Homology of covering spaces of symmetric products of a Riemann surface

by

© Asitha Pitawala
Supervisor: Dr. Thomas Baird

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Department of Mathematics and Statistics

Memorial University

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Abstract

Homology and cohomology are considered valuable algebraic tools for studying topological spaces. Homology groups of the symmetric product of a Riemann surface were determined by I.G.Macdonald [9] in 1962. The main object of this project is finding the cohomology and homology groups of certain covering spaces of these spaces.

Lay summary

Algebraic topology uses algebraic tools such as homology and cohomology to study topological spaces. The $n-$th symmetric product of a compact Riemann space of genus g is a $2n-$dimensional orientable manifold. This thesis deals with the problem of calculating homology and cohomology groups of certain covering spaces of symmetric products of a Riemann surface.

We introduce the covering space as a pullback via the Abel-Jacobi map and use the Euler characteristic to determine the Betti numbers of the covering space. As a summary of results, apart from finding the homology and cohomology, we determine the Poincaré polynomial, Euler characteristic and zeta function for the symmetric product space and its covering space.

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Chapter 1

Introduction

The $n-$th symmetric product of a topological space X is defined as the quotient space $SP^n(X) := X^n/S_n$, where X^n is the $n-$ fold product space and the symmetric group on $n-$letters S_n acts on X^n by permutating factors. If M is a $2-$dimensional manifold, then $SP^n(M)$ is a $2n-$manifold. These manifolds were studied in [2],[10],[9] and they appear in many other papers in different contexts.

A Riemann surface of genus g, Σ_g is a $2-$dimensional manifold with the cell complex structure of one $0-$cell, $2g$ $1-$cells and one $2-$cell.

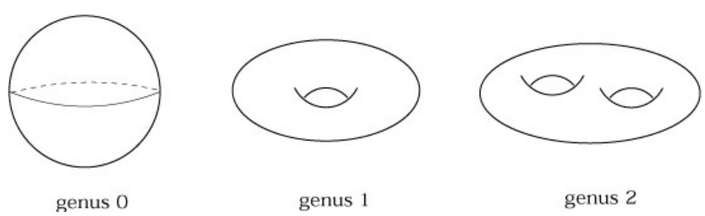


Figure 1.1: Genus g surfaces

The $2n$-dimensional manifold $SP^n(\Sigma_g)$ is the domain of the classical Abel-Jacobi map [1], [6].

$$AJ : SP^n(\Sigma_g) \to J(\Sigma_g)$$

where $J(\Sigma_g)$ is the Jacobian of Σ_g.

The main objective of this thesis is to calculate the homology and cohomology of certain covering spaces of $SP^n(\Sigma_g)$ constructed as pullbacks of covering spaces over $J(\Sigma_g)$. A study of $SP^n(M)$ for closed, even-dimensional manifolds was published by

Hirzebruch [8] and $SP^n(\Sigma_g)$ was studied by Macdonald [9] in 1962.

In the 19^{th} century, homological algebra had its origin through the work of Riemann(1857) and Betti(1871) on homology numbers and then Poincaré in 1895. Emmy Noether's introduction of the homology groups of a space in the first half of the 20^{th} century sparked mathematicians' interest in this approach. From 1940 to 1955, these topologically modified procedures for computing the homology groups were extended to define cohomology. Since then, homology and cohomology have become fundamental tools in algebraic topology.

Given a base point in Σ_g, there is a natural inclusion of $SP^n(\Sigma_g)$ into $SP^{n+1}(\Sigma_g)$. The infinite symmetric space $SP^\infty(\Sigma_g)$ is the colimit of the $SP^n(\Sigma_g)$ under this natural inclusion.

A space X with one nontrivial homotopy group $\pi_n(X) \simeq G$ is called an Eilenberg-MacLane space $K(G, n)$, where G is a group, and for $n \geq 2$ is abelian. In particular, the $K(\mathbb{Z}, n)$ has a natural geometric realization as $SP^\infty(S^n)$. Moreover, $K(G, n)$ plays a main role in Dold and Thom [5], [11] who introduced homotopical decomposition for connected CW-complexes of the infinite symmetric product,

$$SP^\infty(X) \simeq \prod_{n=1}^{\infty} K(H_n(X, \mathbb{Z}), n).$$

The Σ_g satisfies the conditions of the Dold-Thom theorem and opens the possibility to introduce the homology of infinite symmetric space. Note that $SP^\infty(\Sigma_g)$ has a cell complex structure such that every cell of dimension $k \leq n$ lies in $SP^n(\Sigma_g)$. This argument implies that the homology of $SP^n(\Sigma_g)$ and $SP^\infty(\Sigma_g)$ are the same up to degree n; this result was introduced by MacDonald in 1962 [9]. The remaining homology groups given are by the Poincaré Duality Theorem. We present the answer using Poincaré polynomials, which are generating functions for the Betti numbers.

The pullback of a covering space along a continuous function is a covering space. We use this fact to introduce that covering space of $SP^\infty(\Sigma_g)$ as a pullback space, and it has the same homology as $SP^\infty(\Sigma_g)$.

Next we introduce a relation between the covering spaces of $SP^n(\Sigma_g)$ and $SP^\infty(\Sigma_g)$ respectively. Let $\pi : \tilde{Y} \to Y$ be a covering map and Y_n be the $n-$th skeleton of the space Y; then, $\pi^{-1}(Y_n) = \tilde{Y}_n$ is the $n-$th skeleton of the covering space $\tilde{Y}$. This implies that the inclusion of $\widetilde{SP^n(\Sigma_g)}$ into $\widetilde{SP^\infty(\Sigma_g)}$ is a homeomorphism up to the $n-$skeleton and hence, their homology groups are isomorphic up to $n-1$. It remains to calculate the $n-$th homology; we use the Euler characteristic and its properties for covering spaces to find it.

The rest of the homology groups and cohomology groups up to $2n$ are determined by using the Poincaré Duality Theorem. Since the Betti numbers are independent of the field, this also determines the homology over the integer coefficients.

1.0.1 Outline

This thesis presents the homology and cohomology groups of certain covering spaces of the $n-$th symmetric product of a genus g Riemann surface. In particular, we establish the answers over the integer coefficients $\mathbb{Z}$. This section is a synopsis of the subsequent chapters.

Chapter 2 will discuss the algebraic and topological background required for readers who have minimal knowledge of algebraic topology. Then we will describe homology and cohomology in chapter 3.

Chapter 4 will deal with Betti numbers and Poincaré polynomials, which are used to calculate homology and cohomology groups.

The homology, and cohomology of $SP^n(\Sigma_g)$ are presented in chapter 5 . We show all the answers are independent of the field and hence, define the results over integers.

We introduce the Abel-Jacobi map in chapter 6. Subsequently, we will discuss the covering spaces and pullback in chapter 7. Then in chapter 8, we will prove our main result.

In chapter 9 , we summarize our results.

Chapter 2

Background

2.1 Topological background

Definition 2.1.1. A **topological space** (X, τ) is a set X and a collection of open sets τ of X, satisfying the following conditions:

1. $\emptyset$ and X are open,
2. An arbitrary union of open sets is open,
3. Any finite intersection of open sets is open.

Usually, we denote the topological space (X, τ) simply by X. A collection of open sets $\mathfrak{B}$ in a topological space X is called a **basis** if every other open set in X is a union of sets in $\mathfrak{B}$.

Definition 2.1.2. A **continuous map** $f : X \to Y$ between topological spaces is a map of sets for which pre-images of open sets are open. i.e,

$$\text{If} \quad U \subseteq Y \quad \text{is open, then } f^{-1}(U) := \{x \in X \mid f(x) \in U\} \subseteq X \quad \text{is open.}$$

Definition 2.1.3. A **homeomorphism** is a continuous bijection $f : X \to Y$ such that the inverse f^{-1} is also continuous. This is the notion of isomorphism for topological spaces.

Proposition 2.1.1. *Let X, Y and Z be topological spaces.*

1. *The identity map* $Id_X : X \to X$ *is continuous.*

2. *If* $f : X \to Y$ *and* $g : Y \to Z$ *are continuous, then the composition* $g \circ f : X \to Z$ *is continuous.*

3. *Any constant map* $f : X \to Y$ *is continuous.*

2.1.1 Review of Topological Spaces

Definition 2.1.4. Let X be a topological space and $A \subseteq X$ a subset. The **subspace topology** on A is the topology for which $V \subset A$ is open if and only if $V = A \cap U$ for some open set U in X.

The **inclusion map** $i : A \hookrightarrow X$ is continuous with respect to the subspace topology. We have the following special property: a map $f : Y \to A$ is continuous if and only if the composition $i \circ f : Y \to X$ is continuous.

Definition 2.1.5. The **product space** $X \times Y$ of two spaces X and Y is the Cartesian product of sets $X \times Y$, with a basis of open sets of the form $U \times V$ where $U \subset X$ and $V \subset Y$ are both open.

Definition 2.1.6. Let $\{X_\alpha\}$ be an infinite collection of spaces indexed by α. The **coproduct space** is the disjoint union of the sets X_α with $U \subseteq \coprod_\alpha X_\alpha$ and is open if and only if $U \cap X_\alpha$ is open for all α.

The inclusions $i_{\alpha_0} : X_{\alpha_0} \hookrightarrow \coprod_\alpha X_\alpha$ are all continuous. A map $F : \coprod_\alpha X_\alpha \to Y$ is continuous if and only if the composition $F \circ i_\alpha : X_\alpha \to Y$ is continuous for all α.

Definition 2.1.7. An **equivalence relation** on a set X is a relation $\sim$, satisfying for all $x, y \in X$,

1. $x \sim x$

2. $x \sim y$ implies $y \sim x$

3. $x \sim y$ and $y \sim z$ implies $x \sim z$.

