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Groups with restriction on non-normal subgroups

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Introduction

Within the cosmos of group theory, the class of Abelian groups has always been archetypal. As a matter of fact, renouncing commutativity, which we are all accustomed to from the early stages of our mathematical education, often demands a significant increase of sophistication in the arguments used. I believe it is fair to say that the possibility of a deeper and more complete description of the structure of Abelian groups derives from two, among others, of their astounding features: the existence of the torsion subgroup and the fact that all subgroups are normal. As a matter of fact, the former makes distinguishing the study of periodic groups and torsion-free groups a feasible approach: if one is interested in the theory of infinite groups with an attention to finiteness conditions, this is especially interesting when considered together with the fact that periodic Abelian groups are locally finite.
On the other hand, the latter is what really enables, on a more abstract level, to approach studying groups with a *divide et impera* philosophy, which is what often ought to be done if the structure of the group is to be described. In other words, while studying the structure of a group G, one can expect to recover some information about G from what is known about one of its normal subgroups H and the quotient G/H. That is why, for example, groups that have no proper non-trivial normal subgroup, i.e. simple groups, are often conceptualized as atoms in group theory. This is not necessarily intended in the sense that they are building blocks for any other groups, but, in the original etymology sense of the word, as groups that cannot be described by further *breaking them down* into smaller pieces.
I believe it is amazing how often it happens in group theory, that some kind of obstruction to the mathematician's line of reasoning can be localized to a

subgroup H of the studied group G: the argument can then be put forward in the quotient G/H of the group modulo that subgroup, with the hope of then being able to somehow recover information about G itself. With this idea in mind, it is reasonable to expect that groups that are rich in normal subgroups have a structure which is easier to describe.

A branch of group theory has had the class $\mathfrak{A}$ of abelian groups as its archetype in this sense: for classes of groups that are, in some sense, conceptually close to abelianity, how many of the usual arguments for abelian groups can be adapted? Which properties are preserved?

It is, in fact, fairly early in the history of group theory that non-Abelian groups with all subgroups normal were first considered. They were studied in the finite case by Richard Dedekind, culminating in ([7], 1897), and then in the infinite case by Reinhold Baer, with ([3], 1933). In honor of the former, they are nowadays often called Dedekind groups.

The easieast and smallest example of a non-Abelian Dedekind group is the quaternion group, and that is why non-Abelian Dedekind groups are often referred to as Hamiltonian groups, to honor the work of Hamilton in the context of quaternions. The structure of these groups were completely described in the aforementioned papers. In all truth, they were shown to be extremely *close*, in some conceptual sense, to being Abelian, since in any Hamiltonian group G it is always possible to identify a normal subgroup Q isomorphic to the quaternion group, Q is a direct factor in the group and the quotient G/Q is Abelian.

As a matter of fact, at the beginning of their famous 1903 article on minimal-non-Abelian (finite) groups, i.e. non Abelian (finite) groups whose proper subgroups are Abelian, Miller and Moreno (see [40]) write

> *"Several years ago Dedekind and others investigated the groups in which every subgroup is invariant, and found that the theory of these groups presents remarkably few difficulties except such as are involved in Abelian groups".*

What Dedekind had done was a small step outside the class of Abelian groups, but, as Miller and Moreno immediately recognized pointing out Dedekind's work, it was an absolutely crucial step into understanding which ideas from Abelian group theory could be brought outside in the relatively unknown panorama (but not completely, of course, and the definitions of ideas like the commutator subgroup and solubility were already around earlier) of non-Abelian groups. These steps were, among others, at the start a large journey in group theory, investigating what can be said about groups that are not far from being Abelian, in the sense that they share with Abelian groups the property of having a subgroup lattice that is rich in normal subgroups, in some broad sense to be specified from time to time.

Even the abovementioned concept of minimal-non-Abelian groups, which especially in finite group theory are also often called Miller-Moreno groups to honor the authors, was an important step in this direction. In fact, a commonality between the two articles that was pointed out by Miller and Moreno themselves is that both Dedekind groups and minimal-non-Abelian (finite) groups are solvable.

In 1966, then, Romalis and Sesekin introduced and studied metahamiltonian groups, i.e. groups whose non-normal subgroups are Abelian (see [47]). This unified the efforts of Dedekind and Miller and Moreno, because of course this class of groups was large enough to include all hamiltonian groups but also all minimal-non-Abelian groups. They studied these groups in the finite and infinite case and immediately they understood that not all metahamiltonian groups are solvable. As a matter of fact, in more modern terms, we know that a counteraxample to the above statement is given by the so-called Tarski monsters, i.e. simple infinite groups all of whose proper non-trivial subgroups are cyclic of order p, for a given prime p. The existence of these groups was conjectured by the polish matematician Alfred Tarski and then they were actually constructed, with choice of a large enough prime p, by Alexander Yu. Olshanskii in 1979 (see [42]). These groups are of course metahamiltonian and not solvable. Romalis and Sesekin's workaround was to consider locally solvable metahamiltonian groups and within that class of groups they were able to prove solubility and finiteness of the commutator subgroup.

This class of groups proved itself able to capture the attention of many group theorists in the following decades, as it inspired the birth of many similarly defined concepts and papers on related topics, further generalizing metahamiltonianity from many points of view. It is the strange matrimony of two different conditions on subgroups, because in a metahamiltonian group every subgroup is either normal, which is an embedding property of subgroups, or Abelian, which is an absolute property of groups. Many new classes of groups were defined with the same idea, of studying a group in which all subgroups either have some kind of good embedding property or some kind of good absolute property. In this text, in the first chapter we will present many similarly inspired group classes that have been introduced in the last decades and have a fame of their own (and, still, we will only be able to discuss a small part of them), before moving on to an attempt to abstract, generalize and adapt some of the techniques used to a whole collection of group classes defined with the same idea: non-normal subgroups satisfy a property $\mathfrak{X}$.

Of course, if a group G has the property that all non-normal subgroups satisfy a group-theoretical property $\mathfrak{X}$, then the same idea can also be expressed in a logically equivalent way by stating that non-$\mathfrak{X}$-subgroups of G are normal. This opens the door to a different way of generalizing metahamiltonian groups: that is, by substituting the concept of normality in their definition with some weaker embedding property imposed to the non-abelian subgroups. In the second chapter of this text we will explore a couple of ways that this has been done in the past, by restricting the numerosity (resp. the index) of the normalizers of non-abelian groups or and then we will retrace a path similar to the one traced in chapter one, discussing the arguments that can be abstracted, generalized and adapted to deal with groups whose non-$\mathfrak{X}$ subgroups have some restrictions of normalizers.

Among other things, this thesis collects results from two of my published articles (see [22], [23]) together with a couple of unpublished results at the very end, in the last two sections.

Chapter 1

Groups with restrictions on non-normal subgroups

Definition and key properties of metahamiltonian groups

G. M. Romalis and N. F. Sesekin introduced and intensively studied metahamiltonian groups in a series of three articles in 1966, 1968 and 1969 (see [47], [48] and [49]). A group G is called *metahamiltonian* if every non-normal subgroup H of G is Abelian. In symbols, if we denote the class of all Abelian groups with the letter $\mathfrak{A}$ of the Gothic font we can write that G is a metahamiltonian group if and only if

$$\forall H \leq G, H \ntrianglelefteq G \Rightarrow H \in \mathfrak{A}.$$

While a trivial exercise in basic logic, we also state the contrapositive statement, i.e. "G is metahamiltonian if and only if every non-Abelian subgroup H of G is normal" as the two different ways of stating it inspired independent generalizations of the concept later in the mathematical literature.
We will denote the class of all metahamiltonian groups by the letter $\mathfrak{H}$. Of course, any subgroup or homomorphic image of a metahamiltonian group is still a metahamiltonian group: in other words the class is closed with respect to the group class operators S and H, but it is far from being closed with respect to extensions. As a matter of fact, even the direct product $G = H_1 \times H_2$

of two copies H_1, H_2 of the symmetric group S_3 in three elements, which is obviosuly a finite metahamiltonian group, is not a metahamiltonian group.
The class of metahamiltonian groups contains all Abelian, minimal-non-Abelian and hamiltonian groups but we could also say, in some sense, it is fairly larger than those and the dihedral group D_{12} of order 12 is the smallest example of a metahamiltonian group that is not minimal-non-Abelian or hamiltonian.
A natural question leading the investigation into metahamiltonian groups might be, in naive terms, "How far is a metahamiltonian group from being Abelian?". This question might take the specific form of an inquiry of the solubility (or generalized solubility) of metahamiltonian groups.
Of course it is not even the case that all minimal-non-Abelian groups are soluble, as shown by the consideration of Tarski monsters, simple infinite p-groups whose proper subgroups are cyclic of order p, for a fixed prime p. These groups are trivially metahamiltonian but far from being soluble.
They are also 2-generator groups, though, hence they are not even locally soluble. For this reason, some extra assumption when trying to prove solubility for metahamiltonian groups is necessary. What Romalis and Sesekin did at the time was to restrict their study to locally soluble metahamiltonian groups, which is of course enough to exclude Tarski monsters from the discussion. For any group G in this class they were able to prove that G is soluble with derived length at most 3 and that the commutator subgroup is finite of prime power order.
In an article from 1971 (see [6]), Černikov was able to reach the exact same conclusions working within the larger universe of locally graded metahamiltonian groups. Here, a group G is said to be *locally graded* if every finitely generated non-trivial subgroup has a proper subgroup of finite index: this is a relatively weak property shared by locally soluble groups and many other classes of groups defined by a generalized solubility property.
Here, though, we refer to a simplified proof of these results, contained in an article by De Giovanni and De Mari from 2005. However, a couple of lemmas of their own interest are first needed.

Lemma 1.1. *Let G be a locally graded metahamiltonian group with finite* *commutator subgroup. Then the order of G' is a prime power.*

Proof. As the commutator subgroup G' is finite, it is generated by finitely many commutators, so that we can find a finitely generated subgroup E of G such that the commutator subgroup of E is the same as G'. Moreover the factor $E/Z(E)$ of E over its center is finite. This is because E is a finitely generated FC-group. So it is possible to find a subgroup A that is torsion free and it is of finite index inside $Z(E)$.
Since $G' = E' \simeq \frac{E'A}{A}$ we can replace G by E/A in continuing our argument and so we can assume without loss of generality that the group G is itself finite. If we take X to be any Sylow p-subgroup of G, by hypothesis we know that X is either normal in G or abelian.
If X is abelian we have that $N_G(X) = C_G(X)$ and G is p-nilpotent (see for instance 10.1.8 in [46]).
It follows that in both cases G contains a normal non-trivial Sylow subgroup P, and by the Schur-Zassenhaus Theorem there exists a subgroup Q of G such that G is the semi-direct product of P and Q. If the subgroup P is abelian, than the commutator subgroup G' is contained in Q and hence by induction on the order of G we would have that G' has prime-power order.
If, instead, the subgroup P is not abelian then we know that P is a normal subgroup and G/P and G/Q are both Dedekind groups so that G' has order at most 4, so that is is necessarily of prime power order, concluding the proof of the lemma. □

Lemma 1.2. *Let G be a locally graded metahamiltonian group, and let A be a finitely generated torsion-free Abelian normal subgroup of G. Then A is contained in the center $Z(G)$ of G.*

Proof. Let us assume by contradiction that A is not contained in the center of G. This means that there exists an element x of G such that the commutator $[A, x]$ is not trivial. We will now divide the proof in two cases based on the intersection $A \cap \langle x \rangle$.
If $A \cap \langle x \rangle = \{1\}$, as A is torsion-free by hypothesis, it is possible to find an odd prime number p such that $[A^{p^n}, x] \neq \{1\}$ for all positive integers n. This means that for each positive integer n the subgroup $A^{p^n}\langle x \rangle$ is not abelian and

hence is normal in G. Moreover G/A^{p^n} is a Dedekind group for any positive integer n.
As $p > 2$, it follows that $[A, x] \leq A^{p^n}\langle x\rangle$ and so also in $\langle x\rangle$, for all positive integers n. It would follow that $[A, x] = \{1\}$, which is of course a contradiction showing that

$$A \cap \langle x\rangle = \langle x^m\rangle \neq \{1\}$$

necessarily holds instead, for some choice of a positive integer $m \geq 1$.
Let

$$\frac{A}{A \cap \langle x\rangle} = \frac{E}{A \cap \langle x\rangle} \times \frac{B}{A \cap \langle x\rangle}$$

where $\frac{E}{A\cap\langle x\rangle}$ is finite and $\frac{B}{A\cap\langle x\rangle}$ is torsion-free. As A is abelian, all elements in $A \cap \langle x\rangle$ commute with elements in A and in $\langle x\rangle$ so that $A \cap \langle x\rangle$ is contained in $Z(\langle x, A\rangle)$.
Now, $\frac{\langle E,x\rangle}{A\cap\langle x\rangle}$ is a finite group so that $\langle E, x\rangle$ is a group that is finite over its center and by Schur's theorem it follows that $[E, x]$ is a finite subgroup of A (see Theorem 4.12 in Part 1 of [45]).
On the other hand, we know A is a torsion-free group so that $[E, x]$ is necessarily trivial. As A/E is a torsion-free abelian normal subgroup of $\frac{\langle x,A\rangle}{E}$ and $\langle xE\rangle \cap \frac{A}{E} = \{1\}$, we can repeat the argument in the first part of the proof to conclude that $[A, x] \leq E$, so that $[A, x, x] = \{1\}$.
It follows

$$[A, x]^m = [A, x^m] = \{1\},$$

proving that $[A, x]$ is a bounded subgroup of a torsion-free group and so, again, $[A, x]$ is necessarily trivial, which is of course a contradiction, completing the proof of the lemma. □

With the use of the above lemmas, we can finally prove some properties of metahamiltonian groups that are central to our discussion.

