$$

$$T :\rightarrow \begin{cases} \mathcal{X} = -\dfrac{2\beta_0}{3} a\partial_a + \beta_0\phi\partial_\phi \\ F(\phi) = \dfrac{3}{16}\phi^2 \quad V(\phi) = V_0\phi^2, \end{cases} \tag{7.54}$$

$$\mathcal{G} :\rightarrow \begin{cases} \mathcal{X} = \dfrac{2\ell}{3} a\partial_a - \ell\phi\partial_\phi \\ F(\phi) = F_0\phi^2 \qquad V = V_0\phi^2. \end{cases} \tag{7.55}$$

The above suggest that the symmetries fix the equivalence among the three representations of gravity when a scalar field is coupled with R, T, and $\sqrt{\mathcal{G}}$, respectively.

To finalize the approach, let us firstly consider the generator (7.50). Thanks to the system (B.26) in App. B, we can perform the change of variables induced by the Noether symmetry which allows to introduce a cyclic variable into the Lagrangian. The system (B.26) takes the form:

$$\begin{cases} \mathcal{X}z = \dfrac{2\beta_0}{3} a\partial_a z - \beta_0\phi\partial_\phi z = 1 \\ \mathcal{X}u = \dfrac{2}{3} a\partial_a u - \phi\partial_\phi u = 0, \end{cases} \tag{7.56}$$

where z represents the cyclic variable and the minisuperspace of configurations is transformed from $\mathcal{S} = \{a, \phi\}$ to $\mathcal{S}' = \{z, u\}$. A possible solution of the above system is

$$\begin{cases} z = -\dfrac{1}{\beta_0}\ln\phi \\ u = a^{\frac{3}{2}}\phi \end{cases} \quad \rightarrow \quad \begin{cases} \phi = e^{-\beta_0 z} \\ a = u^{\frac{2}{3}} e^{\frac{2\beta_0}{3}z}. \end{cases} \tag{7.57}$$

Replacing the new variables u, z into the Lagrangian (7.6), we get

$$\mathcal{L}_R = -V_0 u^2 + \frac{1}{2}\ell^2 u^2 \dot{z}^2 - \frac{1}{4}\ell u \dot{u}\dot{z} + \frac{1}{4}\dot{u}^2, \tag{7.58}$$

where we set $\ell \equiv \beta_0$ in order to conform the notation to the other examples. Clearly this form of $\mathcal{L}_R$ is cyclic with respect to z. After finding the time-derivatives of the variables as functions of the conjugate momenta, we can easily get the Hamiltonian:

$$\mathcal{H}_R = \pi_u^2 + \frac{4}{7\ell^2}\frac{\pi_z^2}{u^2} + V_0 u^2. \tag{7.59}$$

Classical trajectories (7.12) can be recovered by means of the Hamilton–Jacobi equations after going back to the old variables (7.57). It is worth noticing that, in order to provide a comparison among the three equivalent cases, the Hamiltonian dynamics has been studied for the solution (7.50) only; however, the change of variables coming from the Noether approach can be also found for the other solutions of Noether system.

The next case is the torsion non-minimally coupled to the scalar field, whose Noether's solution is provided by Eq. (7.51) with $s = 0$. With this assumption, the system providing the suitable change of variables is:

$$\begin{cases} -\dfrac{2\beta_0}{3} a\partial_a z + \beta_0 \phi \partial_\phi z = 1 \\ \\ -\dfrac{2}{3} a\partial_a u + \phi\partial_\phi u = 0 \ , \end{cases} \tag{7.60}$$

and a possible solution is the same as before, namely

$$\begin{cases} z = -\dfrac{1}{\beta_0}\ln\phi \\ u = a^{\frac{3}{2}}\phi \end{cases} \qquad \rightarrow \qquad \begin{cases} \phi = e^{-\beta_0 z} \\ a = u^{\frac{2}{3}} e^{\frac{2\beta_0}{3}z}. \end{cases} \tag{7.61}$$

Setting $\beta_0 \equiv \ell$, the new Lagrangian reads

$$\mathcal{L}_T = -\frac{1}{8}\frac{\dot{u}^2}{u} + \frac{1}{2}\ell\dot{u}\dot{z} + \frac{1}{2}\ell^2 u\dot{z}^2 - V_0 u\ . \tag{7.62}$$

Also here, z is the cyclic variable which permits to write the Hamiltonian as

$$\mathcal{H}_T = \frac{\pi_z^2}{2w} + 2\pi_z\pi_u + \frac{3}{16}u\pi_u^2 + V_0 w, \tag{7.63}$$

where π_z, π_u are the conjugate momenta.

Finally, in the Gauss–Bonnet equivalent case, we consider the symmetry generator in Eq. (7.43) to find out Hamiltonian dynamics when the geometry is coupled to the scalar field through the function $F(\mathcal{G}) = \sqrt{\mathcal{G}}$. For $k = 2$, where we have a constant kinetic term, the Lagrangian can be written as:

$$\mathcal{L} = -V_0 a^3\phi^2 + \frac{1}{2}F_0 a^3\sqrt{\mathcal{G}}\phi^2 + 2F_0\phi^2\dot{a}^3\dot{\mathcal{G}})\mathcal{G}^{-\frac{3}{2}} - 8F_0\phi\dot{a}^3\dot{\phi}\mathcal{G}^{-\frac{1}{2}} + \frac{1}{\ell^2}a^3\dot{\phi}^2. \tag{7.64}$$

The condition (B.26) permits to change the minisuperspace variables from $\mathcal{S} = \{a, \phi, \mathcal{G}\}$ to $\mathcal{S}' = \{z, u, \mathcal{G}\}$ and gives rise to the same system of differential equations as Eq. (7.56) and Eq. (7.60), *i.e*

$$\begin{cases} \mathcal{X}z = \dfrac{2\ell}{3}a\partial_a z - \ell\partial_\phi z = 1 \\ \mathcal{X}u = \dfrac{2}{3}a\partial_a u - \partial_\phi u = 0, \end{cases} \tag{7.65}$$

with z being the cyclic variable. One possible solution is

$$\begin{cases} z = -\frac{1}{\ell} \ln \phi \\ u = a^{\frac{3}{2}} \phi \end{cases} \qquad \rightarrow \qquad \begin{cases} \phi = e^{-\ell z} \\ a = u^{\frac{2}{3}} e^{\frac{2\ell}{3} z}. \end{cases} \tag{7.66}$$

Replacing the new variables u, z into the Lagrangian (7.64), we get

$$\mathcal{L}_{\mathcal{G}} = \frac{\mathcal{G}^{-\frac{3}{2}}}{54u} \left[27F_0 \mathcal{G}^2 u^3 - 54\mathcal{G}^{\frac{3}{2}} u^3 (V_0 - \ell^2 \dot{z}^2) + 32F_0 \dot{\mathcal{G}} (\ell u \dot{z} + \dot{u})^3 + 128F_0 \ell \mathcal{G} \dot{z} (\ell u \dot{z} + \dot{u})^3 \right] \tag{7.67}$$

where, as expected, z is cyclic. By a straightforward Legendre transformation, we find the Hamiltonian

$$\begin{aligned} \mathcal{H}_{\mathcal{G}} = \frac{1}{4u^2} \quad & \Big[16\mathcal{G}^2 \pi_{\mathcal{G}}^2 + 8u\mathcal{G}\pi_{\mathcal{G}}\pi_u - 2F_0\sqrt{\mathcal{G}}u^4 + u^2\pi_u^2 \\ & + 4V_0 u^4 + 3 \cdot 2^{\frac{2}{3}} u \left(\frac{\mathcal{G}^{\frac{3}{2}} u \pi_{\mathcal{G}}}{F_0 \ell^3} \right)^{\frac{1}{3}} (\pi_z + \ell \pi_u) \Big]. \end{aligned} \tag{7.68}$$

As final remark, we can state that the three scalar–tensor models analyzed in this chapter are dynamically equivalent, both at the level of equations of motion and of Hamiltonian dynamics. This means that R, T and $\mathcal{G}$ selected by Noether's theorem, leads to the same field equations solutions when they are non–minimally coupled to a dynamical scalar field.

8

Conclusion of Part II

In this part of the thesis, we discussed modifications of the Hilbert–Einstein action including the Gauss–Bonnet topological surface term. First we started by $f(\mathcal{G})$ cosmology, using Noether symmetry approach to find out suitable functions leading to exact cosmological solutions. The existence of symmetries selects the power-law form $f(\mathcal{G}) = f_0\mathcal{G}^k$ and, in 4-dimensions, interesting dynamics can be obtained even in presence of matter fields. We pointed out that GR can be recovered from the Gauss–Bonnet invariant by considering the square root of $\mathcal{G}$ into the action. This is due to the definition of $\mathcal{G}$, namely $\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R^{\mu\nu p\sigma}R_{\mu\nu p\sigma}$, which in a spatially flat FLRW cosmology is comparable with R^2 only, under some limits. From this point of view, GR can be seen as a particular case of $f(\mathcal{G})$ theory, without imposing to recover the scalar curvature as a requirement. Moreover, $f(\mathcal{G})$ function can also mimic the role of Dark Energy, since yields exponential and power-law cosmological solutions also in presence of standard matter. The former can be recovered only in five dimensions or more, while the latter can be found even in four dimensions. However, In four dimensions, de Sitter solutions are possible only adding an extra term $\mathcal{L}_m \sim e^{-3w}$ with $w = -1$. Nonetheless, when the sum $f(\mathcal{G}) = f_0\mathcal{G}^n + f_1\mathcal{G}^k$ is considered, exponential scale factors naturally arise as solutions of the corresponding field equations. Notice that in this latter case, by setting $k = 1/2$, the theory is dynamically equivalent to $f(R, \mathcal{G}) = R + f_0\mathcal{G}^n$ gravity.

Quantum cosmological models was analyzed for the minisuperspace related to the variables a and $\mathcal{G}$. Also in this case, symmetries have a key role for the interpretation of the wave function of the Universe, since allow to find out oscillatory behaviors and then the possibility to apply the Hartle criterion, which states that oscillations mean correlations between variables. This implies the possibility to achieve classical trajectories, that is observables universes.

To address evidences provided by experimental observations in the very early and late time, we also studied the energy conditions for $f(\mathcal{G})$ and $R+f(\mathcal{G})$ gravity, with $f(\mathcal{G})$ selected by Noether's approach. The requirement of energy conditions validity can be useful in extended gravity cosmology in order to check whether the extra geometric contributions can play the role of Dark Energy, but a good choice of cosmographic parameters j,s,q is very worth. We showed that when a function of $\mathcal{G}$ is considered, the energy conditions select a strict validity range of k. Nonetheless, we verified that cosmological slow–roll inflation occurs for several values of k. When the scalar curvature is included in the action, energy conditions suggest that $R+f_0\mathcal{G}^k$ cosmology can describe an accelerated expansion for some values of k. In this latter case, also the value of the coupling constant f_0 must be taken into account. It turns out that when $k=1/2$, all the energy conditions are violated and power-law inflation occurs for a coupling constant f_0 approaching the values $f_0 \sim \pm\sqrt{3/2}$. However, studying the energy conditions in modified theories of gravity is useful to completely discard many theories and to show that the validity of the energy conditions is not as trivial as in GR. In the late-time, modified theories of gravity can explain the anomalous acceleration of the Universe as a curvature effect, which can be intended as a fluid with negative pressure, playing the role of Dark Energy. For this reason, the energy conditions violation in modified theories of gravity does not have to be intended in the same way as GR. While in GR the ordinary matter is the only component in the RHS of the field equations, modified theories of gravity exhibit effective energy density and pressure of the gravitational field. As the ordinary matter must satisfy all the energy conditions identically, some of them can be violated by the effective energy-momentum tensor of the gravitational field.

$f(\mathcal{G})$ gravity was also investigated in a spherically symmetric higher-dimensional background, where Noether symmetry approach selects the same shape of the function provided by the cosmological case. With this choice, analytic static and spherically symmetric solutions can be found, as well as the corresponding horizons and physical singularities. Subsequently, the application of non–local operators of the form $\Box^{-1}$ to the Gauss–Bonnet invariant was considered, pointing out the differences between $f(\Box^{-1}\mathcal{G})$ and $R+f(\Box^{-1}\mathcal{G})$ gravity. The non-local functions was thus selected by Noether's approach, which once again allows to simplify the dynamics. Power–law and exponential scale factors occur as solutions of the field equations when the former action is considered. In the latter case, instead, no solution can be analytically found. This confirm that the additive contribution of the Ricci scalar uselessly complicates the dynamics. In this perspective, considering gravitational actions involving the topological invariant $\mathcal{G}$ seems extremely useful to cure and fix problems that arise from taking into account other curvature invariants. Most of the cosmological solutions found are in agreement with the statement by Deser and Woodard [61] that non-local cosmology can reproduce Dark Energy behavior at IR scales. Among the classes of considered models, the action $S = \int \sqrt{-g} f[\mathcal{G}, \Box^{-1}h(\mathcal{G})]d^4x$ presents an interesting phenomenology because generalizes the analogue $S = \int \sqrt{-g} f[R, \Box^{-1}h(R)]d^4x$ and admits also the possibility to recover GR, corrected with non-local terms.

Finally, we analyzed non-minimal coupling between a scalar field and gravity, taking into account different geometric invariants, namely curvature, torsion, and Gauss–Bonnet scalars respectively. In all cases, the starting action contains three functions of the scalar field, namely the coupling, the kinetic term and the potential. We showed that, by the Noether symmetry approach, it is possible to fix the form of the above functions and exactly solve the dynamics. Furthermore, we demonstrated that if the symmetries coincide, cosmologies coming from curvature, torsion and Gauss–Bonnet gravity are equivalent. In particular, this statement holds as soon as exponential and power–law expansions of the scale factor of the Universe are derived as exact solutions. Interestingly, GR can be recovered in all representations as soon as $R = -T + B$ and $f(\mathcal{G}) = \sqrt{\mathcal{G}}$. According to this result, different theories showing the same symmetries are dynamically equivalent, also if coming from different conceptual foundations.

As a final remark, considering extended Gauss–Bonnet gravity can result useful from several points of view, in particular, for avoiding ghost modes [55] and other pathologies present in GR and in other modified gravity theories. Beside this fact, it seems a natural approach towards the description of quantum fields in curved spaces and, finally, towards quantum gravity.

Part III

CHERN–SIMONS THEORY

9

Chern–Simons Theory: An Overview

A fundamental step in the development of a quantum theory for the gravitational interaction is to merge the formalism of the standard model with the geometric description of the gravitational field. To this purpose, gravity must be treated as a gauge theory. In the covariant formalism, GR can be considered as a diffeomorphism invariant gauge theory, but such an invariance can be recast as a Lorentz invariance in the locally flat tangent space-time. We showed that a similar procedure permits to describe TEGR as a gauge invariant theory with respect to the local translation group. Unfortunately, neither translations nor Lorentz invariance are internal symmetry, and this does not allow to address gravity in a Yang-Mills scheme. Moreover, a theory which well satisfies these requirements at the high energy, must also fit the large–scales results provided by GR in the low energy limit. In this chapter we overview the main features of gauge theories of gravity, finding the corresponding gauge Lagrangians in the flat space-time and showing that no unitary renormalizable Yang–Mills theories of gravity occur in four dimensions. Let us begin by considering a local Lorentz transformation of the form $\Lambda = e^{\frac{i}{2}\omega^{ab}J_{ab}}$, with ω^{ab} being the one-form Lorentz connection. In this case, the covariant derivative of a generic n-form $\mathcal{P}$ provides the $n+1$ form

$$D\mathcal{P} = d\mathcal{P} - \frac{i}{2}\omega^{ab}J_{ab}\mathcal{P} \ , \tag{9.1}$$

so that

$$D \wedge D\mathcal{P} = -\frac{i}{2}R^{ab}J_{ab}\mathcal{P} \quad \text{with} \quad R^{ab} = d\omega^{ab} + \omega^a_c \wedge \omega^{cb} \ . \tag{9.2}$$

R^{ab} is called *Two-Form Curvature* and the second equation of (9.2) is the so called *first Cartan structure equation*. Similarly, thanks to the vielbein[1] postulation, the *Contorsion Tensor* can be defined as:

$$D_{[\mu}e^a_{\nu]} = \partial_{[\mu}e^a_{\nu]} + \omega^a_{[\mu\nu]} = K^a{}_{\mu\nu} \ . \tag{9.3}$$

Notice that the above defined $K^a{}_{\mu\nu}$ is a generalization of the contorsion Tensor (2.20), including the spin connection. The definition of the *Two Form Torsion* by means of the veielbein postulation, provides the *second Cartan structure Equation*:

$$T^a = De^a = de^a + \omega^a_b \wedge e^b \ . \tag{9.4}$$

This formalism manifestly shows that as torsion is the gauge field associated to local translations, curvature is the gauge field related to Lorentz transformations. In those reference frames with vanishing spin connection, the two-form curvature identically vanishes and the space-time structure is described by TEGR. On the other hand, by imposing $T^\alpha = 0$ a torsionless theory ruled by curvature arises.

In Chap. 2 we stressed the equivalence among GR, TEGR and STEGR, showing that the corresponding Lagrangians are formally equivalent among each other in the description of gravity. More precisely they only differ for a boundary term, which does not change the equations of motion. However, as R^{ab} and T^a are the gauge fields linked to Lorentz transformation and translation, respectively, no gauge fields associated to STEGR can be defined.

Therefore, in the development of a gauge theory of gravity, the contribution of non-metricity can be neglected. In the flat tangent space-time, the action must be constructed by means of curvature, torsion, Levi-Civita Tensor and Minkowski metric, which are all invariants with respect to $SO(1, D-1)$. The most general torsionless action containing all these terms and leading to second order field equations is the so called *Lovelock Action*,

[1]Generalization of tetrad fields in higher dimensions.

which reads as [169, 170, 171]:

$$S = \kappa \int_{\mathcal{M}} \sum_{i=0}^{\frac{D}{2}} \alpha_i \mathscr{L}^{(D,i)}, \tag{9.5}$$

with κ dimensionless constant and where $\mathscr{L}^{(D,i)}$ is the *Lovelock Lagrangian*, defined as:

$$\mathscr{L}^{(D,i)} = \epsilon_{a_1,a_2\ldots a_D} R^{a_1 a_2} \wedge R^{a_3 a_4} \wedge \ldots \wedge R^{a_{2i-1} a_{2i}} \wedge e^{a_{2i+1}} \wedge e^{a_{2i+2}} \wedge \ldots \wedge e^{a_D} \ . \tag{9.6}$$

In four dimensions, the Lovelock action turns out to be:

$$\begin{aligned} S_L &= \kappa \int_{\mathcal{M}} \sum_{i=0}^{2} \alpha_i \mathscr{L}^{(4,i)} = \\ &= \kappa \int \epsilon_{abcd} \left(\alpha_0 e^a \wedge e^b \wedge e^c \wedge e^d + \alpha_1 R^{ab} \wedge e^c \wedge e^d + \alpha_2 R^{ab} \wedge R^{cd} \right) . \end{aligned} \tag{9.7}$$

The action can be equivalently expressed in a covariant form as

$$S = \kappa \int \sqrt{-g} \left(\alpha_0 + \alpha_1 R + \alpha_2 \mathcal{G} \right) d^4x, \tag{9.8}$$

namely the Hilbert–Einstein action with cosmological constant plus the Gauss–Bonnet term. Being $\mathcal{G}$ a topological surface term, in four dimensions the Lovelock Lagrangian yields the same field equations as GR. Starting from Eq. (9.6), the free parameters α_i can be suitably set in order to select Lagrangians which are invariant under some gauge transformation. Given the general form of Eq. (9.6), several combinations which yield gauge invariant actions can be found. Nevertheless, this procedure lacks a formal structure and several combinations are missed when proceeding by trial and error.

A straightforward way to overcome this issue is to construct gauge Langrangians by topological invariants. Thus, let E be a $n-$form characteristic class which, by construction, is invariant under local gauge transformations, *i.e.* $\delta E = 0$. If there exists a $D-1$ form $\mathscr{L}$ such that $d\mathscr{L} = E$, then $\mathscr{L}$ is invariant under gauge transformations up to a boundary term. The proof of this statement is straightforward: being E invariant under gauge transformations, the relation

$$\delta E = \delta(d\mathscr{L}) = d(\delta\mathscr{L}) = 0, \tag{9.9}$$

holds. The above suggests that the quantity $\delta\mathscr{L}$ can be written as the exterior derivative of a $D-2$ form B, namely

$$d(\delta\mathscr{L}) = 0 \quad \rightarrow \quad \delta\mathscr{L} = dB. \tag{9.10}$$

This finally shows that $\mathscr{L}$ is invariant up to a total derivative. As $\mathscr{L}$ can be used as a Lagrangian of a topological field theory, E represents a topological surface term which can be found from the condition $E = d\boldsymbol{\mathcal{L}}$. Reversing the argument, gauge Lagrangian can be constructed starting from topological surface terms, whose integration over the space-time provides non-trivial topological invariants. Consequently, the Lagrangian will be invariant up to a boundary term, namely *quasi-gauge invariant*. In general, $D-$dimensional topological surface terms permit to construct $D-1$ dimensional Lagrangians; however, the lack of non-trivial topological surface terms in odd dimensions, yields the impossibility of constructing non-trivial gauge Lagrangians in even dimensions. Such odd-dimensional Lagrangians are called *Chern-Simons Lagrangians* or *Chern-Simons forms*. Chern–Simons theories was firstly introduced by S.S. Chern and J.H. Simons in 1974 [172], with the aim to develop a topological field theory capable of describing all fundamental interactions as Yang–Mills theories. Basic foundations of Chern–Simons theory can be found *e.g.* in Refs. [170, 173, 174, 175, 176, 177, 178] and applications in Refs. [171, 179, 180, 181, 182]. For example, the Chern-Simons 3-form coming from the Pontryagin density $R^{ab}R_{ab}$ is:

$$CS^{P}_{(3)} = \omega^a_b d\omega^b_a + \frac{2}{3}\omega^a_b\omega^b_c\omega^c_a \ . \tag{9.11}$$

Another example is given by the four-dimensional Euler density

$$E_4 = \epsilon_{abc} \left(R^{ab} \wedge T^c + \frac{1}{3l^2} e^a \wedge e^b \wedge T^c \right) ,$$

which turns out to be the exterior derivative of the Chern–Simons Lagrangian invariant with respect to the AdS group:

$$\mathscr{L}_3^{AdS} = \epsilon_{abc} \left(R^{ab} \wedge e^c + \frac{1}{3l^2} e^a \wedge e^b \wedge e^c \right) , \tag{9.12}$$

where T^c is the torsion defined in Eq. (9.4). Note that the presence of torsion in the Euler density does not contradict Lovelock's hypothesis of torsionless Lagrangians. Torsion disappears after performing the exterior derivative, so that the corresponding Chern–Simons form turns out to respect Lovelock assumptions. In general, the $2D-1$-dimensional AdS-invariant Lagrangian reads:

$$\mathscr{L}_{2D-1}^{AdS} = \sum_{i=0}^{D-1} \tilde{\alpha}_i \mathscr{L}^{(2D-1,i)} , \qquad \tilde{\alpha}_i = \frac{(\pm 1)^{i+1} \ell^{2i-D}}{D-2i} \binom{D-1}{i} , \tag{9.13}$$

and the exterior derivative yields the $2D$-dimensional Euler density E_{2D}. In Chaps. 10 and 11 we mainly focus on the five-dimensional Lovelock Lagrangian, with particular interest in those coupling constants yielding the invariance under the local AdS group. The five-dimensional Chern–Simons form is

$$\mathscr{L}_5^{AdS} = \frac{1}{\ell} \epsilon_{abcde} \left(R^{ab} \wedge R^{cd} \wedge e^e + \frac{2}{3\ell^2} R^{ab} \wedge e^c \wedge e^d \wedge e^e + \frac{1}{5\ell^4} e^a \wedge e^b \wedge e^c \wedge e^d \wedge e^e \right) , \tag{9.14}$$

whose exterior derivative provides the following 6-D Euler density [183]:

$$\begin{aligned} E_6 \;=\; & 2R^{abcd} R_{cdef} R^{ef}_{ab} + 8R^{ab}_{cd} R^{ce}_{bf} R^{df}_{ae} + 24R^{abcd} R_{cdbe} R^e_a + 3RR^{abcd} R_{abcd} \\ & +24R^{abcd} R_{ac} R_{bd} + 16R^{ab} R_{bc} R^c_a - 12RR^{ab} R_{ab} + R^3 \, . \end{aligned} \tag{9.15}$$

Notice that the Lagrangian in Eq. (9.14), is contained in the general five-dimensional

Lovelock Lagrangian

$$\mathscr{L}_5^L = \epsilon_{abcde}\left(\alpha_0\, e^a \wedge e^b \wedge e^c \wedge e^d \wedge e^e + \alpha_1 R^{ab} \wedge e^c \wedge e^d \wedge e^e + \alpha_2 R^{ab} \wedge R^{cd} \wedge e^e\right)\ , \tag{9.16}$$

with α_0, α_1, α_2 defined according to the relation (9.13). It is worth remarking that from Eq. (9.6), it is possible to consider several other kind of gauge Lagrangians, belonging to different gauge groups. Most of them are mainly studied due to the results provided in supersymmetry and String Theory [170, 176, 184, 185, 186].

Finally, the four-dimensional *Pontryagin density* $P_4 = dA\,dA$, yields the $U(1)$ invariant three-dimensional Lagrangian

$$\mathscr{L} = AdA, \tag{9.17}$$

where A is the one-form connection. Similarly, from the $SU(4)$ Pontryagin density it is possible to construct the $SU(N)$ invariant 3D Lagrangian, namely

$$\mathscr{L} = \mathrm{Tr}\left(\mathbf{A} \wedge d\mathbf{A} + \frac{2}{3}\mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A}\right), \tag{9.18}$$

which is of particular interest in quantum gravity.

In the next chapters, we start from Eq. (9.14) and Eq. (9.16) and find out exact solutions of the field equations in a cosmological and spherically symmetric space-time. Subsequently, the action (9.17) will be used in Chap. 12 with the aim to construct a three-dimensional massive theory of the electromagnetic interaction. Finally, in Chap. 13, we consider Eq. (9.18) in order to apply the Chern–Simons formalism to biological systems.

10

Chern–Simons Cosmology

In more than four dimensions, the FLRW metric can be extended by including diagonal time-dependent terms. These terms account for new scales factors which, in general, are different than that related to the standard spatial dimensions. We start by assuming that the five-dimensional space-time is not isotropic, extending the FLRW metric to:

$$ds^2 = dx_0^2 - a^2(t)\left[dx_1^2 + dx_2^2 + dx_3^2\right] - b^2(t)dx_4^2\,, \tag{10.1}$$

where x_4 is the fifth dimension, labeled by the scale factor $b(t)$. In order find a solution for the field equations in the extended FLRW interval, we use the first Cartan structure equation to firstly get the two-form curvature. Considering the line element in Eq. (10.1), the set of basis can be chosen as:

$$e^0 = dx^0\,, \quad e^1 = -a(t)dx^1\,, \quad e^2 = -a(t)dx^2\,, \quad e^3 = -a(t)dx^3\,, \quad e^4 = -b(t)dx^4\,, \tag{10.2}$$

with a set of tetrad fields of the form

$$e^a_\mu = \mathrm{diag}\left(1, -a(t), -a(t), -a(t), -b(t)\right). \tag{10.3}$$

The exterior derivatives of the set of independent vectors are:

$$
\begin{aligned}
&de^0 = 0,\\
&de^1 = \partial_0(e^1) \wedge dx^0 = -\dot{a}(t) dx^1 \wedge dx^0 = \frac{\dot{a}}{a} e^1 \wedge e^0,\\
&de^2 = \partial_0(e^2) \wedge dx^0 = -\dot{a}(t) dx^2 \wedge dx^0 = \frac{\dot{a}}{a} e^2 \wedge e^0\,,\\
&de^3 = \partial_0(e^3) \wedge dx^0 = -\dot{a}(t) dx^3 \wedge dx^0 = \frac{\dot{a}}{a} e^3 \wedge e^0\,,\\
&de^4 = \partial_0(e^4) \wedge dx^0 = -\dot{b}(t) dx^4 \wedge dx^0 = \frac{\dot{b}}{b} e^4 \wedge e^0\,.
\end{aligned}
\tag{10.4}
$$

Assuming the absence of torsion and of extra bosonic fields, the second Cartan structure equation (9.4) permits to find the Lorentz connection

$$
\omega^1_0 = \frac{\dot{a}}{a} e^1\,, \quad \omega^2_0 = \frac{\dot{a}}{a} e^2\,, \quad \omega^3_0 = \frac{\dot{a}}{a} e^3\,, \quad \omega^4_0 = \frac{\dot{b}}{b} e^4\,, \quad \omega^0_i = \omega^i_0\,, \tag{10.5}
$$

and its exterior derivatives

$$
\begin{aligned}
d\omega^1_0 \;&=\; e^1 d\left(\frac{\dot{a}}{a}\right) + \frac{\dot{a}}{a} de^1 = \left[\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) + \frac{\dot{a}^2}{a^2}\right] e^1 \wedge e^0 = \frac{\ddot{a}}{a} e^1 \wedge e^0\,,\\
&d\omega^2_0 = \frac{\ddot{a}}{a} e^2 \wedge e^0\,, \quad d\omega^3_0 = \frac{\ddot{a}}{a} e^3 \wedge e^0\,, \quad d\omega^4_0 = \frac{\ddot{b}}{b} e^4 \wedge e^0\,.
\end{aligned}
\tag{10.6}
$$

Using Eq. (9.4), we finally get curvature two-form R^{ab}:

$$R^{ab} = \begin{pmatrix} 0 & \frac{\ddot{a}}{a}e^0 \wedge e^1 & \frac{\ddot{a}}{a}e^0 \wedge e^2 & \frac{\ddot{a}}{a}e^0 \wedge e^3 & \frac{\ddot{b}}{b}e^0 \wedge e^4 \\ -\frac{\ddot{a}}{a}e^0 \wedge e^1 & 0 & \frac{\dot{a}^2}{a^2}e^1 \wedge e^2 & \frac{\dot{a}^2}{a^2}e^1 \wedge e^3 & \frac{\dot{a}\dot{b}}{ab}e^1 \wedge e^4 \\ -\frac{\ddot{a}}{a}e^0 \wedge e^2 & -\frac{\dot{a}^2}{a^2}e^1 \wedge e^2 & 0 & \frac{\dot{a}^2}{a^2}e^2 \wedge e^3 & \frac{\dot{a}\dot{b}}{ab}e^2 \wedge e^4 \\ -\frac{\ddot{a}}{a}e^0 \wedge e^3 & -\frac{\dot{a}^2}{a^2}e^1 \wedge e^3 & -\frac{\dot{a}^2}{a^2}e^2 \wedge e^3 & 0 & \frac{\dot{a}\dot{b}}{ab}e^3 \wedge e^4 \\ -\frac{\ddot{b}}{b}e^0 \wedge e^4 & -\frac{\dot{a}\dot{b}}{ab}e^1 \wedge e^4 & -\frac{\dot{a}\dot{b}}{ab}e^2 \wedge e^4 & -\frac{\dot{a}\dot{b}}{ab}e^3 \wedge e^4 & 0 \end{pmatrix} . \tag{10.7}$$

Let us now focus on the five-dimensional limit of the Lovelock action (9.5), namely

$$S = \kappa \int \epsilon_{abcde} \left(\alpha_0 e^a \wedge e^b \wedge e^c \wedge e^d \wedge e^e + \alpha_1 R^{ab} \wedge e^c \wedge e^d \wedge e^e + \alpha_2 R^{ab} \wedge R^{cd} \wedge e^e \right) . \tag{10.8}$$

By means of the two-form curvature components in Eq. (10.7), we obtain:

$$\begin{aligned} &\epsilon_{abcde} R^{ab} \wedge R^{cd} \wedge e^e = \left[24\frac{\dot{a}^2\ddot{a}}{a^3} + 48\frac{\ddot{a}\dot{a}}{a^2}\frac{\dot{b}}{b} + 24\frac{\dot{a}^2}{a^2}\frac{\ddot{b}}{b} + 24\frac{\dot{a}^3}{a^3}\frac{\dot{b}}{b} \right] e^0 \wedge e^1 \wedge e^2 \wedge e^3 \wedge e^4 \\ &\equiv [\mathcal{G}^{(5)}]\, e^0 \wedge e^1 \wedge e^2 \wedge e^3 \wedge e^4 \end{aligned} \tag{10.9}$$

and

$$\begin{aligned} &\epsilon_{abcde} R^{ab} \wedge e^c \wedge e^d \wedge e^e = -\left[6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) + 2\frac{\ddot{b}}{b} + 6\frac{\dot{a}}{a}\frac{\dot{b}}{b} \right] e^0 \wedge e^1 \wedge e^2 \wedge e^3 \wedge e^4 \\ &\equiv [\mathcal{R}^{(5)}]\, e^0 \wedge e^1 \wedge e^2 \wedge e^3 \wedge e^4 \,. \end{aligned} \tag{10.10}$$