Given any relation R on X, we can generate the 'smallest' equivalence relation $\sim_R$ such that xRy implies $x \sim_R y$. Explicitly, we define $x \sim_R y$ if and only if there exists a finite sequence $\{x_i \in X\}_{i=0}^n$ for $n \geq 0$ satisfying,

1. $x_0 = x$
2. $x_n = y$, and
3. $x_i R x_{i-1}$ or $x_{i-1} R x_i$ for all $i = 1, ..., n$.

Given $x \in X$, the **equivalence class** of x is

$$[x] := \{y \in X \,|\, x \sim y\}$$

The equivalence classes determine a partition of X into disjoint non-empty sets. Notice that $[x] = [y]$ if and only if $x \sim y$. Let $E = X/\sim \;:= \{[x] \,|\, x \in X\}$; then, there is a canonical map

$$Q : X \to E, \quad x \to [x]$$

called the **quotient map**.

Definition 2.1.8. Let X be a topological space and let $\sim$ be an equivalence relation on the underlying set X. The **quotient topology** on E is the topology for which $U \subset E$ is open if and only if $Q^{-1}(U)$ is open in X.

2.1.2 Connectedness and Path-Connectedness

A topological space can be 'separated' if it can break up into at least two open sets; otherwise, we can say that the space is connected.

Definition 2.1.9. Let X be a topological space. A **separation** of X is a pair U, V of disjoint non-empty open subsets of X for which the union is X. The space X is said to be **connected** if there does not exist a separation of X.

In other words, a space X is **connected** if there is no proper subset $A \subset X$ which is both open and closed.

Observe that if $A \subset X$ is both open and closed, then the complement A^c is also both open and closed. There is a natural isomorphism $A \coprod A^c \cong X$. Thus, spaces that are not connected can be decomposed into a disconnected union of non-empty spaces.

Definition 2.1.10. Let I denote the unit interval $[0, 1] \subset \mathbb{R}$ with the Euclidean topology. A space X is called **path-connected** if for any two points $p, q \in X$ there exists a continuous map $\gamma : I \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

Every path-connected space is connected, but the converse is not true in general. Connectedness and path-connectedness are preserved under the following operations:

1. A product of (path-)connected spaces is (path-)connected.
2. The continuous image of a (path-)connected space is (path-)connected.
3. Let $\{U_\alpha\}$ be a covering of X such that each U_α is (path-)connected and the intersection $\cap_\alpha U_\alpha$ is non-empty. Then X is (path-)connected.

2.1.3 Covers and Compactness

Definition 2.1.11. An open (closed) cover of a topological space X is a collection of open (closed) sets $\{U_\alpha\}$ such that the union $\cup_\alpha U_\alpha = X$.

Definition 2.1.12. A space X is called **compact** if every open cover $\{U_\alpha\}$ of X contains a **finite subcover.** That is there exists a finite collection $\{U_1, ..., U_n\} \subseteq \{U_\alpha\}$ such that $\cup_{i=1}^n U_i = X$.

2.1.4 Homotopy and Fundamental groups

Two continuous functions from one topological space to another are called homotopic if one can be continuously deformed into the other when two functions are connected in such a deformation, called homotopy.

Definition 2.1.13. Let X and Y be two topological spaces. A *homotopy* between two continuous functions $f, g : X \to Y$ is a continuous function $H : X \times [0, 1] \to Y$ defined by $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. The notation for this homotopy is $f \simeq g$.

Note that homotopy between two continuous functions is an equivalence relation.

Definition 2.1.14. A *homotopy equivalence* between two topological spaces X and Y is a pair of continuous functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq Id_Y$ and $g \circ f \simeq Id_X$.

2.1.5 Homotopy Groups

Let I^n be the $n-$dimensional unit cube that is the product of $n-$many intervals $I = [0, 1]$. The boundary ∂I^n of I^n is the subspace consisting of at least one coordinate equal to 0 or 1. Let X be a set with a base point $\star \in X$ and consider the set

$$\Omega^n(X) = \{f : I^n \to X \mid f(\partial I^n) = \star\}.$$

We can define the homotopy of two functions in $\Omega^n(X)$ by following definition 2.1.13. Two functions $f_0, f_1 : (I^n, \partial I^n) \to (X, \star)$ in $\Omega^n(X)$ are homotopic if there exists a continuous function $H : (I^{n+1}, \partial I^n \times I) \to (X, \star)$ such that

$$f_0(x_1, ..., x_n) = H(x_1, ..., x_n, 0)$$

and

$$f_1(x_1, ..., x_n) = H(x_1, ..., x_n, 1).$$

Note that $H(\partial I^n \times I) = \star$.

Definition 2.1.15. For a space X with a base point $\star$, define the $n-$th fundamental group $\pi_n(X, \star)$ to be the set of homotopy classes of maps in $\Omega^n(X)$. In other words,

$$\pi_n(X, \star) = \Omega^n(X)/\simeq .$$

Note that $\pi_n(X, \star)$ is a group if $n \geq 1$ and abelian if $n \geq 2$.

Theorem 2.1.2. *Whitehead Theorem*
Let $f : X \to Y$ be a continuous map between connected cell complexes. Then f is a homotopy equivalence if and only if $f_* : \pi_k(X) \to \pi_k(Y)$ is an isomorphism for all $k \geq 1$, where $\pi_k(X)$ and $\pi_k(Y)$ are $k-$th homotopy groups of X and Y for $k \geq 1$, respectively.

2.2 Algebraic background

Definition 2.2.1. A **group** is a set G which is closed under an operation $\circ$, satisfies the following conditions, is denoted as $(G, \circ)$:

1. Identity: there exists $e \in G$ such that $e \circ x = x \circ e = x$ for all $x \in G$
2. Inverse: for every $x \in G$, there exists $i \in G$ such that $x \circ i = i \circ x = e$
3. Associativity: $x \circ (y \circ z) = (x \circ y) \circ z$ for every $x, y, z \in G$

In addition, if $x \circ y = y \circ x$ for all $x, y \in G$, then we say that G is an abelian group.

Definition 2.2.2. A **ring** is a set R, which is closed under two operations $+$ and $\cdot$ such that the $(R, +)$ is an abelian group, the operation $\cdot$ is associative, and satisfies the distributive properties.

Definition 2.2.3. A **field** is a set $\mathbb{F}$ which is closed under two operations $+$ and $\cdot$ such that $\mathbb{F}$ is an abelian group under addition $(\mathbb{F}, +)$, and the set without the additive identity is an abelian group under multiplication $(\mathbb{F} \setminus \{0\}, \cdot)$, and satisfies the distributive property of multiplication over addition.

Definition 2.2.4. Module:
Let R be a ring and 1 be the multiplicative identity. A left R-module ${}_RM$ is an abelian group $(M, +)$ with an operation $\cdot : R \times M \to M$ such that, for all $r, s \in R$ and $x, y \in M$ satisfies:

1. $r \cdot (x + y) = r \cdot x + r \cdot y$
2. $(r + s) \cdot x = r \cdot x + s \cdot x$
3. $(r \cdot s) \cdot x = r \cdot (s \cdot x)$
4. $1 \cdot x = x$

The operation $\cdot$ is called the scalar multiplication and is usually written by juxtaposition, i.e. $r \cdot x = rx$ for $r \in$ R and $x \in M$. The right R-module M_R is defined similarly, except that the ring acts on the right.

Left group actions: An action of a group G on a non-empty set X is a function, $\alpha : G \times X \to X$, that satisfies the axioms:

1. Identity: $\alpha(e, x) = x$
2. Compatibility: $\alpha(h, \alpha(g, x)) = \alpha(hg, x)$

for all $g, h \in G$ and all $x \in X$, where $e \in G$ is the identity. We often shortened $\alpha(x, g)$ to $x \cdot g$ or xg,

1. Identity: $e \cdot x = x$
2. Compatibility: $h \cdot (g \cdot x) = (hg) \cdot x$

Suppose that G acts on X; then, we can define a relation $\sim$ on X by setting $x \sim y$ for $x, y \in X$, iff there exists an element $g \in G$ such that $g \cdot x = y$. Then $\sim$ is an equivalence relation and it gives mutually disjoint equivalence classes called *orbits* of group action. The **orbit** of an element $x \in X$ is the set of elements in X to which x can be moved by the elements of G and is defined as:

$$G \cdot x = \{g \cdot x \,|\, g \in G\}.$$

2.3 Manifolds

2.3.1 Topological manifolds

A **second countable** space is a topological space, the topology of which has a countable base. A **Hausdorff space** is a topological space where for any two distinct points, there exist neighbourhoods of each which are disjoint from each other.

An $n-$dimensional manifold is a topological space that locally looks like $\mathbb{R}^n$. Formally, a topological $n-$manifold is a second countable Hausdorff space in which each point has an open neighbourhood homeomorphic to $\mathbb{R}^n$. A compact manifold is a manifold that is compact as a topological space.

Examples:

- $\mathbb{R}^n$ is an $n-$manifold.
- Any discrete space is a $0-$dimensional manifold.

- The $n-$dimensional sphere S^n is a compact $n-$manifold.
- The $n-$dimensional torus T^n is a compact $n-$manifold.
- Real projective space $\mathbb{R}P^n$ is a $n-$dimensional manifold.
- Complex projective space $\mathbb{C}P^n$ is an $2n-$dimensional manifold.

2.3.2 Smooth manifolds

Let M be a topological manifold. A **chart** on M is a pair (U, φ), where U is an open subset of M and $\varphi : U \to \tilde{U}$ is a homeomorphism from U to an open subset, $\tilde{U} = \varphi(U) \subseteq \mathbb{R}^n$.