Theorem 1.3 (Černikov. 1971, [6] - or - de Giovanni, De Mari, 2005, [13])**.** *Let G be a locally graded metahamiltonian group. Then G is soluble, its derived length is at most 3 and the commutator subgroup G' of G is finite and of prime power order.*

Proof. We first prove that G is soluble. Let $\mathcal{F}$ be the family of all non-Abelian subgroups of G and put

$$X = \bigcap_{F \in \mathcal{F}} F.$$

Clearly each element F of $\mathcal{F}$ is normal in G hence $X \triangleleft G$.
Moreover, the quotient group G/F is Dedekind, since the class of all Abelian groups is closed under subgroups. For this reason $G'' \leq F$ for every $F \in \mathcal{F}$ hence

$$G'' \leq X.$$

As X is the intersection of all non-Abelian subgroups, X is either Abelian or minimal-non-Abelian, since all of its proper subgroups are necessarily Abelian.
Also, X inherits the property of being locally graded by G, so it is either an Abelian group or a locally graded minimal-non-Abelian group. Since locally graded minimal-non-Abelian groups are necessarily finite, in either case G is soluble-by-finite.
Let R be the largest soluble normal subgroup of G. If R is contained in the center, then G is finite over its center. This implies, by Schur's Theorem, that G' is finite. It follows by Lemma 1.1 that G' is of prime-power order, hence soluble.
If the subgroup R is not contained in the center, then there exists an element $x \in G$ such that $[R, x] \neq 1$, hence the soluble subgroup $R\langle x \rangle$ is not Abelian. Since G is metahamiltonian, this means $R\langle x \rangle$ is a normal subgroup of G. As R is the largest soluble normal subgroup of G this means that $x \in R$ necessarily, hence R itself is not Abelian. It follows that R is normal and that G/R is Dedekind, so that G is soluble also in this case.
We have proved that, whatever condition holds for R, G is soluble hence, a posteriori, $R = G$.
In order to prove that G' is finite, it can be assumed by induction on the derived length of G that G'' is finite. By replacing G by G/G'' in our line of reasoning, we may also suppose without loss of generality that G is metabelian. Of course the statement of the theorem is absolutely trivial if G is Abelian,

hence assume G is not Abelian and let E be a finitely generated non-Abelian subgroup of G. Then E is normal in G and G/E is a Dedekind group. E is a finitely generated metabelian group and so it is residually finite (see [45], Part 2, Theorem 9.51).
We claim now that E is either Abelian-by-finite or nilpotent. As a matter of fact, if E is not Abelian-by-finite, then all of its subgroups H of finite index are normal and the quotients E/H are Dedekind. For this reason $\gamma_3(E)$ is contained in the finite residual of E, which is trivial, proving that E is nilpotent. Either way, E satisfies the maximal condition on subgroup and is metabelian, hence it is polycyclic. For this reason G' is finitely generated and we can then assume, without loss of generality, that G itself is finitely generated and so even polycyclic.
With the same reasoning as for E, we prove that G is either Abelian-by-finite or nilpotent and hence it has a torsion-free nilpotent normal subgroup N of finite index.
Let A be a maximal Abelian normal subgroup of N, so that $C_N(A) = A$; on the other hand, A is contained in $Z(N)$ by Lemma 1.2 and hence $N = A$ is Abelian. It follows again from Lemma 1.2 that N lies in $Z(G)$, so that $G/Z(G)$ is finite and hence, by Schur's Theorem, G' is finite.
Once one knows G' is finite, it is of prime power order by Lemma 1.1. □

In [6], Černikov was actually able to prove solubility also for locally graded infinite groups satisfying what he called $\overline{IH}$ property, i.e. groups such that every infinite non-Abelian subgroup is normal. Of course all finite groups satisfy this property, so that one cannot hope to prove solubility or the fact that the commutator is a p-group for some p for all locally graded $\overline{IH}$. Even restricting to locally graded infinite groups with $\overline{IH}$, Černikov showed with examples that the commutator is not necessarily finite and not necessarily a p-group for some prime number p, but he was able to show that locally graded $\overline{IH}$-groups with infinite commutator subgroup are (what we now call) Černikov groups.
As for locally graded metahamiltonian groups, a number of results restricting even further the commutator subgroup were produced in the following

years. As condensation of a series of articles (see [31–37]), Kuzennyi and Levishchenko were able to prove the following result:

Theorem 1.4 (Kuzennyi and Levishchenko, 1989)**.** *Let G be a non-Abelian locally graduated metahamiltonian group. Then its commutator subgroup G' is contained in each non-Abelian subgroup of G and one of the following is true:*

1. *G is a nilpotent metahamiltonian group and if $|G| = p^m$ for some prime number p and $1 < m \in \mathbb{N}$, then the Sylow p-subgroups of G have bounded exponents;*

2. *G is a non-nilpotent metahamiltonian group of the form $G = G' \times H$ and one of the following subcases holds true:*

   2.1. *H is a cyclic extension of a central subgroup of G or*

   2.2. *G' is a minimal normal subgroup that is the commutator subgroup of each non-Abelian subgroup of G.*

3. *G is a non-nilpotent periodic metahamiltonian group of the form $G = G' \times H$ and*

   3.1. *H is a cyclic extension of a central subgroup of G*

   3.2. *G' is a non-Abelian Sylow p-subgroup of order p^3 and if it is not a quaternion group, then it has exponent p.*

   3.3. *G' is a direct factor of each Schmidt*[1] *subgroup of G.*

In particular, it follows that for a soluble group G, one can say that G is metahamiltonian if and only if the commutator subgroup G' of G is contained in every non-Abelian subgroup of G.

An extremely useful property of the class of metahamiltonian groups is that it is local. In other words, the following theorem holds

[1] A subgroup H of a group G is called a *Schmidt* subgroup if H is not nilpotent but every proper subgroup of H is nilpotent.

Theorem 1.5. *Let G be locally graded a group such that all of its finitely generated subgroups are metahamiltonian. Then G is itself a metahamiltonian group.*

Proof. Let X be any non-Abelian subgroup of G and let us consider the family W_X of finitely generated non-Abelian subgroups of X, which is non-empty because Abelian groups form a local class. For any $Y \in W_X$ and $g \in G$, the group Y is a non-Abelian subgroup of the metahamiltonian group $\langle Y, g \rangle$, hence g normalizes Y for every $g \in G$ and so Y is normal in G. For every element x of X there is a finitely generated non-Abelian subgroup Y_x of X containing x, hence

$$\langle Y \mid Y \in W_X \rangle = X$$

and so X is normal in G as it is the join of a family of subgroups that are normal in G. In conclusion, G is metahamiltonian. □

This property of the class of metahamiltonian groups is often useful when stated in the following way: if G is a locally graded group that is not a metahamiltonian group, then it also has a finitely generated subgroup H that is not a metahamiltonian group.

Some classes of generalized metahamiltonian groups

The number of articles and definitions that metahamiltonianity inspired is immeasurable and it is not the scope of this section to list them all. Rather, it will contain a few ideas that generalize metahamiltonianity in a way that will be useful to the narrative plot of the thesis, in ways that will be understood in the next sections. Most of them are to be found in [11], an article by De Falco, de Giovanni and Musella from 2013. They looked at groups that were rich in metahamiltonian subgroups or metahamiltonian homomorphic images for many different interpretations of the word *rich*.
What is perhaps the most natural start for this investigation is to look at groups whose proper subgroups are all metahamiltonian. The authors in [11] proved that locally graded groups with such property are either finite or metahamiltonian. In other words, they proved the following theorem

Theorem 1.6 (De Falco, de Giovanni and Musella, 2013)**.** *Let G be an infinite locally graded group whose proper subgroups are metahamiltonian. Then G is metahamiltonian.*

Proof. Assume for a contradiction that G is not metahamiltonian, so that G must be finitely generated since the class of metahamiltonian groups is local by Theorem 1.5.
Thus G contains a proper subgroup K of finite index such that K is a finitely generated metahamiltonian group, hence polycyclic by Theorem 1.3. It follows that G is polycyclic-by-finite. Let X be a subgroup of G which neither is Abelian nor normal. Then X is contained in a non-normal subgroup H of G of finite index, and H contains a normal subgroup N of G such that G/N is finite and H/N is not Abelian as if this were not the case then H would be Abelian. In particular, the group G/N has a subgroup H/N which is non-Abelian non-normal, hence it is not metahamiltonian. As G is infinite, it is known that the Frattini factor group $G/\Phi(G)$ is likewise infinite (see [38]) and hence there exists a maximal subgroup M of G such that N is

not contained in M. But then $G = MN$ and

$$G/N \simeq M/M \cap N$$

would be a metahamiltonian group. This contradiction proves the statement. □

Finite minimal non-metahamiltonian groups need not be soluble, as the simple example of A_5 shows. However, something else of interest can be said about minimal-non-metahamiltonian groups

Theorem 1.7 (Atlihan, de Giovanni, 2017)**.** *Let G be a minimal-non-hamiltonian group. Then G is generated by at most three elements.*

Proof. As G is not metahamiltonian, it contains a subgroup X which neither is Abelian nor normal. For each non-central element x of X, choose an element $y(x)$ of X such that $[x, y(x)] = 1$. Since X is not normal in G and

$$X = \langle X \setminus Z(X) \rangle = \langle x, y(x) \mid \; x \in X \setminus Z(X) \rangle,$$

there exist elements x of $X \setminus Z(X)$ and g of G such that

$$\langle x, y(x) \rangle^g \neq \langle x, y(x) \rangle.$$

It follows that the subgroup $\langle x, y(x), g \rangle$ is not metahamiltonian, and so

$$G = \langle x, y(x), g \rangle$$

is a 3-generator group. □

The abovementioned example of the alternating group on five elements is a peculiar one, as the group satisfies the property that all non-normal subgroups are either Abelian or minimal-non-Abelian. This class of groups was object of study in the article [2] by de Giovanni and Atlihan. They named this class of groups the class of *parahamiltonian groups*. The authors were able to prove

Theorem 1.8 (Atlihan, de Giovanni, 2017). *Let G be a locally graded parahamiltonian group. Then G' is finite, and its order is divisible by at most three prime numbers.*

Of course A_5 is an example of a locally graded parahamiltonian group whose commutator subgroup has order divided by precisely three prime numbers. Moreover, the same authors proved that

Theorem 1.9 (Atlihan, de Giovanni, 2017). *Let G be a locally graded insoluble parahamiltonian group. Then G is finite and $\pi(G') = \{2, 3, 5\}$*

A different possible interpretation for a "group with many metahamiltonian subgroups" might be looking at groups whose proper subgroups of infinite rank are metahamiltonian. In this investigation, a recent result by M.R. Dixon, M. Evans and H. Smith, contained in [20] is very useful. As a matter of fact, the authors proved that if G is a locally (soluble-by-finite) group whose proper subgroups of infinite rank have finite commutator subgroup, then either G has finite rank or its commutator subgroup G' is finite. This can be used to prove the following theorem.

Theorem 1.10. *Let G be a locally (soluble-by-finite) group of infinite rank whose proper subgroups of infinite rank are metahamiltonian. Then G is metahamiltonian.*

Proof. It follows from Theorem 1.3 that every proper subgroup of G either is finite-by-Abelian or has finite rank, and hence the commutator subgroup G' of G must have finite rank (see [20]).
Let X be any subgroup of G of finite rank. Then the product XG' has likewise finite rank, and so the Abelian factor group G/XG' has infinite rank. It follows that XG' is contained in a proper subgroup of G of infinite rank, and hence X is metahamiltonian.
Then G itself is a metahamiltonian group by Theorem 1.6. □

In [9], the authors dealt with locally graded groups that satisfy the minimal condition on non-metahamiltonian subgroups, i.e. such that there is not any

strictly descending sequence of non-metahamiltonian subgroups.
It is proved in the abovementioned paper that within the universe of locally graded groups the minimal condition on non-metahamiltonian subgroups can occur only in the extreme cases. To understand the proof we need a couple of lemmas we omit the proof of. The first one is due to Zaicev (see [51]).

Lemma 1.11. *Let G be a locally soluble group, and let Γ be a finite group of automorphisms of G. If G is not a Černikov group, then it contains an abelian subgroup A such that $A^\Gamma = A$ and A does not satisfy the minimal condition on subgroups.*

This second lemma can be found as Lemma 4.4 in [9].

Lemma 1.12. *Let G be a locally finite group whose proper subgroups either are metahamiltonian or Černikov groups. Then G is soluble-by-finite.*

We now have the tools required to approach the study of groups satisfying the minimal condition on non-metahamiltonian subgroups, as said above.