While Eq. (10.9) is the five-dimensional extension of the Gauss–Bonnet scalar, Eq. (10.10) represents the five-dimensional expression of the Ricci scalar. Note that the former is not a topological term in five dimensions; however, once considering a four-dimensional space-time, it reduces to

$$\epsilon_{abcd}R^{ab}\wedge R^{cd} = 24\frac{\ddot{a}\dot{a}^2}{a^3}\,, \tag{10.11}$$

which is nothing but $\mathcal{G}^{(5)}$ in the limit $b = 0$. Replacing Eqs. (10.9) and (10.10) in Eq. (10.8), the action can be written as:

$$\begin{aligned} S = \kappa \int \quad & \left\{\alpha_2 \left[24\frac{\dot{a}^2\ddot{a}}{a^3} + 48\frac{\ddot{a}\dot{a}}{a^2}\frac{\dot{b}}{b} + 24\frac{\dot{a}^2}{a^2}\frac{\ddot{b}}{b} + 24\frac{\dot{a}^3}{a^3}\frac{\dot{b}}{b}\right]\right. \\ & \left. -\alpha_1\left[6\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right) + 2\frac{\ddot{b}}{b} + 6\frac{\dot{a}}{a}\frac{\dot{b}}{b}\right] + \alpha_0\right\} e^0\wedge e^1\wedge e^2\wedge e^3\wedge e^4\,. \end{aligned} \tag{10.12}$$

Equivalently, by means of the relation occurring between the wedge product of vielbein fields and the fifth-dimensional volume, the above action takes the form:

$$\begin{aligned} S = \kappa \int |e| \quad & \left\{\alpha_2 \left[24\frac{\dot{a}^2\ddot{a}}{a^3} + 48\frac{\ddot{a}\dot{a}}{a^2}\frac{\dot{b}}{b} + 24\frac{\dot{a}^2}{a^2}\frac{\ddot{b}}{b} + 24\frac{\dot{a}^3}{a^3}\frac{\dot{b}}{b}\right]\right. \\ & \left. -\alpha_1\left[6\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right) + 2\frac{\ddot{b}}{b} + 6\frac{\dot{a}}{a}\frac{\dot{b}}{b}\right] + \alpha_0\right\} d^5x \\ & = \int a^3 b\left\{\alpha_0 + \alpha_1\mathcal{R}^{(5)} + \alpha_2\mathcal{G}^{(5)}\right\} d^5x\,. \end{aligned} \tag{10.13}$$

After integrating by parts the terms containing second derivatives, the point-like cosmological Lagrangian turns out to be:

$$\mathcal{L} = \alpha_0 a^3 b + 6\alpha_1\left(ab\dot{a}^2 + a^2\dot{a}\dot{b}\right) - 8\alpha_2\dot{a}^3\dot{b}\,. \tag{10.14}$$

The Euler–Lagrange equations and the energy condition, are given by

$$\begin{aligned} &\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{a}} = \frac{\partial\mathcal{L}}{\partial a} \to 4\alpha_1 a(\dot{a}\dot{b} + b\ddot{a}) - a^2(\alpha_0 b - 2\alpha_1\ddot{b}) + 2\dot{a}(\alpha_1 b\dot{a} - 8\alpha_2\dot{b}\ddot{a} - 4\alpha_2\dot{a}\ddot{b}) = 0\,, \\ &\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{b}} = \frac{\partial\mathcal{L}}{\partial b} \to \alpha_0 a^3 - 6\alpha_1 a\dot{a}^2 - 6\alpha_1 a^2\ddot{a} + 24\alpha_2\dot{a}^2\ddot{a} = 0\,, \\ &\dot{a}\frac{\partial\mathcal{L}}{\partial\dot{a}} + \dot{b}\frac{\partial\mathcal{L}}{\partial\dot{b}} - \mathcal{L} = 0 \to \alpha_0 a^3 b - 6\alpha_1 ab\dot{a}^2 - 6\alpha_1 a^2\dot{a}\dot{b} + 24\alpha_2\dot{a}^3\dot{b} = 0\,, \end{aligned} \tag{10.15}$$

respectively and yield the following exponential scale factors:

$$
\begin{aligned}
a(t) &= a_0 \exp\left\{ \pm \sqrt{\frac{\alpha_0}{\pm 2\sqrt{9\alpha_1^2 - 6\alpha_0\alpha_2} + 6\alpha_1}\, t} \right\}, \\
b(t) &= b_0 \exp\left\{ \pm \sqrt{\frac{\alpha_0}{\pm 2\sqrt{9\alpha_1^2 - 6\alpha_0\alpha_2} + 6\alpha_1}\, t} \right\},
\end{aligned} \tag{10.16}
$$

with a_0 and b_0 being arbitrary constants. Notice that the only exponential solution admitted by the equations of motion, imposes the relation $a(t) \sim b(t)$. The limit $\alpha_0 = 0$, that is the sum between the scalar curvature and the Gauss–Bonnet term, provides two solutions:

$$
a(t) \sim b(t) \sim \text{Const.} \tag{10.17}
$$

$$
a(t) = a_0 e^{\pm\sqrt{\frac{\alpha_1}{2\alpha_2}}\, t} \sim b(t). \tag{10.18}
$$

The former can be obtained as the limit of Eq. (10.16), after taking the positive sign inside the square root. On the contrary, the latter cannot be recovered and must be computed by assuming a vanishing cosmological constant from the beginning. This is due to the fact that the solution with negative sign in Eq. (10.16) for $\alpha_0 = 0$ turns out to be indeterminate. Five-dimensional GR can be obtained by setting $\alpha_2 = 0$ and the field equations provide

$$
a(t) = a_0 e^{\pm\sqrt{\frac{\alpha_0}{12\alpha_1}}\, t} \sim b(t). \tag{10.19}
$$

The similarity between Eqs. (10.18) and (10.19) suggests that in the former case the Gauss–Bonnet scalar plays the role of an effective cosmological constant. When the contribution of the scalar curvature vanishes, namely when $\alpha_1 = 0$, the Euler–Lagrange equations yield

$$
a(t) = a_0 e^{\pm\left(\frac{\alpha_0}{-24\alpha_2}\right)^{1/4} t} \sim b(t). \tag{10.20}
$$

According to Eqs. (9.13) and (9.14), to get the Chern–Simons five-dimensional Lagrangian we must set

$$\alpha_0 = \frac{1}{5l^4}\,, \qquad \alpha_1 = \frac{2}{3l^2}\,, \qquad \alpha_2 = 1\,, \tag{10.21}$$

so that the scale factors (10.16) become:

$$a(t) = a_0 \exp\left\{\pm\frac{1}{l}\sqrt{\frac{1}{6}\left(1\pm\sqrt{\frac{7}{10}}\right)}\;t\right\}\,, \qquad b(t) = b_0 \exp\left\{\pm\frac{1}{l}\sqrt{\frac{1}{6}\left(1\pm\sqrt{\frac{7}{10}}\right)}\;t\right\}\,. \tag{10.22}$$

Note that the three free parameters occurring in the definition of the general Lovelock Lagrangian, are not constrained by the field equations solution and can be arbitrarily chosen. Thus both an exponential expansion and a bouncing evolution are admitted, depending on the values of the constants α_i. On the other hand, imposing the relations (10.21), the cosmological solution (10.22) suggests that only an accelerated expansion (or contraction) is allowed by the Chern–Simons cosmology.

10.1 Generalization to d+1 Dimensions

Let us consider the extension of Lovelock gravity to $d+1$ dimensions. In light of the result obtained in the previous section, we set $b(t) = a(t)$ from the beginning, and we restore the spatial curvature k. The assumption of a unique scale factor $a(t)$ is needed to obtain suitable field equations, capable of providing analytic solutions in higher dimensions. The line element therefore takes the form

$$ds^2 = dt^2 - \frac{a(t)^2}{1-kr^2}dr^2 - r^2 d\Omega^2_{d-1}, \tag{10.23}$$

where Ω_{d-1} is the $d-1$ sphere, previously defined in Eq. (5.1) as $d\Omega^2_{d-1} = \sum_{j=1}^{d-2} d\theta_j^2 + \sin^2\theta_j d\phi^2$. We only consider the first three terms in the Lovelock $d+1$ dimensional Lagrangian, neglecting all those terms arising in more than five dimensions. This means that the coupling constants α_i, with $i \geq 3$, will be set to zero. Consequently, the only non-vanishing terms in the action (9.5) are the $d+1$ dimensional scalar curvature $\mathcal{R}^{(d+1)}$ and the Gauss–Bonnet scalar $\mathcal{G}^{(d+1)}$. The action, therefore, is comprehended in the general Lovelock action (9.5), with a Lagrangian which generalizes Lagrangian (10.14). Using the $d+1$ dimensional expression of the Ricci scalar and of the Gauss–Bonnet term, namely

$$\mathcal{R}^{(d+1)} = -d\left[2\frac{\ddot{a}}{a} + (d-1)\left(\frac{\dot{a}^2+k}{a^2}\right)\right], \tag{10.24}$$

$$\mathcal{G}^{(d+1)} = d(d-1)(d-2)\left[(d-3)\left(\frac{\dot{a}^4}{a^4} + 2k\frac{\dot{a}^2}{a^4} + \frac{k^2}{a^4}\right) + \frac{4}{3a^3}\frac{d}{dt}\left(\dot{a}^3\right) + 4k\frac{\ddot{a}}{a^3}\right], \tag{10.25}$$

and considering a starting action of the form

$$S = \kappa \int |e| \left[\alpha_0 + \alpha_1 \mathcal{R}^{(d+1)} + \alpha_2 \mathcal{G}^{(d+1)}\right] d^{d+1}x\,, \tag{10.26}$$

the Lagrangian can be written as:

$$\begin{aligned}\mathcal{L} = \frac{r^{d-2}a^{d-4}}{3\sqrt{1-kr^2}} \quad &\Big\{3a^2[a^2\alpha_0 + \alpha_1 d(d-1)(\dot{a}^2-k)] \\ &-\alpha_2 d(d-1)(d-2)(d-3)(\dot{a}^4 + 6k\dot{a}^2 - 3k^2)\Big\}\,.\end{aligned} \tag{10.27}$$

The Gauss–Bonnet term, as confirmed by the above Lagrangian, does not contribute to the equations of motion in less than $3+1$ dimensions, where only the scalar curvature plays a role in the dynamics. Let us first consider the case $k \neq 0$, where no solutions occur if all the coupling constants are simultaneously non-zero. Setting $\alpha_2 = 0$, the theory reduces to the higher-dimensional GR with cosmological constant, in a spatially non-flat universe. The scale factor which solves the equations of motion, in this case turns out to be

$$a(t) = \pm\sqrt{\frac{\alpha_1 k\, d(d-1)}{\alpha_0 - \alpha_0 \coth^2\left[\sqrt{\alpha_0}\left(c_1 + \frac{t}{\sqrt{\alpha_1(d-1)d}}\right)\right]}}, \tag{10.28}$$

which holds as long as $\alpha_0 \neq 0$. When setting $\alpha_0 = 0$ from the beginning, the only solution is

$$a(t) = \sqrt{-k}\, t. \tag{10.29}$$

Another analytical solution occurs for $\alpha_1 = \alpha_0 = 0$, where the Euler–Lagrange equations and the energy condition yield the scale factor

$$a(t) = \sqrt{-k}\, t, \tag{10.30}$$

which is exactly the same as (10.29). Finally, when $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_0 = 0$, the only solution is

$$a(t) = \pm\sqrt{\frac{-\alpha_2 k(d-3)(d-2)}{\alpha_1}} \sinh\left[\sqrt{\alpha_1}\left(\frac{t}{\sqrt{\alpha_2(d-3)(d-2)}} + c_1\right)\right]. \tag{10.31}$$

Comparing Eq. (10.31) with Eq. (10.28), we notice that the Gauss–Bonnet term can play the role of an effective cosmological constant.
Let us now focus on a spatially flat FLRW space-time in higher dimensions. When $k = 0$, the Lagrangian (10.27) reduces to:

$$\mathcal{L} = \frac{r^{d-2}a^{d-4}}{3}\left\{3a^2[a^2\alpha_0 + \alpha_1 d(d-1)\dot{a}^2] - \alpha_2 d(d-1)(d-2)(d-3)\dot{a}^4\right\}. \tag{10.32}$$

The general solution of the related Euler–Lagrange equations can be analytically found only for exponential scale factors of the form

$$a(t) = a_0 \exp\left\{\pm\sqrt{\frac{2\alpha_0}{\pm\sqrt{(d-1)d\left[\alpha_1^2(d-1)d - 4\alpha_0\alpha_2(d-3)(d-2)\right]} + \alpha_1 d(d-1)}\, t}\right\}. \tag{10.33}$$

Notice that in the five-dimensional limit, the scale factor reduces to (10.16), namely

$$a^{(5)}(t) = a_0 \exp\left\{\pm\sqrt{\frac{\alpha_0}{\pm 2\sqrt{9\alpha_1^2 - 6\alpha_0\alpha_2} + 6\alpha_1}\, t}\right\}. \tag{10.34}$$

Let us now separately analyze the subcases not covered by the solution (10.33), namely those cases in which (at least) one of the constants α_i vanishes. In the limit $\alpha_2 = 0$, Lagrangian (10.32) turns into the high-dimensional Hilbert–Einstein Lagrangian with cosmological constant, providing the well known Einstein–de Sitter solution

$$a(t) = a_0 e^{\pm\sqrt{\frac{\alpha_0}{\alpha_1 d(d-1)}}\,t}, \tag{10.35}$$

which is the spatially flat limit of Eq. (10.28).
The case $\alpha_1 = 0$ (analyzed in depth in Chap. 4) yields the following non-trivial solution:

$$a(t) = a_0 \exp\left\{\pm\left[-\frac{\alpha_0}{\alpha_2}\left(\frac{1}{d(d-1)(d-2)(d-3)}\right)\right]^{1/4} t\right\}. \tag{10.36}$$

Finally, by setting $\alpha_0 = 0$, a de Sitter-like scale factor of the form

$$a(t) = a_0 e^{\pm\sqrt{\frac{\alpha_1}{\alpha_2 (d-2)(d-3)}}\,t}, \tag{10.37}$$

solves the field equations. This subcase cannot be directly recovered from Eq. (10.33), whose $\alpha_0 = 0$ limit provides an indeterminate scale factor. Also here, the values assumed by the coupling constants can determine either an exponential expansion or an oscillating solution. Moreover, Eq. (10.37) is the $k = 0$ limit of Eq. (10.31).
Notice that no solutions occur in vacuum when two coupling constants are simultaneously null; therefore, those cases analyzed here are all the possible subcases that can be obtained from Lagrangian (10.32). This can be also inferred from Eqs. (10.29) and (10.30), whose $k = 0$ limits provide $a(t) = 0$.

11

Chern–Simons Black Holes

We now study a spherically symmetric background in Lovelock gravity. As we did in the previous chapter, the five-dimensional case is firstly analyzed. Subsequently, we generalize the treatment to $d+1$ dimensions, where all the coupling constants α_i, with $i > 2$, are neglected. In this way, only the higher-dimensional generalization of the scalar curvature and of the Gauss–Bonnet term are considered into the action. This *ansatz* allows to solve the field equations analytically. For the same reasons mentioned in the previous chapter, we pay main attention to the five-dimensional limit, which needs to be treated separately. On the one hand, for particular combinations of the coupling constants, the five-dimensional Lovelock Lagrangian reduces to the Chern–Simons Lagrangian, invariant under the local AdS group. On the other hand, while in $d+1$ dimensions analytic solutions can be found only under the assumption $P(r) = Q(r)^{-1}$, in five dimensions exact solutions occur without adopting any extra *ansatz*. Let us then consider the metric:

$$ds^2 = P(r)^2 dt^2 - Q(r)^2 dr^2 - r^2 d\theta^2 - r^2 \sin^2(\theta) d\phi^2 - r^2 \sin^2\theta \sin^2\phi \, d\psi^2 \,, \tag{11.1}$$

where $d\theta^2 + \sin^2(\theta)d\phi^2 + \sin^2\theta\sin^2\phi d\psi^2 \equiv d\Omega_3$ is the three-sphere. By choosing the following set of basis

$$\begin{aligned} &e^0 = P(r)dt \,, \quad e^1 = -Q(r)dr \,, \quad e^2 = -r d\theta \,, \\ &e^3 = -r\sin(\theta)d\phi \,, \quad e^4 = -r\sin(\theta)\sin(\phi)d\psi \,, \end{aligned} \tag{11.2}$$

the same computations as the previous chapter yield the expressions of the Gauss–Bonnet term and of the Ricci scalar in the spherically symmetric background

$$\begin{aligned} \mathcal{G}^{(5)} &= -\frac{24P''(r)}{r^2P(r)Q(r)^2}+\frac{24P''(r)}{r^2P(r)Q(r)^4}+\frac{24P'(r)Q'(r)}{r^2P(r)Q(r)^3}-\frac{72P'(r)Q'(r)}{r^2P(r)Q(r)^5} \\ &- \frac{24P'(r)}{r^3P(r)Q(r)^2}+\frac{24P'(r)}{r^3P(r)Q(r)^4}+\frac{24Q'(r)}{r^3Q(r)^3}-\frac{24Q'(r)}{r^3Q(r)^5}\,, \end{aligned} \tag{11.3}$$

$$\mathcal{R}^{(5)} = \frac{2P''(r)}{P(r)Q(r)^2}-\frac{2P'(r)Q'(r)}{P(r)Q(r)^3}+\frac{6P'(r)}{rP(r)Q(r)^2}-\frac{6Q'(r)}{rQ(r)^3}+\frac{6}{r^2Q(r)^2}-\frac{6}{r^2}\,, \tag{11.4}$$

where we set $\theta = \phi = \pi/2$. The Lovelock Lagrangian therefore can be written as:

$$\begin{aligned} \mathcal{L}^{(5)} &= \alpha_0 r^3 P(r)Q(r) - \alpha_1\left(\frac{6r^2P(r)Q'(r)}{Q(r)^2}+6rP(r)Q(r)-\frac{6rP(r)}{Q(r)}\right) \\ &+ \alpha_2\left(\frac{24P(r)Q'(r)}{Q(r)^2}-\frac{24P(r)Q'(r)}{Q(r)^4}\right)\,. \end{aligned} \tag{11.5}$$

Since the values of the coupling constants play a fundamental role in the treatment, in the next subsection we analyze the contribution of each term separately, to finally provide the spherically symmetric solution of the field equations for Chern–Simons and Lovelock gravity.

5D Einstein Gravity

Let us start by setting $\alpha_2 = 0$, so that the Lagrangian reduces to the $4+1$ dimensional Hilbert–Einstein Lagrangian with cosmological constant α_0, namely

$$\mathcal{L} = \alpha_0 r^3 P(r)Q(r) - \alpha_1\left(\frac{6r^2P(r)Q'(r)}{Q(r)^2}+6rP(r)Q(r)-\frac{6rP(r)}{Q(r)}\right)\,. \tag{11.6}$$

The Euler–Lagrange equations

$$\frac{d}{dr}\frac{\partial\mathcal{L}}{\partial P'(r)}=\frac{\partial\mathcal{L}}{\partial P(r)}\quad\rightarrow\quad \alpha_0 r^3 Q(r)=6r\alpha_1\left(\frac{rQ'(r)}{Q(r)^2}+Q(r)-\frac{1}{Q(r)}\right),$$
$$\frac{d}{dr}\frac{\partial\mathcal{L}}{\partial Q'(r)}=\frac{\partial\mathcal{L}}{\partial Q(r)}\quad\rightarrow\quad \alpha_0 r^3 P(r)=6r\alpha_1\left(P(r)-\frac{P(r)}{Q(r)^2}-\frac{rP'(r)}{Q(r)^2}\right),$$

provide the solution

$$P(r)^2=1+\frac{c_1}{r^2}-\frac{\alpha_0}{12\alpha_1}r^2\,,\qquad\qquad Q(r)^2=\frac{1}{1+\frac{c_1}{r^2}-\frac{\alpha_0}{12\alpha_1}r^2}\,,\tag{11.7}$$

where c_1 is an integration constant. The above solution is the five-dimensional extension of an Einstein–de Sitter space-time [187]. Notice that $P(r)$ vanishes when

$$r=r_H\equiv\sqrt{\frac{2\sqrt{3\alpha_1(3\alpha_1+\alpha_0 c_1)}}{\alpha_0}+\frac{6\alpha_1}{\alpha_0}}.\tag{11.8}$$

Moreover, setting $\alpha_0=0$, the horizon turns out to be $r_H=\sqrt{-c_1}$. This means that the quantity $-c_1$ can be intended as a mass term in five dimensions. Therefore, the Bekenstein–Hawking entropy $\mathcal{S}$, in the Einstein–de Sitter five-dimensional space-time, is

$$\mathcal{S}=\frac{8}{3}\pi^2 r_H^3\sim M^{\frac{3}{4}}\,,\tag{11.9}$$

which is exactly the same dependence exhibited by the entropy of a four-dimensional conformal field theory (CFT) [188, 189], because of the AdS/CFT correspondence. The same result is not provided by the five-dimensional Einstein gravity with $\alpha_0=0$, where the entropy behaves like

$$\mathcal{S}\sim M^{\frac{3}{2}}.\tag{11.10}$$

An accurate analysis is relied to Sec. 11.1, where the discussion is extended to $d+1$ dimensions.

Pure Gauss–Bonnet gravity

Let us now consider the case $\alpha_1 = 0$, in which the Lagrangian is only made of the Gauss–Bonnet term and the cosmological constant. The only terms which survive in the Lagrangian after integrating the second derivatives are:

$$\mathcal{L} = \alpha_0 P(r)Q(r)r^3 + \alpha_2 \left(\frac{24P(r)Q'(r)}{Q(r)^2} - \frac{24P(r)Q'(r)}{Q(r)^4} \right) \tag{11.11}$$

and the Euler–Lagrange equations read

$$\begin{aligned} \frac{d}{dr}\frac{\partial \mathcal{L}}{\partial P'(r)} = \frac{\partial \mathcal{L}}{\partial P(r)} \quad &\to \quad 24\alpha_2 \left(Q(r)^2 - 1\right) Q'(r) = \alpha_0 r^3 Q(r)^5 \,, \\ \frac{d}{dr}\frac{\partial \mathcal{L}}{\partial Q'(r)} = \frac{\partial \mathcal{L}}{\partial Q(r)} \quad &\to \quad 24\alpha_2 \left(Q(r)^2 - 1\right) P'(r) = \alpha_0 r^3 P(r) Q(r)^4 \,. \end{aligned} \tag{11.12}$$

There are two classes of solutions coming from the above equations. The first arises by imposing $P(r) = 1/Q(r)$ between the metric components $P(r)$ and $Q(r)$. It reads:

$$\begin{aligned} P(r)^2 &= 1 \pm \sqrt{1 + c_1 - \frac{\alpha_0}{24\alpha_2} r^4} \,, \\ Q(r) &= \frac{1}{P(r)} \,, \end{aligned} \tag{11.13}$$

and the horizon sits at

$$r_H = \left(\frac{24\alpha_2\, c_1}{\alpha_0} \right)^{1/4} \,. \tag{11.14}$$

As we can see from Eq. (11.13), by setting $\alpha_0 = 0$ only trivial solutions occur. This means that, under the assumption $P(r) = 1/Q(r)$, the only non-trivial contribution in five dimensions is provided by the cosmological constant. Moreover, by using the horizon (11.14), the entropy turns out to be

$$\mathcal{S} = \frac{8}{3}\pi^2 \left(\frac{24\alpha_2}{\alpha_0} \right)^{3/4} M^{3/4} \tag{11.15}$$

and the AdS/CFT correspondence is recovered.

Another line element which solves Eqs. (11.12) in five dimensions, can be found without imposing any relation between $P(r)$ and $Q(r)$. In this case, the solution is:

$$P^2(r) = P_0^2\sqrt{48\alpha_2\,(2c_1+1) \mp 4\sqrt{6}\sqrt{\alpha_2\,(24\alpha_2+96\alpha_2 c_1-\alpha_0 r^4)}-\alpha_0 r^4}\,, \tag{11.16}$$

$$Q^2(r) = \frac{2\left(12\alpha_2 \pm \sqrt{6}\sqrt{\alpha_2\,(24\alpha_2+96\alpha_2 c_1-\alpha_0 r^4)}\right)}{\alpha_0 r^4 - 96\alpha_2 c_1}\,, \tag{11.17}$$

where P_0 is an integration constant. The horizon can be found by setting $P(r) = 0$. Four mathematical solutions follow from this imposition, but the only one with physical meaning is

$$r_H = 2\left(\frac{6\alpha_2\, c_1}{\alpha_0}\right)^{1/4}\,. \tag{11.18}$$

Notice that $Q(r_H)$ automatically diverges, though the *ansatz* $P(r) = 1/Q(r)$ is not imposed from the beginning. Also, after redefining the constant c_1, this horizon turns out to be the same as (11.14). This is expected since the solution in Eq. (11.13), coming from the imposition $P(r) = Q(r)^{-1}$, is a particular subcase of Eq. (11.16).
As an example, a more suitable form can be obtained by setting $c_1 = -1/4$, where Eqs. (11.16) and (11.17) reduce to:

$$P^2(r) = P_0^2\sqrt{24\alpha_2 \mp 4\sqrt{6}\sqrt{-\alpha_2\alpha_0 r^4}-\alpha_0 r^4}\,, \tag{11.19}$$

$$Q^2(r) = \frac{24\alpha_2 \pm 2\sqrt{6}\sqrt{-\alpha_2\alpha_0 r^4}}{\alpha_0 r^4 + 24\alpha_2}\,. \tag{11.20}$$

In any case, the metric turns out to be trivially constant when the cosmological constant α_0 is not considered. Moreover, in the limit $r \to \infty$ the asymptotic flatness is not recovered for $P(r)$. However, as we will point out in Sec. 11.1, in higher dimensions the flatness for large radius and the presence of the horizon can be obtained for some combinations of the coupling constants.

To conclude the research for exact solutions in the five-dimensional space-time, we find Euler-Lagrange equations solutions in the most general case, where no coupling constants are neglected in the action (11.5). The equations of motion are:

$$\begin{aligned}
\frac{d}{dr}\frac{\partial\mathcal{L}}{\partial P'(r)} = \frac{\partial\mathcal{L}}{\partial P(r)} \quad &\to \quad Q(r)\left(\alpha_0 r^3 - 6\alpha_1 r\right) - \frac{6\left(\alpha_1 r^2 - 4\alpha_2\right)Q'(r)}{Q(r)^2} + \frac{6\alpha_1 r}{Q(r)} \\
&\quad -\frac{24\alpha_2 Q'(r)}{Q(r)^4} = 0\,, \\
\frac{d}{dr}\frac{\partial\mathcal{L}}{\partial Q'(r)} = \frac{\partial\mathcal{L}}{\partial Q(r)} \quad &\to \quad 6P'(r)\left\{Q(r)^2\left[-\alpha_1 r^2 + 4\alpha_2\right] - 4\alpha_2\right\} \\
&\quad -rP(r)Q(r)^2\left[Q(r)^2\left(\alpha_0 r^2 - 6\alpha_1\right) + 6\alpha_1\right] = 0\,. \qquad (11.21)
\end{aligned}$$

Here, the imposition $P(r) = 1/Q(r)$ is not required to solve the equations of motion. As a matter of fact, the general solution of the Euler–Lagrange equations is:

$$\begin{aligned}
P(r)^2 = P_0^2 \quad & \sqrt{\alpha_0 r^4 - 12\alpha_1 r^2 - 4c_1}\left[\frac{3\sqrt{r^4 uw + 2ux} + \sqrt{3}\left(\alpha_0 x + r^2 w(3\alpha_1 - z)\right)}{-6\alpha_1 + \alpha_0 r^2 + 2z}\right]^{y/2} \\
& \times \left[\frac{6\alpha_1 - \alpha_0 r^2 + 2z}{3\sqrt{r^4 vw + 2vx} + \sqrt{3}\left(r^2 w(z + 3\alpha_1) + \alpha_0 x\right)}\right]^{s/2}\,, \qquad (11.22)
\end{aligned}$$

$$Q(r)^2 = \frac{-3\alpha_1 r^2 + 12\alpha_2 \pm \sqrt{3}\sqrt{8c_1\alpha_2 - 2\alpha_0\alpha_2 r^4 + 3\alpha_1^2 r^4 + 48\alpha_2^2}}{\frac{\alpha_0}{2}r^4 - 6\alpha_1 r^2 - 2c_1}\,, \qquad (11.23)$$

where all the new constants arising in $P(r)^2$ are defined in App. E. By setting $P(r) = 0$

five horizons occur, namely:

$$
\begin{aligned}
r_H &= \sqrt{\pm\frac{2\sqrt{c_1\alpha_0+9\alpha_1^2}}{\alpha_0}+6\frac{\alpha_1}{\alpha_0}}\,, \\
r_H &= \Bigg[\pm\frac{\sqrt{3}\sqrt{-6u^2wx+18\alpha_1^2uw^2x+2uw^2xz^2-12\alpha_1uw^2xz+\alpha_0^2uwx^2}}{3uw-9\alpha_1^2w^2-w^2z^2+6\alpha_1w^2z} \\
&\quad -\frac{\alpha_0wxz}{3uw-9\alpha_1^2w^2-w^2z^2+6\alpha_1w^2z}+\frac{3\alpha_0\alpha_1wx}{3uw-9\alpha_1^2w^2-w^2z^2+6\alpha_1w^2z}\Bigg]^{1/2}\,, \\
r_H &= \sqrt{\frac{-2z+6\alpha_1}{\alpha_0}}\,. \qquad (11.24)
\end{aligned}
$$

They can be obtained by imposing the first, second and third term in Eq. (11.22) to be equal to zero, respectively. Notice that $Q(r)$ diverges only for the first couple of horizons, that is:

$$
Q(r_H)\to\infty \quad \text{where} \quad r_H=\sqrt{\pm\frac{2\sqrt{c_1\alpha_0+9\alpha_1^2}}{\alpha_0}+6\frac{\alpha_1}{\alpha_0}}\,, \qquad (11.25)
$$

with a bekenstein–Hawking entropy of the form

$$
\mathcal{S}=\frac{8}{3}\pi^2\left(\pm\frac{2\sqrt{M\alpha_0+9\alpha_1^2}}{\alpha_0}+6\frac{\alpha_1}{\alpha_0}\right)^{3/2}\,, \qquad (11.26)
$$

which scales as that of a CFT. It is worth discussing the behavior of the general solution in Eq. (11.22) for some particular values of the coupling constants α_i. By setting $\alpha_2=0$, the standard five-dimensional Einstein gravity with cosmological constant is recovered, while in the limit $\alpha_1=0$ Eqs. (11.22) and (11.23) reduce to Eqs. (11.16) and (11.17) (with a proper rescaling of the integration constant c_1). By neglecting the cosmological constant (*i.e.* setting $\alpha_0=0$), the result can be simplified and two different solutions can be analytically found. They read

$$
P_-(r)^2=P_0^2\sqrt{\frac{(2c_1+\alpha_1r^2)^2\left[\sqrt{16\alpha_2(\alpha_2+c_1)+\alpha_1^2r^4}+\alpha_1r^2\right]}{8\alpha_2^2+2\alpha_2\left(\sqrt{16\alpha_2(\alpha_2+c_1)+\alpha_1^2r^4}+4c_1\right)+c_1\left(\sqrt{16\alpha_2(\alpha_2+c_1)+\alpha_1^2r^4}-\alpha_1r^2\right)}}\,, \qquad (11.27)
$$

$$P_+(r)^2 = P_0^2 \sqrt{\frac{8\alpha_2^2 + 2\alpha_2\left(\sqrt{16\alpha_2(\alpha_2+c_1)+\alpha_1^2 r^4} + 4c_1\right) + c_1\left(\sqrt{16\alpha_2(\alpha_2+c_1)+\alpha_1^2 r^4} - \alpha_1 r^2\right)}{\sqrt{16\alpha_2(\alpha_2+c_1)+\alpha_1^2 r^4} + \alpha_1 r^2}} \tag{11.28}$$

$$Q_\pm(r)^2 = \frac{-4\alpha_2 \pm \sqrt{16\alpha_2(\alpha_2+c_1)+\alpha_1^2 r^4} + \alpha_1 r^2}{4c_1 + 2\alpha_1 r^2}. \tag{11.29}$$

Both solutions share the same horizon, namely

$$r_H = \sqrt{\frac{-2c_1}{\alpha_1}}, \tag{11.30}$$

so that the corresponding Bekenstein–Hawking entropy is

$$\mathcal{S} = \frac{16}{3}\pi^2 \sqrt{\frac{2}{\alpha_1^3}} M^{\frac{3}{2}}, \tag{11.31}$$

which behaves differently than that of a CFT, having neglected the cosmological constant α_0.