If (U, φ) and (V, ψ) are two charts such that $U \cap V \neq \emptyset$, then the composite map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is called the **transition map** from φ to ψ. Two charts are said to be **smoothly compatible** if either $U \cap V = \emptyset$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism.

An **atlas** $\mathcal{A}$ for M is a collection of charts the domains for which cover M. An atlas is called a smooth atlas if any two charts in $\mathcal{A}$ are smoothly compatible with each other. A smooth atlas $\mathcal{A}$ on M is **maximal** if it is not properly contained in any larger smooth atlas. A **smooth structure** on M is a maximal smooth atlas.

A **smooth manifold** M is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M. A **differentiable manifold** is a topological manifold M, together with a maximal differentiable atlas on M.

2.3.3 Tangent map

Let M be a differentiable manifold of dimension n and pick a chart (U, φ) on M. For any $x \in M$, suppose that two curves $\gamma_1, \gamma_2 : (-1, 1) \to M$ with $\gamma_1(0) = x = \gamma_2(0)$ are given such that both $\varphi \circ \gamma_1, \varphi \circ \gamma_2 : (-1, 1) \to \mathbb{R}^n$ are differentiable. Then γ_1 and γ_2 are said to equivalent if

$$(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0).$$

This defines an equivalence relation, and equivalence classes of such curves are known as the tangent vectors of M at x. The tangent space of M at x is the set of all tangent vectors of M at x and denoted by T_xM. The equivalence class of a curve γ is denoted by $\gamma^{'}(0)$.

We can define a bijection $D_x\varphi : T_xM \to \mathbb{R}^n$ by

$$D_x\varphi(\gamma^{'}(0)) = \frac{d}{dt}[(\varphi \circ \gamma)(t)]_{t=0}$$

where $\gamma \in \gamma^{'}(0)$. We make T_xM into a vector space by declaring $D_x\varphi$ to be an isomorphism of vector spaces.

2.3.4 Transversality

Definition 2.3.1. Let M, X, Y be differentiable manifolds and let $f : X \to M$ and $g : Y \to M$ be smooth maps. We say that f and g are *transverse,* if, whenever $f(x) = g(y) = m$,

$$D_xf(T_xX) + D_yg(T_yY) = T_mM$$

where D_xf is the derivative of f and T_xX is the tangent space at $x \in X$.

Observe that if dim(X)+ dim$(Y) <$ dim(M), and f, g are transverse, then $f(X) \cap g(Y) = \emptyset$.

Chapter 3

Homology and Cohomology

The homology groups $H_0(X), H_1(X), H_2(X), ...$ of a topological space X are a set of topological invariants of X represented by its homology groups, where the k^{th} homology group $H_k(X)$ describes, informally, the number of $k - \textit{dimensional}$ holes in X. For instance $H_0(X)$ describes the path-connected components of X.

Most of the theorems and proofs in this chapter can be found in the Allen Hatcher's Algebraic Topology book [7].

3.1 Singular Homology

3.1.1 Simplices

Given $n+1$ points $v_0, ..., v_n \in \mathbb{R}^n$ that do not lie in a hyperplane, the simplex $[v_0, ..., v_n]$ is the smallest convex set, where points v_i are the vertices of the simplex.

The standard $n - \textit{simplex}$ Δ^n is the simplex spanned by the zero vector e_0 and the standard basis vectors $e_1, ..., e_n$ in $\mathbb{R}^n$. Thus,

$$\Delta^n := \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1 \quad \text{and} \quad t_i \geq 0 \quad \text{for all} \quad i\}$$

A singular $n - \textit{simplex}$ in a topological space X is a continuous map,

$$\sigma : \Delta^n \to X.$$

A singular $0-simplex$ in X is simply a point in X, and a singular $1-simplex$ is a continuous path in X, etc..

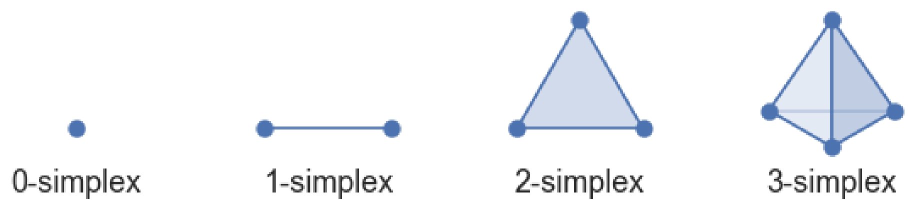


Figure 3.1: Simplices

There is a canonical linear homeomorphism from the standard $n-simplex$ Δ^n onto any other $n-simplex$ $[v_0, ..., v_n]$, preserving the order of vertices, such that $(t_0, ..., t_n) \to \sum_i t_i v_i$. The coefficients t_i are the barycentric coordinates of the point $\sum_i t_i v_i$ in $[v_0, ..., v_n]$.

A face of a simplex is the subsimplex and it can be any nonempty subset of the vertices. We define the face maps for $0 \leq i \leq n$, such that

$$F_n^i : \Delta^{n-1} \to \Delta^n$$

by $F_n^i = [e_0, ..., \hat{e}_i, ..., e_n]$, where the $\hat{e}_i$ means omit e_i.

The $i-th$ face of a singular $n-simplex$ $\sigma : \Delta^n \to X$ is the $(n-1)-simplex$

$$\sigma^{(i)} : \Delta^{n-1} \to X$$

defined by composition with the face map:

$$\sigma^{(i)} = \sigma \circ F_n^i.$$

3.1.2 Chains, Cycles, and Boundaries

Define $S_n(X)$ to be the free abelian group generated by singular $n-simplices$. The elements of $S_n(X)$ are called singular chains and are formal linear combinations of the form

$$\sum_{\sigma} a_\sigma \sigma,$$

where the coefficients $a_\sigma \in \mathbb{Z}$ and the sum is over a finite number of singular $n-$ *simplices* σ.

The boundary map $\partial_n : S_n(X) \to S_{n-1}(X)$ is a homomorphism, defined on singular simplices by

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma^i$$

and extended linearly to all of $S_n(X)$ by the rule:

$$\partial_n \left(\sum_{\sigma} a_\sigma \sigma \right) = \sum_{\sigma} a_\sigma \partial_n(\sigma).$$

We will often drop the subscript and write $\partial = \partial_n$ when it is unlikely to cause confusion.

Proposition 3.1.1. *The composition $\partial_{n-1} \circ \partial_n : S_n(X) \to S_{n-2}(X)$ is the zero map, i.e. dropping subscripts, we write this*

$$\partial^2 = 0.$$

Proof. Since $S_n(X)$ is generated by simplices, it suffices to check that $\partial_{n-1} \circ \partial_n = 0$ for all $n-$ *simplices* σ. It is easily checked that if $0 \leq j < i \leq n$, the free maps satisfy

$$F_n^i \circ F_{n-1}^j = F_n^j \circ F_{n-1}^{i-1}.$$

$$
\begin{aligned}
\partial_{n-1} \circ \partial_n(\sigma) &= \partial_{n-1}(\sum_{i=0}^{n}(-1)^i \sigma^i) \\
&= \sum_{i=0}^{n}(-1)^i \partial_{n-1}(\sigma \circ F_n^i) \\
&= \sum_{i=0}^{n}(-1)^i \sum_{j=0}^{n-1}(-1)^j (\sigma \circ F_n^i \circ F_{n-1}^j) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{n-1}(-1)^{i+j}(\sigma \circ F_n^i \circ F_{n-1}^j) \\
&= \sum_{0 \le i \le j \le n-1}(-1)^{i+j}(\sigma \circ F_n^i \circ F_{n-1}^j) + \sum_{0 \le j \le i \le n-1}(-1)^{i+j}(\sigma \circ F_n^i \circ F_{n-1}^j) \\
&= \sum_{0 \le i \le j \le n-1}(-1)^{i+j}(\sigma \circ F_n^i \circ F_{n-1}^j) - \sum_{0 \le j \le i-1 \le n-1}(-1)^{i-1+j}(\sigma \circ F_n^j \circ F_{n-1}^{i-1}) \\
&= 0
\end{aligned}
$$

□

The group of $n - cycles$ $Z_n(X)$ is the kernel of ∂_n :

$$Z_n(X) := \{\alpha \in S_n(X) \,|\, \partial(\alpha) = 0\}$$

and the group of $n - boundaries$ $B_n(X)$ is the image of ∂_{n+1} :

$$B_n(X) := \{\partial(\beta) \,|\, \beta \in S_{n+1}(X)\}$$

By the Proposition 3.1.1, $B_n(X)$ is a normal subgroup of $Z_n(X)$. The $n-th$ degree singular homology of X is the quotient group:

$$H_n := Z_n(X)/B_n(X).$$

Note that homology is a homotopy invariant, meaning that if $f, g : X \to Y$ are homotopic, then the corresponding induced maps $f_*, g_* : H_*(X) \to H_*(Y)$ are equal.

3.1.3 Homology as a functor

Let $f : X \to Y$ be a continuous map. If σ is an $n-simplex$ for X, then the composition $f \circ \sigma$ is an $n-simplex$ for Y. This defines a homomorphism:

$$S_n(f) : S_n(X) \to S_n(Y), \quad \text{such that}$$

$$S_n(\sigma) = f \circ \sigma,$$

and linearity on S_n implies that,

$$S_n(\sum_\sigma a_\sigma \sigma) = \sum_\sigma a_\sigma f \circ \sigma.$$

Clearly, $S_n(Id_X) = Id_{S_n(X)}$ and $S_n(f \circ g) = S_n(f) \circ S_n(g)$ for composable continuous maps f and g. Thus S_n is a functor from topological spaces to abelian groups.