Theorem 1.13. *Let G be a locally graded group satisfying the minimal condition on subgroups which are not metahamiltonian. Then G is either metahamiltonian or a Cernikov group.*

Proof. Let us start by considering the case of G not being periodic and let us assume for a contradiction that G is not metahamiltonian. As the class of metahamiltonian groups is local, this means there is a finitely generated subgroup E that is not metahamiltonian.
Since G is not periodic, we can find a non-periodic element a in G so that, by substituting E for $\langle E, a\rangle$, we can also assume without loss of generality that E is infinite. Again, replacing G by E, we can also assume without loss of generality that G is finitely generated. As it is also locally graded, we can construct an infinite descending series consisting of subgroups of finite index, and hence by hypothesis, there must be one of those, say H, which is a metahamiltonian subgroup of finite index of G. By Theorem 1.3, we know that H' is finite and so G is finite-by-abelian-by-finite and so also abelian-by-finite. As G is finitely generated, let A be a torsion-free abelian

normal subgroup of G such that G/A is finite. Since G satisfies the minimal condition on non-metahamiltonian subgroups, it contains a minimal-non-metahamiltonian subgroup L, and by Theorem 1.6, we know that L is a finite subgroup. In particular $L \cap A$ is a finite subgroup of a torsion-free group so that it is necessarily trivial.
From this it follows that $LA^{2^{n+1}}$ is properly contained in LA^{2^n} and so an infinite strictly descending sequence of subgroups of G is formed. By our hypotheses, this means there must be a positive integer k such that LA^{2^k} is metahamiltonian, but this is in contradiction with L not being metahamiltonian, so that we have reached a contradiction and G is necessarily metahamiltonian in this case.
Let us now consider the case of a periodic group G. We wish to start by showing that it is also locally finite. As a matter of fact, if one considers a finitely generated subgroup E of G then, as G is locally graded, we can construct an infinite chain

$$E = E_0 > E_1 > \cdots > E_n > E_{n+1} > \cdots$$

of subgroups of E such that the index $|E : E_n|$ is finite for each non-negative integer n. Thus there must be a non-negative integer k such that E_k is metahamiltonian and so E is a periodic, finitely generated and soluble-by-finite group, hence finite.
Now, assume that the statement is false, so that G is neither metahamiltonian nor Černikov. This means the set $\mathcal{M}$ of all subgroups of G which are neither metahamiltonian nor Černikov is not empty. By hypothesis, $\mathcal{M}$ has necessarily a minimal element M. By replacing G with M, we can assume without loss of generality that G is a group whose proper subgroups are metahamiltonian or Černikov, so that we can use Lemma 1.12 to conclude that G is soluble-by-finite. Again, as the class of metahamiltonian is a local class, we can find a finitely generated subgroup X of G which is not metahamiltonian. As we said G is locally finite, X is a finite subgroup. We can now apply Lemma 1.11 to conclude that there exists an abelian subgroup B of G such that $B^X = B$ and B does not satisfy the minimal condition on

subgroups. The socle S of B is obviously infinite, and contains X-invariant subgroups H and K such that $K < H$, $H \cap X = \{1\}$, and the index $|S : K|$ is finite. Then XK is a proper subgroup of G which is neither metahamiltonian nor Černikov, which is in contradiction with our assumption that every proper subgroup of G had one of the two properties. This last contradiction completes the proof of the theorem. □

The above theorem can be applied to the case of groups in which non - metahamitonian subgroups fall into finitely many conjugacy classes. In fact, a group G with this latter property locally satisfies the maximal condition on subgroups, so that a result of D.I. Zaicev shows that if X is any subgroup of G such that $X^g \leq X$ for some element g of G, then $X^g = X$ (see for instance [1], Lemma 4.6.3), and hence G satisfies the minimal condition on non-metahamiltonian subgroups.

Theorem 1.14. *Let G be an infinite locally graded group with finitely many conjugacy classes of non-metahamiltonian subgroups. Then G is itself a metahamiltonian group.*

We now move to considering groups which have, in some sense, many metahamiltonian homomorphic images. Let us start by noticing that argument contained in the proof of Theorem 1.6 also shows that every polycyclic non - metahamiltonian group has a finite homomorphic image which is not metahamiltonian. In other words in a polycyclic group, metahamiltonianity is, in some sense, controlled by the finite homomorphic images. This kind of result, often even under weaker hypotheses than the assumption of dealing with a polycyclic group, is a leitmotiv that is often reoccurring in group theory. For example, it is known that finitely generated hyper-(Abelian or finite) groups such that all of their finite homomorphic images are nilpotent are themselves nilpotent. Here a group G is said to be hyper-(Abelian or finite) if there is a (not necessarily finite) series of subgroups (here, by series, even without specification of the attribute "normal", we mean a sequence of subgroups such that any two consecutive terms are one normal in the other) starting from the trivial subgroup $\{1\}$ and ending at the group G and having

all Abelian or finite factors.
A similar result can be obtained also for metahamiltonian groups, as stated in the following theorem.

Theorem 1.15 (De Falco, de Giovanni and Musella, 2013)**.** *Let G be a finitely generated hyper-(Abelian or finite) group whose finite homomorphic images are metahamiltonian. Then G is metahamiltonian.*

Proof. Assume for a contradiction that G is not metahamiltonian.
Since finitely generated locally graded metahamiltonian groups are polycyclic, with a classical argument due to R. Bear and Zorn's Lemma it follows that G contains a normal subgroup M which is maximal with respect to the condition that G/M is not metahamiltonian. Replacing G by the factor group G/M, it can be assumed without loss of generality that all proper homomorphic images of G are metahamiltonian. On the other hand, by hypothesis there exists a non-trivial normal subgroup N of G which is either finite or Abelian; as G/N is metahamiltonian, we obtain that G is soluble-by-finite and so even soluble. Let A be the smallest nontrivial term of the derived series of G. Then G/A is a finitely generated metahamiltonian group, hence a finitely generated FC-group and so it is central-by-finite. In particular, G is finitely generated metabelian-by-finite and so also residually finite (see [45], Part 2, Theorem 9.51).
We now wish to prove that every subgroup of finite index in G is either normal or Abelian. Let X be a non-normal subgroup of G such that the index $|G : X|$ is finite.
As the core X_G of X has likewise finite index in G, there exists a collection $\{N_i\}_{i\in I}$ of normal subgroups of finite index of G such that $N_i \leq X$ for each i and

$$\bigcap_{i\in I} N_i = \{1\}$$

Then X/N_i is a non-normal subgroup of the metahamiltonian group G/N_i, and hence X/N_i is Abelian for each i. Therefore, the subgroup X satisfies $X \leq N_i'$, so $X' \leq \{1\}$ and hence is itself Abelian. Now we know that every subgroup of finite index in G is either Abelian or normal. If G is not an Abelian-by-finite group, then all of its subgroups of finite index are normal, hence all of it subgroups of finite index are nilpotent. In either case, G is polycyclic, so that the statement of the theorem is known to be true. Nonetheless one might reinforce the conclusion constructively in this way: if X is a non-Abelian subgroup of G, as it is the intersection of the subgroups of finite index that contain X, it is the intersection of normal subgroups. For this reason any non-Abelian subgroup of G is normal and G is metahamiltonian. $\square$

Some classes with restrictions on non-normal subgroups

The mathematical literature is abundant in group classes defined by restricting non-normal subgroups. This section is a small survey citing a few of the major results concerning group classes which have some relevance to the topics here addressed.
A group G is said to be an FC-group (FC stands for finite conjugacy classes) if every element has finitely many conjugates, which is easily seen to be equivalent to the following property

$$\left| \frac{G}{C_G(\langle x \rangle^G)} \right| < +\infty \quad \forall g \in G.$$

FC-groups form a class of groups that is closed under subgroups, homomorphic images and direct products, but not extensions and it is not a local class. All finite-by-Abelian and central-by-finite groups are FC-groups and, in fact, in the opposite direction, finitely generated FC-groups can be shown to be necessarily central-by-finite. Moreover B.H. Neumann studied the following stronger property, called BFC (boundedly finite conjugacy classes),

$$\exists k \in \mathbb{N} : \quad \left| \frac{G}{C_G(\langle x \rangle^G)} \right| < k \quad \forall g \in G,$$

showing that BFC-groups are precisely finite-by-Abelian groups and that there are functions bounding the order of the commutator subgroup in terms of the smallest such k and viceversa. This means every metahamiltonian group is, in fact, also a BFC-group.
On the other hand, the direct product of all the finite dihedral groups D_{2n}, for example, is an FC-group whose commutator subgroup is not even bounded, so that it is of course not a BFC-group or, more so, a metahamiltonian group.
In reality, even if the commutator subgroup of any FC-group can be shown to be locally finite, metahamiltonian groups occupy in some sense a very small part of the class of all FC-groups.
In the next section, we will deal with groups in which non-normal subgroups

are metahamiltonian. It is interesting, then, to cite here what is known of the much larger class of groups whose non-normal subgroups are FC-groups, studied recently in [28].
The first result we quote tells something of the structure non-nilpotent groups with this property that have a Chernikov commutator subgroup.

Theorem 1.16 (Kurdachenko, Otal, Russo and Vincenzi, 2004)**.** *Let G be a non-nilpotent group with Chernikov commutator subgroup K, and let D be the divisible part of K. If every non-FC-subgroup of G is normal and $C_G(D) \neq G$, then $G = DL$, where L is a finite-by-Abelian subgroup of G and $D \cap L$ is a finite G-invariant subgroup. Moreover, every non-normal subgroup of G has finite commutator subgroup.*

The second result specializes to the case of almost FC-groups (or, in other words, FC-by-finite groups) i.e. groups that have an FC-subgroup of finite index, having the abovementioned property. If they are not FC-groups, their commutator subgroup can be strongly restricted.

Theorem 1.17. *Let G be an almost FC-group. If every non-normal subgroup of G is an FC-group, then G either is an FC-group or its commutator subgroup is a Chernikov group.*

A few years later, De Falco, de Giovanni and Musella in [8] elaborated on a smaller class, consisting of groups whose non-normal subgroups have a finite commutator subgroup (i.e. BFC-groups). We start by mentioning two of their lemmas, that have an interest of their own.

Lemma 1.18. *Let G be a locally graded group whose non-normal subgroups have a locally finite commutator subgroup. Then the commutator subgroup G' of G is locally finite.*

Lemma 1.19. *Let G be a locally graded group whose non-normal subgroups have a finite commutator subgroup. Then either G is soluble or the subgroup $G^{(3)}$ is finite*

The above two lemmas converge in the following theorem, regarding groups whose non-normal subgroups have a uniform bound on the size of their commutator subgroups.

Theorem 1.20. *Let k be a positive integer, and let G be a locally graded group whose non-normal subgroups have a finite commutator subgroup of order at most k. Then the commutator subgroup G' of G is finite.*

Classes defined recursively restricting non-normal subgroups

What is contained in this section is a further contribution to the topic obtained by looking at metahamiltonian groups in the general framework of group classes that can be obtained by iterating a restriction on non-normal subgroups. With this intent, we give definitions in the most general fashion, later specializing to Abelianity. Let $\mathfrak{X}$ be any class of groups. We will put $\mathfrak{X}_1 = \mathfrak{X}$, and suppose by induction that a group class $\mathfrak{X}_k$ has been defined for some positive integer k.
We then say that a group G belongs to the class $\mathfrak{X}_{k+1}$ if and only if every non-normal subgroup H of G belongs to $\mathfrak{X}_k$, or in symbols:

$$\forall H \leq G,\, H \ntrianglelefteq G \Rightarrow H \in \mathfrak{X}_k.$$

Of course metahamiltonianity is what is obtained if $\mathfrak{X} = \mathfrak{A}$ and $k = 2$.
In general, a group class $\mathfrak{X}$ is not necessarily contained in $\mathfrak{X}_2$, as shown by the example of the class of simple groups, but this is certainly the case if $\mathfrak{X}$ is a group class that is closed with respect to forming subgroups.
However, $\mathfrak{X}_k$ is closed with respect to forming subgroup for each positive integer $k > 1$ even if $\mathfrak{X}$ is not and the following equality trivially holds by definition

$$\mathfrak{X}_{k+1} = (\mathfrak{X}_k)_2.$$

It follows that if $\mathfrak{X}$ is closed with respect to forming subgroups then $\{\mathfrak{X}_k\}_{k\in\mathbb{N}}$ is an increasing sequence of group classes. In that case, it might me interesting to look at the union of all the classes, hence we define

$$\mathfrak{X}_\infty = \bigcup_{k=1}^{+\infty} \mathfrak{X}_k.$$

Members of the class $\mathfrak{A}_k$ will be called k-hamiltonian groups (for any k). Thus the 1-hamiltonian and 2-hamiltonian groups are precisely the Abelian and the metahamiltonian groups, respectively and here we will also investigate the structure of k-hamiltonian locally graded groups for $k \geq 3$. We will also refer to groups in $\mathfrak{A}_\infty$ as ∞-hamiltonian groups. In the universe of locally graded groups, we will show that ∞-hamiltonian groups are precisely the finite-by-Abelian groups.

The following is a simple example showing that, with the choice $\mathfrak{X} = \mathfrak{A}$ we get, in fact, a strictly increasing sequence of group classes.

Let k be any positive integer, and let $p_1, \dots, p_k$ be pairwise distinct odd prime numbers. If A is a cyclic group of order $p_1 \cdots p_k$ and x is the automorphism of A which inverts all elements, the semidirect product $\langle x \rangle \ltimes A$ is a $(k+1)$-hamiltonian group which is not k-hamiltonian.

It is also clear that $\mathfrak{X}_k \leq \mathfrak{Y}_k$ for all k, whenever $\mathfrak{X}$ and $\mathfrak{Y}$ are arbitrary group classes such that $\mathfrak{X} \leq \mathfrak{Y}$. Notice also that if $\mathfrak{D}$ is the class of Dedekind groups, then the classes $\mathfrak{A}_2$ and $\mathfrak{D}_2$ do not coincide, as shown by the direct product of two copies of the quaternion group Q_8.