In the former case, the asymptotic flatness cannot be recovered, independently of the value of the integration constant P_0. In the latter case, the flatness for large radius occurs by means of the choice $P_0^2 = (\alpha_2)^{-1/2}$.

The five dimensional Chern–Simons gravity, invariant under the local AdS group, can be found as a limit of Eqs. (11.27) and (11.29), imposing $\alpha_2 = 1$, $\alpha_1 = \dfrac{2}{3l^2}$, $\alpha_0 = \dfrac{1}{5l^4}$. In such a case, the point-like spherically symmetric Lagrangian reads:

$$\begin{aligned}\mathcal{L}_{CS}^{(5-AdS)} &= \frac{\kappa}{l}\left[-\frac{2}{3l^2}\left(\frac{6r^2 P(r)Q'(r)}{Q(r)^2} + 6rP(r)Q(r) - \frac{6rP(r)}{Q(r)}\right)\right.\\ &+ \left.\left(\frac{24P(r)Q'(r)}{Q(r)^2} - \frac{24P(r)Q'(r)}{Q(r)^4}\right) + \frac{1}{5l^4}r^3 P(r)Q(r)\right],\end{aligned} \tag{11.32}$$

and the Euler–Lagrange equations yield

$$P(r)^2 = P_0^2 \qquad \sqrt{-4c_1 + \frac{r^4}{5l^4} - \frac{8r^2}{l^2}} \left[\frac{\sqrt{3}\left(\frac{x}{5l^4} - r^2 w\left(-\frac{2}{l^2}+z\right)\right) + 3\sqrt{u\left(r^4 w + 2x\right)}}{\frac{r^2}{5l^4} - \frac{4}{l^2} + 2z} \right]^{y/2}$$
$$\times \left[\frac{-\frac{r^2}{5l^4} + \frac{4}{l^2} + 2z}{\sqrt{3}\left(\frac{x}{5l^4} + r^2 w\left(z+\frac{2}{l^2}\right)\right) + 3\sqrt{v\left(r^4 w + 2x\right)}} \right]^{s/2}, \tag{11.33}$$

$$Q(r)^2 = \frac{\sqrt{24c_1 + \frac{14r^4}{5l^4} + 144} - \frac{2r^2}{l^2} + 12}{-2c_1 + \frac{r^4}{10l^4} - \frac{4r^2}{l^2}}. \tag{11.34}$$

The form of $P(r)$ is formally the same as the case of Lovelock gravity, with different values of the integration constants (see App. E). Also the horizons are formally the same as those in Eq. (11.24). They read:

$$r_H = l\sqrt{20 \pm 2\sqrt{100+5c_1}}\,,$$

$$r_H = \left[\pm \frac{\sqrt{3}\sqrt{-150l^8u^2wx + 50l^8uw^2xz^2 - 200l^6uw^2xz + 200l^4uw^2x + uwx^2}}{5\left(3l^4uw - l^4w^2z^2 - 4l^2w^2z - 4w^2\right)} \right.$$
$$\left. + \frac{2wx}{5l^2\left(3l^4uw - l^4w^2z^2 + 4l^2w^2z - 4w^2\right)} - \frac{wxz}{5\left(3l^4uw - l^4w^2z^2 + 4l^2w^2z - 4w^2\right)} \right]^{1/2},$$

$$r_H = l^2\sqrt{10}\sqrt{l^2z+2}\,. \tag{11.35}$$

The limit of large radius does not provide a flat Minkowski space-time, since the cosmological constant cannot be assumed to vanish. However, it is worth noticing that, for large value of ℓ, the pure Gauss–Bonnet gravity is restored, while for $\ell \ll 1$ we get an Einstein–de Sitter space-time.

As mentioned above, from Eqs. (11.21), another subclass of solutions occurs, constrained by the imposition $P(r) = 1/Q(r)$. This particular five–dimensional solution has already been found and studied in Ref. [190] and reads as:

$$
\begin{aligned}
&P(r)^2 = 1 - \frac{\alpha_1 r^2}{4\alpha_2} \pm \frac{\sqrt{3r^4\left(3\alpha_1^2 - 2\alpha_0\alpha_2\right) + 6\alpha_2 c_1}}{12\alpha_2}\,,\\
&Q(r) = 1/P(r)\,.
\end{aligned}
\tag{11.36}
$$

The two horizons, by means of an appropriate redefinition of the integration constant c_1, are exactly the same as the first couple of Eqs. (11.24). Without the cosmological constant, solution (11.36) takes the form

$$
P(r)^2 = 1 - \frac{\alpha_1 r^2}{4\alpha_2} \pm \frac{\sqrt{9\alpha_1^2 r^4 + 6\alpha_2 c_1}}{12\alpha_2}\,. \tag{11.37}
$$

The Chern–Simons solution can be recovered as a particular limit of Eq. (11.36), namely:

$$
P(r)^2 = 1 - \frac{r^2}{6l^2} \pm \sqrt{\frac{7r^4}{360 l^4} + \frac{c_1}{24}}\,, \tag{11.38}
$$

with horizons sitting at

$$
r_H = l\sqrt{20 \pm \sqrt{5c_1 + 280}}\,. \tag{11.39}
$$

Notice that the solution (11.36) and, hence, also the limit of Chern–Simons gravity, do not admit the asymptotic flatness for large r.

11.1 Generalization to $d+1$ Dimensions

We now extend the previous results to $d+1$ dimensions, finding out analytical solutions under the assumption $P(r) = 1/Q(r)$. In this way, not all the solutions provided by Eq. (11.21) can be recovered under given limits. As a starting point, the five-dimensional line element (11.1) can be extended to:

$$
ds^2 = P(r)^2 dt^2 - Q(r)^2 dr^2 - r^2 d\Omega_{d-1}^2\,. \tag{11.40}
$$

By means of this choice, the Ricci scalar and the Gauss–Bonnet term can be recast in terms of P and Q, as previously written in Eq. (5.3), namely:

$$\begin{aligned}\mathcal{G}^{(d+1)} = \frac{(d-2)(d-1)}{r^4 P Q^5} \quad & \Big\{(d-3)P\left(Q^2-1\right)\left[(d-4)Q^3-(d-4)Q+4rQ'\right] \\ & -4r\Big[(d-3)Q^3P'+rQ^3P''-(d-3)QP'-rQP'' \\ & -rQ^2P'Q'+3rP'Q'\Big]\Big\}\,,\end{aligned} \tag{11.41}$$

$$\mathcal{R}^{(d+1)} = \frac{2r\left\{Q\left[(d-1)P'+rP''\right]-rP'Q'\right\}+(1-d)P\left[(d-2)Q^3+(2-d)Q+2rQ'\right]}{r^2PQ^3}\,. \tag{11.42}$$

Lovelock's point-like Lagrangian can be obtained by integrating out the second derivatives. After some basic computations it takes the form:

$$\begin{aligned}\mathcal{L}^{(d+1)} = \frac{r^{d-5}P}{Q^4} \quad & \Big\{\alpha_0 r^4 Q^5-\alpha_1(d-1)r^2Q^2[(d-2)Q(Q^2-1)+2rQ'] \\ & +\alpha_2(d-3)(d-2)(d-1)(Q^2-1)[(d-4)Q(Q^2-1)+4rQ']\Big\}\,.\end{aligned} \tag{11.43}$$

The general solution of the corresponding Euler–Lagrange equations can be found analytically by imposing $P(r)=Q(r)^{-1}$. Under this assumption, the spherically symmetric solution for $P(r)$ and $Q(r)$ reads as

$$P(r)^2 = 1 \pm \frac{1}{r^{d/2-2}}\sqrt{\frac{c_1}{6\alpha_2\binom{d-1}{d-4}}+r^d\left(\frac{\alpha_1^2}{16\alpha_2^2\binom{d-2}{d-4}^2}-\frac{\alpha_0}{24\alpha_2\binom{d}{d-4}}\right)}-\frac{\alpha_1}{4\alpha_2\binom{d-2}{d-4}}\,r^2\,,$$

$$Q(r) = \frac{1}{P(r)}\,, \tag{11.44}$$

so that when $\alpha_0 = 0$ it reduces to

$$P(r)^2 = 1 \pm \frac{1}{r^{d/2-2}} \sqrt{\frac{c_1}{6\alpha_2 \binom{d-1}{d-4}} + \frac{\alpha_1^2}{16\alpha_2^2 \binom{d-2}{d-4}^2} r^d} - \frac{\alpha_1}{4\alpha_2 \binom{d-2}{d-4}} r^2 \,. \tag{11.45}$$

In these cases, the horizon cannot be found analytically for any value of the integration constant c_1. However, it can be found after expanding the metric up to the first order. By means of the definitions

$$\tilde{c}_1 \equiv \frac{c_1}{6\alpha_2 \binom{d-1}{d-4}} \quad \alpha \equiv \left(\frac{\alpha_1^2}{16\alpha_2^2 \binom{d-2}{d-4}^2} - \frac{\alpha_0}{24\alpha_2 \binom{d}{d-4}} \right) \quad \Lambda \equiv -\frac{\alpha_1}{4\alpha_2 \binom{d-2}{d-4}} , \tag{11.46}$$

Eq. (11.44) can be rewritten as

$$P(r)^2 = 1 \pm \frac{1}{r^{d/2-2}} \sqrt{\tilde{c}_1 + \alpha r^d} + \Lambda\, r^2 \,, \tag{11.47}$$

so that under the assumption $\alpha r^d \ll \tilde{c}_1$, $P(r)^2$ becomes

$$P(r)^2 = 1 - r^{2-\frac{d}{2}} \sqrt{c_1} \left(1 + \frac{\alpha}{2\tilde{c}_1} r^d\right) + \Lambda r^2 = \Lambda \left[\frac{\frac{r^{\frac{d}{2}-2}}{\Lambda} - \frac{\sqrt{c_1}}{\Lambda}\left(1 + \frac{\alpha}{2\tilde{c}_1} r^d\right) + r^{\frac{d}{2}}}{r^{\frac{d}{2}-2}} \right] . \tag{11.48}$$

When $\Lambda \gg r^{\frac{d}{2}-2}$, the horizon can be computed analytically, providing

$$r_H = \left[\frac{2}{\alpha} (-\tilde{c}_1) \left(1 + d^{d/2}\right) \right]^{1/d} = \left[\frac{8\alpha_2 d(d-1)(d-2)^2(d-3)^2}{\alpha_1^2 d(d-1) - 4\alpha_0\alpha_2(d-2)(d-3)} (-c_1) \left(1 + d^{d/2}\right) \right]^{1/d} . \tag{11.49}$$

Considering that the ratio α_2/α_1^2 must be dimensionless, the constant c_1 must have a mass dimension and the horizon is proportional to

$$r_H \sim M^{1/d} \,. \tag{11.50}$$

The Bekenstein–Hawking entropy $\mathcal{S}$, therefore, can be written in terms of r_H as:

$$\mathcal{S} = r_H^{d-1} = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})}\left[\frac{8\alpha_2 d(d-1)(d-2)^2(d-3)^2}{\alpha_1^2 d(d-1) - 4\alpha_0\alpha_2(d-2)(d-3)}\left(1+d^{d/2}\right)\right]^{\frac{d-1}{d}} M^{\frac{d-1}{d}}, \quad (11.51)$$

where Γ is the Euler Gamma function. Comparing Eq. (11.51) with the entropy of a CFT in $d+1$ dimensions, namely [188, 189]

$$\mathcal{S}_{CFT} \sim M^{\frac{d}{d+1}}, \quad (11.52)$$

we see that a D dimensional CFT behaves like a $D+1$ dimensional AdS-invariant theory, in terms of entropy scaling. This is directly liked to the AdS/CFT correspondence and does not hold when $\alpha_0 = 0$. Following the same prescription, now we find exact solutions when $\alpha_2 = 0$, $\alpha_1 = 0$ and show that the entropy scales as that of a CFT only when $\alpha_0 \neq 0$.
The $\alpha_2 = 0$ limit must be found separately and provides the well known high-dimensional Schwarzschild–de Sitter solution

$$P(r)^2 = \frac{1}{Q(r)^2} = 1 + \frac{c_1}{r^{d-2}} - \frac{\alpha_0}{\alpha_1 d(d-1)} r^2\,, \quad (11.53)$$

with a proper redefinition of the constant c_1. The horizon can be found under the assumption $\alpha_0 r^2 \gg 1$, where the element $P(r)^2$ can be approximated to

$$\begin{aligned} P(r)^2 &= 1 + \frac{c_1}{r^{d-2}} - \frac{\alpha_0}{\alpha_1 d(d-1)} r^2 = \frac{\alpha_0}{\alpha_1 d(d-1)}\left(\frac{\frac{\alpha_1 d(d-1)}{\alpha_0}\frac{r^d}{r^2} + \frac{\alpha_1 d(d-1)}{\alpha_0}c_1 - r^d}{r^{d-2}}\right) \\ &\sim \frac{\alpha_0}{\alpha_1 d(d-1)}\left(\frac{\frac{\alpha_1 d(d-1)}{\alpha_0}c_1 - r^d}{r^{d-2}}\right), \end{aligned} \quad (11.54)$$

so that the horizon is

$$r_H \sim \left(\frac{c_1\alpha_1 d(d-1)}{\alpha_0}\right)^{1/d}. \quad (11.55)$$

Setting $\alpha_0 = 0$, the $d+1$ dimensional generalization of Schwarzschild radius can be computed without approximations and turns out to be

$$r_H = (-c_1)^{\frac{1}{d-2}} . \tag{11.56}$$

Identifying the constant c_1 with $-M$, as previously discussed, we notice that the Bekenstein–Hawking entropy $\mathcal{S}$ scales as

$$\mathcal{S} \sim \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})} \left(\frac{\alpha_1 d(d-1)}{\alpha_0} \right)^{\frac{d-1}{d}} M^{\frac{d-1}{d}}, \tag{11.57}$$

for $\alpha \neq 0$ and as

$$\mathcal{S} = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})} M^{\frac{d-1}{d-2}}, \tag{11.58}$$

for $\alpha_0 = 0$. This means that the AdS/CFT correspondence holds as long as $\alpha_0 \neq 0$, as expected.

Finally, assuming $\alpha_1 = 0$, the Lagrangian reduces to a sum of the Gauss–Bonnet term and the cosmological constant, providing [155]:

$$P(r)^2 = 1 \pm \frac{1}{r^{d/2-2}} \sqrt{\frac{c_1}{6\alpha_2 \binom{d-1}{d-4}} - r^d \frac{\alpha_0}{24\alpha_2 \binom{d}{d-4}}} .$$

It is worth stressing out that the above solution turns out to be trivial in less than five dimensions, as expected from the topological nature of $\mathcal{G}$. Moreover, it only holds for $\alpha_0 \neq 0$. Under the same approximations as Lovelock case, the first-order expansion of $P(r)^2$ yields the horizon:

$$r_H \sim \left(\frac{2}{\alpha_0} \right)^{1/d} (-c_1)^{1/d} , \tag{11.59}$$

which means that the Bekenstein–Hawking entropy is proportional to

$$\mathcal{S} \sim \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})} \left(\frac{2}{\alpha_0} \right)^{\frac{d-1}{d}} M^{\frac{d-1}{d}} . \tag{11.60}$$

Also here, the AdS/CFT correspondence is recovered, unlike the case of Gauss–Bonnet gravity with $\alpha_0 = 0$. As a matter of fact, when assuming $\alpha_0 = 0$ from the beginning, the equations of motion yield

$$P(r)^2 = \frac{1}{Q(r)^2} = \left(1 + \frac{c_1}{r^{\frac{d}{2}-2}}\right), \tag{11.61}$$

admitting an horizon at

$$r_H = (-c_1)^{\frac{2}{d-4}}, \tag{11.62}$$

whose corresponding entropy scales as

$$\mathcal{S} = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})} M^{\frac{2d-2}{d-4}}. \tag{11.63}$$

Notice that Eq. (11.44), together with the corresponding subcases above discussed, is the generalization of Eq. (11.36) to $d+1$ dimensions.

12

Application of Chern–Simons Theory to Electromagnetism

In this chapter we outline two famous theories of massive electromagnetism, namely the Proca theory and the Chern-Simons three-dimensional theory, pointing out the differences and the common aspects. Comparing the two theories is particularly interesting, since both yield a wave equations describing massive photons. As the former holds in four dimensions, the latter can be obtained in odd dimensions only. However, adding an extra term in the Proca Lagrangian breaks the conformal invariance of the electromagnetic theory. This does not happen when the three–dimensional Chern–Simons form is considered as a starting Lagrangian, where $U(1)$ gauge invariance is provided. It is worth mentioning that Chern–Simons theory arose as a gauge-invariant theory for the electromagnetic interaction, and only at a later time it was applied to gravity [191, 192]. Let us start by considering the free massless electromagnetic action:

$$S = \int \mathrm{d}\mathbf{A}\mathrm{d}\mathbf{A}, \tag{12.1}$$

where $\mathrm{d}\mathbf{A}$ represents the exterior derivative of the one-form connection $\mathbf{A}$. The U(1) invariant abelian Lagrangian coming from the above action, written in coordinates representation, is:

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}. \tag{12.2}$$

By varying Eq. (12.2) with respect to the potential A^μ, one gets the equation $\Box A^\beta = 0$, with $\Box = \partial_\mu \partial^\mu$. One of the most important properties of the electromagnetic Lagrangian is the gauge invariance under $U(1)$ transformations. As a matter of facts, under the gauge transformation

$$\delta A_\mu = \partial_\mu \Theta, \tag{12.3}$$

the transformed Lagrangian $\delta\mathcal{L}$ reads as:

$$\delta\mathcal{L} = (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\Theta(\partial^\mu \partial^\nu - \partial^\nu \partial^\mu)\Theta = 0. \tag{12.4}$$

Several attempts aim to extend the Lagrangian (12.2) to a more general one describing a massive interaction. One of the most famous is the Proca Lagrangian, in which the introduction of a new term breaks the gauge symmetry and leads to a massive Klein-Gordon equation for the vector field A^μ. The corresponding action is [193]:

$$S = \int \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 A^\mu A_\mu \right) d^4x. \tag{12.5}$$

By varying the action (12.5) with respect to the gauge potential one gets

$$\delta_A S = \int \left(-\frac{1}{2} F^{\mu\nu} \delta_A F_{\mu\nu} + m^2 A^\mu \delta_A A_\mu \right) d^4x = \int \left(\partial_\mu F^{\mu\nu} + m^2 A^\nu \right) \delta_A A_\nu d^4x \tag{12.6}$$

and in the Lorentz gauge, where $\partial_\mu A^\mu = 0$, the above equation becomes:

$$(\Box + m^2) A^\beta = 0, \tag{12.7}$$

so that m can be intended as a mass term. However, as a new massive particle arises, the theory loses the gauge invariance and this yields some shortcomings at the quantum level. Another extension is the odd-dimensional $U(1)$ invariant Chern-Simons theory [194]. Generally, the $D-$dimensional Chern-Simons Lagrangians can be constructed by means of the Chern-Simons $D-$forms, whose exterior derivatives provide a $D+1$-dimensional topological invariant. This makes the theory *quasi*-invariant under gauge transformations, as showed in Chap. 9. In order to get the three-dimensional Chern-Simons field equations for the electromagnetic theory, we start from the three-dimensional $U(1)$ invariant Chern-Simons action

$$S = \int \mathbf{A}\mathrm{d}\mathbf{A}, \tag{12.8}$$

whose exterior derivative provides the four-dimensional Pontryagin density $P_4 = \mathbf{F} \wedge \mathbf{F}$, with $\mathbf{F}$ being the curvature two-form. By computing the exterior derivative $d\mathbf{A}$, the action can be written as:

$$\int \mathbf{A}\mathrm{d}\mathbf{A} = \int (\partial_\mu A_\nu - \partial_\nu A_\mu) A_p \, dx^\mu \wedge dx^\nu \wedge dx^p = \int \epsilon^{\mu\nu p} F_{\mu\nu} A_p \, d^3x. \tag{12.9}$$

By introducing the Chern-Simons form (12.9) in the free electromagnetic Lagrangian (12.2), this latter turns out to be

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m \, \epsilon^{\mu\nu p} F_{\mu\nu} A_p \, . \tag{12.10}$$

Notice that under gauge transformations, the Lagrangian only changes by a total derivative. In fact, under the transformation

$$\delta A_\mu \to \partial_\mu \Theta, \tag{12.11}$$

the Lagrangian variation is:

$$\begin{aligned} \delta\mathcal{L} &= -\frac{1}{2} F_{\mu\nu} (\partial^\mu \partial^\nu - \partial^\nu \partial^\mu)\Theta + \frac{1}{2} m \, \epsilon^{\mu\nu p} [F_{\mu\nu} \partial_p \Theta + A_p (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\Theta] \\ &= \partial_p \left(\frac{1}{2} m \, \epsilon^{\mu\nu p} F_{\mu\nu} \Theta \right). \end{aligned} \tag{12.12}$$

The last equality is motivated by the identity $\partial_p(\epsilon^{\mu\nu p} F_{\mu\nu}) = 0$, coming from the field equations of the theory. The field equations can be obtained by varying the action with respect to the gauge connection, namely:

$$\begin{aligned}\delta_A S &= \int \left\{-\frac{1}{4}\delta_A\left(F^{\mu\nu}F_{\mu\nu}\right) + \frac{1}{2}m\,\epsilon^{\mu\nu p}\delta_A(F_{\mu\nu}A_p)\right\} d^3x \\ &= \int \left\{\partial_\mu F^{\mu\nu} + \frac{1}{2}m\,\epsilon^{\mu p\nu}F_{\mu p}\right\}\delta A_\nu \; d^3x, \end{aligned} \tag{12.13}$$

so that they read

$$\partial_\mu F^{\mu\nu} + \frac{1}{2}m\,\epsilon^{\mu p\nu}F_{\mu p} = 0. \tag{12.14}$$

Using the identity $\partial_\mu\partial_\nu F^{\mu\nu} = 0$, and taking the three-dimensional divergence of Eq. (12.10) we get:

$$\partial_\nu\left(m\,\epsilon^{\mu p\nu}F_{\mu p}\right) = 0, \tag{12.15}$$

which finally provides the result used to obtain Eq. (12.12). After demonstrating the $U(1)$ invariance of the Chern-Simons Lagrangian, by means of Eq. (12.14) it is possible to show that the field equations can be manifestly recast as a Klein-Gordon equation for massive fields. Multiplying Eq. (12.14) by the Levi-Civita symbol, we have

$$\begin{aligned}\epsilon_{\sigma\tau\nu}\partial_\mu F^{\mu\nu} + \frac{1}{2}m\,\epsilon_{\sigma\tau\nu}\epsilon^{\mu p\nu}F_{\mu p} &= \epsilon_{\sigma\tau\nu}\partial_\mu F^{\mu\nu} + \frac{1}{2}m(\delta^\mu_\sigma\delta^p_\tau - \delta^p_\sigma\delta^\mu_\tau)F_{\mu p} \\ &= \epsilon_{\sigma\tau\nu}\partial_\mu F^{\mu\nu} + mF_{\sigma\tau} = 0. \end{aligned} \tag{12.16}$$

Let us evaluate the first term ($\epsilon_{\sigma\tau\nu}\partial_\mu F^{\mu\nu}$):

$$\begin{aligned} &\epsilon_{\sigma\tau\nu}\partial_\mu F^{\mu\nu} = \frac{1}{2}\epsilon_{\sigma\tau\nu}\partial_\mu\left[F^{\mu\nu} - F^{\nu\mu}\right] = \frac{1}{2}\epsilon_{\sigma\tau\nu}\partial_\mu\left[\delta^\mu_\alpha\delta^\nu_\beta F^{\alpha\beta} - \delta^\mu_\beta\delta^\nu_\alpha F^{\alpha\beta}\right] \\ = &\ \frac{1}{2}\epsilon_{\sigma\tau\nu}\epsilon^{\mu\nu\lambda}\epsilon_{\alpha\beta\lambda}\partial_\mu F^{\alpha\beta} = -\frac{1}{2}\left[\delta^\mu_\sigma\delta^\lambda_\tau - \delta^\mu_\tau\delta^\lambda_\sigma\right]\epsilon_{\alpha\beta\lambda}\partial_\mu F^{\alpha\beta} \\ = &\ -\frac{1}{2}\epsilon_{\alpha\beta[\tau}\partial_{\sigma]}F^{\alpha\beta}. \end{aligned} \tag{12.17}$$

Replacing Eq. (12.17) into Eq. (12.16), the field equations take the form:

$$-\frac{1}{2}\epsilon_{\alpha\beta[\tau}\partial_{\sigma]}F^{\alpha\beta} + mF_{\sigma\tau} = 0. \tag{12.18}$$

Taking the divergence and using Eq. (12.14) and Eq. (12.15), field equations finally read:

$$\left(\Box + m^2\right)\left(\epsilon_{\alpha\beta\tau}F^{\alpha\beta}\right) = 0. \tag{12.19}$$

Eq. (12.19) is a Klein-Gordon equation for the vector field $\epsilon_{\alpha\beta\tau}F^{\alpha\beta}$, which finally demonstrates the capability of the Chern-Simons three-dimensional theory to describe massive particles without breaking the symmetry.

13

Application of Chern–Simons Theory to Biological Systems

Genomic strings schematization methods represent one of the most controversial and discussed branches of science. In this scenario, the application of those methods to DNA alignment is still not fully uncover. Several approaches aim to exhaustively predict the evolution of macro molecules, in order to get information regarding their spatial configuration [195, 196, 197, 198]. However, a complete theory capable of predicting the interactions that occur among macro molecules and the corresponding biological implications is still missing.

Macro molecules often interact such that the resulting biological system exhibit a non-trivial topological structure, thus elements which seem to be close to each other might be located even in different chromosomes [199].

Schematization approaches study the interactions among different parts of the same biological system and can be also helpful to predict the probability for the system to evolve towards certain mutations. An example is given by the interaction between proteins and virus genome which, if well described, can lead to a comprehension of the corresponding infection evolution. Standard modeling techniques are mostly based on probability considerations, aimed at outlining the many body interactions by means of statistical mechanics [200, 201, 202].

In this Chapter we test an innovative method for the schematization of biomolecule configurations, based on the topological Chern-Simons theory. It mainly relies on the curvature assumed by biological systems, using the numerical value of the Chern-Simons current, namely the expectation value of the Wilson loop.

Although the application of Chern-Simons gravity to complex systems seems to be unusual, topological field theories are deeply studied in several branches of physics, besides the application to gravitational interaction.

For instance, by means of the Chern-Simons formalism, some stimulating problems of biology have been addressed, such as the presence of knotted DNAs and their interactions with proteins [203]. Yet, in [204] the interactions of unknotted RNAs with knotted proteins have been analyzed in the process of codon and correction of RNA in methil transfer, as well as a general equation to solve the dynamics of knotted proteins has been proposed by Lin and Zewail [205], based on the Wilson loop operator for gene expression with a boundary phase condition.

Here we start from the $SU(N)$-invariant three-dimensional Chern-Simons Lagrangian

$$\mathscr{L}_{CS}^{(3)} = \mathrm{tr}\left[\mathbf{A}\mathrm{d}\mathbf{A} + \frac{2}{3}\mathbf{A}\mathbf{A}\mathbf{A}\right], \tag{13.1}$$

whose exterior derivative yields the $SU(4)$ Pontryagin density $P^{(4)} = \mathrm{tr}\,[F \wedge F]$.

The basic foundations lying behind such an application can be found in [206] and [207], where some of the authors of this paper develop the formal structure of the theory, by applying it to unveil the mechanism of DNA-RNA transcriptions and providing some insights to specifically describe the junk area within the DNA sequence [206]. In [206] the theory is applied to study the docking mechanism of biological macro-molecules, such as the configurational dynamics occurring in protein-protein. Without claiming completeness, in Sec. 13.1 we outline the main properties of the theory, with the aim to test its validity by considering DNA sequences and introducing known mutations. The introduction of a mutation yields a change in the point-like curvature of the given sequence, which may give important information regarding the biological impact that such mutation may have. From the mutated sequence it is possible to infer the frequency/probability of the mutation to occur, as well as to predict the evolution of the system towards a given configuration. In Secs. 13.1.1 and 13.1.2 the formalism is then applied to different strings of KRAS human gene and to COVID-19 virus sequences. In the former case, we apply the model to analyze the mutations of a few region of the KRAS human gene, a gene that acts as on/off switch in cell signaling and, among its functions, controls cell proliferation. When KRAS is mutated, negative signaling is disrupted, with the consequence that cells can continuously proliferate, often degenerating into tumors [208]. In our analysis KRAS sequences with mutations are thus compared with reference sequences, with the aim to use Chern–Simons theory to infer predictions of biological interest. In the latter case we compare sequences of single filament RNA SARS CoV-2 viruses coming from different countries, using Chern-Simons currents to potentially explain the reason why SARS-CoV-2 variants seem to exhibit a higher incidence during the 2020/2021 pandemic.

13.1 Chern-Simons Theory in DNA System

In this section we review the application of Chern-Simons theory to DNA/RNA systems, outlining the main results obtained in [206]. The first step is to use quaternion fields to define a set of Nitrogen Bases over the DNA or RNA; such quaternion fields have unitary norm and belong to $SU(1) \subset \mathbb{H}$. They read:

$$\begin{cases} A_{DNA} := e^{\frac{\pi}{2} i \beta_n} & A_{RNA} := e^{\frac{\pi}{2} j \alpha_n} \\ T_{DNA} := i\, e^{-\frac{\pi}{2} i \beta_n} & U_{RNA} := i\, e^{-\frac{\pi}{2} j \alpha_n} \\ C_{DNA} := j\, e^{i \pi \beta_n} & C_{RNA} := j\, e^{j \pi \alpha_n} \\ G_{DNA} := k\, e^{2\pi i \beta_n} & G_{RNA} := k\, e^{2\pi j \alpha_n}, \end{cases} \tag{13.2}$$

being $[h] \in \mathbb{H}$: $[h] = a + b\,i + c\,j + d\,k$ and $a, b, c, d \in \mathbb{R}$. The connection $\mathbf{A}$ can be thought as a state of the above written nitrogen bases, namely $\mathbf{A} \in \{A, T/U, C, G\}$). It is a one-form connection with values in $SU(2)$, defined over the three-dimensional space of all possible amino acids (formed by three nitrogen bases). Consequently the DNA curvature in the configuration space of nitrogen bases is represented by the two-form curvature $F = d\mathbf{A}$, which in coordinates representation can be written as:

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]} + A_{[\mu} A_{\nu]}. \tag{13.3}$$

Therefore, taking into account the $SU(2)$-invariant Chern-Simons three-dimensional action

$$S^{SU(2)} = \int \mathrm{Tr}\left[\mathbf{A}\mathrm{d}\mathbf{A} + \frac{2}{3}\mathbf{A}\mathbf{A}\mathbf{A}\right], \tag{13.4}$$

it is possible to define the *Chern-Simons current* as the measurable, gauge invariant quantity that can be obtained from the expectation value of the Wilson loop

$$J =< [W(\mathbf{A})] >= \frac{\int \mathcal{D}A\, e^{iS} \Pi_n W(A_n)}{\int \mathcal{D}A\, e^{iS}}. \tag{13.5}$$

Wilson loop is the trace of a path-ordered exponential of the gauge connection and represents the only gauge invariant observable quantity of the theory:

$$W(\mathbf{A}) = \mathrm{tr}\left[\exp\left\{\mathcal{P} \oint \mathbf{A}\right\}\right]. \tag{13.6}$$

They can be obtained from the holonomy of the gauge connection around a given loop and are mainly used in gauge lattice theories and quantum chromodynamics [209, 210, 211, 212]. They have been formerly introduced to address a nonperturbative formulation of quantum chromodynamics [213] but nowadays play an important role in the formulation of loop quantum gravity, particle physics and String Theory.