It is clear that $S_n(f)$ sends $Z_n(X)$ to $Z_n(Y)$ and $B_n(X)$ to $B_n(Y)$, and thus induces a homomorphism between the quotient groups:

$$H_n(f) : H_n(X) \to H_n(Y).$$

3.2 Chain Complexes

A chain complex of abelian groups $C := (C_n, \partial_n)_{n \in \mathbb{Z}}$ is a sequence of abelian groups $(C_n)_{n \in \mathbb{Z}}$ and homomorphisms $\partial_n : C_n \to C_{n-1}$ such that $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$. We usually write a chain complex as:

$$...C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} ... \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0.$$

Typically $C_n = 0$ for all $n < 0$ and where 0 denotes the trivial group.

Let $\text{im}(\partial_{n+1})$ be the image of the boundary map ∂_{n+1} and $\ker(\partial_n)$ be the kernel of the boundary map ∂_n. We define the $n-chains$ $Z_n(C) = \ker(\partial_n)$ and the $n-boundaries$ $B_n(C) = \text{im}(\partial_{n+1})$ and hence, the $n-th$ homology group of the chain complex $H_n(C) = Z_n(C)/B_n(C)$. For $z \in Z_n(C)$, the coset represented by z is denoted by $[z] \in H_n(C)$.

A morphism of chain complexes $f : C \to C'$ is a sequence of homomorphisms $(f_n : C_n \to C'_n)_{n\in\mathbb{Z}}$ that commutes with the boundary maps: $f_{n-1} \circ \partial_n = \partial'_n \circ f_n$ for all n. In other words, the following diagram commutes:

$$\begin{array}{ccc} C_n & \xrightarrow{\partial_n} & C_{n-1} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\ C'_n & \xrightarrow{\partial'_n} & C'_{n-1} \end{array} \tag{3.2.1}$$

A chain map $h : C \to D$ induces homomorphisms in homology $H_q(h) : H(C) \to H(D)$ for all $q \in \mathbb{Z}$ by the rule

$$H_q(h)([z]) = [h_q(z)].$$

Lemma 3.2.1. *Let $h : C \to D$ be a morphism of chain complexes such that $p_i : C_i \to D_i$ is an isomorphism for $i \leq n$.*

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & C_{i+1} & \longrightarrow & C_i & \xrightarrow{\partial_i} & C_{i-1} & \longrightarrow & \ldots & \longrightarrow & 0 \\ & & \downarrow & & \cong\downarrow{\scriptstyle p_i} & & \cong\downarrow{\scriptstyle p_{i-1}} & & & & \\ \ldots & \longrightarrow & D_{i+1} & \longrightarrow & D_i & \xrightarrow{\delta_i} & D_{i-1} & \longrightarrow & \ldots & \longrightarrow & 0 \end{array}$$

Then,

$$H_i(C) \cong H_i(D) \quad \textit{for} \quad i \leq n-1.$$

Proof. 1. First we prove that $p_i(\ker(\partial_i)) = \ker(\delta_i)$.

Take any $x \in \ker(\partial_i)$. Since p_{i-1} is an isomorphism $p_{i-1} \circ \partial_i(x) = 0$ and since the diagram is commutative, $\delta_i \circ p_i(x) = 0$. Hence $p_i(x) \in \ker(\delta_i)$.

However, for any $y \in \ker(\delta_i)$, by the isomorphism and the commutative diagram, given that $\partial_i \circ p_i^{-1}(y) = p_{i-1}^{-1} \circ \delta_i(y) = 0$. Therefore, $y \in p_i(\ker(\partial_i))$.

2. Next we prove that $p_{i-1}(\mathrm{im}(\partial_i)) = \mathrm{im}(\delta_i)$.

Take any $y \in \mathrm{im}(\partial_i)$; then, there exists $x \in C_i$ such that $\partial_i(x) = y$. By the commutative diagram $p_{i-1}(y) = \delta_i \circ p_i(x)$, and by the isomorphism, there exists $z = p_i(x) \in D_i$ such that $p_{i-1}(x) = \delta_i(z)$. Thus, $p_{i-1}(y) \in \mathrm{im}(\delta_i)$.

However, for any $y \in \mathrm{im}(\delta_i)$, there exists $x \in D_i$ such that $y = \delta_i(x)$. By the diagram, $p_{i-1}^{-1}(y) = \partial_i \circ p_i^{-1}(x)$ and by the isomorphism, there exists $z \in C_i$ such that $p_i^{-1}(x) = z$. So, $p_i^{-1}(y) \in \mathrm{im}(\partial_i)$ and hence $y \in p_{i-1}(\mathrm{im}(\partial_i))$.

By Case 1 and Case 2, the p_i is restricted to isomorphisms $\ker(\partial_i) \cong \ker(\delta_i)$ and $\mathrm{im}(\partial_i) \cong \mathrm{im}(\delta_i)$.

Now, by the definition of homology, we have

$$H_i(C, R) = \frac{\ker(\partial_i)}{\mathrm{im}(\partial_{i+1})} \cong \frac{\ker(\delta_i)}{\mathrm{im}(\delta_{i+1})} = H_i(D, R) \quad \text{for} \quad i \leq n-1.$$

□

The following theorem often reduces the problem of calculating homology groups of $\mathbb{Z}$ to calculating homology groups over fields. It is a consequence of the Universal Coefficient Theorem [7].

Theorem 3.2.2. *Let X be a finite cell complex. If for any field $\mathbb{F}$ and all $i \geq 0$ the dimension of $H_i(X, \mathbb{F})$ is independent of $\mathbb{F}$, then $H_i(X, \mathbb{Z})$ is a free abelian group of the same rank as the Betti number 4.1.2.*

3.3 Relative Homology

Let X be a topological space and A be a subset of X. A pair (X, A) gives rise to an inclusion of chain groups $S_n(A) \leq S_n(X)$. Define the relative chain group of the pair to be the quotient group:

$$S_n(X, A) := S_n(X)/S_n(A).$$

The relative chain groups combine to form the relative chain complex:

$$...S_{n+1}(X, A) \xrightarrow{\bar{\partial}_{n+1}} S_n(X, A) \xrightarrow{\bar{\partial}_n} S_{n-1}(X, A) \xrightarrow{\bar{\partial}_{n-1}} ... \xrightarrow{\bar{\partial}_1} S_0(X, A) \xrightarrow{\bar{\partial}_0} 0$$

where the boundary map is defined by the following commutative diagram:

$$\begin{array}{ccc} S_n(X) & \xrightarrow{\partial_n} & S_{n-1}(X) \\ \downarrow & & \downarrow \\ S_n(X,A) & \xrightarrow{\bar{\partial}_n} & S_{n-1}(X,A) \end{array} \tag{3.3.1}$$

where the vertical arrows are quotient maps. Note that $\bar{\partial}_n$ is well defined because ∂_n sends $S_n(A)$ to $S_{n-1}(A)$ and that $\bar{\partial}_n^2 = 0$ because $\partial_n^2 = 0$. It follows that we can define relative cycles and relative boundaries, $Z_n(X, A)$ and $B_n(X, A)$, respectively. Thus, relative homology can be defined as:

$$H_n(X, A) = Z_n(X, A)/B_n(X, A).$$

The quotient morphisms $S_n(X) \to S_n(X, A)$ fit together into a morphism of chain complexes, $j : C(X) \to C(X, A)$ combined with inclusion chain morphism $i : C(A) \to C(X)$. Then we get a commutative diagram:

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & S_{n+1}(A) & \longrightarrow & S_n(A) & \longrightarrow & S_{n-1}(A) & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \\ \ldots & \longrightarrow & S_{n+1}(X) & \longrightarrow & S_n(X) & \longrightarrow & S_{n-1}(X) & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \\ \ldots & \longrightarrow & S_{n+1}(X,A) & \longrightarrow & S_n(X,A) & \longrightarrow & S_{n-1}(X,A) & \longrightarrow & \ldots \end{array}$$

By functoriality, these chain morphisms give rise to homology homomorphisms $H_n(A) \to H_n(X) \to H_n(X, A)$ for all $n \geq 0$. The most important property of relative homology is the existence of a connecting homomorphism:

$$\partial : H_n(X, A) \to H_{n-1}(A), \quad \text{such that} \quad \partial([z]) = [\partial(z)].$$

Definition 3.3.2. A sequence of abelian groups and homomorphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is called exact at B if $\ker(g) = \mathrm{im}(f)$.

Theorem 3.3.1. *The long sequence of homomorphisms*

$$\dots \longrightarrow H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(j)} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \dots \longrightarrow 0$$

is exact (i.e. exact at all groups in the sequence) and is called the long exact homology sequence associated to the pair (X, A).

Let us state the following lemma for later use.

Lemma 3.3.2. *The Five-Lemma*
In a commutative diagram of abelian groups as follows:

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\ \downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \downarrow{\delta} & & \downarrow{\epsilon} \\ A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E' \end{array}$$

If the two rows are exact and α, β, δ *and* ϵ *are isomorphisms, then* γ *is also an isomorphism.*

Proof. A proof can be found in Hatcher 2.1 [7]. □

3.4 Reduced Homology

The reduced homology is a modified version of singular homology. Let X be a space; then there exists a unique map to a point $\epsilon : X \to \{pt\}$. Define the reduced homology as:

$$\tilde{H}_n(X) := \ker H_n(\epsilon).$$

For a non-empty subset A of X, we define the reduced homology as:

$$\tilde{H}_n(X, A) = H_n(X, A).$$

3.5 Cellular Homology

3.5.1 Cell Complexes

Let

$$D^n := \{x \in \mathbb{R}^n \mid |x| \leq 1\}$$

be the unit disk or closed n-cell with the boundary

$$S^{n-1} = \partial D^n := \{x \in \mathbb{R}^n \mid |x| = 1\}.$$

For a topological space X and a continuous map $f : S^{n-1} \to X$, we can construct a new quotient space

$$Y := (X \sqcup D^n)/\sim$$

by the equivalence relation generated by $p \sim f(p)$ for all $p \in S^{n-1}$. We say that Y is obtained from X by attaching an n-cell and the map f is called the attaching map. More generally, if we have a collection of maps $f_\alpha : S^{n-1} \to X$, then

$$Y = (X \sqcup (\sqcup_\alpha D^n_\alpha))/\sim$$

where $p \sim f_\alpha(p)$ for all $p \in S^{n-1}_\alpha$ and α.