It is worth noticing also that if $\mathfrak{X}$ is a group class which is closed with respect to homomorphic images, the same property obviously holds also for every $\mathfrak{X}_k$. What we will now show is that if $\mathfrak{X}$ is a local class, then the same can be said of $\mathfrak{X}_k$ for any positive integer k.

Recall that a group class $\mathfrak{X}$ is local if a group G whose finite subsets are always contained in an $\mathfrak{X}$-subgroup is itself an $\mathfrak{X}$-group. Clearly, a subgroup closed group class $\mathfrak{X}$ is local if and only if it contains all groups whose finitely generated subgroups belong to $\mathfrak{X}$.

Lemma 1.21. *Let $\mathfrak{X}$ be a local group class. Then, for each positive integer k, the class $\mathfrak{X}_k$ is also local.*

Proof. Since $\mathfrak{X}_1 = \mathfrak{X}$, the statement is obvious if $k = 1$. We want to prove the stamente by induction on k. Suppose now that the class $\mathfrak{X}_k$ is local for some positive integer k. As $\mathfrak{X}_{k+1}$ is subgroup closed, it is enough to prove that a group G belongs to $\mathfrak{X}_{k+1}$ provided that all its finitely generated subgroups are $\mathfrak{X}_{k+1}$-groups. Let X be any subgroup of G which is not in $\mathfrak{X}_k$, and let $\mathcal{W}_X$ be the set of all finitely generated subgroups of X which are not contained in an $\mathfrak{X}_k$-subgroup of G. Then $\mathcal{W}_X$ is not empty, because the class

$\mathfrak{X}_k$ is local.
If g is any element of G and $U \in \mathcal{W}_X$ the subgroup $\langle g, U \rangle$ belongs to $\mathfrak{X}_{k+1}$, whence $U^g = U$. It follows that all elements of $\mathcal{W}_X$ are normal in G. Moreover,

$$\langle x, U \rangle \in \mathcal{W}_X$$

for all $x \in X$, and so

$$X = \langle V | V \in \mathcal{W}_X \rangle$$

is likewise normal in G.
Therefore G belongs to $\mathfrak{X}_{k+1}$, and hence $\mathfrak{X}_{k+1}$ is a local class. □

Since the class of Abelian groups is obviously local, the choice $\mathfrak{X} = \mathfrak{A}$ in the above statement gives the following interesting special case, which generalises the well-known Theorem 1.5 for metahamiltonian groups.

Corollary 1.22. *For each positive integer k, the class $\mathfrak{A}_k$ of k-hamiltonian groups is local.*

Let $\mathfrak{X}$ be a group class. A subgroup X of a group G is said to be compressed by $\mathfrak{X}$ if it contains a normal subgroup N of G such that G/N is an $\mathfrak{X}$-group; in this case, such a subgroup N will be called an $\mathfrak{X}$-compressor for X in G. Of course, if the class $\mathfrak{X}$ is closed with respect to homomorphic images, the core X_G of an $\mathfrak{X}$-compressed subgroup X is an $\mathfrak{X}$-compressor for X in G.
It is also clear that in any group the class of finite groups compresses all subgroups of finite index. With regard to this definition, we may prove a useful lemma

Lemma 1.23. *Let $\mathfrak{X}$ be a group class, and let X and Y be subgroups of a group G such that X is not normal in G and $Y \leq X$. If Y is compressed in G by the class $\mathfrak{X}_k$ for some integer $k > 1$, then Y is compressed in X by $\mathfrak{X}_{k-1}$.*

Proof. Let N be an $\mathfrak{X}_k$-compressor for Y in G. Then X/N is a non-normal subgroup of the $\mathfrak{X}_k$-group G/N, and hence it belongs to $\mathfrak{X}_{k-1}$, which means that Y is compressed by $\mathfrak{X}_{k-1}$ in X. □

Let $\mathfrak{X}$ be any subgroup closed group class such that every finitely generated hyper-(Abelian or finite) group whose subgroup of finite index are compressed by $\mathfrak{X}$, belongs to $\mathfrak{X}$ and it is polycyclic-by-finite: we will call such a class a *Robinson class*. Of course if $\mathfrak{X}$ is also closed under homomorphic images then the definition of a Robinson class can be stated in a simpler way. If so, indeed, a group class $\mathfrak{X}$ is a Robinson class if and only if every finitely generated hyper-(Abelian or finite) group G whose finite homomorphic images lie in $\mathfrak{X}$ is itself a polycyclic-by-finite $\mathfrak{X}$-group. With Theorem 1.15, De Falco, de Giovanni and Musella have proven that the class of metahamiltonian groups, or $\mathfrak{A}_2$, with our new notation, is a Robinson class. The concept of a Robinson class arises from abstraction of a series of results that have been produced in the history of group theory, and owes its name to a result by Robinson (see, for example, [45]) proving that the class $\mathfrak{N}$ of nilpotent groups is a Robinson class. As a matter of fact, for every natural number c, his argument can be easily adapted to proving that $\mathfrak{N}_c$ is a Robinson class, where by $\mathfrak{N}_c$ we mean the class consisting of all nilpotent groups whose nilpotency class at most c. Further corollary of that, even though provable in other ways, is the fact that the class $\mathfrak{A}$ of Abelian groups is a Robinson class. Consequence of the next theorem, stated in a more general form, is the fact that k-hamiltonian groups form a Robinson class $\mathfrak{A}_k$ for any choice of the natural number k, extending the abovementioned result from [11].

Theorem 1.24. *Let $\mathfrak{X}$ be a Robinson class of groups. For any positive integer k, the class $\mathfrak{X}_k$ is also a Robinson class.*

Proof. The statement is obvious if $k = 1$. Suppose now by induction on k that $\mathfrak{X}_k$ is a Robinson class for some positive integer k, and let G be any finitely generated hyper- (Abelian or finite) group in which all subgroups of finite index are compressed by $\mathfrak{X}_{k+1}$.

If X is any non-normal subgroup of finite index of G, it follows from Lemma 1.23 that every subgroup of finite index of X is compressed in X by the Robinson class $\mathfrak{X}_k$; then the finitely generated hyper-(Abelian or finite) group X belongs to $\mathfrak{X}_k$, and in particular it is polycyclic-by-finite. On the other hand, if all subgroups of finite index of G are normal, then every finite homomorphic image of G is nilpotent, and so it follows from Robinson's theorem that G itself is nilpotent and hence also polycyclic. Therefore G is polycyclic-by-finite in any case. Let H be any non-normal subgroup of G. Since H is the intersection of a collection of subgroups of finite index of G, it is contained in a subgroup K of finite index which is not normal in G. Thus K belongs to $\mathfrak{X}_k$ and so H is an $\mathfrak{X}_k$-group, because $\mathfrak{X}_k$ is subgroup closed. Therefore G belongs to $\mathfrak{X}_{k+1}$ and hence $\mathfrak{X}_{k+1}$ is a Robinson class. □

Let $\mathfrak{X}$ be any class of groups. Recall that a minimal-non-$\mathfrak{X}$ group G is a group that does not belong to the class $\mathfrak{X}$ but such that all of its proper subgroups do. We will say that a class $\mathfrak{X}$ is *accessible* if any locally graded group whose proper subgroups belong to $\mathfrak{X}$ is either an $\mathfrak{X}$-group or a finite group. In other words $\mathfrak{X}$ is an accessible class if every locally graded minimal-non-$\mathfrak{X}$ is finite. Of course if $\mathfrak{X}$ is a class of groups containing the class of finite groups, this would mean that $\mathfrak{X}$ is an accessible class whenever examples of locally graded minimal-non-$\mathfrak{X}$ groups do not exists.
We have already mentioned the fact that Abelian groups form an accessible class and proved with Theorem 1.6 that the same can be said for metahamiltonian groups. The following is a lemma that we prove with the intent of generalizing the above result, i.e. to prove that k-hamiltonian groups for an accessible class $\mathfrak{A}_k$ for any choice of the natural number k.

Lemma 1.25. *Let $\mathfrak{X}$ be a subgroup closed group class and let G be a group in the class $\mathfrak{X}_k$ for some integer $k > 1$. Then all proper subgroups of G'' belong to $\mathfrak{X}_{k-1}$.*

Proof. Let X be any subgroup of G which is not in $\mathfrak{X}_{k-1}$. Then all subgroups of G containing X are normal, so that G/X is a Dedekind group and hence $G'' \leq X$. For this reason, G'' is contained in the intersection of all non-$\mathfrak{X}_{k-1}$ subgroups of G. It follows, then, that all proper subgroups of G'' belong to group class $\mathfrak{X}_{k-1}$. □

As stated above, a consequence of the following theorem, which will be stated in a more general form, is the fact that k-hamiltonian groups for an accessible class $\mathfrak{A}_k$ for any choice of the natural number k.

Theorem 1.26. *Let X be a Robinson class consisting of soluble-by-finite groups, which is local and closed with respect to homomorphic images. If $\mathfrak{X}$ is accessible, then for each positive integer k the class $\mathfrak{X}_k$ is also accessible.*

Proof. Assume for a contradiction that the statement is false and let k be the smallest positive integer such that there exists an infinite locally graded group G which is not an $\mathfrak{X}_k$ group while all its proper subgroups belong to $\mathfrak{X}_k$.
Clearly, $k > 1$ since $\mathfrak{X}$ is accessible by hypothesis. Moreover, G is finitely generated because $\mathfrak{X}_k$ is a local class by Lemma 1.21, and so G contains a proper subgroup X of finite index.
By an iterated application of Lemma 1.25, there is a positive integer n such that the subgroup $X^{(n)}$ either is minimal-non-$\mathfrak{X}_h$ for some $h < k$ or belongs to $\mathfrak{X}$. In the first case $X^{(n)}$ is finite by the minimal assumption on k and so it follows that the subgroup X is soluble-by-finite in any case, implying that the group G itself is soluble-by-finite.
Since $\mathfrak{X}_k$ is a Robinson class by Theorem 1.24, the group G contains a normal subgroup N of finite index such that G/N is not in $\mathfrak{X}_k$. On the other hand, the Frattini factor group $G/\Phi(G)$ is infinite by a result of Lennox (see [38]), and so there exists a maximal subgroup M of G such that $G = MN$. It follows that

$$G/N \simeq M/M \cap N$$

is an $\mathfrak{X}_k$-group, and this contradiction completes the proof of the theorem. □

Our next point is to show that locally graded k-hamiltonian groups are always soluble-by-finite, for any choice of the natural number k. We will obtain such a result as corollary of a more general statement, i.e. the fact that for any k the property of being a subclass of the class of soluble-by-finite groups is inherited from $\mathfrak{X}$ to $\mathfrak{X}_k$ for any natural number k in the case of an accessible class $\mathfrak{X}$.

Lemma 1.27. *Let $\mathfrak{X}$ be a subgroup closed group class such that $\mathfrak{X}_k$ is accessible for each positive integer k. If $\mathfrak{X}$ consists of soluble-by-finite groups, then all groups in $\mathfrak{X}_\infty$ are soluble-by-finite.*

Proof. Assume for a contradiction that the statement is false and let k be the smallest positive integer such that the class $\mathfrak{X}_k$ contains a group G which is not soluble-by-finite.
Then $k > 1$ and it follows from Lemma 1.25 that all proper subgroups of G'' belong to $\mathfrak{X}_{k-1}$. As the class $\mathfrak{X}_{k-1}$ is accessible, G'' either is finite or belongs to $\mathfrak{X}_{k-1}$. Either way, G'' is soluble-by-finite, which is of course a contradiction proving the statement of the theorem. □

Corollary 1.28. *Let k be a positive integer and let G be a locally graded k-hamiltonian group. Then G is soluble-by-finite.*

Proof. Of course by the Theorem 1.26 we know that the group class $\mathfrak{A}_k$ is accessible for each positive integer k and it follows from Lemma 1.27 that all groups in $\mathfrak{A}_\infty$ are soluble-by-finite. Hence the statement is proved. □

In section 1 we said that locally graded metahamiltonian groups have finite commutator subgroup, are soluble of derived length at most 3 and that the commutator subgroup is of prime power order. We will deal with the first of these three properties later, but for now let us look at the last two, which fail for locally graded k-hamiltonian groups whenever $k > 2$. Of course, since the class of 3-hamiltonian groups contains the class of parahamiltonian groups, we already know of the example of A_5 as a perfect group that is 3-hamiltonian. But even when one considers soluble 3-hamiltonian groups, it is not true that their derived length is at most 3 and their commutator subgroup is of prime-power order.
The simplest example to show this is the group $G = GL(2,3)$, the general linear group of two-by-two matrices with non-zero determinant over the field with three elements. In this case G is a group of order $48 = 2^4 \times 3$ which is soluble of derived length 4 and it has commutator subgroup isomorphic to the special linear group $SL(2,3)$ of order $24 = 2^3 \times 3$, hence not a p-group

for any prime p. However, G is easily proven a 3-hamiltonian group even without knowing well the structure of the general linear group, because any subgroup H of G that is not normal in G must have as its order $|H|$ a divisor of 48 smaller than 24, i.e. $|H| \leq 16$. It is trivial to see, in turn, that H is necessarily metahamiltonian, as a non-normal subgroup of H has order at most 4.

What we want to show now is that soluble k-hamiltonian groups have a bound on the derived length, but again we will obtain this result as corollary of a slightly more general statement about $(\mathfrak{A}^n)_k$ where by $\mathfrak{A}^n$ we mean the class of soluble groups whose derived length does not exceed n.