The choice of the three-dimensional action is the key point of the method: standard biology suggests that nitrogen bases combine each other in triplets, and therefore form a three-dimensional discrete space of configurations that can be described by means of the Chern-Simons three forms. Any point of the space is thus labeled by a given triplet. Sixty-four possible combinations arise after combining the four nitrogen bases in triplets, and correspond to the combinations occurring in the genetic code.

By means of Eq. (13.2), it is possible to define a discrete superstate of configurations, in which the nitrogen bases represent the dynamical variables, so that the genetic code is labeled by the Chern-Simons currents only. After some calculations the curvature spectrum of the genetic can be obtained [206], as reported in Table **II**.

Table II: *Value of Chern-Simons current for the triplets of the genetic code.*

Amino acid	CS Current	Amino acid	CS Current	Amino acid	CS Current	Amino acid	CS Current
Phe (UUU)	0.7071	Ser (UCU)	0.0534	Tyr (UAU)	0.0214	Cys (UGU)	0.0122
Phe (UUC)	0.5000	Ser (UCC)	0.0495	Tyr (UAC)	0.0205	Cys (UGC)	0.0118
Leu (UUA)	0.3717	Ser (UCA)	0.0460	Sto (UAA)	0.0197	Sto (UGA)	0.0115
Leu (UUG)	0.2887	Ser (UCG)	0.0429	Sto (UAG)	0.0189	Trp (UGG)	0.0112
Leu (CUU)	0.2319	Pro (CCU)	0.0402	His (CAU)	0.0182	Arg (CGU)	0.0109
Leu (CUC)	0.1913	Pro (CCC)	0.0377	His (CAC)	0.0175	Arg (CGC)	0.0106
Leu (CUA)	0.1612	Pro (CCA)	0.0354	Gin (CAA)	0.0169	Arg (CGA)	0.0103
Leu (CUG)	0.1382	Pro (CCG)	0.0334	Gin (CAG)	0.0163	Arg (CGG)	0.0010
Ile (AUU)	0.1201	Thr (ACU)	0.0316	Asn (AAU)	0.0157	Ser (AGU)	0.0098
Ile (AUC)	0.1057	Thr (ACC)	0.0299	Asn (AAC)	0.0152	Ser (AGC)	0.0096
Ile (AUA)	0.0939	Thr (ACA)	0.0284	Lys (AAA)	0.0147	Arg (AGA)	0.0093
Met (AUG)	0.0841	Thr (ACG)	0.0270	Lys (AAG)	0.0142	Arg (AGG)	0.0091
Val (GUU)	0.0759	Ala (GCU)	0.0257	Asp (GAU)	0.0138	Gly (GGU)	0.0089
Val (GUC)	0.0690	Ala (GCC)	0.0245	Asp (GAC)	0.0134	Gly (GGC)	0.0087
Val (GUA)	0.0630	Ala (GCA)	0.0234	Glu (GAA)	0.0129	Gly (GGA)	0.0085
Val (GUG)	0.0579	Ala (GCG)	0.0224	Glu (GAG)	0.0126	Gly (GGG)	0.0083

The same analysis can be also pursued by considering the amino acids, so that the genetic code is equivalently described by 21 different Chern-Simons currents. The simplest way to construct a curvature spectrum with respect to amino acids, is to take the average values of the Chern-Simons currents which refer to triplets coding for the same amino acid. Chern-Simons currents for the amino acids are listed in Table **III**.

Table III: *Value of Chern-Simons current for the amino acids.*

Amino acid	CS Current	Amino acid	CS Current	Amino acid	CS Current	Amino acid	CS Current
Phe (F)	0.60355	Ser (S)	0.0352	His (H)	0.01785	Giu (E)	0.01275
Leu (L)	0.2305	Pro (P)	0.036675	Gin (Q)	0.0166	Cys (C)	0.012
Ile (I)	0.106567	Thr (T)	0.029225	Asn (N)	0.01545	Trp (W)	0.0112
Met (M)	0.0841	Ala (A)	0.024	Lys (K)	0.01445	Arg (R)	0.01005
Val (V)	0.06645	Tyr (Y)	0.02095	Asp (D)	0.0136	Gly (G)	0.0086

In summary, the theory deals with a discrete configuration space made of all possible amino acids, namely all possible ways in which triplets can combine each other (64 ways). To a given point of the space, it corresponds only one amino acid, so that the nitrogen bases represent the coordinates of the given point. Hence, an observable quantity can be assigned to this latter, so that its curvature can be numerically quantified. For a better visualization, Fig. **6**. shows that different points in the linear structure of the configuration space have different curvature, which can be used to infer the position toward which the system is most likely to evolve, as well as the attractors in the docking between proteins.

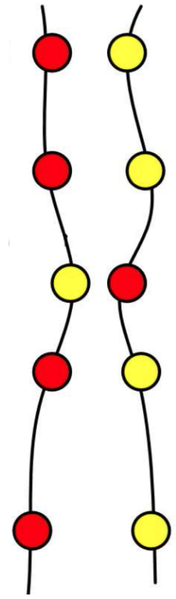

Figure 6: *Visual representation of the configuration space curvature. Circles in the two sequences represent amino acids, each of which have a proper curvature provided by the Chern-Simons current. Points belonging to the left sequence can be attracted to points of the right sequence (or* vice versa*), depending on the point-like curvature.*

Notice that the formalism permits to assign a numerical value to each component of the genetic code, finding a point by point correspondence between triplets and curvature. Such a curvature of the DNA is the key parameter of our approach, as it may provide several predictions about the docking between two different parts of DNA or between DNA and RNA. The genomic curvature can be also used to find out those positions having highest probability to exhibit a mutation. The introduction of the mutation, indeed, leads to a local variation of the curvature, whose value might suggest the clinic importance and the impact of the corresponding disease. Moreover, the curvature spectrum can provide important insights regarding the evolution of the genomic strings: those points with highest curvature are the best candidates to evolve towards a more stable configuration, making the entire sequence more uniform in the configuration space of all the possible triplets.

13.1.1 Chern-Simons Current in Mutated KRAS Human Gene

The first application of the above described method, is focused on the comparison between mutated DNA and standard DNA sequences. In particular, first we consider the KRAS human gene, whose details are reported in App. A. It is located in the 12th chromosome, from the base 25,205,246 to 25,250,929 and represents one of the most mutated human genes [214, 215, 216]. Then we introduce some known mutations into the original sequence, yielding a change in the Chern-Simons current. Being the current linked to the curvature of the DNA, the configuration space made of nitrogen bases changes the point-like curvature whereas a mutation is introduced.
By means of physical considerations, we theoretically expect the mutation to level out the graph, providing slighter variations of the current with respect to the original sequence. In analogy with other physical systems, the curved point is surrounded by a non-equilibrium region, which in turn tends to mutate in order to reach a minimum free energy state.
Moreover, this prescription is in agreement with the general criterion which governs thermodynamic transformations, according to which any spontaneous transformations must minimize the Gibbs free energy. This statement can be simply proved by considering the definition of the Gibbs free energy $\mathcal{G}$, that is

$$\mathcal{G} = U - TS + pV, \tag{13.7}$$

with p being the pressure, V the volume, T the temperature, S the entropy and U the free energy. Neglecting the contribution of p and setting $T =$ const. (as standard for biological systems), it turns out that for the system to undergo a spontaneous transformation, the entropy must increase as the free energy must decrease. This latter can be thought as the expectation value of the Hamiltonian of the system, which includes potential and kinetic energies. Therefore, requiring the Gibbs free energy to decrease spontaneously is equivalent to require the gravitational potential to decrease spontaneously. This means that as the system evolves toward a configuration with $\Delta\mathcal{G} < 0$, the potential energy decreases. By applying these considerations to the formalism developed in Sec. 13.1, a spontaneous transformation must yield an evolution of the system toward a flat regions in the configuration space.

For these reasons, mutations of DNA/RNA sequences occur to render the graph slighter and to bring the general state toward an equilibrium configuration. Reversing the argument, those mutations which make the sequence more peaked than the original one, are supposed to occur less frequently, since they lead to a higher free energy configuration. Therefore, significant variation should not occur in flat regions of the curvature spectrum, which are closer to an equilibrium state. The result of the analysis in KRAS human gene via Chern-Simons current method is reported in Fig. **7a** [217, 218].

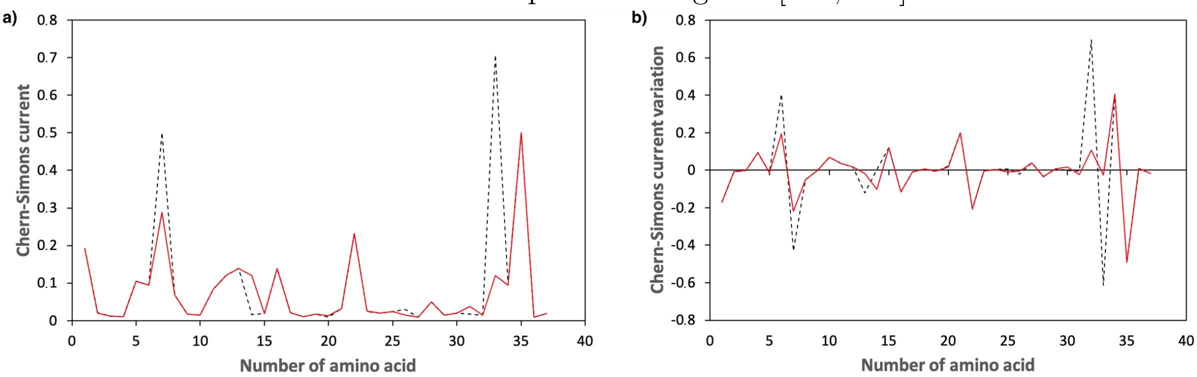

Figure 7: *Chern-Simons current in KRAS human gene. Figure 7a shows the comparison between the original sequence (black dashed line) and the mutated one (red solid line), while Figure 7b shows the Chern-Simons current variation, obtained comparing the point-like differences between contiguous points of the original and mutated sequences. The region considered is 25,245,274 - 25,245,384 of the 12th chromosome.*

Most significant mutations occur in the regions comprised between the 5th and the 15th amino acid, and between the 30th and the 35th. Further details are reported in App. A. As expected by the free energy minimization argument, the mutations occur whereas the curvature is most peaked, providing a smoother general trend, with respect to the original one. Notice, however, that mutations are not directly correlated to peaks, but rather to curvature gradients, namely they are mostly located near those points whose curvature is very much higher (or lower) with respect to a contiguous point. By computing the differences between contiguous points, it is possible to associate mutations to peaks, as reported in Fig. **7b**.

In the same region of the twelfth chromosome, another set of mutations occurs.

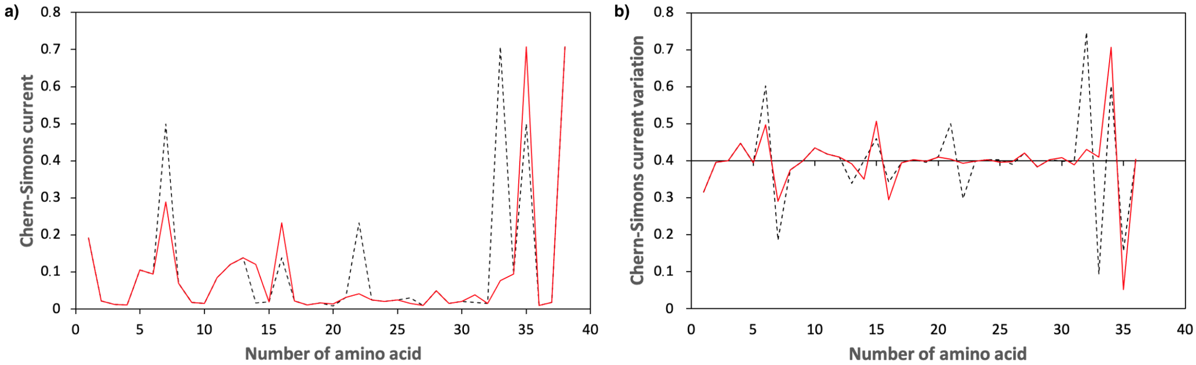

Figure 8: *Chern-Simons current in KRAS human gene. Figure 8a shows the comparison between the original sequence (black dashed line) and the mutated one (red solid line), while Figure 8b shows the Chern-Simons current variation, obtained comparing the point-like differences between contiguous points of the original and mutated sequences. The region considered is 25,245,274 - 25,245,384 of the 12th chromosome.*

Fig. **7** and Fig. **8** refer to the same region of KRAS, though different mutations are introduced in the two cases. More precisely, mutations occurring in these selected regions are split in two different sets, in order to facilitate reading and visualizing the curvature spectrum. It is worth noticing that even in this case, a mutation corresponds to each peak, as theoretically inferred. Moreover, the mutated sequence makes the overall trend smoother than the original one, in agreement with theoretical predictions. To confirm this result, two other different regions of human KRAS are analyzed in Figs. **9-10**, where the original sequences are again compared with the corresponding mutated. As we did before, this latter can be obtained by replacing nitrogen bases (or amino acids) in the original sequence. These replacements are carefully chosen according to the database BioMuta. Also in this case, further details can be found in App. A.

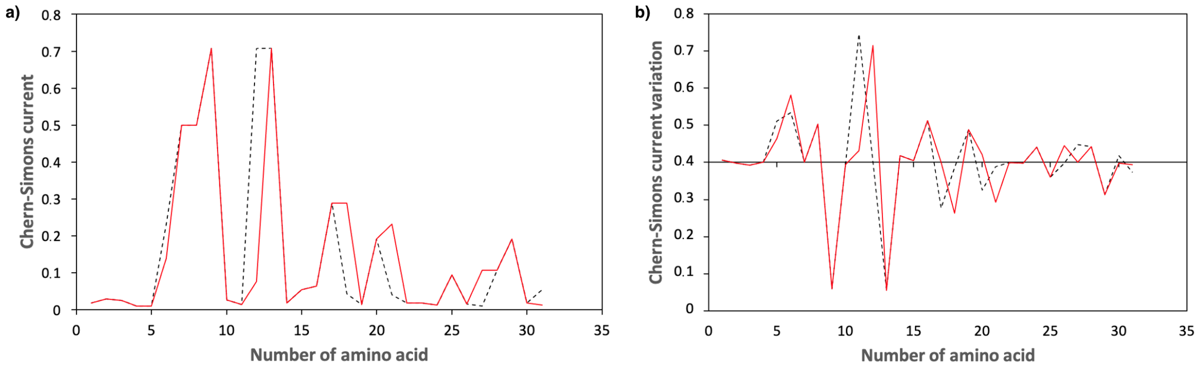

Figure 9: *Chern-Simons current in KRAS human gene. Figure 9a shows the comparison between the original sequence (black dashed line) and the mutated one (red solid line), while Figure 9b shows the Chern-Simons current variation, obtained comparing the point-like differences between contiguous points of the original and mutated sequences. The region considered is 25,215,468 - 25,215,560 of the 12th chromosome.*

The last region analyzed, corresponding to the region 25,227,263-25,227,379 of the 12th chromosome, yields the graph in Fig. **10**.

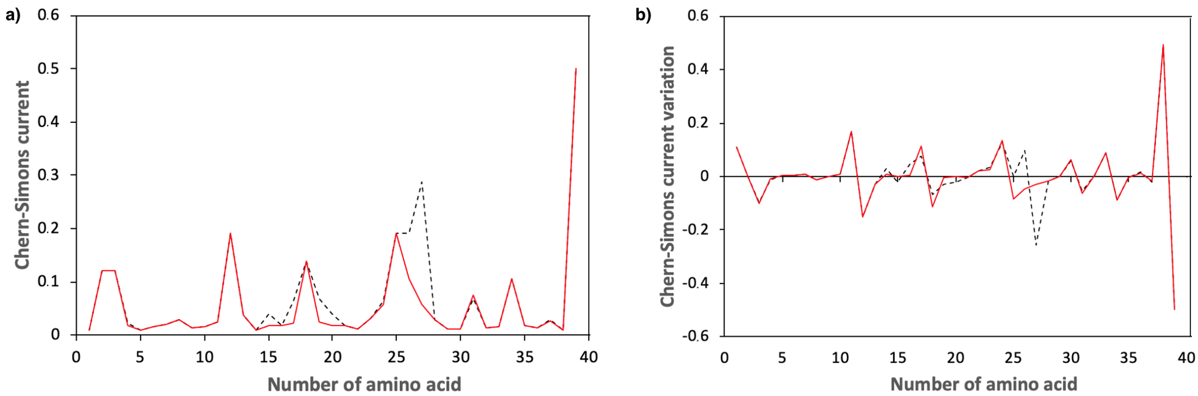

Figure 10: *Chern-Simons current in KRAS human gene. Figure 10a shows the comparison between the original sequence (black dashed line) and the mutated one (red solid line), while Figure 10b shows the Chern-Simons current variation, obtained comparing the point-like differences between contiguous points of the original and mutated sequences. The region considered is 25,227,263-25,227,379 of the 12th chromosome.*

Notice that in both cases the mutations occur where the sequence is peaked, in agreement with theoretical predictions. This is particularly evident in the former case (Fig. **9**), where almost all peaks correspond to a mutation. Moreover, the introduction of the mutations has the effect to avoid abrupt differences in the overall trend of the curvature spectrum. On the contrary, well known mutations may occur also in flat regions of sequences with no peak in the Chern-Simons current values. This nay be due to other factors that induce mutations, not taken into account in our model at the moment, where we basically rely on an argument based on the curvature gradient variation and free energy minimization.

13.1.2 Chern-Simons Current in Mutated COVID-19 Sequences

In this subsection we discuss the results provided by the applications of Chern-Simons formalism to different variants of SARS-CoV-2 virus. The S glycoprotein is a Class I fusion protein, composed by two subunits (S1,S2) [219]; the S1 subunit contains the receptor binding domain (RBD), directly binding to the main receptor human angiotensin-converting enzyme 2 (hACE2) and determinant for both host range and cellular tropism [220]; the S2 subunit is directly involved in membrane fusion and virus endocytosis [221, 222]. Receptor binding triggers conformational changes; specifically, host proteases (such as furin) will mediate its functional transition by cleaving the interface between the two subunits (S1, S2). Additionally, the RBDs of SARS-CoV and SARS-CoV-2 are highly similar, despite few key residues, appearing to enhance the transmissibility of the novel CoV [223, 224]. The spike glycoprotein is the main inducer for neutralizing antibodies [225]; unwillingly, it shows the highest mutation rate among SARS-CoV-2 proteins [226, 227], and a variable glycosylation can create novel CTL epitopes, possibly altering hACE2 binding and accessibility to proteases and neutralizing antibodies [221, 228].

The purpose here is to find a correlation in terms of Chern-Simons current among the mutations of the sequences, a correlation that could possibly give insights aiming at localizing and predicting mutation sites in the new variants of the virus. We analyze eleven strings, which underwent mutations with respect to the original sequence of SARS-CoV-2, firstly detected in Wuhan at the end of 2019. They all correspond to the same RNA region and was selected according to Fig. **11**. In particular, we compare the difference of Chern-Simons currents, considering variants from Asia, Europe, Oceania and North America. Specifically, sequence 19A is the first one which arose in Wuhan and have been spreading during the initial 2020 outbreak; 19B is the first detected variant in China; 20A dominated mostly in Europe from march 2020, to subsequently spreading out globally; 20B and 20C are variants of 20A which mainly spread in the early 2020; finally, 20D, 20E, 20F, 20G, 20H, 20I occurred on summer 2020 as variants of 20B, 20C and 20A. Among them, 20I and 20H are English and south-African variants. To be more precise, we used the tool Nextclade, yielding the graph of Fig. **11**. This figure shows the aforementioned evolution of the sequences (https://github.com/nextstrain/ncov/blob/master/defaults/clades.tsv).

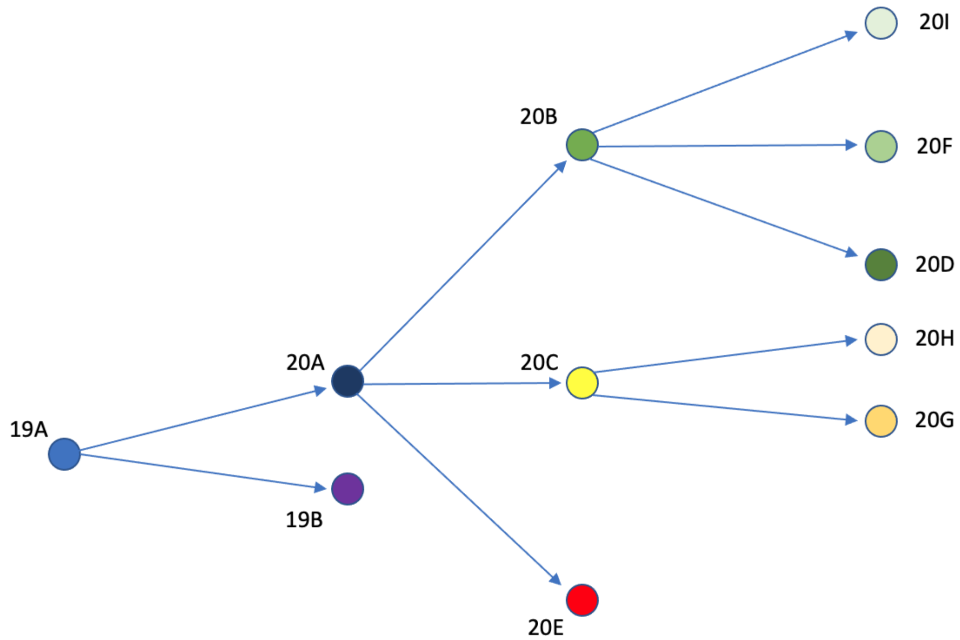

Figure 11: *Evolution of the first-detected Wuhan sequence (19A) to other variants which spread out during the 2020 pandemic.*

Mutations of the triplets which caused the occurrence of variants are reported in App. A. In our analysis, because of the large amount of nitrogen bases, we only compute the difference of Chern-Simons currents between the original sequence and the mutated one.

Specifically, we consider the slope of the current for each mutation, namely the number

$$\text{Slope} = \frac{\text{Mutated Seq.} - \text{Original Seq.}}{\text{Original Seq.}}. \tag{13.8}$$

In the absence of a curvature spectrum, the slope provides a numerical value aimed at confirming the results obtained in the previous section. Specifically, high values of the slope represents a large discrepancy between the original sequence and the mutated one in the curvature spectrum, while lower values account for small differences. Even though the general trend is inevitably missing, the approach may provide significant results regarding the one-to-one correspondence between different couples of strings. We perform the one-to-one comparison between contiguous sequences (showed in Fig. **11**), with the aim to find out a correlation between slopes and mutations. Each variant is compared with the corresponding predecessor, so that no comparison is carried on between sequences which are not directly evolving from one another, according to Fig. **11**. For example, sequence 19A is not compared with 20I, as well as 20D is not compared with 20H.

The analysis shows that mutations occur with highest probability where the slope (as defined in Eq. (13.8)) of Chern–Simons current assumes extreme values, namely when its modulus is extremely high or extremely low[1].

This means that even those mutations which do not cause significant current variations can support variants. In particular, the one-to-one comparison between the original and the corresponding mutated sequences shows that 60% of mutations corresponds to extreme values of current. Such percentage increases up to 80% if we consider only those mutations which will effectively spread out (denoted in italic bold and highlighted in light yellow), as showed in App. A, Figs. **13-23**. Consequently, this statistic can be used to point out which occurred mutation of the sequence can be more likely to evolve in a real, spread out variant of the virus. To be more precise, once we know the position of a given

[1]As reported at the beginning of App. A, we define current variations as "low" if they are comprehended in the range [-11%,11%], and as "high" if they are $>$ 100% or $<$ - 100%. Also notice that there is no upper limit to the modulus of the current variation, since it represents the percentage of current increase with respect to surrounding points

mutation, Chern–Simons currents can allow to predict which type of triplets will arise from such mutation. In particular, as suggested by the analysis, the mutated sequences should exhibit mutations whose related Chern-Simons currents provide extremely high or extremely low percentage variations, with respect to the original ones. Therefore, we do not expect the sequence to evolve such that mutations cause intermediate values of current variations; rather, if the position of the mutation is known, we expect the triplet to mutate towards those possible configurations whose Chern-Simons current is either very close or very far from the initial one (in terms of percentage). This means that from a given triplet we can select a set of possible mutations, namely those which cause either high or low current variations.

The above results constitute a part of the analysis of SARS-Cov-2 virus, which mainly relies on the evolution of given sequences towards mutated configurations. As mentioned above, this first part turns out to be useful to restrict all possible mutations within a given range, but can provide suitable information only if the position of the mutation is known *a priori*. From this point of view, no information regarding the mutation position can be provided. Now, in the next part, we use Chern-Simons formalism to select regions where mutations are most likely to occur.

With the aim to link the currents with the probability to exhibit mutations, we separately analyzed only those sequences which generate variants, *i.e.* 19A, 20A, 20B and 20C. Specifically, as we can infer from Fig. **11**, 19A generates 19B and 20A; 20A generates 20B, 20C and 20E; 20C generates 20H and 20C. Similarly to the previous analysis of KRAS human gene, we aim to relate the curvature spectrum with the likelihood to find out mutations. To this purpose, we calculated the Chern–Simons currents of 19A, 20A, 20B and 20C sequences and computed the current variations in those points affected by known mutations. Specifically, let n be the position of a given mutation along the sequence and j_n the corresponding Chern–Simons current. The normalized current variations are computed according to the formulas:

$$\text{Variation } (\%)_1 = \frac{j_{n+1} - j_n}{j_n} \tag{13.9}$$

and

$$\text{Variation } (\%)_2 = \frac{j_n - j_{n-1}}{j_{n-1}}. \tag{13.10}$$

This means that we are investigating the current variations where the mutations occur, with respect to the previous and the subsequent points, respectively. The comparison between these values calculated for the triplets affected by mutations and the surrounding points can be used to relate the current variation with RNA mutations.

This prescription is suggested by the analysis performed on human KRAS regions, where it turns out that points far from the equilibrium state in the curvature spectrum are the best candidates to provide mutations. Here, given the large amount of amino acids, the curvature spectrum cannot be compute entirely. For this reason, we only focused on noticeable mutations, namely preferred points which exhibit known triplet variations.

The analysis again shows that mutations mostly occur where the current variation, as calculated in Eqs. (13.9) and (13.9), is high-valued. More precisely, in a set of 125 total mutations, 59% of them (74/125, see Tables **X-XIII**) are are located in points where the curvature undergoes abrupt variations. This percentage increases up to 69%, if only noticeable mutations which had more impact in the development of the corresponding variants are considered. Indeed, among 25 mutations with the greatest impact in generating the variants, 17 exhibit high percentage variations of current with respect to surroundings points. These results are reported in App. A, Tables **X-XIII**.

This result can be explained based on the achievements of the previous section, where non-equilibrium points turned out to be best candidates to provide nitrogen bases mutations. More precisely, large values of the current variations account for peaked regions, which tend to evolve to a lower curvature, that is a lower current. Reversing the argument, large variations of current are exhibited by points which are far from the minimum of energy, which is supposed to occur where the trend is constant.

In this framework, the application of Chern-Simons theory to DNA/RNA systems such as SARS-CoV-2 or KRAS, can give important information about the positions where the mutation is more likely to manifest. The consequent biological impact naturally follows, since this prediction can be used to prevent the occurrence of variants or to know in advance the probability for the sequence to evolve towards another configuration.

Taking into account these results, let us evaluate the spike region of SARS-CoV-2 virus only, with the aim to analyze the tertiary structure. In particular, we rely on the interaction points reported Ref. [229], according to which the amino acids of the spike protein are interact as reported in Fig. **12** [2].

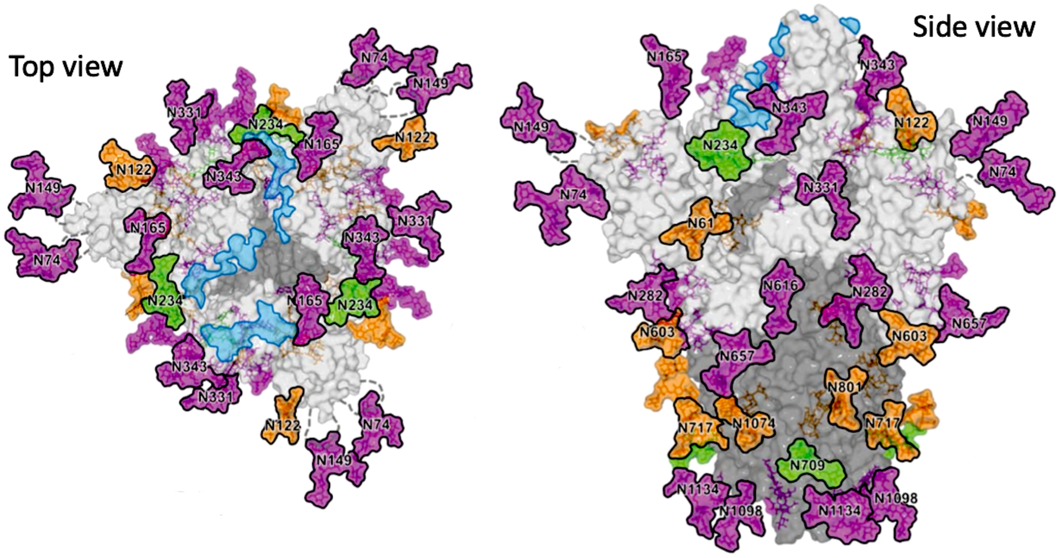

Figure 12: *Tertiary structure of the spike protein of SARS-CoV-2 virus. Green, orange and pink colors refer to the oligomannose content. Specifically, glycan sites labeled in green contain 80-100% of oligommannose, those labeled in orange 30-79% and those labeled in pink 0-29%. Light blue denotes ACE2 binding sites.*

In light of the results provided by Ref. [229], we analyze 11 contact points, namely 22 corresponding amino acids. The features of these latter, such as position, current or percentage variation with respect to the surrounding triplets are reported in Table **XIV**.

[2]Numbers refer to the positions on the spike protein only

We considered 22 sites and calculated the Chern-Simons current variation of each amino acid with respect to the surrounding points in the linear structure. Beside the first amino acid (position 19), none of them is affected by known mutations. As a first remark, it is interesting to observe that the percentage of large variations in those sites which are not affected by mutations is 7/21, namely 33%. Note that such a percentage is quite lower than the previously discussed one, which is of the order of 72%. This confirms that Chern-Simons current variations is high-valued whereas mutations occur. Moreover, these seven sites which undergo large percentage variations are oligomannose-type, as pointed out in Ref. [229]. This, in principle, could be the reason of such large values. For instance, the high value of current variation in position 234 might be due to the proximity of the site with ACE-2, or to the high percentage of glycosylation occurring in such amino acid.
As a secondary result, we also considered the Chern-Simons current variations of surrounding points with respect to the tertiary structure. Details of the analysis are reported in Table **XV**. Notice that the docking points have same or similar values of current, which means low percentage variation. This is expected from a physical point of view, since those points with same curvature tend to interact in order to reach a stabler configuration. Also here, the analogy with gravitational interaction is simply understood. Reversing the argument, Chern-Simons current can be used, in future works, to investigate the tertiary structure of a given protein. In particular, those points with same or similar current are most likely to interact each other. This application can prove to be profitable in biology and bioinformatics, where the shape of the molecule is needed to extrapolate information regarding the evolution of a given system.

14

Conclusion of Part III

In the first chapters of this part we applied the formalism of the Chern–Simons theory to cosmology and spherical symmetry. Specifically, we showed that the five-dimensional Lovelock gravity admits exact cosmological solutions, with exponential scale factors. Depending on the values of the coupling constants, both de Sitter-like solutions and bouncing space-times are allows by Lovelock cosmology. The restriction to Chern–Simons gravity sets the values of the coupling constants such that exponential scale factors with complex exponents are forbidden. Then we generalized the prescription by including the spatial curvature k in the starting line element. We found the solutions of the corresponding equations of motion and showed that in the limit $k = 0$, only exponential solutions occur. In Table **IV** the cosmological solutions are outlined.