A cell complex (CW-complex) is a space obtained by iteratively attaching $n-$cells. That is to say, a $0-$dimensional cell complex is a discrete space, and an n-dimensional cell complex X_n is a space obtained by attaching n-cells to $(n-1)-$dimensional cell complex X_{n-1}. An infinite dimensional cell complex is defined as a colimit.

A subspace $A \subseteq X$ is called a subcomplex if it is a closed union of cells. For a given subcomplex $A \subseteq X$, the quotient space X/A defined by identifying all points in A with each other is naturally a cell complex, called a quotient complex of X.

The subcomplex $X_n \subseteq X$, consisting of all cells of dimension $\leq n$, is called the n-skeleton of X.

Proposition 3.5.1. *If $A \subseteq X$ is a subcomplex, then $H_n(X, A) \cong \tilde{H}_n(X/A)$ and we have a long exact sequence in homology*

$$\ldots \longrightarrow \tilde{H}_{n+1}(X/A) \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow \tilde{H}_n(X/A) \longrightarrow H_{n-1}(A) \longrightarrow \ldots \longrightarrow 0.$$

Lemma 3.5.2. *Let X be a cell complex and $C \subset X$ a compact subspace. Then C is contained within the union of finitely many cells of X.*

Lemma 3.5.3. *If X is a cell complex, then:*

1. *$H_k(X_n, X_{n-1})$ is zero if $k \neq n$ and is a free abelian group with generators corresponding to the n-cells when $k = n$.*

2. *$H_k(X_n) = 0$ for $k > n$. Thus $H_k(X) = 0$ for $k > dim(X)$.*

3. *The inclusion $i : X_n \hookrightarrow X$ induces an isomorphism $H_k(i) : H_k(X_n) \to H_k(X)$ for $k < n$.*

Proof. By Proposition 1.2, we have an isomorphism $H_k(X_n, X_{n-1}) = \tilde{H}_k(X_n/X_{n-1})$ and X_n/X_{n-1} is a wedge of spheres indexed by the $n - cell$ of X. Property 1 follows.

Property 2 is proven by induction, and is clearly true for $n = 0$. Now suppose it has been proven for $n - 1$. The long exact sequence of the pair contains

$$... \longrightarrow H_k(X_{n-1}) \longrightarrow H_k(X_n) \longrightarrow H_k(X_n, X_{n-1}) \longrightarrow ...$$

where both $H_k(X_{n-1}) = H_k(X_n, X_{n-1}) = 0$ for $k > n$ by induction and property 1. Thus $H_k(X_n) = 0$ as well.

To prove property 3, consider the exact sequence

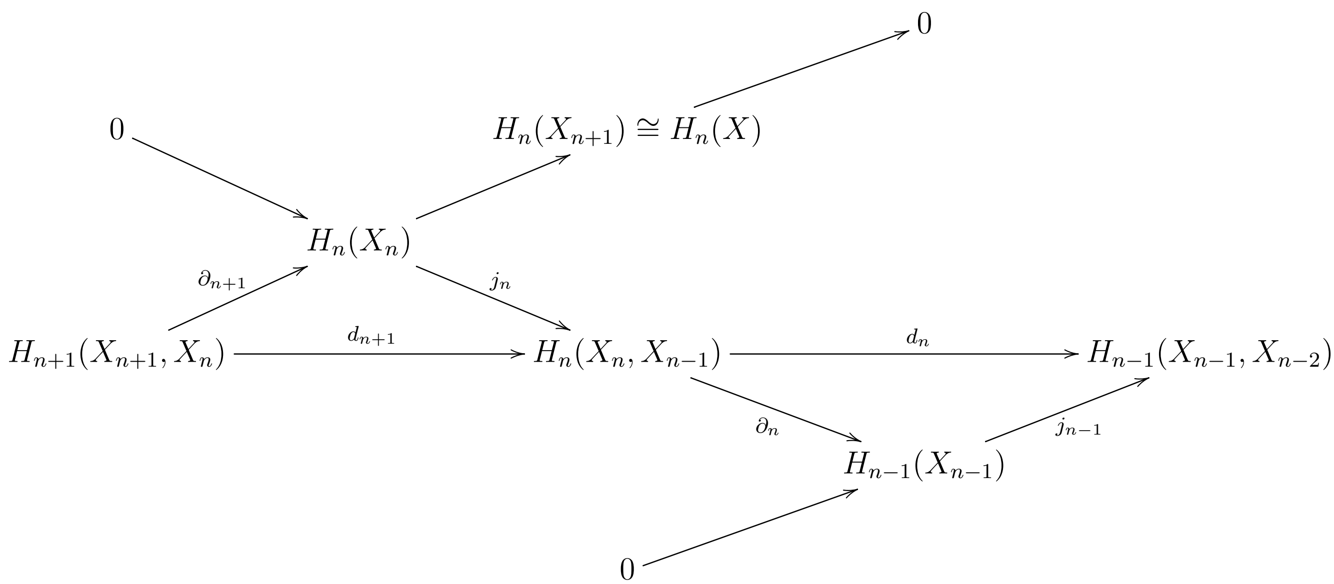
$$H_{k+1}(X_{n+1}, X_n) \longrightarrow H_k(X_n) \longrightarrow H_k(X_{n+1}) \longrightarrow H_k(X_{n+1}, X_n).$$

By property 1, the groups on the end vanish for $k < n$ so $H_k(X_n) \cong H_k(X_{n+1})$. Repeating this argument, we get

$$H_k(X_n) \cong H_k(X_{n+1}) \cong H_k(X_{n+2}) \cong ...$$

which suffices if X is finite dimensional. For the infinite dimensional case, observe that Lemma 1.1 implies that every chain $S_k(X)$ must be in the image of $S_k(X_n)$ for some n (since the union of images of simplices occurring in the chain is a compact subset of X). Thus every cycle $Z_k(X)$ arises as the image of a cycle in $Z_k(X_n)$ for some n, and every boundary $B_k(X)$ arises as the image of a boundary in $B_k(X_n)$ for some n. Thus, the result follows. □

Let X be a CW complex. Using Lemma 3.5.3, we can define a homomorphism $d_n : H_n(X_{n+1}, X_n) \to H_{n-1}(X_{n-1}, X_{n-2})$ by the commutative diagram



(3.5.1)

where d_{n+1} and d_n are defined as the compositions $j_n \circ \partial_{n+1}$ and $j_{n-1} \circ \partial_n$. Note that the diagonal maps occur in the long exact sequences of pairs and hence, the composition $d_n \circ d_{n+1} = 0$. Thus $(H_n(X_n, X_{n-1}), d_n)_{N \in \mathbb{Z}}$ forms a chain complex, called the cellular chain complex. The homology of the cellular chain complex is called the cellular homology.

Theorem 3.5.4. *The cellular homology groups are naturally isomorphic to the singular homology groups.*

Proof. From the diagram above, we may identify $H_n(X) \cong H_n(X_n)/\mathrm{im}(\partial_{n+1})$. Since j_n is injective, it maps $\mathrm{im}(\partial_{n+1})$ isomorphically onto $\mathrm{im}(j_n \partial_{n+1}) = \mathrm{im}(d_{n+1})$ and by exactness, $H_n(X_n)$ isomorphically onto $\mathrm{im}(j_n) = \ker(\partial_n)$. Finally, j_{n-1} is injective, $\ker(\partial_n) = \ker(d_n)$ and hence $H_n(X) \cong \ker(d_n)/\mathrm{im}(d_{n+1})$. □

3.5.2 Examples for homology

Example 3.5.2. Genus g Riemann surface Σ_g is constructed by attaching a 2-cell to a wedge of $2g$ circles using an attaching map $\prod_{i=1}^{g} [a_i, b_i]$. The cellular chain complex is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \quad \text{with both} \quad d_1 = d_2 = 0$$

and it follows that the result is:

$$H_k(\Sigma_g, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, 2, \\ \mathbb{Z}^{2g} & \text{if } k = 1, \\ 0 & otherwise. \end{cases}$$

Example 3.5.3. $n - dim$ sphere is defined by attaching an $n-$cell to a $0-$cell, and

$$H_k(S^n, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, n, \\ 0 & otherwise. \end{cases}$$

Example 3.5.4. The infinite complex projective plane is a cell complex with one cell in each even degree

$$H_k(\mathbb{C}P^\infty, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, even, \\ 0 & \text{if } k = odd. \end{cases}$$

Example 3.5.5. The real projective plane has a cell complex structure with one cell in each dimension. However, the homology groups are dependent on the field.

$$H_k(\mathbb{R}P^n, \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & \text{if } 0 \leq k \leq n, \\ 0 & otherwise. \end{cases}$$

$$H_k(\mathbb{R}P^n, \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } k = 0 \text{, and } k = n \text{ if } n \text{ is odd,} \\ 0 & otherwise. \end{cases}$$

$$H_k(\mathbb{R}P^n, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{, and } k = n \text{ odd,} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } 0 < k < n \text{, and } k \text{ odd,} \\ 0 & otherwise. \end{cases}$$

3.6 Cohomology Groups

Homology groups $H_n(X)$ are the result of forming a chain complex $... \to C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\cdot} ...$ of singular, simplicial, or cellular chains, and then take the homology groups of this chain complex, Z_n/B_n, as we defined. To obtain the cohomology groups $H^n(X)$, we dualize this chain complex as follows:

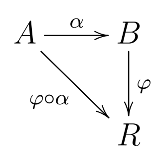
Let R be a commutative ring and consider a usual chain complex C of free abelian groups,

$$... \to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\cdot} ..$$

To dualize this complex we replace each chain group C_n by its dual cochain group $C_n^* = Hom(C_n, R)$, the group of homomorphisms $C_n \to R$, and we replace each boundary map $\partial : C_n \to C_{n-1}$ by its dual coboundary map $\delta = \partial^* : C_{n-1}^* \to C_n^*$.