Lemma 1.29. *Let G be a soluble group in the class $(\mathfrak{A}^n)_k$, where n and k are positive integers. Then G has derived length at most $n + 3(k-1)$.*

Proof. In proving the result by the induction principle, it can obviously be assumed that $k > 1$.

Then it follows from Lemma 1.25 that all proper subgroups of G'' belong to the class $(\mathfrak{A}^n)_{k-1}$ and hence by induction G''' has derived length at most $n + 3(k-2)$. Therefore the derived length of G is at most $n + 3(k-2) + 3 = n + 3(k-1)$ and the result is proved. □

Corollary 1.30. *Let k be a positive integer, and let G be a k-hamiltonian locally soluble group. Then G is soluble of derived length at most $3k - 2$.*

Proof. Of course if G is a k-hamiltonian locally soluble group, then by Lemma 1.29 there is a bound on the derived length of any of its finitely generated subgroups, hence the same bound on the derived length holds for the group itself. □

The above corollary is not expected to give an optimal bound to the derived lenght of a k-hamiltonian group and in fact it does not even give the known optimal bound for the derived length of metahamiltonian groups, with choice of $k = 2$.

We can now prove the main theorem of this section, showing that locally graded k-hamiltonian groups, as k ranges in the set of natural numbers, exhaust the class of finite-by-Abelian groups.

Theorem 1.31. *A locally graded group G has a finite commutator subgroup if and only if it belongs to the class $\mathfrak{A}_\infty$.*

Proof. Let us start by proving that every $\mathfrak{A}_\infty$ group that is locally graded has finite commutator subgroup: we will do so by contradiction. Assume there is a $\mathfrak{A}_k$ group G that does not have finite commutator subgroup and that k is the smallest integer for which such a group exists. Of course $k > 1$ and, as a matter of fact, we even know $k > 2$ because of the theory of metahamiltonian groups. Since G is a locally graded k-hamiltonian group, it is soluble-by-finite by Lemma 1.27. Also, since the class of $(k-1)$-hamiltonian groups is a local class a G is not a $(k-1)$-hamiltonian group, there must be a finitely generated subgroup E of G that is not $(k-1)$-hamiltonian. This means E is normal in G and G/E is a Dedekind group, so that

$$\left(\frac{G}{E}\right)' \simeq \frac{G'}{G' \cap E}$$

is finite. E is a finitely generated $(k-1)$-hamiltonian group, so that it is nilpotent-by-finite for the argument in Theorem 1.24. For this reason, E is a group satisfying the maximal condition on subgroups and hence G' is a finitely generated subgroup. Now, let us consider the largest locally finite normal subgroup T of G. Since G' is finitely generated but infinite, this means that

$$\left(\frac{G}{T}\right)' \simeq \frac{G'T}{T}$$

is infinite, so that G/T is still a counterexample to the thesis of the Theorem and so we can assume, without loss of generality, that G does not have non-trivial locally finite normal subgroups.

Now, for Lemma 1.25, all proper subgroups of G'' belong to $\mathfrak{A}_{k-1}$ so that G'' is either finite or $(k-1)$-hamiltonian. For this reason G''' is in any case finite. But G is free of non-trivial locally finite normal subgroups, so that G''' has to be trivial and G'' is an Abelian subgroup. But, again, G'' is not locally finite, so that it has to be torsion-free Abelian.

Let X be any non-normal subgroup of G'. Then X belongs to $\mathfrak{A}_{k-1}$, which means X' is a finite subgroup of G'' which, again, in turn, means it is necessarily trivial. So any non-normal subgroup of G' is Abelian. It follows that G' is a metahamiltonian group and so G'' is finite, which implies that G'' is trivial. This has the same consequence as before on G', which is then a torsion-free Abelian subgroup of G and for this reason, if one chooses a non-normal subgroup Y of G, since it is $\mathfrak{A}_{k-1}$, it satisfies $Y' = \{1\}$, as it is a finite subgroup of the torsion-free Abelian subgroup G'. This would imply that G is itself a metahamiltonian group. This is a contradiction because, as we know by Theorem 1.3, metahamiltonian groups have finite commutator subgroup and G' is infinite. This contradiction shows that all locally graded k-hamiltonian groups have a finite commutator subgroup.

Let us now show the converse is also true. Assume that G is a group such that G' is finite and let m be the number of (not necessarily distinct) prime numbers in the prime decomposition of the order of G'.

Consider an arbitrary finite **non-normal chain** of G, i.e. a sequence of subgroups $\{G_i\}_{i\leq t}$ of G such that

$$G_1 \ntriangleleft G_2 \ntriangleleft \ldots \ntriangleleft G_t,$$

so that obviously

$$G_1 \lneq G_2 \lneq \ldots \lneq G_t.$$

For any $0 \leq i \leq j \leq t$, if we have

$$G_i \cap G' = G_j \cap G'$$

then we would have

$$[G_i, G_j] \leq G_j \cap G' \leq G_i \cap G' \leq G_i$$

which would mean $G_i \triangleleft G_j$, so that $i = j$,

For this reason

$$G_1 \cap G' \lneq G_2 \cap G' \lneq \ldots \lneq G_t \cap G'.$$

This means that $t \leq m$, because m is of course a bound on the lenght of any

chain of subgroups in the subgroup lattice of G'.
So we have proved that G is a subgroup in which any non-normal chain of subgroup has length at most m. We will now prove by induction on m that a group with such a bound is necessarily $(m+1)$-hamiltonian. Of course this is true for $m = 1$, because a group whose subgroups are all normal is a Dedekind group, hence a metahamiltonian group. Assume this is true for a natural number m and consider a group G in which non-normal chains have a bound of $m+1$ on the length. This means that any non-normal subgroup of G has a bound of m on the length of non-normal chains, hence it is a $(m+1)$-hamiltonian group by induction hypothesis. This proves G is a $(m+2)$-hamiltonian group, completing the proof by induction and hence the whole theorem, as it shows that any group such that G' is finite is in the class $\mathfrak{A}_\infty$. □

Chapter 2

Groups with restrictions on normalizers

Groups with few normalizers of non-Abelian subgroups

In every metahamiltonian group G, of course, the normalizer $N_G(H)$ of a non-Abelian subgroup H is G as the subgroup H is normal in G by definition of metahamiltonianity. For this reason, in a metahamiltonian group G, the set

$$\{N_G(H) : H \leq G, H \notin \mathfrak{A}\}$$

is trivially the singleton of G.
In [14], the authors studied groups in which the set of normalizers of non-Abelian subgroups is finite, i.e.

$$|\{N_G(H) : H \leq G, H \notin \mathfrak{A}\}| < +\infty.$$

Notice that this does not at all automatically imply that there is finitely many non-Abelian non-normal subgroups of G or that non-Abelian subgroups are in some sense close to being normal. It can be interpreted, though, as a restriction on the set of non-Abelian subgroups as a whole, making it close to being a set of normal subgroups.
They proved the following result, extending part of Theorem 1.3 to this larger

class of groups

Theorem 2.1. *Let G be a locally graded group such that*

$$|\{N_G(H) : H \leq G, H \notin \mathfrak{A}\}| < +\infty.$$

Then the commutator subgroup G' of G is finite.

To show the proof of this Theorem we need a few lemmas, dealing more generally with the case of a locally graded group such that

$$|\{N_G(H) : H \leq G, H \notin \mathfrak{A}, \ H \text{ infinite}\}| < +\infty.$$

The following lemma proves the existence of a subgroup of finite index in which every infinite non-Abelian subgroup is subnormal of defect at most 2.

Lemma 2.2. *Let G be a group with finitely many normalizers of infinite non-Abelian subgroups. Then G contains a characteristic subgroup of finite index M such that $N_M(X)$ is normal in M for each infinite non-Abelian subgroup X of M.*

Using this lemma, we show we are actually working within the universe of soluble-by-finite and locally max groups.

Lemma 2.3. *Let G be a locally graded group with finitely many normalizers of infinite non-Abelian subgroups. Then G is soluble-by-finite and locally satisfies the maximal condition on subgroups.*

Proof. By Lemma 2.2, the group G contains a characteristic subgroup of finite index M in which all normalizers of infinite non-Abelian subgroups are normal. Let X be any infinite non-Abelian subgroup of M. Then every subgroup of M containing X is subnormal with defect at most 2, so that both groups

$$N_M(X)/X \text{ and } M/N_M(X)$$

are nilpotent of class at most 3 (see [39], Theorem 1); in particular, $M^{(6)}$ is contained in X, and hence every proper subgroup of $M^{(6)}$ either is finite or

Abelian.
Thus $M^{(6)}$ either is soluble or finite (see [6], Theorem 1.1) and so G is soluble-by-finite.
In order to prove that G locally satisfies the maximal condition on subgroups, we may obviously suppose that G is finitely generated, so that also M is finitely generated. If every subgroup of finite index of M is subnormal, the group M is nilpotent (see [45] Part 2, Theorem 10.51) and G satisfies the maximal condition.
Assume now that M contains a non-subnormal subgroup of finite index H. It follows from the definition of M that H either is finite or Abelian, so that also in this case M is finitely generated and Abelian-by-finite. This means that M satisfies the maximal condition on subgroups and hence G also satisfies the maximal condition on subgroups. □

The following lemma, of which we omit the technical proof, shows that Abelian torsion-free normal subgroups of groups satisfying our property are also necessarily central

Lemma 2.4. *Let G be a group with finitely many normalizers of infinite non-Abelian subgroups, and let A be an Abelian torsion-free normal subgroup of G. Then A is contained in the centre of G.*

Lemma 2.5. *Let G be a torsion-free locally graded group with finitely many normalizers of non-Abelian subgroups. Then G is Abelian.*

The above lemmas are useful in proving the commutator subgroup is periodic

Lemma 2.6. *Let G be a locally graded group with finitely many normalizers of infinite non-Abelian subgroups. Then the commutator subgroup G' of G is periodic.*

Proof. We will start by proving that a group G satisfying our hypothesis has a torsion subgroup containing all of its periodic elements.
With the aim of proving that the elements of finite order of G form a subgroup, we may obviously suppose that $G = \langle x, y \rangle$, where x and y are periodic

elements.
It follows from Lemma 2.3 that G satisfies the maximal condition on subgroups and contains a soluble normal subgroup of finite index L. Moreover, if N is the largest periodic normal subgroup of G, replacing G by the factor group G/N, it is possible to assume, without loss of generality, that G does not have periodic non-trivial normal subgroups.
Now, let A be the smallest non-trivial term of the derived series of L. The subgroup A is a torsion-free abelian normal subgroup of G, and hence it is contained in the center $Z(G)$ of G by Lemma 2.4.
By induction on the derived length of L, we have that G/A is finite, so that G is finite over its center and hence G' is finite by Schur's theorem,
From this it follows that $G = \langle x, y\rangle$ itself is finite and we have finally proved that the set T of all elements of finite order of G is actually a subgroup and the factor group G/T is Abelian by Lemma 2.5. Therefore G' is periodic. □

Proof of Theorem 2.1. The group G is soluble-by-finite by Lemma 2.3. Of course the thesis of the Theorem is immediately proved if G is metahamiltonian so that, in what follows, it can be assumed that G is not metahamiltonian. Let $N_G(X_1), \ldots, N_G(X_k)$ be the proper normalizers of non-Abelian subgroups of G. We will divide the proof in two cases according to whether they cover the group G or not. Suppose first that the set

$$N_G(X_1) \cup \ldots \cup N_G(X_k)$$

is properly contained in G. Let x be an element of

$$G \setminus (N_G(X_1) \cup \ldots \cup N_G(X_k)).$$

Then each subgroup of G containing x either is Abelian or normal. Clearly, the element x cannot be in the center of G so that $xg \neq gx$ for some $g \in G$. This implies that $\langle x, g\rangle$ is normal in G and $G/\langle x, g\rangle$ is a metahamiltonian group. It follows that

$$G'\langle x, g\rangle/\langle x, g\rangle$$

is finite, so that G' is finitely generated. Now, G' is a soluble-by-finite group that is finitely generated and periodic by Lemma 2.6 so that G' is necessarily finite, reaching the conclusion of the proof of the theorem in this case.
Suppose now that

$$G = N_G(X_1) \cup \ldots \cup N_G(X_k).$$

Then it follows from a result of B.H. Neumann that we can omit from the union any subgroup of infinite index (see [45] Part 1, Lemma 4.17), and hence

$$G = N_G(X_{i_1}) \cup \ldots \cup N_G(X_{i_t}),$$

where the index $|G : N_G(X_{i_j})|$ is finite for all $j = 1, \ldots, t$.
In this case, we will prove the result by induction on the number k of proper normalizers of non-Abelian subgroups, knowing the result is true for $k = 0$ by 1.3.
Clearly, each $N_G(X_{i_j})$ has less than k proper normalizers of non-Abelian subgroups, and so by induction it can be assumed that $N_G(X_{i_j})'$ is finite. By Dietzmann's Lemma also the normal closure

$$E = \langle N_G(X_{i_1})', ..., N_G(X_{i_t})' \rangle^G$$

is finite, and the factor group G/E has a finite covering consisting of Abelian subgroups.
Therefore G/E is central-by-finite (see [45] Part 1, Theorem 4.16). Again, a famous theorem from Schur allows to deduce that G' is finite, concluding the proof of the theorem. □

Of course, this theorem gains relevance if we show that there exists non-metahamiltonian groups that satisfy its hypotheses. To give an example of such a group, consider the dihedral group

$$D = \langle a, x \,|\, a^8 = x^2 = 1,\, ax = xa^{-1} \rangle$$

of order 16 and an infinite extraspecial 2-group E, i.e. a 2-group whose center is cyclic of order 2 and such that the quotient of E over its center is a non-trivial elementary Abelian 2-group. Let G be the direct product of D and E in which the two subgroups $\langle a^4 \rangle$ and $E' = Z(E)$ of order 2 are identified, and let X be any non-Abelian subgroup of G. Then it can be shown that $X \cap G' \neq \{1\}$, so that $a^4 \in X$ and hence E is contained in the normalizer $N_G(X)$. Therefore G has finitely many normalizers of non-Abelian subgroups, because it has at most as many as there are subgroups of D, but of course G is not metahamiltonian, as the subgroup

$$F := \langle x, E \rangle$$

shows. As a matter of fact, F is not Abelian because it contains the non-Abelian group E and it is non-normal because $\langle x \rangle$ is not normal in D.