On the other hand, the study of Lovelock gravity in a spherically symmetric background, showed that several space-times can be obtained from the general $d+1$-dimensional Lagrangian (11.43). In five dimensions, analytic solutions can be found without imposing any relation between the components of the line element $P(r)$ and $Q(r)$. Similarly to the cosmological case, Ads-invariant Chern–Simons gravity can be obtained through an appropriate choice of the coupling constants. We extended the treatment to $d+1$ dimensions, and after imposing $P(r) = 1/Q(r)$ we found exact solutions. It is worth remarking that not all the solutions are physically relevant and, often, a careful selection of the free parameters is necessary. On the other hand, in some other cases, physical solutions cannot be recovered regardless of the values assumed by the integration constants. Spherically symmetric solutions are outlined in Table **V**.

The main purpose is to show that most of the theories coming from the general Lovelock action can be perfectly suitable theories in more than four dimensions. This is worth *e.g.* in view of the AdS/CFT correspondence, according to which the (d+1)-dimensional AdS-invariant Lovelock action should be equal to a d-dimensional CFT. Although in four dimensions GR perfectly fits the current observations at Solar System scales, in more dimensions there are several other candidates capable of describing a higher-dimensional universe.

In the final chapters, we showed that Chern–Simons theory can be also applied to electromagnetic theory and biological systems. The former application provides a massive wave equations, while the latter is based on a newly proposed theory which aims to treat DNA/RNA systems under the formalism of a topological theory.

Therefore, we tested the validity of the Chern-Simons theory as a method to schematize the DNA and RNA sequences. After briefly overviewing the general formalism, we studied the KRAS human gene, introducing some known mutations with the aim to compare the reference sequence with the mutated one. To develop the formalism, the nitrogen bases can be recast as quaternion fields, which combine in triplets as suggested by standard biology. These triplets form a three-dimensional space of configurations that can be described by means of the Chern-Simons three form. The expectation value of the only observable of the theory, the Wilson Loop, provides the so called Chern-Simons current. This latter gives a point-like information of the curvature of the genetic code, and can be used in order to compute the curvature spectrum of a given string. If some triplet of the initial sequence undergoes a change due *e.g.* to the replacement of a nitrogen basis, the point-like curvature changes accordingly. Therefore, the introduction of some mutations, yields a variation in the Chern-Simons current. The difference between the original and the mutated sequence can be used to infer where DNA-DNA (or DNA-RNA) interactions take place, or to predict the evolution probability towards a given configuration.

The result of the analysis of four different regions showed that common features are shared by all strings. Specifically, in almost all cases, to any peaked regions it corresponds a known mutation, which often yields a new slighter curvature spectrum with respect to the reference one. This can be theoretically motivated by physical considerations: the most peaked regions represent non-equilibrium points, which tend to evolve towards a stabler configuration of minimum energy. Consequently, it follows that the variations in the curvature spectrum must occur in those regions with higher curvature, and the effect of the variation must provide an alignment of curved points with the rest of the graph. This means that mutations avoid abrupt variations in the overall trend, making neighbors point to have similar values of current. As mentioned above, this happens for the best part of cases; however, DNA and RNA evolution can also depend on other extrinsic factors that cannot be taken into account by this method. The application of Chern–Simons theory to DNA system, indeed, only relies on the intrinsic curvature assumed by biological systems in the configuration space made of nitrogen bases.

The same prescription is then applied for more than 20K bases of COVID-19 virus, coming from different countries. Due to intrinsic peculiarities of RNA viruses, mutation, recombination, and re-assortment events are likely to occur, furtherly complicating genomic analyses, with divergence and recombination. Using a genome wide approach, Bobay et al. examined SARS-CoV-2 RNA, observing that recombination events account for approximately 40% of the polymorphisms, and gene exchange occurs only within strains of the same subgenus (Sarbecovirus). Moreover, frequent mutations tend to increase the likelihood of convergent mutations, in regions exposed to a major positive selection, causing sequences analogies that could be misinterpreted as recombination, and introduce new diversifying mutations which might accumulate, masking past recombination events. [230]. Genomic sequences of various SARS-CoV-2 strains from all the world are available on specific platforms (eg. GISAID) and increasingly monitored to timely track SARS-CoV-2 variants [231]; as large databases and systematic sequencing are required, irregular sampling in time and space represents a crucial limitation. Genetic diversity observed in SARS-CoV-2 populations across distinct geographic areas suggests independent events of SARS-CoV-2 introduction occurred, with few exceptions including China, being the original source, and, to a lesser extent, the early-involved Italy [232]. Quantitatively, amino acid mutations were found to be significantly higher in SARS-CoV-2 genomes in Europe (43.07%) than in Asia (38.08%) and North America (29.64%) [231].

Unfortunately, due to the large amount of amino acids, it is not possible to compute the overall trend of the curvature spectrum, which is necessary in order to adopt the same considerations as KRAS human gene. However, the analysis shows that most mutations occur where the slope of the Chern–Simons current takes extremely high values, which accounts for peaked regions in the curvature spectrum. This result can be explained considering the principle of minimum energy, according to which peaked points tend to evolve towards a stabler configuration. On the other hand, we note that a few mutations are also exhibited in correspondence of low current values. This may happen because some regions with low current values, namely having a small curvature and being rather flat, often are the border with areas with steep gradients of the current value denoting high curvature. Then, in some cases even regions with very small curvature may be affected by a close instability due to the presence of a current gradient nearby and this cause the occurrence of a mutation. By comparing low current variations listed in Figs. **13-23** with Tables **X-XIII**, it turns out that 47% of points which exhibit low current variations between mutated and original sequences, are unstable due to the presence of a current gradient nearby.

As a final remark, the importance of the applications here discussed is twofold. On the one hand, this method represents a first step aimed at comparing the Chern-Simons theory with other known DNA schematization methods. On the other hand, it tests the capability of a topological theory in schematizing DNA interactions/mutations.

Table IV: *Cosmological solutions in Lovelock and Chern–Simons gravity.*

Case	α_0	α_1	α_2	k	**Scale Factor**	Dimension
Einstein–de Sitter	$\neq 0$	$\neq 0$	0	$\neq 0$	$a(t) = \pm\sqrt{\dfrac{\alpha_1 k\, d(d-1)}{\alpha_0 - \alpha_0 \coth^2\left[\sqrt{\alpha_0}\left(c_1 + \frac{t}{\sqrt{\alpha_1 (d-1) d}}\right)\right]}}$	Any
				0	$a(t) = a_0 e^{\pm\sqrt{\frac{\alpha_0}{\alpha_1 d(d-1)}}\, t}$	
					$a(t) = a_0 e^{\pm\sqrt{\frac{\alpha_0}{12\alpha_1}}\, t}$	5
Pure Gauss–Bonnet	$\neq 0$	0	$\neq 0$	0	$a(t) = a_0 \exp\left\{\pm\sqrt[4]{\dfrac{-\alpha_0}{d(d-1)(d-2)(d-3)\alpha_2}}\, t\right\}$ $a(t) = b(t)$	Any
					$a(t) = a_0 \exp\left\{\pm\sqrt[4]{\dfrac{-\alpha_0}{24\alpha_2}}\, t\right\}$	5
	0	0	$\neq 0$	$\neq 0$	$a(t) = \sqrt{-k}\, t$	Any
				0	$a(t) \sim$ Const.	
					$a(t) \sim$ Const.	5
Lovelock	$\neq 0$	$\neq 0$	$\neq 0$	0	$a(t) = a_0 \exp\left\{\pm\sqrt{\dfrac{2\alpha_0}{\pm\sqrt{(d-1)d\left[\alpha_1^2 (d-1)d - 4\alpha_0\alpha_2 (d-3)(d-2)\right]} + \alpha_1 d(d-1)}}\, t\right\}$	Any
					$a(t) = a_0 \exp\left\{\pm\sqrt{\dfrac{\alpha_0}{\pm 2\sqrt{9\alpha_1^2 - 6\alpha_0\alpha_2} + 6\alpha_1}}\, t\right\}$	5
	0	$\neq 0$	$\neq 0$	$\neq 0$	$a(t) = \pm\sqrt{\dfrac{-\alpha_2 k (d-3)(d-2)}{\alpha_1}} \sinh\left[\sqrt{\alpha_1}\left(\dfrac{t}{\sqrt{\alpha_2 (d-3)(d-2)}} + c_1\right)\right]$	Any
				0	$a(t) = a_0 \exp\left\{\pm\sqrt{\dfrac{\alpha_1}{\alpha_2\,(d-2)\,(d-3)}}\, t\right\}$	
					$a(t) = a_0 \exp\left\{\pm\sqrt{\dfrac{\alpha_1}{2\alpha_2}}\, t\right\}$	5
Chern–Simons	$\dfrac{1}{5l^4}$	$\dfrac{2}{3l^2}$	1	0	$a(t) = a_0 \exp\left\{\pm\frac{1}{l}\sqrt{\frac{1}{6}\left(1 \pm \sqrt{\frac{7}{10}}\right)}\, t\right\}$	

Table V: *Spherically symmetric solutions in Lovelock and Chern–Simons gravity.*

Case	α_0	α_1	α_2	$P(r)^2,\quad Q^2(r)$	Dimension
Einstein–de Sitter	$\neq 0$	$\neq 0$	0	$P(r)^2 = 1/Q(r)^2 = 1 + \frac{c_1}{r^{d-2}} - \frac{\alpha_0}{\alpha_1 d(d-1)} r^2$	Any
				$P(r)^2 = 1/Q(r)^2 = 1 + \frac{c_1}{r^2} - \frac{\alpha_0}{12\alpha_1} r^2$	5
Pure Gauss–Bonnet	$\neq 0$	0	$\neq 0$	$P(r)^2 = 1/Q(r)^2 = 1 \pm \frac{1}{r^{d/2-2}} \sqrt{\frac{c_1}{6\alpha_2\binom{d-1}{d-4}} - r^d \frac{\alpha_0}{24\alpha_2\binom{d}{d-4}}}$	Any
				$P(r)^2 = 1/Q(r)^2 = 1 \pm \sqrt{1 + c_1 - \frac{\alpha_0}{24\alpha_2} r^4}$	5
				$P^2(r) = P_0^2 \sqrt{48\alpha_2\left(2c_1+1\right) \mp 4\sqrt{6}\sqrt{\alpha_2\left(24\alpha_2 + 96\alpha_2 c_1 - \alpha_0 r^4\right)} - \alpha_0 r^4} \qquad Q^2(r) = \frac{2\left(12\alpha_2 \pm \sqrt{6}\sqrt{\alpha_2\left(24\alpha_2 + 96\alpha_2 c_1 - \alpha_0 r^4\right)}\right)}{\alpha_0 r^4 - 96\alpha_2 c_1}$	
	0	0	$\neq 0$	$P(r)^2 = 1/Q(r)^2 = 1 + \frac{c_1}{r^{\frac{d}{2}-2}}$	Any
				Const.	5
Lovelock	$\neq 0$	$\neq 0$	$\neq 0$	$P(r)^2 = 1/Q(r)^2 = 1 \pm \frac{1}{r^{d/2-2}} \sqrt{\frac{c_1}{6\alpha_2\binom{d-1}{d-4}} + r^d \left(\frac{\alpha_1^2}{16\alpha_2^2\binom{d-2}{d-4}^2} - \frac{\alpha_0}{24\alpha_2\binom{d}{d-4}}\right)} - \frac{\alpha_1}{4\alpha_2\binom{d-2}{d-4}} r^2$	Any
				$P(r)^2 = 1/Q(r)^2 = 1 - \frac{\alpha_1 r^2}{4\alpha_2} \pm \frac{\sqrt{3r^4\left(3\alpha_1^2 - 2\alpha_0\alpha_2\right) + 6\alpha_2 c_1}}{12\alpha_2}$	5
				$P(r)^2 = P_0^2 \sqrt{-4c_1 + \alpha_0 r^4 - 12\alpha_1 r^2} \left[\frac{3\sqrt{r^4 uw + 2ux} + \sqrt{3}\left(\alpha_0 x - r^2 w(-3\alpha_1 + z)\right)}{-6\alpha_1 + \alpha_0 r^2 + 2z}\right]^{y/2} \left[\frac{6\alpha_1 - \alpha_0 r^2 + 2z}{3\sqrt{r^4 vw + 2vx} + \sqrt{3}\left(r^2 w(z + 3\alpha_1) + \alpha_0 x\right)}\right]^{s/2}$ $Q(r)^2 = \frac{-3\alpha_1 r^2 + 12\alpha_2 \pm \sqrt{3}\sqrt{8c_1\alpha_2 - 2\alpha_0\alpha_2 r^4 + 3\alpha_1^2 r^4 + 48\alpha_2^2}}{\frac{\alpha_0}{2} r^4 - 6\alpha_1 r^2 - 2c_1}$	
	0	$\neq 0$	$\neq 0$	$P(r)^2 = 1/Q(r)^2 = 1 \pm \frac{1}{r^{d/2-2}} \sqrt{\frac{c_1}{6\alpha_2\binom{d-1}{d-4}} + \frac{\alpha_1^2}{16\alpha_2^2\binom{d-2}{d-4}^2} r^d} - \frac{\alpha_1}{4\alpha_2\binom{d-2}{d-4}} r^2$	Any
				$P(r)^2 = 1/Q(r)^2 = 1 - \frac{\alpha_1 r^2}{4\alpha_2} \pm \frac{\sqrt{9\alpha_1^2 r^4 + 6\alpha_2 c_1}}{12\alpha_2}$	5
				$P(r)^2 = P_0^2 \left(2c_1 + \alpha_1 r^2\right) \sqrt{\frac{\sqrt{16\alpha_2(\alpha_2 + c_1) + \alpha_1^2 r^4} + \alpha_1 r^2}{8\alpha_2^2 + 2\alpha_2\left(\sqrt{16\alpha_2(\alpha_2 + c_1) + \alpha_1^2 r^4} + 4c_1\right) + c_1\left(\sqrt{16\alpha_2(\alpha_2 + c_1) + \alpha_1^2 r^4} - \alpha_1 r^2\right)}},$ $Q(r)^2 = \frac{-4\alpha_2 - \sqrt{16\alpha_2(\alpha_2 + c_1) + \alpha_1^2 r^4} + \alpha_1 r^2}{4c_1 + 2\alpha_1 r^2}$	
				$P(r)^2 = P_0^2 \sqrt{\frac{8\alpha_2^2 + 2\alpha_2\left(\sqrt{16\alpha_2(\alpha_2 + c_1) + \alpha_1^2 r^4} + 4c_1\right) + c_1\left(\sqrt{16\alpha_2(\alpha_2 + c_1) + \alpha_1^2 r^4} - \alpha_1 r^2\right)}{\sqrt{16\alpha_2(\alpha_2 + c_1) + \alpha_1^2 r^4} + \alpha_1 r^2}}$ $Q(r)^2 = \frac{-4\alpha_2 + \sqrt{16\alpha_2(\alpha_2 + c_1) + \alpha_1^2 r^4} + \alpha_1 r^2}{4c_1 + 2\alpha_1 r^2}$	
Chern–Simons	$\frac{1}{5l^4}$	$\frac{2}{3l^2}$	1	$P(r)^2 = 1/Q(r)^2 = 1 - \frac{r^2}{6l^2} \pm \sqrt{\frac{7r^4}{360l^4} + \frac{c_1}{24}}$	
				$P(r)^2 = P_0^2 \sqrt{-4c_1 + \frac{r^4}{5l^4} - \frac{8r^2}{l^2}} \left[\frac{\sqrt{3}\left(\frac{x}{5l^4} - r^2 w\left(-\frac{2}{l^2} + z\right)\right) + 3\sqrt{u\left(r^4 w + 2x\right)}}{\frac{r^2}{5l^4} - \frac{4}{l^2} + 2z}\right]^{y/2} \left[\frac{-\frac{r^2}{5l^4} + \frac{4}{l^2} + 2z}{\sqrt{3}\left(\frac{x}{5l^4} + r^2 w\left(z + \frac{2}{l^2}\right)\right) + 3\sqrt{v\left(r^4 w + 2x\right)}}\right]^{s/2}$ $Q(r)^2 = \frac{\sqrt{24c_1 + \frac{14r^4}{5l^4} + 144} - \frac{2r^2}{l^2} + 12}{-2c_1 + \frac{r^4}{10l^4} - \frac{4r^2}{l^2}}$	

Epilogue

In this thesis we investigate applications of topological invariants in physics, with particular attention to gravity and biological systems. As the former is more standard and straightforward, the latter represents an interesting task which deserves to be further developed in forthcoming works. Summing up, we first considered cosmological and spherically symmetric applications of the Gauss–Bonnet topological invariants, starting from $f(\mathcal{G})$ gravity up to non-local and scalar-tensor theories. In the second part of this work we dealt with Chern–Simons theory, finding out exact higher dimensional black holes and cosmological solutions. Then we showed that Chern–Simons theories can be also considered in the framework of other fundamental interactions, such as electromagnetic theory in odd dimensions. Specifically, a massive wave equation arises form this latter application and the gauge invariance is preserved. Finally, we discussed an unconventional application, that is the application to biological systems. Relying on previous works [206, 207], we applied the formalism of such topological field theory to DNA/RNA systems. First we studied the KRAS human gene, comparing original sequences with mutated ones. Mutations with higher impact was then selected in the database BioMuta. The analysis showed that mutations occur in correspondence of peaked regions of the curvature spectrum. In particular, mutations aim to avoid abrupt variations of the curvature, and the overall spectrum goes towards a state of higher equilibrium. From this point of view, the analogy with gravitational interaction (and, in general, with all physics branches) is straightforward. Then we considered the application to SARS-CoV-2 strings, comparing variants of COVID-19 coming from different countries. The sequencing of more than 20K bases showed that most of mutations occur where the slope of the Chern–Simons current takes extremely high values, which accounts for peaked regions in the curvature spectrum. This result can be explained considering the principle of minimum energy, according to which peaked points tend to evolve towards a stabler configuration. On the other hand, note that few mutations are exhibited in correspondence of low currents. This is due to the influence of neighbors points with higher curvature, which influence their surroundings and cause the mutation. Predictions inferred in the study of KRAS human gene are confirmed even in this case.

Generally, in this work we set out to show that the features of topological invariants can be used in several contexts and can provide important results in different frameworks. Relying on the topology of the system, regardless of the point-like geometry, topological invariants are of particular interest for all those systems which manifest a non-trivial structure. This is the case *e.g.* of modified theories of gravity, which often exhibit hardly solvable field equations. Though exact solutions often cannot be found analytically, topological theories can be helpful to settle issues suffered by GR in high regimes.

In future works we aim to consider astrometric and cosmological data provided by experiments, in order to constraint the free parameters occurring in Gauss–Bonnet and Chern-Simons gravity. This can be pursued by comparing exact solutions coming from the above theories, with *e.g.* the fundamental plane of galaxies, Plank data of the early Universe evolution, Mass-Radius correlation of neutron stars, late-time Universe expansion and so on. Furthermore, other topological terms can be investigated in this context, as well as Kretschmann scalar or Pontryagin density. The main purpose is to test whether the applications to modified gravity can reduce the complexity of the equations of motion. Regarding Chern–Simons gravity, other extensions can be considered, including torsion or higher order terms. The application to biological system, finally, must be further developed by focusing on other DNA regions. From this point of view, another perspective is to compare the point by point curvature of sequences in order to study the docking among biological large molecules. Docking might occur in those regions having similar values of curvature, that is Chern–Simons current.

In order to corroborate this method and to study other possible implications, we aim to make a comparison with some standard modeling techniques, based on a probabilistic vision of the biomolecules interaction. The intrinsic probabilistic aspect of these techniques is necessary in order to handle systems made of a huge amount of particles. As the evolution of a particle can be well described by quantum or classical mechanics, many-bodies interactions are not such straightforward. Therefore, a statistical view of the biomolecule configurations is needed to infer the evolution of the system. Merging the two different approaches (the one lying behind Chern–Simons Current with the more conventional ones coming from bioinformatic), can implement the nowadays knowledge of the biological scenario. On the one hand, using topological field theories to describe DNA configuration can provide the exact position in which the mutation takes place. On the other hand, once the position of the mutation is identified, bioinformatics can predict the probabilistic evolution and the clinical impact of such mutation. Moreover, also the docking between macro-molecules can be further developed, since the probabilistic vision provided by bioinformatic techniques can be combined with the prediction given by topological field theories. In this regard, the final purpose is to develop a coherent scheme capable of predicting where and when a disease could manifest.

Appendices

A

Sequences used in Chap. 13

A.1 Mutations in KRAS Sequence

KRAS HUMAN

SOURCE FOR THE SEQUENCES: Genome Browser

SOURCE FOR THE MUTATIONS: BioMuta

ORIGINAL SEQUENCE 1: Chr12: 25,245,274 - 25,245,384

CUCUAUUGUUGGAUCAUAUUCGUCCACAAAAUGAUUCUGAAUUAGCUG
UAUCGUCAAGGCACUCUUGCCUACGCCACCAGCUCCAACUACCACAAG
UUUAUAUUCAGUCAU

First set of mutations (Fig. 7)

Table VI: *Comparison between reference triplet and mutated one in KRAS, Chr12: 25,245,274 - 25,245,384.*

Position	Ref. Base	Mutation	Ref. Amino	Mutation	Current Variation (%)	Initial CS current	Mutated CS current
25,245,279	UAU	UAA	Y	Stop	-7.944	0.0214	0.0197
25,245,294	UUC	UUG	F	L	-42.260	0.5	0.2887
25,245,314	AAU	AUU	N	I	664.968	0.0157	0.1201
25,245,332	GGC	GAC	G	D	54.023	0.0087	0.0134
25,245,342	GCC	GCU	A	A	4.898	0.0245	0.0257
25,245,350	ACC	AAC	T	N	-49.164	0.0299	0.0152
25,245,365	CAC	CCC	H	P	115.429	0.0175	0.0377
25,245,370	UUU	AUU	F	I	-83.015	0.7071	0.1201

CUCUAAUGUUGGAUCAUAUUGGUCCACAAAAUGAUUCUGAUUUAGCUG
UAUCGUCAAGACACUCUUGCUUACGCCAACAGCUCCAACUACCCCAAG
AUUAUAUUCAGUCAU

Second set of mutations (Fig. 8)

Table VII: *Comparison between reference triplet and mutated one in KRAS, Chr12: 25,245,274 - 25,245,384.*

Position	Ref. Base	Mutation	Ref. Amino	Mutation	Current Variation (%)	Initial CS current	Mutated CS current
25,245,294	UUC	UUG	F	L	-42.26	0.5	0.2887
25,245,314	AAU	AUU	N	I	664.968	0.0157	0.1201
25,245,321	CUG	CUU	L	L	67.8	0.1382	0.2319
25,245,332	GGC	GAC	G	D	54.023	0.0087	0.0134
25,245,338	CUU	CCU	L	P	-82.665	0.2319	0.0402
25,245,350	ACC	AAC	T	N	-49.164	0.0299	0.0152
25,245,365	CAC	CCC	H	P	115.429	0.0175	0.0377
25,245,370	UUU	GUU	F	V	-89.266	0.7071	0.0759
25,245,378	UUC	UUU	F	F	41.42	0.5	0.7071

CUCUAUUGUUGGAUCAUAUUGGUCCACAAAAUGAUUCUGAUUUAGCUU
UAUCGUCAAGACACUCCUGCCUACGCCAACAGCUCCAACUACCCCAAG
GUUAUAUUUAGUCAU

ORIGINAL SEQUENCE 2: Chr12: 25,215,468 - 25,215,560

CACACAGCCAGGAGUCUUUUCUUCUUUGCUGAUUUUUUUCAAUCUGUA
UUGUCGGAUCUCCCUCACCAAUGUAUAAAAAGCAUCCUCCACUCU

Third set of mutations (Fig. 9)

Table VIII: *Comparison between reference triplet and mutated one in KRAS, Chr12: 25,215,468 - 25,215,560.*

Position	Ref. Base	Mutation	Ref. Amino	Mutation	Current Variation (%)	Initial CS current	Mutated CS current
25,215,485	CUU	CUG	L	L	-40.405	0.2319	0.1382
25,215,501	UUU	GUU	F	V	-89.266	0.7071	0.0759
25,215,520	UCG	UUG	S	L	572.96	0.0429	0.2887
25,215,529	CCU	CUU	P	L	476.866	0.0402	0.2319
25,215,539	UGU	UGC	C	C	-3.279	0.0122	0.0118
25,215,547	AGC	AUC	S	I	1001.042	0.0096	0.1057
25,215,559	UCU	UGU	S	C	-77.154	0.0534	0.0122

CACACAGCCAGGAGUCUGUUCUUCUUUGCUGAUGUUUUUCAAUCUGUA
UUGUUGGAUCUCCUUCACCAAUGCAUAAAAAUCAUCCUCCACUGU

ORIGINAL SEQUENCE 3: Chr12: 25,227,263-25,227,379

AGUAUUAUUUAUGGCAAAUACACAAAGAAAGCCCUCCCCAGUCCUCAU
GUACUGGUCCCUCAUUGCACUGUACUCCUCUUGACCUGCUGUGUCGAG
AAUAUCCAAGAGACAGGUUUC

Fourth set of mutations (Fig. 10)

Table IX: *Comparison between reference triplet and mutated one in KRAS, Chr12: 25,227,263-25,227,379.*

Position	Ref. Base	Mutation	Ref. Amino	Mutation	Current Variation (%)	Initial CS current	Mutated CS current
25,227,272	UAU	CAU	Y	H	-14.953	0.0214	0.0182
25,227,306	CCU	CAU	P	H	-54.726	0.0402	0.0182
25,227,312	GUA	GCA	V	A	-62.857	0.063	0.0234
25,227,318	GUC	GCC	V	A	-64.493	0.069	0.0245
25,227,321	CCU	CAU	P	H	-54.726	0.0402	0.0182
25,227,334	GUA	GUG	V	V	-8.095	0.063	0.0579
25,227,338	CUC	AUC	L	I	-44.746	0.1913	0.1057
25,227,341	UUG	GUG	L	V	-79.945	0.2887	0.0579
25,227,355	GUC	GUU	V	V	10	0.069	0.0759
25,227,373	ACA	ACG	T	T	-4.93	0.0284	0.027

AGUAUUAUUCAUGGCAAAUACACAAAGAAAGCCCUCCCCAGUCAUCAU
GCACUGGCCCAUCAUUGCACUGUGCUCAUCGUGACCUGCUGUGUUGAG
AAUAUCCAAGAGACGGGUUUC

Mutations in SARS-CoV-2 Sequences

Comparison Between Original Sequences and Mutated Ones

The pie graphs of **Figs. 13-23** show the percentage of large and small values of current variations; large variations ($> 100\% \vee < -100\%$) are labeled by light blue squares, small variations ([-11%;11%]) by solid red, other intermediate values by grey lines.

Position	19A Triplet	19A Current	19B Triplet	19B Current	Current Variation (%)
2840	***AGC***	0.0096	***AGT***	0.0098	2
3607	TTT	0.7071	TTG	0.2887	-59
5829	CTG	0.1382	TTG	0.2887	109
5866	ATG	0.0841	GTG	0.0579	-31
5933	TCT	0.0534	TTT	0.7071	1224
9294	TTT	0.7071	TTC	0.5	-29
9697	***TAT***	0.0214	***TAC***	0.0205	-4
9762	TAC	0.0205	CAC	0.0175	-15

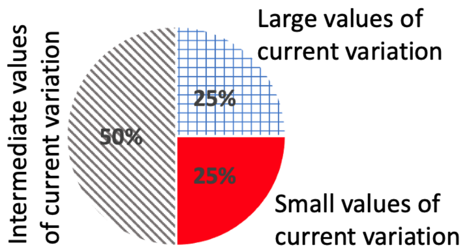

Figure 13: *Comparison between 19A and 19B sequences, with related Chern-Simons current and percentage variation.*

Position	19A Triplet	19A Current	20A Triplet	20A Current	Current Variation (%)
925	TTC	0.5	TTT	0.7071	41
3607	TTT	0.7071	TTG	0.2887	-59
3840	AAA	0.0147	AAG	0.0142	-3
4716	***CTA***	0.1612	***TTA***	0.3717	131
7714	***GAT***	0.0138	***GGT***	0.0089	-36
8847	CAG	0.0163	GAG	0.0126	-23
9697	TAT	0.0214	TAC	0.0205	-4
9762	TAC	0.0205	CAC	0.0175	-15

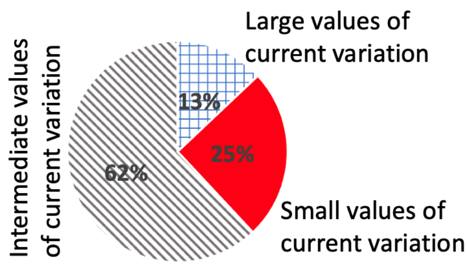

Figure 14: *Comparison between 19A and 20A sequences, with related Chern-Simons current and percentage variation.*

Position	20A Triplet	20A Current	20B Triplet	20B Current	Current Variation (%)
3840	AAG	0.0142	AAA	0.0147	4
6590	CCC	0.0377	CTC	0.1913	407
8036	GAC	0.0134	GAT	0.0138	3
8847	GAG	0.0126	CAG	0.0163	29
9516	GGC	0.0087	GGT	0.0089	2
9540	***AGG***	0.0091	***AAA***	0.0147	62
9541	GGA	0.0085	CGA	0.0103	21

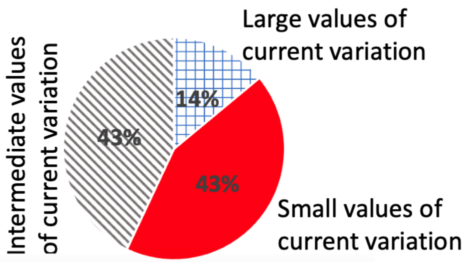

Figure 15: *Comparison between 20A and 20B sequences, with related Chern-Simons current and percentage variation.*

Position	20A Triplet	20A Current	20C Triplet	20C Current	Current Variation (%)
266	***ACC***	0.0299	***ATC***	0.1057	254
2130	GCT	0.0257	GTT	0.0759	195
3840	AAG	0.0142	AAA	0.0147	4
6098	ACA	0.0284	ATA	0.0939	231
6161	ACG	0.027	ATG	0.0841	211
6773	TGA	0.0115	TTA	0.3717	3132
8434	***AGA***	0.0093	***ATA***	0.0939	910
8437	CCA	0.0354	CTA	0.1612	355
8847	GAG	0.0126	CAG	0.0163	29

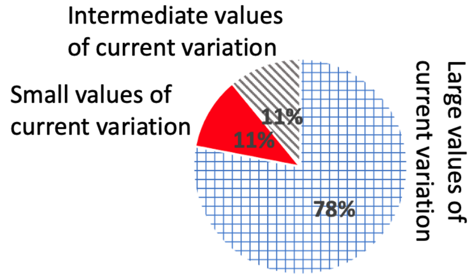

Figure 16: *Comparison between 20A and 20C sequences, with related Chern-Simons current and percentage variation.*

Position	20A Triplet	20A Current	20E Triplet	20E Current	Current Variation (%)
61	GTT	0.0759	GTC	0.069	-9
2008	ACC	0.0299	ACT	0.0316	6
3840	AAG	0.0142	AAA	0.0147	4
6800	CTA	0.1612	CCA	0.0354	-78
6998	CGT	0.0109	CCT	0.0402	269
7069	TGA	0.0115	TTA	0.3717	3132
7322	***GCT***	0.0257	***GTT***	0.0759	195
9546	AGA	0.0093	AAA	0.0147	58
9557	GCT	0.0257	GTT	0.0759	195
9708	GAC	0.0134	GAT	0.0138	3
9795	GTA	0.063	TTA	0.3717	490

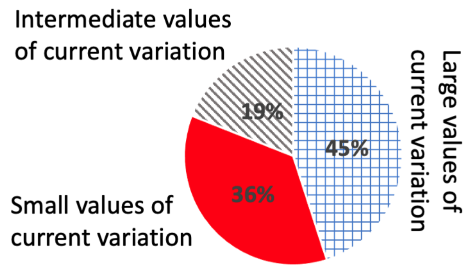

Figure 17: *Comparison between 20A and 20E sequences, with related Chern-Simons current and percentage variation.*