The reason why δ goes in the opposite direction, increasing rather than decreasing dimension, is purely formal: For a homomorphism $\alpha : A \to B$, the dual homomorphism $\alpha^* : Hom(B, R) \to Hom(A, R)$ is defined by $\alpha^*(\varphi) = \varphi\alpha$, so α^* sends $B \xrightarrow{\varphi} R$ to the composition $A \xrightarrow{\alpha} B \xrightarrow{\varphi} R$.

That is



The dual homomorphisms obviously satisfy $(\alpha\beta)^* = \beta^*\alpha^*$, $\mathbb{I}^* = \mathbb{I}$, and $0^* = 0$. In particular, since $\partial\partial = 0$, it follows that $\delta\delta = 0$, according to the following cochain complex

$$... \leftarrow C_{n+1}^* \xleftarrow{\delta} C_n^* \xleftarrow{\delta} C_{n-1}^* \leftarrow ...$$

and the cohomology group $H^n(X; R)$ can be defined as the homology group. We can now define cochains $Z^n := \ker \delta_{n+1}$, coboundaries $B^n := \mathrm{im}\delta_n$, and cohomology

$$H^n := Z^n/B^n.$$

Definition 3.6.1. Let R be a commutative ring. The singular cohomology of a pair of spaces (X, A), denoted by $H^n(X, A; R)$ for $n \geq 0$, is the cohomology of the singular cochain complex

$$\dots \longrightarrow S^{n-1}(X,A;R) \longrightarrow S^{n}(X,A;R) \longrightarrow S^{n+1}(X,A;R) \longrightarrow \dots$$

obtained by dualizing the singular chain complex of X.

Theorem 3.6.1. *Let X have a finite cell complex. Then, for any field $\mathbb{F}$,*

$$H_k(X,\mathbb{F}) \cong H^k(X,\mathbb{F}) \quad \textit{for} \quad k \in \mathbb{N}.$$

Proof. Let us consider the cellular chain and cochain complexes of X :

$$\dots \to C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \xrightarrow{\partial_{i-1}} \dots$$

$$\dots \leftarrow C^*_{i+1} \xleftarrow{\delta_{i+1}} C^*_i \xleftarrow{\delta_i} C^*_{i-1} \xleftarrow{\delta_{i-1}} \dots$$

with $C_i = H_i(X_i, X_{i-1}) \cong \mathbb{F}^{N_i}$, where N_i is the number of $i - cells$ and $C^*_i = Hom(C_i;\mathbb{F}) = Hom(\mathbb{F}^{N_i};\mathbb{F}) \cong \mathbb{F}^{N_i}$ have the same dimension. Also, since $\delta_i = \partial_i^T$, we have $rank(\partial_i) = rank(\delta_i)$. In other words, the dimensions of the image of ∂_i and δ_i are the same.

By the definition of homology, we have $H_i(X,\mathbb{F}) = \ker(\partial_i)/\mathrm{im}(\partial_{i+1})$ and hence,

$$\dim(H_i(X,\mathbb{F})) = \dim(\ker(\partial_i)) - \dim(\mathrm{im}(\partial_{i+1})).$$

Now, by using the Rank-Nullity Theorem,

$$\dim(H_i(X,\mathbb{F})) = \dim(C_i) - \mathrm{rank}(\partial_i) - \mathrm{rank}(\partial_{i+1}).$$

Similarly, by the definition of cohomology, we have

$$\dim(H^i(X,\mathbb{F})) = \dim(C^*_i) - \mathrm{rank}(\delta^i) - \mathrm{rank}(\delta^{i+1}),$$

and hence $\dim(H_i(X,\mathbb{F})) = \dim(H^i(X,\mathbb{F}))$.

This result leads us to complete the proof. □

3.6.1 The cup product

A graded R-algebra A^* is a direct sum of $R-$modules

$$A^* = \bigoplus_{i\in\mathbb{Z}} A^i$$

equipped with a multiplication $A^i \times A^j \to A^{i+j}$ which is distributive with respect to addition. The elements of $A^i \subseteq A^*$ are called homogeneous of degree i. We say A^* is a graded commutative if the multiplication satisfies

$$m \cdot n = (-1)^{i+j} n \cdot m$$

for homogeneous elements $m \in A^i$ and $n \in A^j$.

Theorem 3.6.2. *There is a multiplication called the cup product that makes the direct sum*

$$H^*(X;R) := \bigoplus_{i=0}^{\infty} H^i(X;R)$$

into a graded commutative, associative R-algebra for which $H^i(X;R)$ has degree i.

We begin by defining the cup product at the level of cochains,

$$S^p(X,A;R) \times S^q(X,A;R) \to S^{p+q}(X,A;R), \qquad (\varphi,\psi) \to \varphi \cup \psi$$

defined on a $(p+q)-$simplex σ by

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma \circ [e_0, ..., e_p])\psi(\sigma \circ [e_p, ..., e_{p+q}]),$$

where $[e_0, ..., e_p] : \Delta^p \to \Delta^{p+q}$ and $[e_p, ..., e_{p+q}] : \Delta^q \to \Delta^{p+q}$ are affine simplices. The cup product is both associative and distributive with respect to addition.

Lemma 3.6.3. *The cup product satisfies the Leibniz rule*

$$\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^p \varphi \cup \delta\psi$$

for $\varphi \in S^p(X)$ and $\psi \in S^p(X)$.

Consequently, it implies that the product of two cocycles is a cocycle. Also, if the product of a cocycle with a coboundary is a coboundary, because

$$\varphi \cup \delta\psi = \pm\delta(\varphi \cup \psi) \pm \delta\varphi \cup \psi = \pm\delta(\varphi \cup \psi) \quad \text{if} \quad \delta\varphi = 0$$

and

$$\delta\varphi \cup \psi = \delta(\varphi \cup \ \psi) \pm \varphi \cup \delta\psi = \delta(\varphi \cup \psi) \quad \text{if} \quad \delta\psi = 0.$$

It follows that the cup product descends a map

$$\cup : H^p(X; R) \times H^q(X; R) \to H^{p+q}(X; R)$$

which is both associative and bilinear with respect to the $R-$module structure. Thus the cup product makes the direct sum

$$H^*(X; R) := \bigoplus_{i=0}^{\infty} H^i(X; R)$$

into a graded, associative $R-$algebra. There is a multiplicative identity, denoted by 1, which is represented by the $0-$cocycle that sends every $0-$simplex to 1.

3.6.2 The cap product and Poincaré Duality

The cap product is a bilinear map that takes as input a singular $n-$chain and singular $k-$cochain, and outputs an $(n-k)-$chain for $n \geq k$:

$$\cap : S_n(X; R) \times S^k(X; R) \to S_{n-k}(X; R).$$

Given an $n-$simplex $\sigma : \Delta^n \to X$ and a cochain $\psi \in S^k(X; R)$, the cap product is dual to the cup product in the sense that if $\varphi \in S^{n-k}(X; R)$ is a cochain, then

$$\varphi(\sigma \cap \psi) = (\psi \cup \varphi)(\sigma),$$

so that the homomorphism

$$\psi \cup () : S^{n-k}(X; R) \to S^n(X; R)$$

is the transpose of the linear map

$$() \cap \psi : S_n(X;R) \to S_{n-k}(X;R).$$

Theorem 3.6.4. *The cap product determines a bilinear map,*

$$\cap : H_n(X;R) \times H^k(X;R) \to H_{n-k}(X;R)$$

by the rule

$$[\alpha] \cap [\psi] = [\alpha \cap \psi].$$

Lemma 3.6.5. *If* $\alpha \in S_n(X)$ *and* $\psi \in S^k(X)$ *then*

$$\partial(\alpha \cap \psi) = (-1)^k(\partial\alpha \cap \psi - \alpha \cap \delta\psi).$$

Theorem 3.6.6. ***Poincaré Duality Theorem:*** *Let* M *be a closed,* $R-$*oriented* $n-$*manifold with fundamental class* $[M] \in H_n(M;R)$*. The cap product with respect to* $[M]$ *defines an isomorphism*

$$[M] \cap () : H^k(M;R) \xrightarrow{\cong} H_{n-k}(M;R).$$

3.7 The Kůnneth Formula

For two given spaces, we may form the product space $X \times Y$, which comes equipped with projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$. These can be used to define the cross product:

$$H^p(X;R) \times H^q(Y;R) \to H^{p+q}(X \times Y;R)$$

given by $a \times b = \pi_1^\star(a) \cup \pi_2^\star(b)$.