Classes of groups with few normalizers

Exactly as metahamiltonianity was a source of inspiration to define some new group classes with the position that subgroups that do not satisfy some (absolute or relative) group-theoretical property θ are all normal, something similar has been done by replacing normality with this weaker condition: we may ask that the set of normalizers of non-θ subgroups is finite. Let us notice that this is a condition imposed on the whole collection of non-θ subgroups. In other words, we are not asking that a large portion of them is normal nor that they have large normalizers, but that the collection of all of their normalizers is small.

Before trying to discuss a general approach to studying similarly defined group classes, let us present a few of the key results regarding some notable examples of group classes defined in a similar fashion in the mathematical literature. Most of those can now be found in surveys (see [23] or [13]).

Of course, if a group G satisfies some property θ that is inherited by subgroups, or if G is a group in which the set of normalizers of all of its subgroups is finite, then these are of course two "extreme" cases of groups in which non-θ subgroups have finitely many normalizers. A strong property of groups in which the set of normalizers of all subgroups is finite is that the factor $G/Z(G)$ of G over its center is finite. In fact, in 1980, Polovickiĭ proved that if the set of normalizers of abelian subgroups is finite, then G is finite over its center (see [43]).

The kind of theorem we might expect to see, then, if we are to be inspired from these two extreme cases is something along the lines of: "A group G such that subgroups not satisfying θ have finitely many normalizers is either a group in which all subgroups satisfy θ or a group that is finite over the center/has finite commutator subgroup".

An interesting generalization of the class of groups whose non-abelian subgroups have finitely many normalizers can be obtained by replacing abelianity with (local) nilpotency. This has been done in [17] (see also [4] by B. Bruno and R. E. Phillips regarding groups in which non-normal subgroups are locally nilpotent).

Their work was within the universe of $\mathcal{W}$-groups, so let us give the definition of this group class. Here a group G is said to belong to the class $\mathcal{W}$ if all of its finitely generated non-nilpotent subgroup have a finite non-nilpotent homomorphic image. Of course this class trivially contains locally nilpotent groups, but it can also be shown to contain all locally (soluble-by-finite) groups (see [45], part 2, Theorem 10.51) and all linear groups (see [50]).
In [17], the authors proved

Theorem 2.7. *Let G be a $\mathcal{W}$-group with finitely many normalizers of non-(locally nilpotent) subgroups. Then either G is locally nilpotent or its commutator subgroup G' is finite.*

Of course, as all finite groups satisfy the hypotheses of the above theorem, we cannot expect to give a bound for the derived length for the soluble ones, as it can be done for metahamiltonian groups, for instance. We remark that if G is finite-by-abelian, is locally nilpotent subgroups are actually nilpotent, so that the above theorem has an interesting corollary.

Corollary 2.8. *Let G be a $\mathcal{W}$-group with finitely many normalizers of non-(locally nilpotent) subgroups. If G is not locally nilpotent, then it has finitely many normalizers of non-nilpotent subgroups.*

Moreover, the same authors proved

Theorem 2.9. *Let G be a $\mathcal{W}$-group with finitely many normalizers of infinite non- (locally nilpotent) subgroups. If G is not locally nilpotent, then either it is a Černikov group or its commutator subgroup G' is finite.*

By considering the non-abelian extension of a Prüfer p-group with a group of order 2 we have an example of a group that is not locally nilpotent nor finite-by-abelian, but it satisfies the hypotheses of the theorem, showing that the option of G being a Černikov group cannot be removed from the statement of the theorem.
Another idea was to consider nilpotency of bounded class instead of local nilpotency. This has been done by F. de Giovanni, M. Trombetti and me in [23]. To show the proof of the theorem, we cite a lemma whose proof can be found in [19].

Lemma 2.10. *Let c be a positive integer, and let G be a locally graded group whose non-normal subgroups are nilpotent of class at most c. Then the subgroup $\gamma_{c+1}(G)$ is finite.*

Theorem 2.11. *Let c be a positive integer, and let G be a locally graded group such that the set of normalizers of subgroups which are not nilpotent of class at most c is finite. Then the subgroup $\gamma_{c+1}(G)$ is finite.*

Proof. By Lemma 2.10, we may assume that G contains subgroups which neither are normal nor nilpotent of class at most c, otherwise the thesis is easily reached. Let

$$N_G(X_1), \dots, N_G(X_t)$$

be all the normalizers of subgroups of G which are neither normal nor nilpotent of class at most c. For each $i = 1, \dots, t$, the group $N_G(X_i)$ has less than t normalizers of subgroups that are not nilpotent of class at most c, and hence, if we argue by induction on the number t of normalizers of subgroups that are not nilpotent of class at most c, we may suppose that the subgroup $\gamma_{c+1}(N_G(X_i))$ is finite. Clearly, the subgroup $N_G(X_i)$ has only finitely many conjugates in G, and so the index

$$|G : N_G(N_G(X_i))| < \infty.$$

As

$$\gamma_{c+1}(N_G(X_i)) \triangleleft N_G(N_G(X_i))$$

it follows that also the subgroup $\gamma_{c+1}(N_G(X_i))$ has finitely many conjugates in G, and hence its normal closure $\gamma_{c+1}(N_G(X_i))^G$ is finite by Dietzmann's Lemma.

Thus

$$L = \langle \gamma_{c+1}(N_G(X_1)), \dots, \gamma_{c+1}(N_G(X_t)) \rangle^G$$

is a finite normal subgroup of G.

Now, if we take X/L to be a subgroup of the factor group G/L which is not nilpotent of class at most c, we have that X cannot be contained in $N_G(X_i)$ for any $i = 1, ..., t$, because $N_G(X_i)L/L$ is nilpotent of class at most c, and hence

$$X \triangleleft G.$$

In other words, all non-normal subgroups of G/L are nilpotent of class at most c, so that we are in the hypotheses of Lemma 2.10 and so $\gamma_{c+1}(G)/L$. As L is also finite, we conclude $\gamma_{c+1}(G)$ is finite, and the statement is proved. □

Other results with similar spirit were also obtained in [23]. For instance, we studied groups in which cyclic subgroups have finitely many normalizers, which are in some sense conceptually close to having all cyclic subgroups normal (i.e. Dedekind groups).

Theorem 2.12. *Let G be a group such that the set of normalizers of cyclic subgroups is finite. Then the factor group $G/Z(G)$ of G over its center is finite.*

Proof. As the statement is obvious when all subgroups of G are normal, we may suppose that G contains at least one cyclic non-normal subgroup. Let

$$N_G(\langle x_1\rangle), \ldots, N_G(\langle x_1\rangle)$$

be all proper normalizers of cyclic subgroups of G. If g is any element of the set

$$G \setminus [N_G(\langle x_1\rangle) \cup \ldots \cup N_G(\langle x_1\rangle)]$$

the normalizer $N_G(\langle g\rangle)$ cannot be a proper subgroup of G, so that $\langle g\rangle$ is a normal subgroup of G and in particular g belongs to the FC-centre F of G, that is the set of elements with finitely many conjugates. Thus

$$G = F \cup [N_G(\langle x_1\rangle) \cup \ldots \cup N_G(\langle x_1\rangle)]\,.$$

Then it follows from a result of B.H. Neumann that we can omit from the union any subgroup of infinite index (see [45] Part 1, Lemma 4.17). To avoid complicating the notation, let us assume without loss of generality that all of the sets in the above union are subgroups of finite index in G.
Now, if we take $\langle x\rangle$ to be any cyclic non-normal subgroup of G, of course the

group $N_G(\langle x\rangle)$ has at least one less proper normalizer of cyclic subgroups than G. In other words, it has strictly less than t, so that by arguing by induction on the number t of proper normalizers of cyclic subgroups, we can conclude $N_G(\langle x\rangle)$ is finite over its center. By combining what we know, $N_G(\langle x\rangle)$ is a central-by-finite subgroup of finite index in G, so that it is contained in its FC-center F, hence

$$G = F \cup [N_G(\langle x_1\rangle) \cup \ldots \cup N_G(\langle x_1\rangle)] = F$$

and G is an FC-group.
This, in turn, means all of the normalizers of cyclic subgroups have finite index. Since the proper normalizers of cyclic subgroups are also central-by-finite, it follows G is abelian-by-finite, therefore G, being an FC-group, is also central-by-finite (see, for instance, [5]). □

A nice consequence of the theorem above is that if G is a group having finitely many normalizers of cyclic subgroups, then even the set of normalizers of all subgroups of G is finite.
Moving to another example, non-periodic groups whose non-periodic subgroups have finitely many normalizers have been studied in [10] by M. De Falco, F. de Giovanni and C. Musella, proving the following theorem.

Theorem 2.13. *Let G be a group with finitely many normalizers of non-periodic subgroups. Then either G is periodic or the factor group $G/Z(G)$ is finite.*

In particular, a group satisfying the hypotheses of the above theorem, has always G' periodic.
Yet another example is given by groups whose non-T-subgroups have finitely many normalizers. Here a T-group is a group in which normality is a transitive relation, or, in symbols

$$K \triangleleft H \triangleleft G \Rightarrow K \triangleleft G.$$

The class of T-groups is a peculiar one as it contains all Dedekind groups, which are rich in normal subgroup and all simple groups, which have none.

A technical difficulty of the class of T-groups is that subgroups of T-groups are not necessarily T-groups, as A_4 is an example of a group that is not a T-group, but any group can be embedded into a simple group, and simple groups are T-groups. For this reason, when studying T-groups, one also introduces the class of $\overline{T}$-groups which is the class of groups such that all of their subgroups are T-groups. Regarding T-groups see also [27] and [44]
The following theorem was proven in [18] (see also [23]) about groups in which non-T-subgroups have finitely many normalizers.

Theorem 2.14. *Let G be a soluble group with finitely many normalizers of subgroups that do not have the T-property. Then either G is a $\overline{T}$-group or its commutator subgroup G' is finite.*

The last two examples we cite are a little different in nature, as they deals with groups in which subnormal (resp. non-subnormal) subgroups have finitely many normalizers. Here the collection of subgroups whose normalizers we put a restriction on is a collection of subgroups having (resp. failing) an embedding property. We cite it here as it in strong connection to T-groups.
To understand the result we cite, proved in [15], we recall the definition of the Wielandt subgroup. The Wielandt subgroup $\omega(G)$ of a group G is defined as the intersection of all normalizers of subnormal subgroups of G. In other words, every subnormal subgroup H of G that is contained in $\omega(G)$ is normal in it. Clearly, $\omega(G)$ has the G-property, and indeed G is a T-group if and only if it coincides with its Wielandt subgroup. For this reason the "size" (meaning finite vs infinite) of $G/\omega(G)$ can be considered as a measure of conceptually how far is the group G from enjoying the T property. Some results confirm this naive intuition is useful.
For instance, it is known that if G is a group satisfying the minimal condition on subnormal subgroups, then $G/\omega(G)$ is finite (see [45] Part 1, Theorem 5.49, and also [21] for a recent generalization of this result). The next result reported here was proved in [15], and shows that periodic soluble groups with finitely many normalizers of subnormal subgroups are not too far from being T-groups.