Position	20B Triplet	20B Current	20D Triplet	20D Current	Current Variation (%)
1247	***ACT***	0.0316	***ATT***	0.1201	280
1306	AAG	0.0142	AAT	0.0157	11
2148	AAC	0.0152	AAT	0.0157	3
3279	***GGT***	0.0089	***AGT***	0.0098	10
3892	TGT	0.0122	TGC	0.0118	-3
4425	***ACA***	0.0284	***ATA***	0.0939	231
4993	ACA	0.0284	ATA	0.0939	231
6299	ATT	0.1201	ACT	0.0316	-74
6479	TCA	0.046	TCG	0.0429	-7
6590	CTC	0.1913	CCC	0.0377	-80
7120	ACC	0.0299	ATC	0.1057	254
7823	***ACC***	0.0299	***ACT***	0.0316	6
8036	GAT	0.0138	GAC	0.0134	-3
8081	CTT	0.2319	CTC	0.1913	-18
9516	GGT	0.0089	GGC	0.0087	-2
9571	ATG	0.0841	ATT	0.1201	43

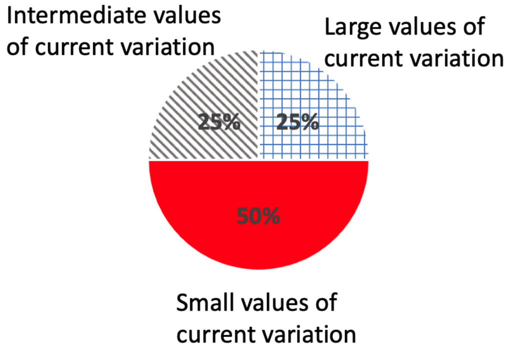

Figure 18: *Comparison between 20B and 20D sequences, with related Chern-Simons current and percentage variation.*

Position	20B Triplet	20B Current	20F Triplet	20F Current	Current Variation (%)
301	***ATT***	0.1201	***TTT***	0.7071	489
2426	***ACT***	0.0316	***ACC***	0.0299	-5
5462	***CGC***	0.0106	***CTC***	0.1913	1705
6098	***ACA***	0.0284	***ATA***	0.0939	231
6590	CTC	0.1913	CCC	0.0377	-80
7577	***AGC***	0.0096	***AAC***	0.0152	58
7713	***CAG***	0.0163	***CAA***	0.0169	4
8036	GAT	0.0138	GAC	0.0134	-3
9516	GGT	0.0089	GGC	0.0087	-2

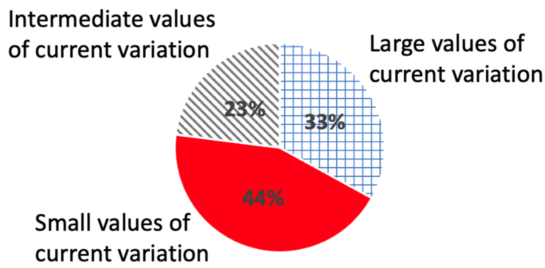

Figure 19: *Comparison between 20B and 20F sequences, with related Chern-Simons current and percentage variation.*

Position	20B Triplet	20B Current	20I Triplet	20I Current	Current Variation (%)
217	TCC	0.0495	TCT	0.0534	8
1002	ACT	0.0316	ATT	0.1201	280
1668	ATT	0.1201	ATA	0.0939	-22
1709	GCT	0.0257	GAT	0.0138	-46
1908	TTC	0.5	TTT	0.7071	41
2231	ATA	0.0939	ACA	0.0284	-70
4805	***CCG***	0.0334	***CTG***	0.1382	314
5006	***ACC***	0.0299	***ATC***	0.1057	254
5305	CTT	0.2319	CCT	0.0402	-83
5785	AGC	0.0096	GGC	0.0087	-9
6590	CTC	0.1913	CCC	0.0377	-80
7170	GTC	0.069	ATC	0.1057	53
7601	***AAT***	0.0157	***TAT***	0.0214	36
7670	GCT	0.0257	GAT	0.0138	-46
7781	CCT	0.0402	CAT	0.0182	-55
7816	ACA	0.0284	ATA	0.0939	231
8036	GAT	0.0138	GAC	0.0134	-3
8082	TCA	0.046	GCA	0.0234	-49
8218	GAC	0.0134	CAC	0.0175	31
9237	TCA	0.046	TTA	0.3717	708
9262	TAG	0.0189	TAT	0.0214	13
9283	GTA	0.063	GTG	0.0579	-8

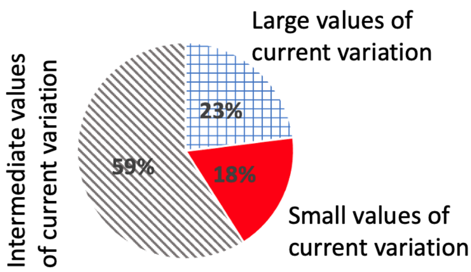

Figure 20: *Comparison between 20B and 20I sequences, with related Chern-Simons current and percentage variation.*

Position	20C Triplet	20C Current	20G Triplet	20G Current	Current Variation (%)
220	CTG	0.1382	TTG	0.2887	109
555	ACT	0.0316	ACC	0.0299	-5
1323	ACA	0.0284	ACC	0.0299	5
1978	CCC	0.0377	CCT	0.0402	7
2130	GTT	0.0759	GCT	0.0257	-66
3353	*CTT*	0.2319	*TTT*	0.7071	205
4236	AAC	0.0152	AAT	0.0157	3
5168	TAG	0.0189	TAT	0.0214	13
6054	*CTA*	0.1612	*CTG*	0.1382	-14
6092	TTG	0.2887	TTT	0.7071	145
6098	ATA	0.0939	ACA	0.0284	-70
6129	CTG	0.1382	TTG	0.2887	109
6161	ATG	0.0841	ACG	0.027	-68
6773	TTA	0.3717	TGA	0.0115	-97
7331	ATA	0.0939	ATT	0.1201	28
7620	GCA	0.0234	TCA	0.046	97
8437	CTA	0.1612	CCA	0.0354	-78
8549	*GTG*	0.0579	*TTG*	0.2887	399
8556	TTT	0.7071	TTC	0.5	-29
8897	GAT	0.0138	TAT	0.0214	55
9234	*GTC*	0.069	*GTT*	0.0759	10
9404	CCT	0.0402	TCT	0.0534	33
9536	*CCA*	0.0354	*CTA*	0.1612	355
9726	*CAG*	0.0163	*CTG*	0.1382	748

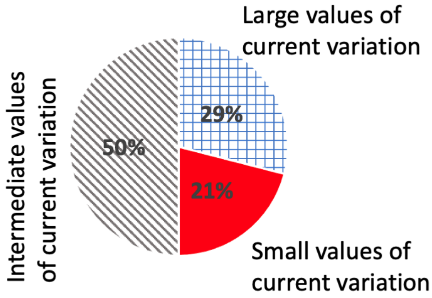

Figure 21: *Comparison between 20C and 20G sequences, with related Chern-Simons current and percentage variation.*

Position	20C Triplet	20C Current	20H Triplet	20H Current	Current Variation (%)
810	ACA	0.0284	ACT	0.0316	11
1656	AAG	0.0142	AAT	0.0157	11
2047	CCA	0.0354	CTA	0.1612	355
2130	GTT	0.0759	GCT	0.0257	-66
2454	GTT	0.0759	GTC	0.069	-9
2597	AAT	0.0157	AGT	0.0098	-38
3354	AAG	0.0142	AGG	0.0091	-36
3378	GTG	0.0579	GTT	0.0759	31
3451	GAC	0.0134	GAT	0.0138	3
5475	GTA	0.063	GTG	0.0579	-8
6098	ATA	0.0939	ACA	0.0284	-70
6161	ATG	0.0841	ACG	0.027	-68
6398	CGT	0.0109	CAT	0.0182	67
6773	TTA	0.3717	TGA	0.0115	-97
7118	CTT	0.2319	TTT	0.7071	205
7167	GCT	0.0257	GTT	0.0759	195
7180	GAT	0.0138	GCT	0.0257	86
7315	GAT	0.0138	GGT	0.0089	-36
7517	AAG	0.0142	AAT	0.0157	11
7584	*GAA*	0.0129	*AAA*	0.0147	14
7601	*AAT*	0.0157	*TAT*	0.0214	36
7801	GCA	0.0234	GTA	0.063	169
8437	CTA	0.1612	CCA	0.0354	-78
8548	CAG	0.0163	TAG	0.0189	16
8732	CTG	0.1382	TTG	0.2887	109
9331	CAT	0.0182	TAT	0.0214	18
9542	ACT	0.0316	ATT	0.1201	280
9809	AGT	0.0098	ATT	0.1201	1126

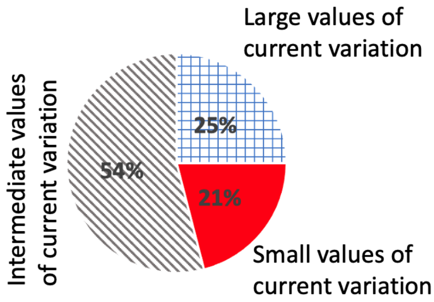

Figure 22: *Comparison between 20C and 20H sequences, with related Chern-Simons current and percentage variation.*

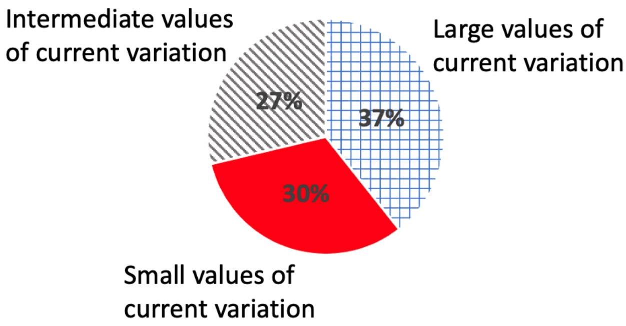

Figure 23: *Details of the whole set of mutations occurring in all sequences.*

Chern-Simons Current Variations in the Surroundings of Expected Mutations

Table X: Chern-Simons currents and their corresponding percentage variations (with respect to the surrounding points) in 19A sequence of SARS-CoV-2 virus. Large values are highlighted in red.

19A Mutations				
Position	19A Triplet	19A Current	Variation (%) with respect to previous position	Variation (%) with respect to subsequent position
925	TTC	0.5	836	-96
2840	***AGC***	0.0096	6213	70
3607	TTT	0.7071	0	-97
3840	AAA	0.0147	-97	1201
4716	***CTA***	0.1612	821	-90
5829	CTG	0.1382	362	68
5866	ATG	0.0841	436	-62
5933	TCT	0.0534	299	596
7714	***GAT***	0.0138	-15	450
8847	CAG	0.0163	-37	889
9294	TTT	0.7071	412	-96
9697	TAT	0.0214	-9	-31
9762	TAC	0.0205	53	39

Table XI: Chern-Simons currents and their corresponding percentage variations (with respect to the surrounding points) in 20A sequence of SARS-CoV-2 virus. Large values are highlighted in red.

20A Mutations				
Position	**20A Triplet**	**20A Current**	**Variation (%) with respect to previous position**	**Variation (%) with respect to subsequent position**
61	GTT	0.0759	772	280
266	***ACC***	0.0299	123	1572
2008	ACC	0.0299	90	-63
2130	GCT	0.0257	0	195
3840	AAG	0.0142	-97	1247
6098	ACA	0.0284	-51	-38
6161	ACG	0.027	-78	98
6590	CCC	0.0377	-55	67
6773	TGA	0.0115	-19	1302
6800	CTA	0.1612	927	-57
6998	CGT	0.0109	-87	67
7069	TGA	0.0115	-88	6049
7322	***GCT***	0.0257	-40	1346
8036	GAC	0.0134	-21	243
8434	***AGA***	0.0093	-98	141
8437	CCA	0.0354	-85	-58
8847	CAG	0.0163	-51	1179
9516	GGC	0.0087	0	13
9540	***AGG***	0.0091	-7	272
9546	AGA	0.0093	-64	804
9557	GCT	0.0257	-89	1023
9708	GAC	0.0134	-6	10
9795	GTA	0.063	273	-78

Table XII: Chern-Simons currents and their corresponding percentage variations (with respect to the surrounding points) in 20B sequence of SARS-CoV-2 virus. Large values are highlighted in red.

20B Mutations				
Position	**20B Triplet**	**20B Current**	**Variation (%) with respect to previous position**	**Variation (%) with respect to subsequent position**
217	TCC	0.0495	-83	-74
301	***ATT***	0.1201	1191	-91
1002	ACT	0.0316	0	280
1247	***ACT***	0.0316	145	-55
1306	AAG	0.0142	-3	81
1668	ATT	0.1201	125	-88
1709	GCT	0.0257	0	-41
1908	TTC	0.5	2236	-94
2148	AAC	0.0152	-93	-3
2231	ATA	0.0939	-22	28
2426	***ACT***	0.0316	11	280
3279	***GGT***	0.0089	-29	37
3892	TGT	0.0122	-28	466
4425	***ACA***	0.0284	33	-71
4805	***CCG***	0.0334	120	89
4993	ACA	0.0284	0	-57
5006	***ACC***	0.0299	-84	1143
5305	CTT	0.2319	717	-93
5462	***CGC***	0.0106	-28	334
5785	AGC	0.0096	-90	3772
6098	***ACA***	0.0284	-51	-38
6299	ATT	0.1201	586	15
6479	TCA	0.046	-71	708
6590	CTC	0.1913	127	-67
7120	ACC	0.0299	5	-69
7170	GTC	0.069	279	-23
7577	***AGC***	0.0096	8	196
7601	***AAT***	0.0157	-50	-43
7670	GCT	0.0257	-79	-48
7713	***CAG***	0.0163	-24	-45
7781	CCT	0.0402	-25	-75
7816	ACA	0.0284	-25	-45
7823	***ACC***	0.0299	-61	-5
8036	GAT	0.0138	-19	233
8081	CTT	0.2319	119	-80
8082	TCA	0.046	-80	-76
8218	GAC	0.0134	-53	13
9237	TCA	0.046	124	-38
9262	TAG	0.0189	97	-22
9283	GTA	0.063	37	-71
9516	GGT	0.0089	2	10
9571	ATG	0.0841	472	-37

Table XIII: Chern-Simons currents and their corresponding percentage variations (with respect to the surrounding points) in 20C sequence of SARS-CoV-2 virus. Large values are highlighted in red.

20C Mutations				
Position	20C Triplet	20C Current	Variation (%) with respect to previous position	Variation (%) with respect to subsequent position
220	CTG	0.1382	717	-90
555	ACT	0.0316	145	-19
810	ACA	0.0284	-55	-46
1323	ACA	0.0284	-20	-53
1656	AAG	0.0142	-3	-21
1978	CCC	0.0377	33	42
2047	CCA	0.0354	141	95
2130	GTT	0.0759	195	0
2454	GTT	0.0759	522	-79
2597	AAT	0.0157	-79	72
3353	***CTT***	0.2319	1533	-94
3354	AAG	0.0142	-94	435
3378	GTG	0.0579	26	542
3451	GAC	0.0134	-82	-32
4236	AAC	0.0152	-96	0
5168	TAG	0.0189	-73	-41
5475	GTA	0.063	-25	358
6054	***CTA***	0.1612	821	-42
6092	TTG	0.2887	3004	-93
6098	ATA	0.0939	62	-81
6129	CTG	0.1382	931	-93
6161	ATG	0.0841	-30	-37
6398	CGT	0.0109	-93	478
6773	TTA	0.3717	2518	-57
7118	CTT	0.2319	1377	-88
7167	GCT	0.0257	41	265
7180	GAT	0.0138	-98	10
7315	GAT	0.0138	27	1286
7331	ATA	0.0939	165	-91
7517	AAG	0.0142	67	746
7584	***GAA***	0.0129	-83	-31
7601	AAT	0.0157	-50	-43
7620	GCA	0.0234	29	51
7801	GCA	0.0234	163	-45
8437	CTA	0.1612	-30	-91
8548	CAG	0.0163	-93	255
8549	***GTG***	0.0579	255	45
8556	TTT	0.7071	339	-80
8732	CTG	0.1382	-72	-24
8897	GAT	0.0138	13	191
9234	***GTC***	0.069	143	22
9331	CAT	0.0182	-97	786
9404	CCT	0.0402	-92	-74
9542	ACT	0.0316	272	69
9536	***CCA***	0.0354	12	-75
9809	AGT	0.0098	-91	491

Table XIV: List of amino acids of 19A sequence in the spike protein, with corresponding positions, Chern-Simons currents and their variations with respect to surrounding positions. Listed amino acid are those involved in forming the tertiary structure, according to Ref. [229].

Position	19A		Variation (%) with respect to	Variation (%) with respect to
	Triplets	Currents	the previous position	the subsequent position
17	AAT	0.0157	-79	1377
61	AAT	0.0157	-68	383
74	AAT	0.0157	-47	-43
122	AAC	0.0152	-3	69
149	AAC	0.0152	0	-3
165	AAT	0.0157	0	-25
234	AAC	0.0152	-87	595
282	AAT	0.0157	22	-46
331	AAT	0.0157	-61	665
343	AAC	0.0152	-98	61
603	AAT	0.0157	-45	101
616	AAC	0.0152	-80	-22
657	AAC	0.0152	-78	0
709	AAT	0.0157	-71	-3
717	AAT	0.0157	-45	4404
801	AAT	0.0157	-98	4404
1074	AAC	0.0152	7	3189
1098	AAT	0.0157	-66	-45
1134	AAC	0.0152	-78	0
1158	AAT	0.0157	11	16
1173	AAT	0.0157	-87	64
1194	AAT	0.0157	-96	-18

Table XV: Interaction between surrounding amino acids with respect to the tertiary structure, with corresponding Chern-Simons currents and percentage variations.

Contact position	First amino	Second amino	First amino current	Second amino current	Percentage Variation
74-149	AAT	AAC	0.0157	0.0152	-3.185
165-234	AAT	AAC	0.0157	0.0152	-3.185
122-149	AAC	AAC	0.0152	0.0152	0
234-343	AAC	AAC	0.0152	0.0152	0
282-603	AAT	AAT	0.0157	0.0157	0
331-343	AAT	AAC	0.0157	0.0152	-3.185
603-657	AAT	AAC	0.0157	0.0152	-3.185
657-1074	AAC	AAC	0.0152	0.0152	0
1098-1134	AAT	AAC	0.0157	0.0152	-3.185
122-343	AAC	AAC	0.0152	0.0152	0

B

Noether Symmetry Approach

The Noether symmetry approach is widely used to deal with cosmologies coming from different theories of gravity. For example, in [153, 233, 234, 235, 236, 237, 238], the approach is applied to $f(R)$ gravity. In [239, 240, 241, 242, 243, 244], extended $f(T)$ TEGR models have been discussed in cosmology and spherical symmetry. In [74, 245], the Noether theorem has been used to study $f(R, \mathcal{G})$ dynamics. Scalar-tensor actions have been studied in [167, 246, 247, 248, 249], where the coupling and the potential are found by symmetry considerations. The basic formulation of the Noether theorem for dynamical systems is in turn presented in [250, 251, 252], where the foundations of the approach are outlined.

Noether symmetries are a subclass of Lie point symmetries, applied to dynamical systems described by a Lagrangian density. Noether's theorem affirms that if

$$\mathcal{X} = \xi \partial_t + \eta^i \partial_{q^i} \,, \tag{B.1}$$

is a generator of infinitesimal point transformations, then the Lagrangian density is invariant under $\mathcal{X}$ if and only if

$$X^{[1]} \mathcal{L} + \dot{\xi} \mathcal{L} = \dot{g} \,, \tag{B.2}$$

with g being a function of the affine parameter t and the generalized coordinates q^i and

$X^{[1]}$ is the first prolongation of Noehter's vector, defined as:

$$X^{[1]} = \xi\frac{\partial}{\partial t} + \eta^i\frac{\partial}{\partial q^i} + \eta^{i\,[1]}\frac{\partial}{\partial \dot{q}^i} = \xi\frac{\partial}{\partial t} + \eta^i\frac{\partial}{\partial q^i} + (\dot{\eta}^i - \dot{q}^i\dot{\xi})\frac{\partial}{\partial \dot{q}^i}\,. \tag{B.3}$$

Generally, for Lagrangian involving higer-order derivatives, it is possible to use the n-prolongation of the Noether vector, which has the form

$$X^{[n]} = \xi\frac{\partial}{\partial t} + \eta^i\frac{\partial}{\partial q^i} + \eta^{i\,[1]}\frac{\partial}{\partial \dot{q}^i} + ... + \eta^{i\,[n]}\frac{\partial}{\partial \frac{d^n q^i}{dt^n}}\,, \tag{B.4}$$

with

$$\eta^{i\,[n]} = \frac{d\eta^{i\,[n-1]}}{dt} - \dot{\xi}\frac{d^n q^i}{dt^n}\,. \tag{B.5}$$

Here the parameter t represents any affine parameter and it is chosen depending on the symmetry of the space-time.

It is easy to extend Eqs. (B.2) and (B.3) to a general Lagrangian density $\mathscr{L}$ that depends on x^μ parameters, such as the space-time coordinates. Specifically, the prolongation (B.3) becomes

$$X^{[1]} = \xi^\mu\partial_\mu + \eta^i\frac{\partial}{\partial q^i} + (\partial_\mu\eta^i - \partial_\mu q^i\partial_\nu\xi^\nu)\frac{\partial}{\partial(\partial_\mu q^i)} \tag{B.6}$$

and Noether's theorem (B.2)

$$X^{[1]}\mathscr{L} + \partial_\mu\xi^\mu\mathscr{L} = \partial_\mu g^\mu\,. \tag{B.7}$$

In more details, let us consider the following transformation

$$\mathscr{L}(x^\mu, q^i, \partial_\mu q^i) \to \mathscr{L}(\tilde{x}^\mu, \tilde{q}^i, \partial_\mu\tilde{q}^i)\,, \tag{B.8}$$

where transformation of x^μ and q^i are given by:

$$\begin{cases} \tilde{x}^\mu = x^\mu + \epsilon\xi^\mu(x^\mu, q^i) + O(\epsilon^2)\,, \\ \tilde{q}^i = q^i + \epsilon\eta^i(x^\mu, q^i) + O(\epsilon^2)\,. \end{cases} \tag{B.9}$$

The derivatives of the generalized coordinates q^i transform as

$$\frac{d\tilde{q}^i}{d\tilde{x}^\mu} = \frac{dq^i + \epsilon d\eta^i}{dx^\mu + \epsilon d\xi^\mu} = \left(\frac{dq^i}{dx^\mu} + \epsilon\frac{d\eta^i}{dx^\mu}\right)\left(1 + \epsilon\frac{d\xi^\nu}{dx^\nu}\right)^{-1} \sim \left(\frac{dq^i}{dx^\mu} + \epsilon\frac{d\eta^i}{dx^\mu}\right)\left(1 - \epsilon\frac{d\xi^\nu}{dx^\nu}\right), \tag{B.10}$$

which at the first order in ϵ have the form

$$\frac{d\tilde{q}^i}{d\tilde{x}^\mu} = \frac{dq^i}{dx^\mu} + \epsilon\left(\frac{d\eta^i}{dx^\mu} - \frac{dq^i}{dx^\mu}\frac{d\xi^\nu}{dx^\nu}\right) + O(\epsilon^2) = \partial_\mu q^i + \epsilon\left(\partial_\mu \eta^i - \partial_\mu q^i \partial_\nu \xi^\nu\right) + O(\epsilon^2)\,. \tag{B.11}$$

From Eq. (B.9) and Eq. (B.11) we can construct the generator of these transformations, that reads

$$\mathcal{X} = \xi^\mu \partial_\mu + \eta^i \frac{\partial}{\partial q^i}\,. \tag{B.12}$$

Now, if the transformations (B.9) and (B.11) hold, the equations of motion, *i.e.* the Euler-Lagrange equations, are invariant, and thus there exists a function $g^\mu = g^\mu(x^\mu, q^i)$ such that the following condition holds

$$\frac{d\tilde{x}^\mu}{dx^\mu}\tilde{\mathscr{L}} = \mathscr{L} + \epsilon\partial_\mu g^\mu\,. \tag{B.13}$$

The derivative of Eq. (B.13) with respect to ϵ provides

$$\tilde{\mathscr{L}}\frac{\partial}{\partial\epsilon}\frac{d\tilde{x}^\mu}{dx^\mu} + \frac{d\tilde{x}^\mu}{dx^\mu}\frac{\partial\tilde{\mathscr{L}}}{\partial\epsilon} = \partial_\mu g^\mu\,, \tag{B.14}$$

and can be explicitly calculated by means of the relation (B.9), which yields

$$\frac{d\tilde{x}^\mu}{dx^\mu} = \frac{\partial\tilde{x}^\mu}{\partial x^\mu} + \frac{\partial\tilde{x}^\mu}{\partial q^i}\partial_\mu q^i = 1 + \epsilon\frac{\partial\xi^\mu}{\partial q^i}\partial_\mu q^i\,, \tag{B.15}$$

$$\frac{\partial}{\partial\epsilon}\frac{d\tilde{x}^\mu}{dx^\mu} = \frac{d}{dx^\mu}\left(\frac{\partial\tilde{x}^\mu}{\partial\epsilon}\right) = \partial_\mu\xi^\mu\,, \tag{B.16}$$

$$\frac{\partial\tilde{\mathscr{L}}}{\partial\epsilon} = \frac{\partial\tilde{\mathscr{L}}}{\partial\tilde{x}^\mu}\frac{\partial\tilde{x}^\mu}{\partial\epsilon} + \frac{\partial\tilde{\mathscr{L}}}{\partial\tilde{q}^i}\frac{\partial\tilde{q}^i}{\partial\epsilon} + \frac{\partial\tilde{\mathscr{L}}}{\partial(\partial_\mu\tilde{q}^i)}\frac{\partial(\partial_\mu\tilde{q}^i)}{\partial\epsilon}\,. \tag{B.17}$$

With the help of Eq. (B.11), we can replace Eqs. (B.15), (B.16) and (B.17) into Eq.

(B.14) and obtain

$$\left[\xi^\mu\partial_\mu + \eta^i\frac{\partial}{\partial q^i} + (\partial_\mu\eta^i - \partial_\mu q^i\partial_\nu\xi^\nu)\frac{\partial}{\partial(\partial_\mu q^i)} + \partial_\mu\xi^\mu\right]\mathscr{L} = \partial_\mu g^\mu\,, \tag{B.18}$$

that is nothing else but (B.7). It is worth noticing that the associated Noether integral, which is the conserved quantity, is given by

$$j^\mu = -\frac{\partial\mathscr{L}}{\partial(\partial_\mu q^i)}\eta^i + \frac{\partial\mathscr{L}}{\partial(\partial_\mu q^i)}\partial_\nu q^i\,\xi^\nu - \mathscr{L}\xi^\mu + g^\mu\,. \tag{B.19}$$

In particular, for spherical symmetry, where the metric only depends on r, Eqs. (B.7) and (B.6) acquire the form:

$$X^{[1]} = \xi(r,q^i)\partial_r + \eta^i(r,q^i)\frac{\partial}{\partial q^i} + [\partial_r\eta^i(r,q^i) - \partial_r q^i\partial_r\xi(r,q^i)]\frac{\partial}{\partial(\partial_r q^i)}\,, \tag{B.20}$$

$$X^{[1]}\mathscr{L} + \partial_r\xi(r,q^i)\mathscr{L} = \partial_r g(r,q^i)\,, \tag{B.21}$$

while in cosmology they reduce to Eqs. (B.3) and (B.2). It is worth analyzing a particular subcase of Noether's theorem, by means of which it is possible to select internal symmetries. To this purpose, let us focus on the specific condition (B.2), only involving the affine parameter t. By setting $\xi = g = 0$, we get the non-extended Noether vector X, which provides symmetries not depending on the coordinates transformation. In such a case, Eq. (B.2) and the corresponding conserved quantity can be rewritten as:

$$\left[\eta^i\frac{\partial}{\partial q^i} + \dot\eta^i\frac{\partial}{\partial\dot q^i}\right]\mathcal{L} = 0 \quad \rightarrow \quad J = \eta^i\frac{\partial\mathcal{L}}{\partial\dot q^i}\,. \tag{B.22}$$

The former equation can be recast in terms of Lie derivative as

$$L_X\mathcal{L} = 0\,. \tag{B.23}$$

It means that the Lie derivative of the Lagrangian along the flux of vector X, vanishes identically if such a Lagrangian contains symmetries. Thus the vector field X can be used to introduce a cyclic variable into the system, by means of a methodical procedure. Indeed, by considering the transformation

$$q^i \rightarrow Q^i(q^j)\,, \tag{B.24}$$

and the *inner derivative* of the new variables Q^i

$$i_X dQ^i \equiv \delta q^j \frac{\partial Q^i}{\partial q^j}\,, \tag{B.25}$$

the non-extended Noether vector can be written in terms of the variables Q^i as

$$X' = (i_X dQ^k)\frac{\partial}{\partial Q^k} + \frac{\partial(i_X dQ^k)}{\partial t}\frac{\partial}{\partial \dot{Q}^k}.$$

Imposing

$$i_X dQ^1 = 1\,, \qquad \text{and} \qquad i_X dQ^i = 0\,, \qquad i \neq 1, \tag{B.26}$$

the infinitesimal generator of the variable Q^1 is constant and the conserved quantity is

$$J = \partial_{\dot{Q}^i}\mathcal{L} = \pi_{Q^1}\,, \tag{B.27}$$

where π_{Q^1} is the conserved momentum. In this way, the conjugate momentum related to Q^1 is a constant of motion and, therefore, Q^1 is a cyclic variable. Summing up, the relations (B.26) allow to replace a variable with the corresponding integral of motion.

C

Canonical Quantization of Gravity

A quantum description of gravity needs to be developed under the Hamiltonian point of view, as a first step. In this way, in analogy to quantum mechanics, one can impose the commutation relations to the quantized Hamiltonian and find the wave function of the Universe. It cannot be intended as a standard wave function with the same meaning as any other QFT. The main reason is due to the standard probabilistic interpretation of the wave function as a probability amplitude, whose squared modulus integrated over the space provides the probability to get a certain configuration. Such an interpretation requires many copies of the same system to make sense, otherwise the concept of probability itself stops being valid. This cannot be applied to gravity and cosmology, since we do not have a final theory of quantum gravity and a self-consistent interpretation of probability for the space-time. Nevertheless, although the meaning of the wave function is still unclear, many interpretations have been given over the years. For instance, according to the so called *Many World Interpretation*, the wave function comes from quantum measurements that are simultaneously realized in different universes without, therefore, showing any collapse of the wave function as in standard quantum mechanics [253]. Another interpretation was provided by Hawking, according to whom the wave function is supposed to be related to the probability for the early Universe to develop towards our classical Universe [12, 13, 254].

In this scheme of interpretation, J. B. Hartle proposed a criterion to gain information from the wave function, based on its trend in the late-time. Specifically, according to the Hartle Criterion, the wave function must have an oscillating behavior in the classically permitted area, namely whereas it describes our classical Universe [12]. In principle, thanks to the Hartle Criterion and in the WKB approximation, it is possible to write the wave function in terms of the classical action S_0 as $\psi \sim e^{iS_0}$. In this way, thanks to the Hamilton–Jacobi equations, we recover the same equations of motion provided by the cosmological Lagrangian and, therefore, the same trajectories. These results make quantum cosmology an important connection point between classical and quantum gravity; while waiting for a complete theory of quantum gravity, the particular application to cosmology represents a sort of interpretative model capable of reducing the infinite-dimensional superspace coming from the ADM formalism to minisuperspaces where the equations of motion can be interpreted and, eventually, integrated. However, quantum cosmology does not aim to provide UV and IR quantum corrections, since cannot settle *e.g.* the renormalizability problem or the lack of a Yang-Mills description. Therefore, the canonical quantization of gravity is not a complete theory, but only aims to solve part of the high-energy issues arising in standard GR. Without the claim of completeness, let us discuss the basic foundations of the ADM formalism, whose main features can be found in Refs. [255, 256, 257].