The cross product is bilinear, so it determines a homomorphism

$$H^p(X;R) \otimes H^q(Y;R) \to H^{p+q}(X \times Y;R), \quad a \otimes b \mapsto a \times b. \tag{3.7.1}$$

Theorem 3.7.1. *If both $H^p(X;R)$ and $H^q(Y;R)$ are free $R-$modules for all degrees of p and q respectively, then there is an isomorphism:*

$$H^n(X \times Y; R) \cong \bigoplus_{p+q=n} H^p(X;R) \otimes_R H^q(Y;R)$$

defined by adding up the natural homomorphism 3.7.1.

A proof in the case when X and Y are cell complexes can be found in the Hatcher, section 3.2 [7].

Chapter 4

Betti Numbers and Poincaré Polynomials

4.1 Betti Numbers

4.1.1 Fundamental theorem of finitely generated abelian groups

Let G be a finitely generated abelian group. Then it decomposes as follows:

$$G \cong \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times ... \times \mathbb{Z}_{n_s} \quad \text{for some integers} \quad r, n_1, n_2, ..., n_s,$$

and uniquely satisfying the following conditions:

1. $r \geq 0$ and $n_i \geq 2$ for all i,
2. $n_{i+1} \mid n_i$ for $1 \leq i \leq s-1$.

Then,

1. the integer r in the decomposition is called the free rank of G,
2. $n_1, n_2, ..., n_s$ are called invariant factors of G,
3. the decomposition is called the invariant factor decomposition of G.

Definition 4.1.1. We say that a group G is finitely generated if there exists a finite set $S \subseteq G$ such that every element of G can be written as a product of finitely many elements of S and the inverses of such elements.

Definition 4.1.2. For each positive integer r, let $\mathbb{Z}^r = \mathbb{Z} \times \mathbb{Z} \times ... \times \mathbb{Z}$ be the direct product of r copies of $\mathbb{Z}$. Then $\mathbb{Z}^r$ is called the free abelian group of order r.

Definition 4.1.3. For each positive integer n, we call $\mathbb{Z}_n = \frac{\mathbb{Z}}{n\mathbb{Z}}$ the cyclic group of order n.

4.1.2 Betti Numbers

Definition 4.1.4. Let X be a topological space and the abelian group $H_k(X)$ be the n^{th} homology group of X. Then for a non-negative integer k, the k^{th} Betti number $b_k(X)$ of X is the dimension of $H_k(X)$, i.e.,

$$H_k(X, \mathbb{F}) = \mathbb{F}^{b_k} \quad \text{for a field } \mathbb{F}. \tag{4.1.5}$$

The Betti numbers depends on the field $\mathcal{F}$ and only through the characteristic of $\mathcal{F}$. If the homology groups are torsion-free, then the Betti numbers are independent of the filed.

Example 4.1.6. $n - dim$ sphere.
Consider

$$H_k(S^n, \mathbb{F}) = \begin{cases} \mathbb{F} & \text{if } k = 0, n, \\ 0 & otherwise. \end{cases}$$

Therefore $b_0(S^n) = b_n(S^n) = 1$ and all the other Betti numbers are 0.

Example 4.1.7. Genus g Riemann surface.
Consider

$$H_k(\Sigma_g, \mathbb{F}) = \begin{cases} \mathbb{F} & \text{if } k = 0, 2, \\ \mathbb{F}^{2g} & \text{if } k = 1, \\ 0 & otherwise. \end{cases}$$

Therefore $b_0(\Sigma_g) = b_2(\Sigma_g) = 1, \quad b_1(\Sigma_g) = 2g$ and all the other Betti numbers are 0.

Example 4.1.8. The infinite complex projective plane.
Consider

$$H_k(\mathbb{C}P^\infty, \mathbb{F}) = \begin{cases} \mathbb{F} & \text{if } k = 0, even, \\ 0 & \text{if } k = odd. \end{cases}$$

Therefore $b_n(\mathbb{C}P^\infty) = \begin{cases} 1 & \text{if } k = 0, even, \\ 0 & \text{if } k = odd. \end{cases}$

Example 4.1.9. The real projective plane.

$$H_k(\mathbb{R}P^n, \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & \text{if } 0 \leq k \leq n, \\ 0 & otherwise. \end{cases}$$

$$H_k(\mathbb{R}P^n, \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } k = 0 \text{ and, } k = n \text{, odd} \\ 0 & otherwise. \end{cases}$$

Therefore, over the field $\mathbb{Z}_2$, $b_n(\mathbb{R}P^n) = \begin{cases} 1 & \text{if } 0 \leq k \leq n, \\ 0 & otherwise, \end{cases}$

and over $\mathbb{Q}$, $b_n(\mathbb{R}P^n) = \begin{cases} 1 & \text{if } k = 0 \text{, and } k = n \text{ odd,} \\ 0 & otherwise. \end{cases}$

4.2 Poincaré Series

Definition 4.2.1. For a fixed coefficient field $\mathbb{F}$, define the Poincaré polynomial $P_X(t)$ of a topological space X to be the generating power series of its Betti numbers.

$$i.e. \quad P_X(t) = \sum_i b_i t^i,$$

where b_i is the dimension of $H_i(X, \mathbb{F})$ as a vector space of $\mathbb{F}$ (or the i^{th} Betti number of X).

Example 4.2.2. $n - dim$ sphere.
By Example 4.1.6, the Betti numbers are $b_0(S^n) = b_n(S^n) = 1$ and all the other Betti numbers are 0. Therefore $P_{S^n}(t) = 1 + t^n$.

Example 4.2.3. Genus g Riemann surface.
By Example 4.1.7, the Betti numbers are $b_0(\Sigma_g) = b_2(\Sigma_g) = 1, \quad b_1(\Sigma_g) = 2g$ and all the other Betti numbers are 0. Therefore $P_{\Sigma_g}(t) = 1 + 2gt + t^2$.

Example 4.2.4. The infinite complex projective plane.
By Example 4.1.8, the Betti numbers are $b_k(\mathbb{C}P^\infty) = \begin{cases} 1 & \text{if } k = 0, even, \\ 0 & \text{if } k = odd. \end{cases}$.

Therefore $P_{\mathbb{C}P^\infty}(t) = 1 + t^2 + t^4 + t^6 + ...$

Theorem 4.2.1. *Let X and Y be two topological spaces. Then the Poincaré polynomial of the tensor product $X \times Y$ can be written as:*

$$P_{X\times Y}(t) = P_X(t)P_Y(t).$$

This theorem follows from the *Künneth Theorem* for fields.

Example 4.2.5.

$$P_{S^1\times S^1}(t) = P_{S^1}(t)P_{S^1}(t) = (1+t)(1+t) = 1 + 2t + t^2.$$

Example 4.2.6.

$$P_{(S^1)^{2g}}(t) = P_{S^1\times S^1\times\ldots\times S^1}(t) = P_{S_1}(t)P_{S_1}(t)...P_{S_1}(t) = (1+t)^{2g}.$$

Example 4.2.7.

$$\begin{aligned} P_{\mathbb{C}P^\infty\times(S^1)^{2g}}(t) &= P_{\mathbb{C}P^\infty}(t)P_{(S^1)^{2g}}(t) \\ &= (1 + t^2 + t^4 + t^6 + ...)(1+t)^{2g} \\ &= (1 + t^2 + t^4 + t^6 + ...)\sum_{l=0}^{2g}\binom{2g}{l}t^l \\ &= \sum_{l=0}^{\infty}\sum_{i=0}^{2g}\binom{2g}{i}t^{i+2l}. \end{aligned}$$

Chapter 5

The Symmetric Product Space and The Infinite Symmetric Product Space

5.1 The symmetric product space $SP^n(\Sigma_g)$

The symmetric product of a topological space X can be thought of as a set of finite unordered $n - tuples$ drawn from space X. Key to this construction is that the symmetric group S_n acts naturally on the product space X^n by permuting elements, namely,

$$\sigma(x_1, x_2, ..., x_n) = (x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)}) \text{ for all } \sigma \in S_n \text{ and } (x_1, x_2, ..., x_n) \in X^n.$$

Definition 5.1.1. Let X be a topological space. For any natural number n, the n^{th} symmetric product of X is the orbit space:

$$SP^n(X) = X^n/S_n$$

of the natural permutation action described above, where $X^n = X \times X \times ... \times X$ is the product space.

In case $X = \Sigma_g$, then $SP^n(\Sigma_g)$ may be interpreted as the space of all effective divisors of order n. In addition to that, $SP^n(\Sigma_g)$ serves as the domain of the classical

Abel-Jacobi map:

$$AJ_n : SP^n(\Sigma_g) \to Jac(\Sigma_g),$$

which we will explain in Chapter 7. In this case $SP^n(\Sigma_g)$ is a complex manifold of dimension n (a real manifold of dimension $2n$).

5.2 The infinite symmetric product space $SP^\infty(X)$

Let X be a topological space and $* \in X$ be a given base point. Then, there is an embedding

$$SP^n(X) \hookrightarrow SP^{n+1}(X)$$

given by

$$j(x_1, x_2, ..., x_n) = (x_1, x_2, ..., x_n, *).$$

Thus $SP^n(X)$ can naturally considered as a subset of $SP^{n+1}(X)$ and this is given a sequence

$$SP^1(X) \subset SP^2(X) \subset ... \subset SP^n(X) \subset SP^{n+1}(X) \subset ... \tag{5.2.1}$$

that allows defining an infinite symmetric space.

Let $X_1 \hookrightarrow X_2 \hookrightarrow ...$ be a sequence of inclusions. Consider the union $X = \cup X_n$ and define the topology of X by the rule that $U \subseteq X$ is open if $U \cap X_n$ is open for all n. If the X_n are a sequence of cell complexes, that is, X_n is a subcell complex of X_{n+1}, then X is also a cell complex. We call X the colimit of the sequence.