Theorem 2.15. *Let G be a periodic soluble group with finitely many normalizers of infinite subnormal subgroups. Then the factor group $G/\omega(G)$ is finite*

On the other hand, on a dual topic, in [16] the authors proved

Theorem 2.16. *Let G be a group with finitely many normalizers of subgroups that are not subnormal. Then every non-subnormal subgroup of G has only finitely many conjugates.*

We will now discuss a general approach to the study of group classes defined in the aforementioned fashion and then apply it to the case of groups with finitely many normalizers of non-k-hamiltonian subgroups: in particular we will obtain as corollary that groups whose non-metahamiltonian subgroups have finitely many normalizers have finite commutator subgroup.
Let $\mathfrak{X}$ be a class of groups and let us define a new class $\overline{\mathfrak{X}}_2$ by saying a group G is in the class $\overline{\mathfrak{X}}_2$ if and only if the set of normalizers of non-$\mathfrak{X}$ subgroups of G is finite.
We can iterate this idea and, upon defining

$$\overline{\mathfrak{X}}_1 := \mathfrak{X}$$

and assuming $\overline{\mathfrak{X}}_k$ is already defined, we can define the class $\overline{\mathfrak{X}}_{k+1}$ by stating that a group G is in the class if and only if the following holds:

$$|\{N_G(H) : \ H \leq G, \, H \notin \overline{\mathfrak{X}_k}\}| < +\infty.$$

Of course, we would have that, using the notation introduced in Chapter 1 for k-hamiltonian groups,

$$\mathfrak{X}_k \subseteq \overline{\mathfrak{X}}_k.$$

Lemma 2.17. *If $\mathfrak{X}$ is an accessible class of soluble-by-finite groups, then groups in $\overline{\mathfrak{X}}_2$ are also soluble-by-finite. If, additionally, $\mathfrak{X}$ is a class of groups locally satisfying the maximal condition on subgroups, then the same can be said for groups in the class $\overline{\mathfrak{X}}_2$.*

Proof. Consider a non-$\mathfrak{X}$ subgroup H of G and define:

$$M(H) = \bigcap_{\alpha \in \mathrm{Aut}(G)} N_G(N_G(H))^\alpha$$

As the normalizers of non-$\mathfrak{X}$ are finite in number, they have finitely many conjugates and so $N_G(N_G(H))$ and all of its images under automorphisms have finite index in G.
$M(H)$ is then a finite intersection of subgroups of finite index and so it is a characteristic subgroup of G of finite index.
Starting from H_1 and H_2 such that $N_G(H_1)$ and $N_G(H_2)$ coincide gives rise to the same subgroup $M(H_1) = M(H_2)$ hence

$$M = \bigcap_{G \geq H \notin \mathfrak{X}} M(H)$$

is still a characteristic subgroup of finite index in G.
M has the property that if H is a non-$\mathfrak{X}$ subgroup of M, then it is subnormal of defect at most 2 in M. That is because M contains the normalizer of its normalizer. So if you take any non-$\mathfrak{X}$ subgroup H of G both $M/N_M(X)$ and $N_M(X)/X$ have all subgroups subnormal of defect at most 2 and so they are soluble of derived length at most 3. This means that $M^{(6)}$ has all proper subgroups in $\mathfrak{X}$ and so it is either $\mathfrak{X}$ or finite, proving G soluble-by-finite.
To prove the second part of the stament of the lemma, assume that G is finitely generated, so that M is a finitely generated soluble-by-finite group. If all of its subgroups of finite index are subnormal, then this means it is nilpotent and hence polycyclic, showing that M satisfies the maximal condition on subgroups and so the same can be said of G. If, instead, M has a non-subnormal subgroup of finite index, by definition of M this means that M is $\mathfrak{X}$-by-finite and, again, it satisfies the maximal condition on subgroups because of our hypothesis on the class $\mathfrak{X}$, which again implies the condition holds also in G, which is a finite extension of M. This concludes the proof of the lemma. □

Now, it would possible to prove with similar methods that for any accessible, local and Robinson class $\mathfrak{X}$ that is contained in the class of finite-by-abelian groups, also the classes $\overline{\mathfrak{X}}_k$ are classes of finite-by-abelian groups. This result would apply, for example, to the class $\mathfrak{N}_c$ of nilpotent groups of nilpotency class bounded by c so that, in particular, it applies to the class of all abelian groups. It also applies to the class of all k-hamiltonian groups for any integer k, generalizing a result from the previous section. As a consequence, a group with finitely many normalizers of non-k-hamiltonian subgroups has finite commutator subgroup. This means such a group is also h-hamiltonian for some h in light of our main theorem from [22]: this is actually logically equivalent to having finite commutator subgroup for locally graded groups. The arguments for proving this are very similar to what has been done in [22], so that, for the sake of variety, what we now show is a different way to directly prove the relation between having finitely many normalizers of non-k-hamiltonian subgroups and being h-hamiltonian for some h.

Theorem 2.18. *Let k be a positive integer, and let G be a group which has only finitely many normalizers of subgroups that are not k-hamiltonian. Then G is h-hamiltonian for some positive integer h.*

Proof. Of course, it can be assumed that G is not $(k+1)$-hamiltonian, so that it contains a subgroup which neither is normal nor k-hamiltonian. Let

$$N_G(X_1), \dots, N_G(X_t)$$

be all normalizers of subgroups of G which neither are normal nor k-hamiltonian. We will prove the statement of the theorem by induction on the number t of such normalizers. Our assumption that G is not $(k+1)$-hamiltonian is equivalent to having dealt with the case $t = 0$.
For each index $i = 1, \dots, t$, the group $N_G(X_i)$ has obviously less than t proper normalizers of subgroups that are not k-hamiltonian, and so by induction on t we may suppose that it is h_i-hamiltonian for some positive integer h_i. Put

$$h = \max\{k, h_1, \dots, h_t\},$$

and let X by any subgroup of G which neither is normal nor k-hamiltonian. Then

$$X \leq N_G(X) = N_G(X_i)$$

for some $i \leq t$, and so X is an h_i-hamiltonian group. Therefore all non-normal subgroups of G are h-hamiltonian, and hence G is $(h+1)$-hamiltonian, hence concluding the proof of the theorem. □

This theorem has, as a corollary, the abovementioned property of finiteness of the commutator subgroup.

Corollary 2.19. *Let k be a positive integer, and let G be a locally graded group which has only finitely many normalizers of subgroups that are not k-hamiltonian. Then the commutator subgroup G' of G is finite.*

Classes of groups with many almost normal subgroups

A subgroup H of a group G is said to be **almost normal** in G if H has only a finite number of conjugates in G. Since there is an obvious bijection between the conjugacy class of H in G and the cosets of the normalizer $N_G(H)$ in G, we can state almost-normality as a condition on the normalizer of H. In fact, H is almost normal in G if the normalizer $N_G(H)$ of H has finite index in G.

A well known result of B. H. Neumann [41] states

Theorem 2.20. *Let G be a group. The factor group $G/Z(G)$ of G over its center is finite if and only if all of its subgroups are almost normal, i.e.*

$$\forall H \leq G, \quad |G : N_G(H)| < \infty,$$

This also has the peculiar consequence that if

$$\forall H \leq G, \quad |G : N_G(H)| < \infty$$

holds, then there is also a positive integer k such that

$$\forall H \leq G, \quad |G : N_G(H)| < k$$

also holds. In other words groups in which subgroups have finite conjugacy classes are the same as groups in which subgroups have boundedly finite conjugacy classes, which is different from what happens by imposing finiteness of the conjugacy classes of elements.

This same thesis was also obtained by L. A. Kurdachenko and V. V. Pylaev [29] for groups with the minimal condition on subgroups which are not almost normal:

Theorem 2.21. *Let G be a group with the minimal condition on non almost normal subgroups. Then the factor group $G/Z(G)$ of G over its center is finite.*

Moreover, it is clear that FC-groups (i.e. groups with finite conjugacy

classes of elements) are precisely those groups whose finitely generated subgroups are almost normal. Groups in which infinitely generated subgroups are almost normal were named, for this reason, anti-FC groups and studied in [26]. In the same paper, also groups whose non-cyclic subgroups are almost normal were studied and completely classified.
We now turn to a general discussion of groups in which the set of normalizers of subgroups not satisfying a given property is finite.
Let $\mathfrak{X}$ be any class of groups that is closed under considering subgroups and homomorphic images (an SH-closed class of groups, using terminology from [45]) and d a natural number.
We define

$$\mathfrak{X}_{d,1} := \mathfrak{X}$$

while for any natural number $k > 1$ we say that a group G belongs to the class $\mathfrak{X}_{d,k}$ iff

$$\forall H \leq G, H \notin \mathfrak{X}_{d,k-1} \Rightarrow |G : N_G(H)| \leq d.$$

Of course we have $\mathfrak{X}_{d_1,k} \subseteq \mathfrak{X}_{d_2,k}$ whenever $d_1 < d_2$ and $\mathfrak{X}_{d,k}$ is again an SH-closed class for any d and k.
If $\mathfrak{X} = \mathfrak{A}$ and $d = 1$, we get the classes of k-hamiltonian groups. In general, with $d = 1$ we obtain classes of groups in which non-$\mathfrak{X}$ subgroups are normal, which we have already studied in a previous section.
Our first result is to prove that locality of a class of groups $\mathfrak{X}$ is inherited by all classes of groups $\mathfrak{X}_{d,k}$.

Lemma 2.22. *Let $\mathfrak{X}$ be a local class of groups. Then, for any d and k natural numbers, $\mathfrak{X}_{d,k}$ is a local class of groups.*

Proof. We will prove the result (i.e $\mathfrak{X}_{d,k} = L\mathfrak{X}_{d,k}\ \forall d$) by induction on k. The statement is trivial for $k = 1$, so that we can assume it to be true for k and try to prove it for $k + 1$, so let us consider a group G all of whose finitely generated subgroups are $\mathfrak{X}_{d,k+1}$.
Let $H \leq G$, $H \notin \mathfrak{X}_{d,k}$ and finitely generated. Our first step is to prove that H has finitely many conjugates in G and to do so we start with noticing that, under our hypothesis, for any finitely generated subgroup K of G that

contains H we have that H has at most d conjugate subgroups in K.
Now, if H had $d+1$ different conjugate subgroups

$$H^{g_1}, \dots, H^{g_{d+1}}$$

in G, one could consider the finitely generated subgroup

$$F = \langle H, H^{g_1}, \dots, H^{g_{d+1}}, g_1, \dots, g_{d+1} \rangle$$

and H would have more than d conjugate subgroups in F, which is a contradiction. So we have $|G : N_G(H)| < d$ for all finitely generated non-$\mathfrak{X}_{d,k}$ subgroups of G.
Take any non-$\mathfrak{X}_{d,k}$ subgroup K of G whose smallest set of generators has infinite cardinality κ. By transfinite induction on κ, we can assume all non-$\mathfrak{X}_{d,k}$ subgroups which are generated by strictly less than κ elements have at most d conjugates.
Since $\mathfrak{X}_{d,k}$ is local by induction hypothesis, K can be seen as the union of a family of non-$\mathfrak{X}_{d,k}$ subgroups K_α, indexed in the set of ordinals α strictly smaller than κ and having a generating set of cardinality smaller than κ.
Consider any set

$$\{g_1, \dots, g_{d+1}\}$$

of $d+1$ elements in G and the subgroups

$$K_\alpha^{g_1}, \dots, K_\alpha^{g_{d+1}}.$$

For any fixed α, these $d+1$ subgroups cannot all be distinct and so there will be two indices $i, j \in \{1, \dots, d+1\}$ such that

$$K_\alpha^{g_i} = K_\alpha^{g_j}$$

for a set $\mathcal{A}$ of indices α with cardinality κ.
This implies $K^{g_i} = K^{g_j}$ so that K cannot have any more than d different conjugates.
As K is any non-$\mathfrak{X}_{d,k}$ subgroup, we have proven that $G \in \mathfrak{X}_{d,k+1}$. □

We now want to prove that if $\mathfrak{X}$ is a Robinson class (i.e. finitely generated hyper(abelian-or-finite) groups with all finite homomorphic images in $\mathfrak{X}$ are polycyclic-by-finite and $\mathfrak{X}$), then the same can be said for $\mathfrak{X}_{d,k}$ for all d and k.
To do so, we will need the following lemma, proving in particular that the class of groups with a bound on the size of conjugacy classes of subgroups is a Robinson class.

Lemma 2.23. *Let G be a polycyclic-by-finite group and H be a subgroup of G such that every subgroup of finite index of G containing H has at most d conjugates. Then H has at most d conjugates.*
In particular, the class of all groups with a given bound on the size of conjugacy classes of subgroups is a Robinson class.

Proof. Let us consider $d+1$ conjugates of H,

$$H_1, \dots, H_{d+1}.$$

This means that, if $\mathcal{F}$ is the set of normal subgroups of G having finite index, for any $F \in \mathcal{F}$ we have $H_iF = H_jF$ for some $i \neq j$, with $i, j \in \{1, \dots, d+1\}$. We can then consider a finite covering $\{\mathcal{F}_{ij}\}_{i,j \leq n}$ of $\mathcal{F}$, where $\mathcal{F}_{ij}$ is the set of subgroups F in $\mathcal{F}$ such that $H_iF = H_jF$.
We now want to prove that one of the $\mathcal{F}_{ij}$ has the property that for every $A \in \mathcal{F}$ there exists $B \in \mathcal{F}_{ij}$ such that $B \leq A$.
Assume by contradiction that this is not the case. Then, for each choice of i and j we have a subgroup $A_{ij} \in \mathcal{F}$ such that no subgroups of $\mathcal{F}_{ij}$ are contained in it. Now consider

$$\overline{A} = \bigcap_{i,j} A_{ij}.$$

Since it is a finite intersection, it still is a normal subgroup of finite index, hence it belongs to one of the $\mathcal{F}_{ij}$, which is of course a contradiction, because

$$\mathcal{F}_{ij} \ni \overline{A} \leq A_{ij}.$$

This means there is a choice of i^* and j^* in $\{1,\dots,d+1\}$ such that $\mathcal{F}_{i^*j^*}$ has the desired property.
Now, notice that

$$K \subseteq \bigcap_{F\in\mathcal{F}_{i^*j^*}} KF \subseteq \bigcap_{K\leq L\lhd G, |G/L|<+\infty} L = K$$

The second inclusion holds because of the proved property for $\mathcal{F}_{i^*j^*}$.
In particular

$$H_{i^*} = \bigcap_{F\in\mathcal{F}_{i^*j^*}} H_{i^*}F = \bigcap_{F\in\mathcal{F}_{i^*j^*}} H_{j^*}F = H_{j^*}$$

which means that H has at most d conjugates. □

Lemma 2.24. *Let $\mathfrak{X}$ be a Robinson class. Then $\mathfrak{X}_{d,k}$ is a Robinson class.*