In the general form, the Hilbert–Einstein action includes the extrinsic curvature tensor of the three-dimensional spatial surface K_{ij}, the cosmological constant Λ and the four-dimensional scalar curvature R. It reads as [258, 259]:

$$S = \frac{1}{2}\int_V \sqrt{-g}[R-2\Lambda]d^4x + \int_{\partial V} \sqrt{h}K \; dx^3 \,. \tag{C.1}$$

Here V represents the manifold considered and ∂V the three-dimensional spatial surface, while the scalar K is defined through the extrinsic curvature tensor K_{ij} as $K = h^{ij}K_{ij}$, where h_{ij} is the spatial metric. Dealing with the three-dimensional surface is important in sight of the (3+1) decomposition of the metric $g_{\mu\nu}$, so that the spatial coordinates account for the dynamical degrees of freedom evolving in the time-line. According to the foliament method, a coordinates transformation $X^\alpha \to X'^\alpha$ can be seen as a transformation of hypersurfaces with local coordinates x^i. To any coordinate x^0 it corresponds a space-like hypersurface $x^0 = k$, and the variation of x^0 provides the foliament required. Moreover, to any point of the hypersurface, it corresponds a three-dimensional vectors basis X_i^α, tangent to the surface and perpendicular to the normal surface vector n^ν. In light of these considerations, the following relations hold:

$$g_{\mu\nu} X_i^\mu n^\nu = 0\,, \qquad g_{\mu\nu} n^\mu n^\nu = -1\,. \tag{C.2}$$

The first equation establishes the orthonormality between the vector n^ν and the set of coordinates X^α, while the second is nothing but the parallelism condition between two unitary vectors n^ν and n^μ. The *Deformation Tensor* can be defined as the time derivative of the coordinates X^α:

$$N^\alpha = \dot{X}^\alpha = \partial_0 X^\alpha(x^0, x^i)\,, \tag{C.3}$$

and can be decomposed in the basis of tangent and orthonormal vectors by means of the *Lapse Function* N^i and the *Shift Function* N:

$$N^\alpha = N n^\alpha + N^i X_i^\alpha\,. \tag{C.4}$$

With these definitions in mind, the metric tensor can be written in terms of N and N^i as:

$$g_{\mu\nu} = \begin{pmatrix} -(N^2 - N_i N^i) & N_j \\ N_j & h_{ij} \end{pmatrix}. \tag{C.5}$$

Neglecting the cosmological constant Λ and the integral over the three-dimensional surface, the Lagrangian becomes:

$$\mathscr{L} = \frac{1}{2}\sqrt{h} N \left(K^{ij} K_{ij} - K^2 + {}^{(3)}R \right) + t.d.\,, \tag{C.6}$$

where ${}^{(3)}R$ stands for the intrinsic three-dimensional curvature and h is the determinant of the three-dimensional metric h_{ij}. Given the Lagrangian of the theory and considering that the dynamical degrees of freedom are N, N^i and h_{ij}, the conjugate momenta can be written as:

$$
\begin{aligned}
&\pi \equiv \frac{\delta \mathscr{L}}{\delta \dot{N}} = 0\,, \quad \pi^i \equiv \frac{\delta \mathscr{L}}{\delta \dot{N}_i} = 0\,, \\
&\pi^{ij} \equiv \frac{\delta \mathscr{L}}{\delta \dot{h}_{ij}} = \frac{\sqrt{h}}{2}\left(K h^{ij} - K^{ij}\right),
\end{aligned} \tag{C.7}
$$

so that, by Legendre transforming the Lagrangian (C.6), the Hamiltonian density turns out to be

$$
\mathscr{H} = \pi^{ij}\dot{h}_{ij} - \mathscr{L}\,, \tag{C.8}
$$

satisfying the constraints

$$
\begin{cases}
\dot{\pi} = -\{\mathcal{H}, \pi\} = \dfrac{\delta \mathcal{H}}{\delta N} = 0\,, \\
\dot{\pi}^i = -\{\mathcal{H}, \pi^i\} = \dfrac{\delta \mathcal{H}}{\delta N_i} = 0\,,
\end{cases} \tag{C.9}
$$

where $\mathcal{H} = \int \mathscr{H}\, d^3x$.

As usual method to quantize the theory, the first step consists in transforming the dynamical variables into operators and the Poisson brackets in commutators. Therefore, by rewriting the momenta definitions (C.7) as

$$
\hat{\pi} = -i\frac{\delta}{\delta N}\,, \quad \hat{\pi}^i = -i\frac{\delta}{\delta N_i}\,, \quad \hat{\pi}^{ij} = -i\frac{\delta}{\delta h_{ij}}\,, \tag{C.10}
$$

the following commutation relations hold:

$$
\begin{cases}
[\hat{h}_{ij}(x), \hat{\pi}^{kl}(x')] = i\;\delta^{kl}_{ij}\;\delta^3(x - x')\,, \\
\delta^{kl}_{ij} = \frac{1}{2}(\delta^k_i \delta^l_j + \delta^l_i \delta^k_j)\,, \\
[\hat{h}_{ij}, \hat{h}_{kl}] = 0\,, \\
[\hat{\pi}^{ij}, \hat{\pi}^{kl}] = 0\,.
\end{cases} \tag{C.11}
$$

Finally, the first relation of Eq. (C.9), in the canonical quantization scheme, becomes

$$\hat{\mathcal{H}}|\psi>=0\,. \tag{C.12}$$

The above equations, after considering the form of the Hamiltonian as a function of dynamical variables and momenta, leads to a Schroedinger-like equation of the form

$$\left(D^2-\frac{1}{4}\sqrt{h}\,{}^{(3)}R\right)|\psi>=0\,, \tag{C.13}$$

called WDW equation. In Eq. (C.13), ψ is the wave function of the Universe, which depends on the spatial metric h_{ij} and describing the evolution of the gravitational field. The operator D^2, is defined as

$$D^2=\frac{1}{\sqrt{h}}\left(h_{ik}h_{jl}+h_{il}h_{jk}-h_{ij}h_{kl}\right)\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}}\,. \tag{C.14}$$

In non-relativistic quantum mechanics, the scalar product $\int\psi^*\psi\;dx^3$ is everywhere positive, so that an infinite-dimensional Hilbert space can be defined. The main problem related to the wave function of the Universe is that it is not possible to define an everywhere positive scalar product, due to the hyperbolic nature of Eq.(C.13). As a consequence, no probabilistic meanings can be assigned to the wave function.

Nevertheless, the wave function may represent an important quantity capable of giving information about the early stages of the Universe and of explaining the nowadays cosmic evolution. Regarding the latter point, the oscillating wave function in the minisuperspaces allows to recover the Hartle Criterion.

D

Gauge-Invariance and Field Equations of the Chern–Simons SU(N)-Invariant Action

We show how to perform the variation of the $SU(N)$ invariant Chern-Simons action. It turns out that only a two-dimensional boundary term survives after varying the action with respect to the gauge connection, thus proving the gauge-invariance of the theory. The Chern–Simons $SU(N)$-invariant three-dimensional action

$$S = \int Tr \left[\mathbf{A} d\mathbf{A} + \frac{2}{3} \mathbf{A}\mathbf{A}\mathbf{A} \right], \tag{D.1}$$

can be equivalently written in coordinates representation as

$$S = Tr \left[\int \epsilon^{ijk} \left(A_i \partial_j A_k + \frac{2}{3} A_i A_j A_k \right) d^3x \right]. \tag{D.2}$$

The second term is the so called *Wess-Zumino-Witten* term, whose variation is:

$$\delta \left(\epsilon^{ijk} A_i A_j A_k \right) = 3 \epsilon^{ijk} \delta A_i A_j A_k. \tag{D.3}$$

Action (D.1) is holographically dual to the 2-dimensional Wess-Zumino-Witten model, which describes propagating strings on the given group. Furthermore, considering the relation

$$\epsilon^{ijk} A_j A_k = -\epsilon^{ijk} \partial_j A_k, \tag{D.4}$$

and replacing Eq. (D.4) into Eq. (D.3), this latter yields:

$$\delta \left(\epsilon^{ijk} A_i A_j A_k \right) = 3\epsilon^{ijk} \delta A_i \partial_j A_k. \tag{D.5}$$

On the other hand, the variation of the first term of Eq. (D.1) can be recast as:

$$\delta S_{AdA} = Tr \left[\int \left(\epsilon^{ijk} \delta A_i \partial_j A_k + \epsilon^{ijk} A_i \partial_j \delta A_k \right) d^3x \right]. \tag{D.6}$$

Integrating the quantity $\epsilon^{ijk} A_i \partial_j \delta A_k$, two contributions arise, which can be understood as a boundary and a bulk term, respectively. Therefore Eq. (D.6) takes the form:

$$\delta S_{AdA} = Tr \left[\int \epsilon^{ijk} \delta A_i \partial_j A_k \, d^3x + \int_{\mathcal{B}} e^{ik} A_i \delta A_k \, d^2x - \int \epsilon^{ijk} \partial_j A_i \delta A_k d^3x \right]. \tag{D.7}$$

By using the anti-symmetric property of the Levi-Civita tensor, the action becomes

$$\delta S_{AdA} = Tr \left[2 \int \epsilon^{ijk} \delta A_i \partial_j A_k \, d^3x + \int_{\mathcal{B}} e^{ik} A_i \delta A_k \, d^2x \right]. \tag{D.8}$$

Merging Eq. (D.5) and Eq. (D.8), it turns out that the final variation only yields the term on the boundary:

$$\delta S = Tr \left[\int_{\mathcal{B}} e^{ik} A_i \delta A_k \, d^2x \right], \tag{D.9}$$

which makes the action *quasi*-gauge Invariant. In order to find the field equations on $SU(N)$, the action must be varied with respect to the gauge connection $\mathbf{A}$. First, notice that this latter can be generally written in terms of the group element g as

$$A_i = g^{-1} \partial_i g. \tag{D.10}$$

Thus, the variation of the connection with respect to g reads

$$\delta A_i = -g^{-1}\delta g\, A_i + g^{-1}\partial_i \delta g. \tag{D.11}$$

The variation of the Chern-Simons action can be split in two contributions, namely

$$\delta S = \delta S_{AdA} + \delta S_{AAA}, \tag{D.12}$$

that will be separately evaluated in what follows.

1) Term A dA

In coordinates representation, the first term of Eq. (D.12) can be written as

$$\delta S_{AdA} = Tr\left[\int \epsilon^{ijk}\delta A_i \partial_j A_k\, d^3x + \int \epsilon^{ijk} A_i \partial_j \delta A_k\, d^3x\right]. \tag{D.13}$$

It can be easily showed that the first term of the RHS in Eq. (D.13) vanishes identically. To this purpose, notice that the identity (D.11) yields

$$\int \epsilon^{ijk}\delta A_i \partial_j A_k\, d^3x = \int \epsilon^{ijk}\left(-g^{-1}A_i\partial_j A_k\,\delta g + g^{-1}\partial_j A_k \partial_i \delta g\right) d^3x. \tag{D.14}$$

Integrating out the pure boundary $\partial_i \delta g$ and using again the relation (D.10), the above variation provides:

$$\begin{aligned}\int \epsilon^{ijk}\delta A_i \partial_j A_k\, d^3x &= \int \epsilon^{ijk}\left(-g^{-1}A_i\partial_j A_k + g^{-1}\partial_i g g^{-1}\partial_j A_k\right)\delta g\, d^3x = \\ &= \int \epsilon^{ijk}\left(-g^{-1}A_i\partial_j A_k + g^{-1}A_i\partial_j A_k\right)\delta g\, d^3x = 0.\end{aligned} \tag{D.15}$$

Therefore, the only term surviving in Eq. (D.13) is:

$$\delta S_{AdA} = Tr\left[\int \epsilon^{ijk} A_i \partial_j \delta A_k\, d^3x\right], \tag{D.16}$$

which can be made explicit by means of the relation (D.16), providing

$$\begin{aligned}&\int \epsilon^{ijk} A_i \partial_j \delta A_k \, d^3x = \\ &= \int \epsilon^{ijk} A_i g^{-1} \left(\partial_j g \, g^{-1} A_k \delta g - \partial_j A_k \delta g - A_k \partial_j \delta g - \partial_j g \, g^{-1} \partial_k \delta g \right) d^3x.\end{aligned} \tag{D.17}$$

Using Eqs. (D.10) and (D.4), it turns out that the last two terms in the above equation cancel out, so that we finally have

$$\delta S_{AdA} = Tr \left[\int \epsilon^{ijk} A_i \partial_j \delta A_k \, d^3x \right] = Tr \left[-2 \int \epsilon^{ijk} g^{-1} A_k \partial_i A_j \, \delta g \, d^3x \right]. \tag{D.18}$$

2) Term AAA

In order to show that the variation of the Wess-Zumino-Witten term yields a boundary term, let us consider Eq. (D.3), by means of which the quantity δS_{AAA} takes the form:

$$\delta S_{AAA} = Tr \left[2 \int \epsilon^{ijk} \delta A_i A_j A_k \, d^3x \right]. \tag{D.19}$$

The above equation can be equivalently written in terms of the element g, as

$$\delta S_{AAA} = Tr \left[2 \int \epsilon^{ijk} g^{-1} \left(-A_i \delta g + \partial_i \delta g \right) A_j A_k \, d^3x \right]. \tag{D.20}$$

Integrating by parts and neglecting the boundary term, the variation of the action yields

$$\begin{aligned}\delta S_{AAA} &= Tr \left[2 \int \epsilon^{ijk} \left(-g^{-1} A_i A_j A_k - \partial_i (g^{-1} A_j A_k) \right) \delta g \, d^3x \right] = \\ &= Tr \left[2 \int \epsilon^{ijk} g^{-1} \left(-A_i A_j A_k + A_i A_j A_k - \partial_i A_j A_k - \partial_i A_k A_j \right) \delta g \, d^3x \right],\end{aligned} \tag{D.21}$$

which vanishes due to the anti-symmetric property of the Levi-Civita symbol. Merging Eqs. (D.18) and (D.21), the variation of the total Chern-Simons three-dimensional action finally reads

$$\delta S_{CS} = Tr\left[-2\int \epsilon^{ijk} g^{-1} A_k \partial_i A_j \,\delta g \, d^3x\right] = 0. \tag{D.22}$$

E

Values of Constants in Chap. 11

Table XVI: *Definitions of the parameters occurring in the general five-dimensional solutions of Lovelock and Chern–Simons gravity.*

	Lovelock Gravity	**Chern–Simons Gravity**
w	$3\alpha_1^2 - 2\alpha_0\alpha_2$	$\frac{14}{15l^4}$
z	$\sqrt{c_1\alpha_0 + 9\alpha_1^2}$	$\sqrt{\frac{20+c_1}{5l^4}}$
u	$\sqrt{c_1\alpha_0\alpha_1^2 + 6\alpha_1^2 w - 2\alpha_1 z\, w + 4\alpha_0^2\alpha_2^2}$	$\frac{2}{15}\sqrt{\frac{149+5c_1-14\sqrt{5}\sqrt{20+c_1}}{l^8}}$
y	$\frac{(-z\alpha_1 + w)}{u}$	$\frac{7-\sqrt{5}\sqrt{20+c_1}}{\sqrt{149+5c_1-14\sqrt{5}\sqrt{20+c_1}}}$
s	$-\frac{z\,\alpha_1 + w}{v}$	$\frac{-7-\sqrt{5}\sqrt{20+c_1}}{\sqrt{149+5c_1+14\sqrt{5}\sqrt{20+c_1}}}$
v	$\sqrt{c_1\alpha_0\alpha_1^2 + 6\alpha_1^2 w + 2\alpha_1 z\, w + 4\alpha_0^2\alpha_2^2}$	$\frac{2}{15}\sqrt{\frac{149+5c_1+14\sqrt{5}\sqrt{20+c_1}}{l^8}}$
x	$4\alpha_2(c_1 + 6\alpha_2)$	$4(c_1+6)$

Bibliography

[1] K. Akiyama *et al.* [Event Horizon Telescope], "First M87 Event Horizon Telescope Results. V. Physical Origin of the Asymmetric Ring," Astrophys. J. Lett. **875** (2019) no.1, L5

[2] B. P. Abbott *et al.* [LIGO Scientific and Virgo], "Observation of Gravitational Waves from a Binary Black Hole Merger," Phys. Rev. Lett. **116** (2016) no.6, 061102

[3] A. Bosma, "21-cm line studies of spiral galaxies. 2. The distribution and kinematics of neutral hydrogen in spiral galaxies of various morphological types.," Astron. J. **86** (1981), 1825

[4] M. H. Goroff and A. Sagnotti, "The Ultraviolet Behavior of Einstein Gravity," Nucl. Phys. B **266** (1986) 709. doi:10.1016/0550-3213(86)90193-8

[5] P. Horava, "Quantum Gravity at a Lifshitz Point," Phys. Rev. D **79** (2009), 084008

[6] K. S. Stelle, "Renormalization of Higher Derivative Quantum Gravity," Phys. Rev. D **16** (1977), 953-969

[7] B. S. DeWitt, "Quantum Theory of Gravity. 2. The Manifestly Covariant Theory," Phys. Rev. **162** (1967) 1195.

[8] J. Engle, R. Pereira and C. Rovelli, "The Loop-quantum-gravity vertex-amplitude," Phys. Rev. Lett. **99** (2007), 161301

[9] S. Alexandrov and E. R. Livine, "SU(2) loop quantum gravity seen from covariant theory," Phys. Rev. D **67** (2003), 044009

[10] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Int. J. Theor. Phys. **38** (1999), 1113-1133

[11] H. P. Nilles, "Supersymmetry, Supergravity and Particle Physics," Phys. Rept. **110** (1984), 1-162

[12] J. B. Hartle and S. W. Hawking, "Wave Function of the Universe," Phys. Rev. D **28** (1983) 2960 [Adv. Ser. Astrophys. Cosmol. **3** (1987) 174].

[13] S. W. Hawking, "The Quantum State of the Universe," Adv. Ser. Astrophys. Cosmol. **3** (1987), 236-255

[14] J. J. Halliwell and S. W. Hawking, "The Origin of Structure in the Universe," Adv. Ser. Astrophys. Cosmol. **3** (1987), 277-291

[15] A. Vilenkin, "Quantum Creation of Universes," Phys. Rev. D **30** (1984) 509.

[16] F. Bajardi, D. Vernieri and S. Capozziello, "Bouncing Cosmology in f(Q) Symmetric Teleparallel Gravity," Eur. Phys. J. Plus **135** (2020) no.11, 912

[17] A. Lasenby, C. Doran and S. Gull, "Gravity, gauge theories and geometric algebra," Phil. Trans. Roy. Soc. Lond. A **356** (1998) 487

[18] F. W. Hehl, P. Von Der Heyde, G. D. Kerlick and J. M. Nester, "General Relativity with Spin and Torsion: Foundations and Prospects," Rev. Mod. Phys. **48** (1976) 393.

[19] F. Gronwald and F. W. Hehl, "On the gauge aspects of gravity," gr-qc/9602013.

[20] E. W. Mielke and P. Baekler, "Topological gauge model of gravity with torsion," Phys. Lett. A **156** (1991) 399.

[21] M. Gasperini, "Spin Dominated Inflation in the Einstein-cartan Theory," Phys. Rev. Lett. **56** (1986) 2873.

[22] Y. N. Obukhov, "Poincare gauge gravity: Selected topics," Int. J. Geom. Meth. Mod. Phys. **3** (2006) 95

[23] J. M. Pawlowski, M. Reichert, C. Wetterich and M. Yamada, "Higgs scalar potential in asymptotically safe quantum gravity," Phys. Rev. D **99** (2019) no.8, 086010

[24] F. Bezrukov, M. Y. Kalmykov, B. A. Kniehl and M. Shaposhnikov, "Higgs Boson Mass and New Physics," JHEP **1210** (2012) 140

[25] M. Niedermaier and M. Reuter, "The Asymptotic Safety Scenario in Quantum Gravity," Living Rev. Rel. **9** (2006) 5.

[26] R. Percacci, "Asymptotic Safety," In *Oriti, D. (ed.): Approaches to quantum gravity* 111-128

[27] T. P. Sotiriou and S. Liberati, "Metric-affine f(R) theories of gravity," Annals Phys. **322** (2007), 935-966

[28] S. Fay, R. Tavakol and S. Tsujikawa, "f(R) gravity theories in Palatini formalism: Cosmological dynamics and observational constraints," Phys. Rev. D **75** (2007), 063509

[29] T. P. Sotiriou, "Constraining f(R) gravity in the Palatini formalism," Class. Quant. Grav. **23** (2006), 1253-1267

[30] C. Barragan, G. J. Olmo and H. Sanchis-Alepuz, "Bouncing Cosmologies in Palatini f(R) Gravity," Phys. Rev. D **80** (2009), 024016

[31] S. Capozziello and M. De Laurentis, "Extended Theories of Gravity," Phys. Rept. **509** (2011) 167-321

[32] A. K. Mishra, M. Rahman and S. Sarkar, "Black Hole Topology in $f(R)$ Gravity," Class. Quant. Grav. **35** (2018) no.14, 145011

[33] S. Capozziello and M. De Laurentis, "The dark matter problem from f(R) gravity viewpoint," Annalen Phys. **524** (2012) 545.

[34] G. J. Olmo, "Palatini Approach to Modified Gravity: f(R) Theories and Beyond," Int. J. Mod. Phys. D **20** (2011) 413

[35] S. Capozziello, M. De Laurentis and V. Faraoni, "A Bird's eye view of f(R)-gravity," Open Astron. J. **3** (2010) 49

[36] T. P. Sotiriou and V. Faraoni, "f(R) Theories Of Gravity," Rev. Mod. Phys. **82** (2010) 451

[37] Y. F. Cai, S. Capozziello, M. De Laurentis and E. N. Saridakis, "f(T) teleparallel gravity and cosmology," Rept. Prog. Phys. **79** (2016) no.10, 106901

[38] M. R. Setare and N. Mohammadipour, "Can $f(T)$ gravity theories mimic ΛCDM cosmic history," JCAP **1301** (2013) 015

[39] K. Bamba, S. Capozziello, M. De Laurentis, S. Nojiri and D. SÃfÂ¡ez-GÃfÂ³mez, "No further gravitational wave modes in $F(T)$ gravity," Phys. Lett. B **727** (2013) 194

[40] N. Tamanini and C. G. Boehmer, "Good and bad tetrads in f(T) gravity," Phys. Rev. D **86** (2012) 044009

[41] Y. Zhang, H. Li, Y. Gong and Z. H. Zhu, "Notes on $f(T)$ Theories," JCAP **1107** (2011) 015

[42] R. Aldrovandi and J. G. Pereira, "Teleparallel Gravity : An Introduction," Fundam. Theor. Phys. **173** (2013).

[43] J. J. Halliwell, "Scalar Fields in Cosmology with an Exponential Potential," Phys. Lett. B **185** (1987) 341.

[44] J. P. Uzan, "Cosmological scaling solutions of nonminimally coupled scalar fields," Phys. Rev. D **59** (1999) 123510

[45] H. Mohseni Sadjadi, "Notes on teleparallel cosmology with nonminimally coupled scalar field," Phys. Rev. D **87** (2013) 064028

[46] C. Rubano and P. Scudellaro, "On some exponential potentials for a cosmological scalar field as quintessence," Gen. Rel. Grav. **34** (2002) 307

[47] F. Bajardi and S. Capozziello, "$f(\mathcal{G})$ Noether cosmology," Eur. Phys. J. C **80** (2020) no.8, 704

[48] M. Benetti, S. Santos da Costa, S. Capozziello, J. S. Alcaniz and M. De Laurentis, "Observational constraints on Gauss-Bonnet cosmology," Int. J. Mod. Phys. D **27** (2018) no.08, 1850084

[49] J. L. Blázquez-Salcedo, F. S. Khoo and J. Kunz, "Quasinormal modes of Einstein-Gauss-Bonnet-dilaton black holes," Phys. Rev. D **96** (2017) no.6, 064008

[50] H. Shabani and A. H. Ziaie, "Static vacuum solutions on curved space-times with torsion," Int. J. Mod. Phys. A **33** (2018) no.16, 1850095

[51] S. Capozziello and C. Stornaiolo, "Torsion tensor and its geometric interpretation," Annales de la Fondation Louis de Broglie **32**, (2007)

[52] T. P. Sotiriou, “f(R) gravity, torsion and non-metricity,” Class. Quant. Grav. **26** (2009) 152001

[53] S. Capozziello, “Curvature quintessence,” Int. J. Mod. Phys. D **11** (2002) 483

[54] A. A. Starobinsky, “A New Type of Isotropic Cosmological Models Without Singularity,” Phys. Lett. B **91** (1980) 99 [Phys. Lett. **91B** (1980) 99] [Adv. Ser. Astrophys. Cosmol. **3** (1987) 130].

[55] S. Capozziello and F. Bajardi, “Gravitational waves in modified gravity,” Int. J. Mod. Phys. D **28** (2019) no.05, 1942002

[56] L. Modesto, L. Rachwal and I. L. Shapiro, “Renormalization group in super-renormalizable quantum gravity,” Eur. Phys. J. C **78** (2018) no.7, 555

[57] E. T. Tomboulis, “Renormalization and unitarity in higher derivative and nonlocal gravity theories,” Mod. Phys. Lett. A **30** (2015) no.03n04, 1540005.

[58] L. Buoninfante, A. S. Koshelev, G. Lambiase and A. Mazumdar, “Classical properties of non-local, ghost- and singularity-free gravity,” JCAP **09** (2018), 034

[59] L. Buoninfante, G. Lambiase and A. Mazumdar, “Ghost-free infinite derivative quantum field theory,” Nucl. Phys. B **944** (2019), 114646

[60] S. Nojiri and S. D. Odintsov, “Modified non-local-F(R) gravity as the key for the inflation and dark energy,” Phys. Lett. B **659** (2008), 821-826

[61] S. Deser and R. P. Woodard, “Nonlocal Cosmology,” Phys. Rev. Lett. **99** (2007), 111301

[62] T. Koivisto, “Dynamics of Nonlocal Cosmology,” Phys. Rev. D **77** (2008), 123513

[63] R. P. Woodard, “Nonlocal Models of Cosmic Acceleration,” Found. Phys. **44** (2014), 213-233

[64] G. Calcagni, M. Montobbio and G. Nardelli, “A Route to nonlocal cosmology,” Phys. Rev. D **76** (2007), 126001

[65] A. S. Koshelev and S. Y. Vernov, “On bouncing solutions in non-local gravity,” Phys. Part. Nucl. **43** (2012), 666-668

[66] A. D. Linde, "A New Inflationary Universe Scenario: A Possible Solution of the Horizon, Flatness, Homogeneity, Isotropy and Primordial Monopole Problems," Adv. Ser. Astrophys. Cosmol. **3** (1987), 149-153

[67] A. H. Guth, "The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems," Phys. Rev. D **23** (1981) 347 [Adv. Ser. Astrophys. Cosmol. **3** (1987) 139].

[68] Y. Shtanov, J. H. Traschen and R. H. Brandenberger, "Universe reheating after inflation," Phys. Rev. D **51** (1995) 5438

[69] R. Allahverdi, R. Brandenberger, F. Y. Cyr-Racine and A. Mazumdar, "Reheating in Inflationary Cosmology: Theory and Applications," Ann. Rev. Nucl. Part. Sci. **60** (2010) 27

[70] P. B. Greene, "Inflationary reheating and fermions," AIP Conf. Proc. **478** (1999) no.1, 72

[71] S. Bahamonde, M. Marciu and P. Rudra, "Generalised teleparallel quintom dark energy non-minimally coupled with the scalar torsion and a boundary term," JCAP **1804** (2018) no.04, 056

[72] S. Bahamonde, M. Zubair and G. Abbas, "Thermodynamics and cosmological reconstruction in $f(T, B)$ gravity," Phys. Dark Univ. **19** (2018) 78

[73] S. Bahamonde, S. Capozziello, M. Faizal and R. C. Nunes, "Nonlocal Teleparallel Cosmology," Eur. Phys. J. C **77** (2017) no.9, 628

[74] S. Capozziello, M. De Laurentis and K. F. Dialektopoulos, "Noether symmetries in Gauss-Bonnet-teleparallel cosmology," Eur. Phys. J. C **76** (2016) no.11, 629

[75] F. Bajardi and S. Capozziello, "Noether Symmetries and Quantum Cosmology in Extended Teleparallel Gravity," doi:10.1142/S0219887821400028

[76] L. Jarv, M. Runkla, M. Saal and O. Vilson, "Nonmetricity formulation of general relativity and its scalar-tensor extension," Phys. Rev. D **97** (2018) no.12, 124025

[77] M. Rünkla and O. Vilson, "Family of scalar-nonmetricity theories of gravity," Phys. Rev. D **98** (2018) no.8, 084034

[78] T. Harko, T. S. Koivisto, F. S. N. Lobo, G. J. Olmo and D. Rubiera-Garcia, "Coupling matter in modified Q gravity," Phys. Rev. D **98** (2018) no.8, 084043

[79] L. Heisenberg, "A systematic approach to generalisations of General Relativity and their cosmological implications," Phys. Rept. **796** (2019) 1

[80] S. Capozziello, M. De Laurentis, R. Farinelli and S. D. Odintsov, "Mass-radius relation for neutron stars in f(R) gravity," Phys. Rev. D **93** (2016) no.2, 023501

[81] K. F. Dialektopoulos, D. Borka, S. Capozziello, V. Borka Jovanović and P. Jovanović, "Constraining nonlocal gravity by S2 star orbits," Phys. Rev. D **99** (2019) no.4, 044053

[82] V. Borka Jovanović, S. Capozziello, P. Jovanović and D. Borka, "Recovering the fundamental plane of galaxies by $f(R)$ gravity," Phys. Dark Univ. **14** (2016) 73

[83] M. D'Onofrio *et al.*, "The hybrid solution for the Fundamental Plane," Mon. Not. Roy. Astron. Soc. **435** (2013) 45

[84] S. Capozziello, V. F. Cardone and A. -Troisi, "Low surface brightness galaxies rotation curves in the low energy limit of r**n gravity: no need for dark matter?," Mon. Not. Roy. Astron. Soc. **375** (2007) 1423

[85] M. R. Setare and N. Mohammadipour, "Cosmological viability conditions for $f(T)$ dark energy models," JCAP **1211** (2012) 030

[86] C. Charmousis and J. F. Dufaux, "General Gauss-Bonnet brane cosmology," Class. Quant. Grav. **19** (2002) 4671

[87] S. Dutta, M. M. Panja and S. Chakraborty, "A scalar field dark energy model: Noether symmetry approach," Gen. Rel. Grav. **48** (2016) no.4, 54.

[88] M. B. Green, J. H. Schwarz and E. Witten, " Superstring Theory. Vol. 1: Introduction," Cambridge Monographs On Mathematical Physics, Cambridge Univ. Press, 1987.

[89] M. B. Green, J. H. Schwarz and E. Witten, " Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies And Phenomenology," Cambridge Monographs On Mathematical Physics, Cambridge Univ. Press, 1987.

[90] J. Polchinski, “String theory. Vol. 1: An introduction to the bosonic string," Cambridge University Press, 1998.

[91] J. Polchinski, “ String theory. Vol. 2: Superstring theory and beyond," Cambridge University Press, 1998.

[92] K. Becker, M. Becker and J. H. Schwarz, “String theory and M-theory: A modern introduction," Cambridge University Press, 2006.

[93] T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, “ Modified Gravity and Cosmology," Phys. Rept. **513** (2012), 1.

[94] T. Han, J. D. Lykken and R. J. Zhang, “On Kaluza-Klein states from large extra dimensions," Phys. Rev. D **59** (1999), 105006.

[95] C. Rovelli and F. Vidotto, “Covariant Loop Quantum Gravity : An Elementary Introduction to Quantum Gravity and Spinfoam Theory," Cambridge University Press, 2014.

[96] A. Ashtekar and P. Singh, “Loop Quantum Cosmology: A Status Report," Class. Quant. Grav. **28** (2011), 213001.

[97] C. Rovelli, “ Loop quantum gravity," Living Rev. Rel. **1** (1998), 1.

[98] E. Kiritsis and G. Kofinas, “ Horava-Lifshitz Cosmology," Nucl. Phys. B **821** (2009), 467.

[99] R. G. Cai, L. M. Cao and N. Ohta, “Topological Black Holes in Horava-Lifshitz Gravity," Phys. Rev. D **80** (2009), 024003.

[100] T. P. Sotiriou, “ Horava-Lifshitz gravity: a status report," J. Phys. Conf. Ser. **283** (2011), 012034.

[101] S. Mukohyama, “ Horava-Lifshitz Cosmology: A Review," Class. Quant. Grav. **27** (2010), 223101.

[102] N. Arkani-Hamed, S. Dimopoulos, G. Dvali and G. Gabadadze, “Nonlocal modification of gravity and the cosmological constant problem," arXiv: hep-th/0209227 (2002)

[103] L. Modesto and S. Tsujikawa, "Non-local massive gravity," Phys. Lett. B **727** (2013), 48.