Proof. Let us prove the result for all d by induction on k, the result being trivial for $k = 1$.
Let us consider a group G that is finitely generated, hyper(abelian-or-finite) and with all finite homomorphic images in $\mathfrak{X}_{d,k+1}$.
If there is a subgroup H of finite index with more than $d + 1$ conjugates, then, for any subgroup K of finite index in H, we have that

$$G/(H_G \cap K_G)$$

is a $\mathfrak{X}_{d,k+1}$ group in which

$$H/(H_G \cap K_G)$$

has more than d conjugates, so that

$$H/(H_G \cap K_G) \in \mathfrak{X}_{d,k}.$$

By induction hypothesis, since H is a finitely generated, hyper(abelian-or-finite) group whose finite homomorphic images are $\mathfrak{X}_{d,k}$, H is polycyclic-by-finite and $\mathfrak{X}_{d,k}$, showing that also G is polycyclic-by-finite.
On the other hand, if there is no subgroup H of finite index with more than $d + 1$ conjugates, then $G^{d!}$ (which is finitely generated because it has finite

index in G, given our hypotheses on G) normalizes all of its subgroups of finite index, hence being nilpotent, proving G to be polycylic-by-finite.
Let us also observe that we have in fact proved that subgroups of finite index of G are either $\mathfrak{X}_{d,k}$ or they have at most d conjugates.
In conclusion, either way, G is polycyclic-by-finite. Now consider any subgroup H of G. It can be seen as the intersection of the subgroups of finite index in G that contain it. If H is not $\mathfrak{X}_{d,k}$, then all subgroups of finite index containing H have at most d conjugates, so that we are in the hypotheses of the previous lemma and H has at most d conjugates, proving $G \in \mathfrak{X}_{d,k+1}$. □

Lemma 2.25. *Let $\mathfrak{X}$ be an SH-closed, local and accessible Robinson class of soluble-by-finite groups. For any choice of natural numbers k and d, also $\mathfrak{X}_{d,k}$ is an accessible class of soluble-by-finite groups.*

Proof. Let us prove the result by induction on k. Consider a locally graded group G that is minimal-non-$\mathfrak{X}_{d,k+1}$, so that it must be finitely generated and hence there must be a proper normal subgroup X of finite index in G. X is of course $\mathfrak{X}_{d,k+1}$.
If X is also $\mathfrak{X}_{d,k}$ we already know the group G is soluble-by-finite, so, temporarily, with the aim of showing that G is soluble-by-finite we can assume without loss of generality that X and even X' are not $\mathfrak{X}_{d,k}$. This means that there is a subgroup $H \le X'$ that is finitely generated and not belonging to the class $\mathfrak{X}_{d,k}$. Now, X/H^X is a group in which all subgroups have boundedly finite conjugacy classes, so that it has finite commutator subgroup. Since $H \le X'$, we have that $H^X \le X'$ and so

$$\left(\frac{X}{H^X}\right)' = \frac{X'H^X}{H^X} = \frac{X'}{H^X} \text{ is finite.}$$

Since H^X is generated by finitely many conjugates of H, which is a finitely generated subgroup, then H^X is also a finitely generated subgroup and the same can be said of X'.
Moreover $X'H^X/H^X$ is finite of order smaller than $f(d)$ for any non-$\mathfrak{X}_{d,k}$ subgroup H of X', so that $(X')^{[f(d)]!}$ is contained in the intersection of all non-$\mathfrak{X}_{d,k}$-subgroups of X', hence $(X')^{[f(d)]!}$ is either $\mathfrak{X}_{d,k}$ or minimal-non-$\mathfrak{X}_{d,k}$:

either way, $(X')^{[f(d)]!}$ is soluble-by-finite.
Since $(X')/(X')^{[f(d)]!}$ is a finitely generated locally graded ("locally graded" is inherited to quotients over a soluble-by-finite group) bounded group, it is finite so that X' and hence G are soluble-by-finite in any case.
Since $\mathfrak{X}_{d,k+1}$ is a Robinson class, G has a finite homomorphic image G/N which is not $\mathfrak{X}_{d,k+1}$. Assume by contradiction G is infinite.
A result by Lennox shows that $G/\Phi(G)$ is infinite, so that some maximal subgroup M of G not containing N must exist and $G = MN$, so that $G/N = MN/N = M/(M \cap N)$ is $\mathfrak{X}_{d,k+1}$ which is absurd. □

What we want to prove now is that if $\mathfrak{X}$ is a local subclass of $\mathfrak{F}\mathfrak{A}$, then the same can be said of $\mathfrak{X}_{d,k}$ for any d and k.
To do so, we first need to observe that, with such a choice of $\mathfrak{X}$ any finitely generated soluble-by-finite group in $\mathfrak{X}_{d,k}$ is minimax.
As P. Kropholler proved, for a finitely generated soluble group to be minimax it is necessary and sufficient to not have any sections isomorphic to the standard wreath product of a cyclic group of prime order with an infinite cyclic group.
So, the next lemma is what is needed.

Lemma 2.26. *Let $\mathfrak{X}$ be a local subclass of the class of finite-by-abelian groups. The standard wreath product $G = \langle a \rangle \wr \langle b \rangle$ of a cyclic group of prime order with an infinite cyclic group does not belong to $\mathfrak{X}_{d,k}$ for any positive integers d and k.*

Proof. If $G = \langle a \rangle \wr \langle b \rangle$ with $o(a) = p$ and $o(b) = \infty$, then $H = \langle a, b^p \rangle$ is a self-normalizing subgroup of infinite index that is isomorphic to the whole group G.
Arguing by contradiction, assume G to be $\mathfrak{X}_{d,k}$. Then H would have to be $\mathfrak{X}_{d,k-1}$ and so the same could be said for G. This would allow to inductively prove $G \in \mathfrak{X} \subseteq \mathfrak{F}\mathfrak{A}$, which is absurd. □

Theorem 2.27. *Let $\mathfrak{X}$ be a SH-closed, local and accessible Robinson class of finite-by-abelian groups, i.e. $\mathfrak{X} \subseteq \mathfrak{F}\mathfrak{A}$, and let d and k be positive integers and G be a locally graded group in $\mathfrak{X}_{d,k}$. Then G is in the class $\mathfrak{F}\mathfrak{A}$ of finite-by-abelian groups.*

Proof. We will prove the result for all d by induction on k.
The statement is trivial for $k = 1$, so we assume for it to be true for k and prove it for $k + 1$.

Part 1: It is not restrictive to assume G finitely generated.
If G' is $\mathfrak{X}_{d,k-1}$, we know by induction hypothesis that G'' is finite and so, in this case, we reason modulo G'' and assume that G is metabelian and G' is abelian.
For any finitely generated non-$\mathfrak{X}_{d,k}$ subgroup K of G we have that K^G is finitely generated and $X = G'K^G$ is soluble and finitely generated. This is because G/K^G has all subgroups almost normal and so it has finite commutator subgroup. Of course, then, by the previous lemma, X (and hence G') is minimax. For this reason, there is a finitely generated subgroup A such that G'/A is periodic and so such a subgroup K can be chosen with $A \leq K'$, hence

$$A \leq X' \leq G' \leq X.$$

Since G'/X' is finite, there exists a finitely generated subgroup

$$L/X' \leq X/X'$$

such that $L' = G'$, therefore $E = XL$ is a finitely generated subgroup of G such that $E' = G'$.
If G' is not $\mathfrak{X}_{d,k}$ such a subgroup K can be chosen inside G' by locality of the class $\mathfrak{X}_{d,k}$ and so $X = G'K^G = G'$ is finitely generated.
We have proved that in any case, it is not restrictive to assume G is finitely generated with the purpose of proving G' finite.

Part 2: Conclusion. Of course G^m, where $m = d!$, normalizes all non-$\mathfrak{X}_{d,k}$ subgroups. This mean G^m is a group in which non-$\mathfrak{X}_{d,k}$ subgroups are normal. Now, remember $\mathfrak{X}_{d,k}$ is a SH-closed, local, accessible Robinson class of finite-by-abelian groups by induction.
Now, if $\mathfrak{Y}$ is such a class, then locally graded groups in which non-normal subgroups are $\mathfrak{Y}$ have finite commutator subgroup, so that $(G^m)'$ is finite. G/G^m is bounded and locally graded (because this property is inherited from the whole group by quotients over finite-by-abelian normal subgroups) so that

it is finite.
Our group G is then finite-by-abelian-by-finite and finitely generated, hence it is abelian-by-finite. Let us choose a torsion-free abelian normal subgroup A of finite index.
Assume, by contradiction, the statement is false so that G is finitely generated with infinite commutator subgroup and it has an element x that is not an FC-element. As $\langle A, x\rangle$ would still be a counteraxample, we rename $G = \langle A, x\rangle$.
As $[A^n, x]$ is infinite for any n, $\langle A_n, x\rangle$ is, by induction hypothesis, not a $\mathfrak{X}_{d,k}$ subgroup and hence it is normalized by A^m, $m \leq n$.
Now

$$\frac{\langle x\rangle}{A\cap\langle x\rangle} = \bigcap \frac{\langle x, A^n\rangle}{A\cap\langle x\rangle} = \frac{\bigcap\langle x, A^n\rangle}{A\cap\langle x\rangle} \lhd \frac{\langle x, A^m\rangle}{A\cap\langle x\rangle}$$

which is absurd, because it would imply $\langle x\rangle$ to be normal in a subgroup of finite index in G and so x to be an FC-element. This proves G is finite-by-abelian. □

Groups whose non-$\mathfrak{X}$-subgroups are subnormal of bounded defect

Throughout all of this section, let $\mathfrak{X}$ be an SH-closed class of groups and let d be a positive integer. We define a new class of groups, $\mathfrak{X}^d$ by saying that a group G belongs to $\mathfrak{X}^d$ if

$$\forall H \leq G, \quad H \notin \mathfrak{X} \Rightarrow H \underset{\leq d}{\mathrm{sn}} G$$

where by $H \underset{\leq d}{\mathrm{sn}} G$ we mean that H is a subnormal subgroup of G of defect at most d.
It is easy to see that also $\mathfrak{X}^d$ is a SH-closed class of groups.

Lemma 2.28. *If $\mathfrak{X}$ is a Robinson class, then the same can be said of $\mathfrak{X}^d$ for all positive integers.*

Proof. Let G be a finitely generated hyper(abelian-or-finite) group whose finite homomorphic images all lie in $\mathfrak{X}^d$. If it has a $\mathfrak{X}$-subgroup of finite index X, then X is polycyclic-by-finite and so is G. If it does not have such a subgroup, then all of its finite homomorphic images are groups such that all of their subgroups are subnormal of bounded defect. For a Theorem by Roseblade, this means they are nilpotent of bounded class. Since nilpotency is a Robinson class, this means that G is nilpotent. Either way, we concluded G is polycyclic-by-finite.

Now, let H be a subgroup of G. For a famous result by Malcev, it is the intersection of subgroups of finite index containing it. If it is contained in a $\mathfrak{X}$-group, since $\mathfrak{X}$ is S-closed, then also H is a $\mathfrak{X}$-group. Otherwise, it is the intersection of a family of subnormal subgroups whose defect is bounded and hence the same can be said of H, proving that G is a $\mathfrak{X}^d$-group. □

Lemma 2.29. *Let G be a $\mathfrak{X}^d$-group. Then G has a normal subgroup X that is either $\mathfrak{X}$ or minimal-non-$\mathfrak{X}$ and such that G/X is soluble of derived length bounded in terms of d.*

Proof. Let H be any subgroup of G that is not an $\mathfrak{X}$-group. Since every subgroup that contains H^G is not a $\mathfrak{X}$-group, G/H^G is a group in which every subgroups is subnormal of bounded defect d, hence $\gamma_{\rho(d)}(G) \leq H^G$, where ρ is the function in the famous Roseblade Theorem and hence also the term of the derived series $G^{(\rho(d))} \leq H^G$. Now, define X_1 to be the intersection of H^G as H varies in the set all of non-$\mathfrak{X}$ subgroups of G. Of course X_1 is a normal subgroup of G and the factor $G^{(\rho(d))} \leq X_1$. Now, consider any subgroup $H_1 \leq X_1$ that is not a $\mathfrak{X}$-group. Since $H_1 \underset{\leq d-1}{\text{sn}} H_1^G$, we have also that $H_1 \underset{\leq d-1}{\text{sn}} X_1$. This means X_1 is a $\mathfrak{X}^{d-1}$-group. From here on, we can repeat the construction inside it to find a subgroup X_2 which is $\mathfrak{X}^{d-2}$ and it contains $X_1^{(\rho(d-1))}$, so it also contains $G^{(\rho(d)+\rho(d-1))}$. If we repeat this construction $d-1$ times we reach the definition of X_{d-1}, which is a $\mathfrak{X}^1$-group, i.e. a group in which non-$\mathfrak{X}$-subgroups are normal. By intersecting all of its non-$\mathfrak{X}$ subgroups one last time, we find a minimal-non-$\mathfrak{X}$-group X_d. Hence the subgroup $X = G^{\left(\sum_{i=1}^{d} \rho(i)\right)}$ is such that the factor of G over it is soluble of derived length bounded in terms of d and it is contained (not necessarily strictly) in a minimal-non-$\mathfrak{X}$ group, concluding the proof. □

As a consequence, if $\mathfrak{X}$ is an accessible class of soluble-by-finite groups, e.g. the class of metahamiltonian groups, then $\mathfrak{X}^d$ groups are soluble-by-finite.

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