[104] L. Modesto, " Super-renormalizable Gravity," *doi* : 10.1142/9789814623995_0098 (2013)

[105] D. Lovelock, "The Einstein tensor and its generalizations," J. Math. Phys. **12** (1971) 498.

[106] M. Montesinos, R. Romero and B. Díaz, "Symmetries of first-order Lovelock gravity," Class. Quant. Grav. **35** (2018) no.23, 235015

[107] A. De Felice and S. Tsujikawa, " Construction of cosmologically viable $f(\mathcal{G})$ dark energy models," Phys. Lett. B **675** (2009) 1,

[108] K. Uddin, J. E. Lidsey and R. Tavakol, "Cosmological scaling solutions in generalised Gauss-Bonnet gravity theories," Gen. Rel. Grav. **41** (2009) 2725,

[109] A. De Felice, D. F. Mota and S. Tsujikawa, "Matter instabilities in general Gauss-Bonnet gravity," Phys. Rev. D **81** (2010) 023532

[110] S. C. Davis, "Solar system constraints on f(G) dark energy,"

[111] S. Chakraborty and S. SenGupta, "Spherically symmetric brane in a bulk of $f(R)$ and Gauss-Bonnet gravity," Class. Quant. Grav. **33** (2016) no.22, 225001

[112] J. de Boer, M. Kulaxizi and A. Parnachev, "AdS(7)/CFT(6), Gauss-Bonnet Gravity, and Viscosity Bound," JHEP **1003** (2010) 087

[113] A. V. Astashenok, S. D. Odintsov and V. K. Oikonomou, " Modified Gauss - Bonnet gravity with the Lagrange multiplier constraint as mimetic theory," Class. Quant. Grav. **32** (2015) no.18, 185007,

[114] M. De Laurentis and A. J. Lopez-Revelles, "Newtonian, Post Newtonian and Parameterized Post Newtonian limits of $f(R, G)$ gravity," Int. J. Geom. Meth. Mod. Phys. **11** (2014) 1450082

[115] V. A. Rubakov and M. E. Shaposhnikov, "Do We Live Inside a Domain Wall?," Phys. Lett. **125B** (1983) 136.

[116] I. Antoniadis, “A Possible new dimension at a few TeV,” Phys. Lett. B **246** (1990) 377.

[117] L. Randall and R. Sundrum, “A Large mass hierarchy from a small extra dimension,” Phys. Rev. Lett. **83** (1999) 3370

[118] L. Randall and R. Sundrum, “An Alternative to compactification,” Phys. Rev. Lett. **83** (1999) 4690

[119] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, “The Hierarchy problem and new dimensions at a millimeter,” Phys. Lett. B **429** (1998) 263

[120] D. Lovelock, “The Einstein tensor and its generalizations," J. Math. Phys. **12** (1971) 498.

[121] S. Capozziello, C. Corda and M. F. De Laurentis, “Massive gravitational waves from f(R) theories of gravity: Potential detection with LISA,” Phys. Lett. B **669** (2008), 255-259

[122] D. Glavan and C. Lin, “Einstein-Gauss-Bonnet Gravity in Four-Dimensional Space-time,” Phys. Rev. Lett. **124**, no.8, 081301 (2020) [arXiv:1905.03601 [gr-qc]].

[123] M. Gurses, T. Ç. Şişman and B. Tekin, “Comment on ”Einstein-Gauss-Bonnet Gravity in 4-Dimensional Space-Time”,” Phys. Rev. Lett. **125**, no.14, 149001 (2020) [arXiv:2009.13508 [gr-qc]].

[124] S. Santos Da Costa, F. V. Roig, J. S. Alcaniz, S. Capozziello, M. De Laurentis and M. Benetti, “Dynamical analysis on $f(R,\mathcal{G})$ cosmology,” Class. Quant. Grav. **35** (2018) no.7, 075013

[125] K. Andrew, B. Bolen and C. A. Middleton, “Solutions of Higher Dimensional Gauss-Bonnet FRW Cosmology,” Gen. Rel. Grav. **39** (2007) 2061

[126] M. Ivanov and A. Toporensky, “Cosmological dynamics of fourth order gravity with a Gauss-Bonnet term,” Grav. Cosmol. **18** (2012), 43-53

[127] S. Capozziello and A. De Felice, “f(R) cosmology by Noether’s symmetry,” JCAP **08** (2008), 016

[128] S. Capozziello, P. Martin-Moruno, and C. Rubano, “Dark energy and dust matter phases from an exact $f(R)$-cosmology model,” Phys. Lett. B **664** (2008) 12

[129] S. Nojiri and S. D. Odintsov, “Modified Gauss-Bonnet theory as gravitational alternative for dark energy," Phys. Lett. B **631** (2005) 1.

[130] S. Nojiri, S. D. Odintsov and M. Sasaki, “Gauss-Bonnet dark energy," Phys. Rev. D **71** (2005) 123509

[131] M. De Laurentis, M. Paolella and S. Capozziello, “Cosmological inflation in $F(R,\mathcal{G})$ gravity,” Phys. Rev. D **91** (2015) 083531.

[132] J. Sultana and D. Kazanas, “A no-hair theorem for spherically symmetric black holes in R^2 gravity,” Gen. Rel. Grav. **50** (2018) no.11, 137

[133] S. Nojiri and S. D. Odintsov, “The Final state and thermodynamics of dark energy universe,” Phys. Rev. D **70** (2004), 103522

[134] S. Nojiri and S. D. Odintsov, “Introduction to modified gravity and gravitational alternative for dark energy,” eConf **C0602061** (2006), 06

[135] S. Capozziello, V. F. Cardone, H. Farajollahi and A. Ravanpak, “Cosmography in f(T)-gravity,” Phys. Rev. D **84** (2011), 043527

[136] S. Capozziello, F. S. N. Lobo and J. P. Mimoso, “Generalized energy conditions in Extended Theories of Gravity,” Phys. Rev. D **91** (2015) no.12, 124019

[137] J. Santos, J. S. Alcaniz, M. J. Reboucas and F. C. Carvalho, “Energy conditions in f(R)-gravity,” Phys. Rev. D **76** (2007), 083513

[138] N. M. Garcia, T. Harko, F. S. N. Lobo and J. P. Mimoso, “Energy conditions in modified Gauss-Bonnet gravity,” Phys. Rev. D **83** (2011), 104032

[139] A. A. Starobinsky, “Dynamics of Phase Transition in the New Inflationary Universe Scenario and Generation of Perturbations,” Phys. Lett. B **117** (1982), 175-178

[140] M. De Laurentis, M. Paolella and S. Capozziello, “Cosmological inflation in $F(R,\mathcal{G})$ gravity,” Phys. Rev. D **91** (2015) no.8, 083531

[141] A. Albrecht and P. J. Steinhardt, “Cosmology for Grand Unified Theories with Radiatively Induced Symmetry Breaking,” Adv. Ser. Astrophys. Cosmol. **3** (1987),

158-161

[142] K. Freese, J. A. Frieman and A. V. Olinto, “Natural inflation with pseudo - Nambu-Goldstone bosons,” Phys. Rev. Lett. **65** (1990), 3233-3236

[143] D. Polarski and A. A. Starobinsky, “Spectra of perturbations produced by double inflation with an intermediate matter dominated stage,” Nucl. Phys. B **385** (1992), 623-650

[144] A. D. Linde, “Hybrid inflation,” Phys. Rev. D **49** (1994), 748-754

[145] D. Rapetti, S. W. Allen, M. A. Amin and R. D. Blandford, “A kinematical approach to dark energy studies,” Mon. Not. Roy. Astron. Soc. **375** (2007), 1510-1520

[146] F. Bajardi, S. Capozziello and D. Vernieri, “Energy Conditions in $f(\mathcal{G})$ Gravity,” To Be Submitted

[147] C. Bogdanos, C. Charmousis, B. Gouteraux and R. Zegers, “Einstein-Gauss-Bonnet metrics: Black holes, black strings and a staticity theorem,” JHEP **0910** (2009) 037

[148] N. Deppe, C. D. Leonard, T. Taves, G. Kunstatter and R. B. Mann, “Critical Collapse in Einstein-Gauss-Bonnet Gravity in Five and Six Dimensions,” Phys. Rev. D **86** (2012) 104011

[149] S. Ray, “Birkhoff’s theorem in Lovelock gravity for general base manifolds,” Class. Quant. Grav. **32** (2015) no.19, 195022

[150] R. Zegers, “Birkhoff’s theorem in Lovelock gravity,” J. Math. Phys. **46** (2005) 072502

[151] G. Dotti, J. Oliva and R. Troncoso, “Static solutions with nontrivial boundaries for the Einstein-Gauss-Bonnet theory in vacuum,” Phys. Rev. D **82** (2010) 024002

[152] C. Bogdanos, “Extensions of Birkhoff’s theorem in 6D Gauss-Bonnet gravity,” AIP Conf. Proc. **1241** (2010) no.1, 521.

[153] S. Capozziello, A. Stabile and A. Troisi, “Spherically symmetric solutions in f(R)-gravity via Noether Symmetry Approach,” Class. Quant. Grav. **24** (2007) 2153,

[154] S. Bahamonde, S. Capozziello and K. F. Dialektopoulos, “Constraining Generalized Non-local Cosmology from Noether Symmetries,” Eur. Phys. J. C **77**, no. 11, 722 (2017) [arXiv:1708.06310 [gr-qc]].

[155] F. Bajardi, K. F. Dialektopoulos and S. Capozziello, "Higher Dimensional Static and Spherically Symmetric Solutions in Extended Gauss–Bonnet Gravity," Symmetry **12** (2020) no.3, 372

[156] A. O. Barvinsky, "Aspects of Nonlocality in Quantum Field Theory, Quantum Gravity and Cosmology," Mod. Phys. Lett. A **30**, no.03n04, 1540003 (2015) [arXiv:1408.6112 [hep-th]].

[157] E. Elizalde, S. D. Odintsov, E. O. Pozdeeva and S. Y. Vernov, "De Sitter and power-law solutions in non-local Gauss-Bonnet gravity," Int. J. Geom. Meth. Mod. Phys. **15** (2018) no.11, 1850188

[158] F. Bajardi, S. Capozziello and D. Vernieri, "Non-local curvature and Gauss–Bonnet cosmologies by Noether symmetries," Eur. Phys. J. Plus **135** (2020) no.12, 942

[159] S. Capozziello, E. Elizalde, S. Nojiri and S. D. Odintsov, "Accelerating cosmologies from non-local higher-derivative gravity," Phys. Lett. B **671** (2009), 193-198

[160] S. Capozziello, G. Lambiase, M. Sakellariadou and A. Stabile, "Constraining models of extended gravity using Gravity Probe B and LARES experiments," Phys. Rev. D **91** (2015) no.4, 044012

[161] G. Allemandi, M. Capone, S. Capozziello and M. Francaviglia, "Conformal aspects of Palatini approach in extended theories of gravity," Gen. Rel. Grav. **38** (2006) 33.

[162] F. Bajardi and S. Capozziello, "Equivalence of nonminimally coupled cosmologies by Noether symmetries," Int. J. Mod. Phys. D **29** (2020) no.14, 2030015

[163] S. Capozziello and R. de Ritis, "Minisuperspace and Wheeler-DeWitt equation for string dilaton cosmology," Int. J. Mod. Phys. D **2** (1993), 373-379

[164] S. Capozziello, R. de Ritis and C. Rubano, "String dilaton cosmology with exponential potential," Phys. Lett. A **177** (1993), 8-12

[165] S. Capozziello and R. de Ritis, "Scale factor duality and general transformations for string cosmology," Int. J. Mod. Phys. D **2** (1993) 367.

[166] S. Capozziello, G. Gionti, S.J. and D. Vernieri, "String duality transformations in $f(R)$ gravity from Noether symmetry approach," JCAP **1601**, 015 (2016)

[167] S. Capozziello, S. Nesseris and L. Perivolaropoulos, "Reconstruction of the Scalar-Tensor Lagrangian from a LCDM Background and Noether Symmetry," JCAP **0712** (2007) 009

[168] G. Kofinas and E. N. Saridakis, "Teleparallel equivalent of Gauss-Bonnet gravity and its modifications," Phys. Rev. D **90** (2014) 084044.

[169] B. Cvetković and D. Simić, "5D Lovelock gravity: new exact solutions with torsion," Phys. Rev. D **94**, no.8, 084037 (2016) [arXiv:1608.07976 [gr-qc]].

[170] J. Zanelli, "Lecture notes on Chern-Simons (super-)gravities. Second edition (February 2008)," In Proceedings, 7th Mexican Workshop on Particles and Fields (MWPF 1999): Merida, Mexico, November 10-17, 1999, 2005. [arXiv:hep-th/0502193 [hep-th]].

[171] F. Bajardi, S. Capozziello, D. Vernieri, "Cosmological and Spherically Symmetric Solutions of Lovelock Gravity in D-Dimensions," Submitted

[172] S. S. Chern and J. Simons, "Characteristic forms and geometric invariants," Annals Math. **99** (1974), 48-69

[173] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, "N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals," JHEP **10** (2008), 091

[174] E. Witten, "Quantum Field Theory and the Jones Polynomial," Commun. Math. Phys. **121** (1989), 351-399

[175] D. T. Son and A. O. Starinets, "Minkowski space correlators in AdS / CFT correspondence: Recipe and applications," JHEP **09** (2002), 042

[176] A. Achucarro and P. K. Townsend, "A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories," Phys. Lett. B **180** (1986), 89

[177] A. Borowiec, L. Fatibene, M. Ferraris and M. Francaviglia, "Covariant Lagrangian formulation of Chern-Simons and BF theories," Int. J. Geom. Meth. Mod. Phys. **3** (2006), 755-774

[178] L. Fatibene, M. Francaviglia and S. Mercadante, “Covariant formulation of Chern-Simons theories,” Int. J. Geom. Meth. Mod. Phys. **2** (2005), 993-1008

[179] R. G. Cai and K. S. Soh, “Topological black holes in the dimensionally continued gravity,” Phys. Rev. D **59** (1999), 044013

[180] L. Smolin and C. Soo, “The Chern-Simons invariant as the natural time variable for classical and quantum cosmology,” Nucl. Phys. B **449** (1995), 289-316

[181] S. Giombi, S. Minwalla, S. Prakash, S. P. Trivedi, S. R. Wadia and X. Yin, “Chern-Simons Theory with Vector Fermion Matter,” Eur. Phys. J. C **72** (2012), 2112

[182] L. Susskind, “The Quantum Hall fluid and noncommutative Chern-Simons theory,” [arXiv:hep-th/0101029 [hep-th]].

[183] J. Oliva and S. Ray, “Classification of Six Derivative Lagrangians of Gravity and Static Spherically Symmetric Solutions,” Phys. Rev. D **82**, 124030 (2010) [arXiv:1004.0737 [gr-qc]].

[184] G. F. Chapline and N. S. Manton, “Unification of Yang-Mills Theory and Supergravity in Ten-Dimensions,” Phys. Lett. B **120** (1983), 105-109

[185] M. Benna, I. Klebanov, T. Klose and M. Smedback, “Superconformal Chern-Simons Theories and AdS(4)/CFT(3) Correspondence,” JHEP **09** (2008), 072

[186] A. H. Chamseddine, “Topological gravity and supergravity in various dimensions,” Nucl. Phys. B **346** (1990), 213-234

[187] N. Ahmed and H. Rafat, “Deformation Retract and Folding of the 5D Schwarzchild Field,” [arXiv:1405.1057 [physics.gen-ph]].

[188] Y. S. Myung, “Entropy of the three-dimensional Schwarzschild-de Sitter black hole,” Mod. Phys. Lett. A **16**, 2353 (2001) [arXiv:hep-th/0110123 [hep-th]].

[189] O. Aharony and T. Banks, “Note on the quantum mechanics of M theory,” JHEP **03**, 016 (1999) [arXiv:hep-th/9812237 [hep-th]].

[190] C. Garraffo and G. Giribet, “The Lovelock Black Holes,” Mod. Phys. Lett. A **23** (2008), 1801-1818

[191] S. Carlip, S. Deser, A. Waldron and D. K. Wise, “Cosmological Topologically Massive Gravitons and Photons,” Class. Quant. Grav. **26** (2009), 075008

[192] R. Jackiw, “Chern-Simons violation of Lorentz and PCT symmetries in electrodynamics,” Comments Mod. Phys. A **1** (1999), 1-9

[193] L. Heisenberg, “Generalization of the Proca Action,” JCAP **05** (2014), 015

[194] S. M. Carroll, G. B. Field and R. Jackiw, “Limits on a Lorentz and Parity Violating Modification of Electrodynamics,” Phys. Rev. D **41** (1990), 1231

[195] P. Baldi and S. Branak, “Bioinformatics, the machine learning approach," Second edition, The MIT Press, Cambridge (USA) - London (UK), 2001,

[196] M. Waterman, “Introduction to Computational Biology," Maps, Sequences and Genomes, Chapman Hall, 1995,

[197] D. Gusfield, “Algorithms on Strings, Trees and Sequences: Computer Science and computational Biology," Cambridge University Press, Cambridge, 1997

[198] S. Vinga and J. Almeida, “Alignment-Free Sequence Comparison: A Review," Bioinformatics **19**(3) (2003), 513-523

[199] J. E. Phillips, V. G. Corces “Master weaver of the genome," Cell. **137**(7) (2009), 1194-211.

[200] T. Lengauer, M. Rarey “Computational methods for biomolecular docking," Curr. Opin. Struct. Biol. **6** (1996), 402-406.

[201] R.J. Morris, R.J. Najmanovich, A. Kahraman, J.M. Thornton “Real spherical harmonic expansion coefficients as 3D shape descriptors for protein binding pocket and ligand comparisons," Bioinformatics **21**(2005), 2347-55.

[202] A. Kahraman, R.J. Morris, R.A. Laskowski, J.M. Thornton “Shape variation in protein binding pockets and their ligands," J. Mol. Biol. **368** (2007), 283-301.

[203] P. Dabrowski-Tumanski, J.I. Sulkowska “Topological knots and links in proteins," Proc. Natl. Acad. Sci. **114** (2017), 3415-3420

[204] M. Lachner, T. Jenuwein “The many faces of histone lysine methylation," Curr. Opin. Cell. Biol. **14** (2002), 286-98

[205] M. Lin, A. Zewail "Protein folding – simplicity in complexity," Ann. Phys. **524** (2012), 379-391

[206] S. Capozziello, R. Pincak, K. Kanjamapornkul and E. N. Saridakis, "The Chern-Simons Current in Systems of DNA-RNA Transcriptions," Annalen Phys. **530** (2018) no.4, 1700271

[207] S. Capozziello and R. Pincak, "The Chern–Simons current in time series of knots and links in proteins," Annals Phys. **393** (2018), 413-446

[208] O. Kranenburg "The KRAS Oncogene," Biochimica et Biophysica Acta **1756** (2005), 81-82.

[209] J. M. Maldacena, "Wilson loops in large N field theories," Phys. Rev. Lett. **80** (1998), 4859-4862

[210] G. P. Lepage and P. B. Mackenzie, "On the viability of lattice perturbation theory," Phys. Rev. D **48** (1993), 2250-2264

[211] Y. Aoki, Z. Fodor, S. D. Katz and K. K. Szabo, "The QCD transition temperature: Results with physical masses in the continuum limit," Phys. Lett. B **643** (2006), 46-54

[212] T. Appelquist and R. D. Pisarski, "High-Temperature Yang-Mills Theories and Three-Dimensional Quantum Chromodynamics," Phys. Rev. D **23** (1981), 2305

[213] K. G. Wilson, "Confinement of Quarks," Phys. Rev. D **10** (1974), 2445-2459

[214] D.J. Hartman, J.M. Davison, T.J. Foxwell, M.N. Nikiforova, S.I. Chiosea, "Mutant allele-specific imbalance modulates prognostic impact of KRAS mutations in colorectal adenocarcinoma and is associated with worse overall survival," International Journal of Cancer **131** (8) (2012), 1810-7

[215] S. Misale, R. Yaeger, S. Hobor, E. Scala, M. Janakiraman, D. Liska, E. Valtorta, R. Schiavo, M. Buscarino, G. Siravegna, K. Bencardino, A. Cercek, C.T. Chen, S. Veronese, C. Zanon, A. Sartore-Bianchi, M. Gambacorta, M. Gallicchio, E. Vakiani, V. Boscaro, E. Medico, M. Weiser, S. Siena, F. Di Nicolantonio, D. Solit, A. Bardelli, "Emergence of KRAS mutations and acquired resistance to anti-EGFR therapy in

colorectal cancer," Nature **486** (7404) (2012), 532-6

[216] A. Liévre, J.B. Bachet, D. Le Corre, V. Boige, B. Landi, J.F. Emile, J.F. Coté, G. Tomasic, C. Penna, M. Ducreux, P. Rougier, F. Penault-Llorca, P. Laurent-Puig, “KRAS mutation status is predictive of response to cetuximab therapy in colorectal cancer," Cancer Research **66** (8) (2006), 3992-5

[217] R. Benedetti, F. Bajardi, S. Capozziello, V. Carafa, M. Conte, M. R. Del Sorbo, A. Nebbioso, M. Singh, H. G. Stunnenberg and M. Valadan, *et al.* “Different approaches to unveil biomolecule configurations and their mutual interactions,” 10.1080/00032719.2020.1716241, [arXiv:2002.02364 [q-bio.BM]].

[218] F. Bajardi, C. Altucci, L.Altucci, R. Benedetti, S. Capozziello, M. Del Sorbo and G. Franci, “DNA Mutations Description via Chern-Simons Current," Submitted to Scientific Reports (2021)

[219] Y. Yuan Y, D. Cao, Y. Zhang, J. Ma, J. Qi, Q. Wang, G. Lu, Y. Wu, J. Yan, Y. Shi, X. Zhang, G.F. Gao, “Cryo-EM structures of MERS-CoV and SARS-CoV spike glycoproteins reveal the dynamic receptor binding domains,” Nat Commun. **8** (2017) 15092

[220] R. Yan, Y. Zhang, Y. Li, L. Xia, Y. Guo, Q. Zhou, “Structural basis for the recognition of SARS-CoV-2 by full-length human ACE2,” Science **367** (2020) 6485

[221] A.C. Walls, Y.J. Park, M.A. Tortorici, A. Wall, A.T. McGuire, D. Veesler, “Structure, Function, and Antigenicity of the SARS-CoV-2 Spike Glycoprotein,” Cell **181** (2020) 281-292

[222] X. Ou, Y. Liu, X. Lei, P. Li, D. Mi, L. Ren, L. Guo, R. Guo, T. Chen, J. Hu, Z. Xiang, Z. Mu, X. Chen, J. Chen, K. Hu, Q. Jin, J. Wang, Z. Qian, “Characterization of spike glycoprotein of SARS-CoV-2 on virus entry and its immune cross-reactivity with SARS-CoV,” Nat. Commun. **11** (2020) 1620

[223] R. Lu *et al.*, “Genomic characterisation and epidemiology of 2019 novel coronavirus: implications for virus origins and receptor binding,” Lancet **395** (2020) 10224

[224] D. Wrapp, N. Wang, K.S. Corbett, J.A. Goldsmith, C.L. Hsieh, O. Abiona, B.S.

Graham, J.S. McLellan, “Cryo-EM structure of the 2019-nCoV spike in the prefusion conformation,” Science **367** (2020) 6483

[225] C.A. de Haan, L. Kuo, P.S. Masters, H. Vennema, P.J. Rottier, “Coronavirus particle assembly: primary structure requirements of the membrane protein,” J. Virol. **72** (1998) 6838-50

[226] S.M. Lokman, M. Rasheduzzaman, A. Salauddin, R. Barua, A.Y. Tanzina, M.H. Rumi, M.I. Hossain, A. Siddiki, A. Mannan, M.M. Hasan, “Exploring the genomic and proteomic variations of SARS-CoV-2 spike glycoprotein: A computational biology approach,” Infect. Genet. Evol. **84** (2020) 104389

[227] S. Kumar, V.K. Maurya, A.K. Prasad, M.L.B. Bhatt, S.K. Saxena, “Structural, glycosylation and antigenic variation between 2019 novel coronavirus (2019-nCoV) and SARS coronavirus (SARS-CoV),” Virusdisease **31** (2020) 13-21

[228] J. Lan, J. Ge, J. Yu, S. Shan, H. Zhou, S. Fan, Q. Zhang, X. Shi, Q. Wang, L. Zhang, X. Wang, “Structure of the SARS-CoV-2 spike receptor-binding domain bound to the ACE2 receptor,” Nature **581** (2020) 215-220

[229] Y. Watanabe, J. D. Allen, D. Wrapp, J. S. McLellan, M. Crispin, “Site-specific glycan analysis of the SARS-CoV-2 spike," American Association for the Advancement of Science **369** (2020), 6501

[230] L.M. Bobay, A.C. O’Donnell, H. Ochman, “Recombination events are concentrated in the spike protein region of Betacoronaviruses,” PLoS Genet. **16** (2020) e1009272

[231] M.R. Islam, M.N. Hoque, M.S. Rahman, A. Alam, M. Akther, J.A. Puspo, S. Akter, M. Sultana, K.A. Crandall, M.A. Hossain, “Genome-wide analysis of SARS-CoV-2 virus strains circulating worldwide implicates heterogeneity,” Sci. Rep. **10** (2020) 14004

[232] L. van Dorp, D. Richard, C.C.S. Tan, L.P. Shaw, M. Acman, F. Balloux, “No evidence for increased transmissibility from recurrent mutations in SARS-CoV-2,” Nat. Commun. **11** (2020) 5986

[233] S. Capozziello and A. De Felice, “f(R) cosmology by Noether’s symmetry,” JCAP

0808 (2008) 016

[234] H. R. Fazlollahi, "F(R) cosmology via Noether symmetry and Λ-Chaplygin Gas like model," Phys. Lett. B **781** (2018) 542

[235] M. Sharif and I. Nawazish, "Wormhole geometry and Noether symmetry in $f(R)$ gravity," Annals Phys. **389** (2018) 283

[236] S. Capozziello, N. Frusciante and D. Vernieri, "New Spherically Symmetric Solutions in f(R)-gravity by Noether Symmetries," Gen. Rel. Grav. **44** (2012) 1881

[237] B. Vakili, "Noether symmetry in f(R) cosmology," Phys. Lett. B **664** (2008) 16

[238] A. Paliathanasis, M. Tsamparlis and S. Basilakos, "Constraints and analytical solutions of $f(R)$ theories of gravity using Noether symmetries," Phys. Rev. D **84** (2011) 123514

[239] S. Capozziello, M. De Laurentis and R. Myrzakulov, "Noether Symmetry Approach for teleparallel-curvature cosmology," Int. J. Geom. Meth. Mod. Phys. **12** (2015) no.09, 1550095

[240] A. Paliathanasis, S. Basilakos, E. N. Saridakis, S. Capozziello, K. Atazadeh, F. Darabi and M. Tsamparlis, "New Schwarzschild-like solutions in f(T) gravity through Noether symmetries," Phys. Rev. D **89** (2014) 104042

[241] S. Basilakos, S. Capozziello, M. De Laurentis, A. Paliathanasis and M. Tsamparlis, "Noether symmetries and analytical solutions in f(T)-cosmology: A complete study," Phys. Rev. D **88** (2013) 103526

[242] K. Atazadeh and F. Darabi, "$f(T)$ cosmology via Noether symmetry," Eur. Phys. J. C **72** (2012) 2016

[243] H. Mohseni Sadjadi, "Generalized Noether symmetry in f(T) gravity," Phys. Lett. B **718** (2012) 270

[244] H. Wei, X. J. Guo and L. F. Wang, "Noether Symmetry in $f(T)$ Theory," Phys. Lett. B **707** (2012) 298

[245] S. Capozziello, M. De Laurentis and S. D. Odintsov, "Noether Symmetry Approach in Gauss–Bonnet Cosmology," Mod. Phys. Lett. A **29** (2014) no.30, 1450164

[246] A. Borowiec, S. Capozziello, M. De Laurentis, F. S. N. Lobo, A. Paliathanasis, M. Paolella and A. Wojnar, “Invariant solutions and Noether symmetries in Hybrid Gravity,” Phys. Rev. D **91** (2015) no.2, 023517

[247] A. Paliathanasis, M. Tsamparlis, S. Basilakos and S. Capozziello, “Scalar-Tensor Gravity Cosmology: Noether symmetries and analytical solutions,” Phys. Rev. D **89** (2014) no.6, 063532

[248] S. Capozziello, M. De Laurentis and S. D. Odintsov, “Hamiltonian dynamics and Noether symmetries in Extended Gravity Cosmology,” Eur. Phys. J. C **72** (2012) 2068

[249] Y. Zhang, Y. g. Gong and Z. H. Zhu, “Noether Symmetry Approach in multiple scalar fields Scenario,” Phys. Lett. B **688** (2010) 13

[250] M. Tsamparlis and A. Paliathanasis, “The geometric nature of Lie and Noether symmetries,” Gen. Rel. Grav. **43** (2011) 1861.

[251] K. F. Dialektopoulos and S. Capozziello, “Noether Symmetries as a geometric criterion to select theories of gravity,” Int. J. Geom. Meth. Mod. Phys. **15** (2018) no.supp01, 1840007

[252] S. Capozziello, R. De Ritis, C. Rubano and P. Scudellaro, “Noether symmetries in cosmology,” Riv. Nuovo Cim. **19N4** (1996), 1-114

[253] R. Bousso and L. Susskind, “The Multiverse Interpretation of Quantum Mechanics,” Phys. Rev. D **85** (2012), 045007

[254] A. Vilenkin, “The Interpretation of the Wave Function of the Universe,” Phys. Rev. D **39** (1989), 1116

[255] S. W. Hawking and G. T. Horowitz, “The Gravitational Hamiltonian, action, entropy and surface terms,” Class. Quant. Grav. **13** (1996), 1487-1498

[256] K. Kuchar, “Geometry of Hyperspace. 1.,” J. Math. Phys. **17** (1976), 777-791

[257] A. Ashtekar, “New Hamiltonian Formulation of General Relativity,” Phys. Rev. D **36** (1987), 1587-1602

[258] B. S. DeWitt, “Quantum Theory of Gravity. 1. The Canonical Theory,” Phys. Rev. **160** (1967), 1113-1148.

[259] T. Thiemann, “Modern canonical quantum general relativity,” [arXiv:gr-qc/0110034 [gr-qc]].

[260] S. Capozziello, R. De Ritis and P. Scudellaro, “Noether’s symmetries in quantum cosmology,” Int. J. Mod. Phys. D **3** (1994) 609.

[261] S. Capozziello and R. de Ritis, “Noether’s symmetries and exact solutions in flat nonminimally coupled cosmological models,” Class. Quant. Grav. **11** (1994) 107.

[262] S. Capozziello, R. De Ritis and P. Scudellaro, “Noether’s symmetries in nonflat cosmologies,” Nuovo Cim. B **109** (1994) 159.

[263] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept. **323** (2000), 183-386

[264] T. Padmanabhan, “Dark energy and gravity,” Gen. Rel. Grav. **40** (2008), 529-564

[265] L. Barack, V. Cardoso, S. Nissanke, T. P. Sotiriou, A. Askar, C. Belczynski, G. Bertone, E. Bon, D. Blas and R. Brito, *et al.* “Black holes, gravitational waves and fundamental physics: a roadmap,” Class. Quant. Grav. **36** (2019) no.14, 143001

[266] P. Bull, Y. Akrami, J. Adamek, T. Baker, E. Bellini, J. Beltran Jimenez, E. Bentivegna, S. Camera, S. Clesse and J. H. Davis, *et al.* “Beyond ΛCDM: Problems, solutions, and the road ahead,” Phys. Dark Univ. **12** (2016), 56-99

[267] E. Witten, “Chern-Simons gauge theory as a string theory,” Prog. Math. **133** (1995), 637-678

[268] H. Ooguri and C. Vafa, “Knot invariants and topological strings,” Nucl. Phys. B **577** (2000), 419-438

[269] L. Avilés, P. Mella, C. Quinzacara and P. Salgado, “Some cosmological solutions in Einstein-Chern-Simons gravity,” [arXiv:1607.07137 [gr-qc]].

[270] F. Gomez, P. Minning and P. Salgado, “Standard cosmology in Chern-Simons gravity,” Phys. Rev. D **84** (2011), 063506

[271] C. Molina, P. Pani, V. Cardoso and L. Gualtieri, "Gravitational signature of Schwarzschild black holes in dynamical Chern-Simons gravity," Phys. Rev. D **81** (2010), 124021

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