Definition 5.2.2. Let X be a topological space. The infinite symmetric product of X is the colimit

$$SP^\infty(X) \simeq colim SP^n(X)$$

according to this sequence.

Theorem 5.2.1. *Let Σ_g be a compact Riemann surface of genus g. Then there is a homotopy equivalence such that*

$$SP^\infty(\Sigma_g) \cong \mathbb{C}P^\infty \times (S^1)^{2g}. \tag{5.2.3}$$

Theorem 5.2.1 is a consequence of the Dold-Thom Theorem 5.2.2, which we now explain.

5.2.1 Eilenberg-Maclane Space

Given a positive integer n and a group G (necessarily abelian if $n \geq 2$) then a connected topological space X is called an Eilenberg-Maclane space of the type $K(G, n)$, if it has n^{th} homotopy group

$$\pi_n(X) \simeq G$$

and

$$\pi_i(X) = 0 \quad \text{for} \quad i \neq n.$$

The homotopy type of a CW complex $K(G, n)$ is uniquely determined by G and n. Moreover, $K(G, n)$ is a cell complex structure and it is unique up to homotopy equivalence.

Example 5.2.4. The unit circle S^1 with $G = \mathbb{Z}$:

$$K(\mathbb{Z}, 1) \simeq S^1.$$

Example 5.2.5. The infinite dimensional real projective space $\mathbb{R}P^\infty$ with $G = \mathbb{Z}_2$:

$$K(\mathbb{Z}_2, 1) \simeq \mathbb{R}P^\infty.$$

Example 5.2.6. The infinite dimensional complex projective space $\mathbb{C}P^\infty$ with $G = \mathbb{Z}$:

$$K(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty.$$

5.2.2 Dold-Thom Theorem

Theorem 5.2.2. *Let X be a connected cell complex. Then, there is a homotopy equivalence*

$$SP^\infty(X) \simeq \prod_{n=1}^{\infty} K(H_n(X, \mathbb{Z}), n). \tag{5.2.7}$$

5.2.3 Proof of Theorem 5.2.1

Note that Σ_g is a connected cell complex and

Remark 5.2.3. $H_n(\Sigma_g, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 2 \\ \mathbb{Z}^{2g} & \text{if } n = 1 \\ 0 & \textit{otherwise.} \end{cases}$

Now the Dold-Thom theorem implies:

$$\begin{aligned} SP^{\infty}(\Sigma_g) &\simeq \prod_{n=1}^{\infty} K(H_n(\Sigma_g, \mathbb{Z}), n) \\ &= K(H_1(\Sigma_g, \mathbb{Z}), 1) \times K(H_2(\Sigma_g, \mathbb{Z}), 2) \\ &= K(\mathbb{Z}^{2g}, 1) \times K(\mathbb{Z}, 2) \\ &= (S^1)^{2g} \times \mathbb{C}P^{\infty}. && \text{(by Ex.5.2.4 and Ex.5.2.6)} \end{aligned}$$

5.3 The Betti numbers and Homology of $SP^{\infty}(\Sigma_g)$

By Theorem 5.2.1 and Theorem 4.2.1, we have the Poincaré polynomial for $SP^{\infty}(\Sigma_g)$ as:

$$\begin{aligned} P_{SP^{\infty}(\Sigma_g)}(t) &= P_{\mathbb{C}P^{\infty} \times (S^1)^{2g}}(t) \\ &= (1+t)^{2g}(\frac{1}{1-t^2}) \\ &= \sum_{l=0}^{\infty} \sum_{i=0}^{2g} \binom{2g}{i} t^{i+2l}. \end{aligned}$$

Note that the $k-th$ Betti number b_k of a space is the coefficient of t^k of its own Poincaré polynomial. For example, consider b_4, has the following combinations to be $i + 2l = 4$,

i	0	2	4
j	2	1	0

Table 5.1: i and l values for b_4

and hence, $b_4 = \binom{2g}{0} + \binom{2g}{2} + \binom{2g}{4}$. So,

k	b_k
0	$b_0 = \binom{2g}{0}$
1	$b_1 = \binom{2g}{1}$
2	$b_2 = \binom{2g}{0} + \binom{2g}{2}$
3	$b_3 = \binom{2g}{1} + \binom{2g}{3}$
4	$b_4 = \binom{2g}{0} + \binom{2g}{2} + \binom{2g}{4}$
5	$b_5 = \binom{2g}{1} + \binom{2g}{3} + \binom{2g}{5}$
6	$b_6 = \binom{2g}{0} + \binom{2g}{2} + \binom{2g}{4} + \binom{2g}{6}$
7	$b_7 = \binom{2g}{1} + \binom{2g}{3} + \binom{2g}{5} + \binom{2g}{7}$
$\vdots$	$\vdots$

Table 5.2: Classification of Betti numbers of $SP^\infty(\Sigma_g)$.

In general,

$$b_k(SP^\infty(\Sigma_g)) = \begin{cases} \sum_{i=0}^{k/2} \binom{2g}{2i} & \text{if } k = 0, even, \\ \sum_{i=0}^{(k-1)/2} \binom{2g}{2i+1} & \text{if } k = odd. \end{cases} \tag{5.3.1}$$

Recall that the homology of a space X,

$$H_k(X, \mathbb{F}) = \mathbb{F}^{b_k} \quad \text{for a field } \mathbb{F}.$$

and hence, as a conclusion, we have:

$$H_k(SP^\infty(\Sigma_g), \mathbb{F}) = \begin{cases} \mathbb{F}^{\sum_{i=0}^{k/2} \binom{2g}{2i}} & \text{if } k = 0, even, \\ \mathbb{F}^{\sum_{i=0}^{(k-1)/2} \binom{2g}{2i+1}} & \text{if } k = odd. \end{cases} \tag{5.3.2}$$

5.4 The relationship between $SP^n(\Sigma_g)$ and $SP^\infty(\Sigma_g)$

As we described in subsection 5.2, $SP^\infty(\Sigma_g)$ is the colimit of $SP^n(\Sigma_g)$. $SP^\infty(\Sigma_g)$ has a cell complex structure [9] for which the $SP^n(\Sigma_g)$ are subcomplexes such that the natural inclusion $i : SP^n(\Sigma_g) \hookrightarrow SP^\infty(\Sigma_g)$ is an isomorphism up to the n-skeletons:

$$(SP^n(\Sigma_g))_n \cong (SP^\infty(\Sigma_g))_n.$$

This inclusion determines an isomorphism between cellular chain complexes $p_i : C_i(SP^n(\Sigma_g)) \to C_i(SP^\infty(\Sigma_g))$ for $i \leq n$.

Theorem 5.4.1. *Let Σ_g be a compact Riemann surface of genus g with an n^{th} symmetric product space $SP^n(\Sigma_g)$ and infinite symmetric product space $SP^\infty(\Sigma_g)$. Then,*

$$H_k(SP^n(\Sigma_g), \mathbb{F}) \cong H_k(SP^\infty(\Sigma_g), \mathbb{F}) \qquad for \quad k = 0, 1, .., n-1. \tag{5.4.1}$$

Proof. The proof follows directly by Lemma 3.2.1. □

In fact, I.G MacDonald [9] proved that 5.4.1 is also an isomorphism for $k = n$.

5.4.1 The Homology and Betti numbers of $SP^n(\Sigma_g)$

Theorem 5.4.1 allows us to find homology groups of $SP^n(\Sigma_g)$ and Equation 5.3.2 implies that

$$H_k(SP^n(\Sigma_g), \mathbb{F}) = \begin{cases} \mathbb{F}^{\sum_{i=0}^{k/2} \binom{2g}{2i}} & \text{if } k = even, \\ \mathbb{F}^{\sum_{i=0}^{(k-1)/2} \binom{2g}{2i+1}} & \text{if } k = odd \end{cases} \tag{5.4.2}$$

for $k = 0, 1, 2, ..., n$.

To calculate the rest, from $n+1, ..., 2n$ of the homology groups, we need to use Theorem 3.6.1 and the Poincaré duality.

Since $SP^n(\Sigma_g)$ has a finite cell complex structure, by Theorem 3.6.1:

$$H_k(SP^n(\Sigma_g), \mathbb{F}) \cong H^k(SP^n(\Sigma_g), \mathbb{F}) \quad \text{for} \quad k \in \mathbb{N}.$$

Recall that the $n-th$ symmetric product space $SP^n(\Sigma_g)$ of a compact Riemann surface Σ_g of genus g is a closed $R-orientable$ $2n$-dimensional manifold and satisfies the Poincaré duality theorem. Thus we have

$$H^k(SP^n(\Sigma_g), \mathbb{F}) \cong H_{2n-k}(SP^n(\Sigma_g), \mathbb{F}).$$

Therefore, the conclusion is:

$$H_k \cong H^k \cong H_{2n-k} \cong H^{2n-k}.$$

As results:

- Poincaré polynomial

$$P_t(SP^n(\Sigma_g)) = \sum_{i+k\leq n} \binom{2g}{i} t^{i+2k}; \tag{5.4.3}$$

- Euler characteristic

$$\chi(SP^n(\Sigma_g)) = (-1)^n \binom{2g-2}{n}; \tag{5.4.4}$$

- In addition to these results, we introduce the zeta function [9], $\zeta(u,t)$, which is the power series of Poincaré polynomials of each symmetric product:

$$\zeta(u,t) = \sum_{n=0}^{\infty} P_t(SP^n(\Sigma_g))u^n = \frac{(1+ut)^{2g}}{(1-u)(1-t^2u)}. \tag{5.4.5}$$

Moreover, all the Betti numbers are independent of the field, $SP^n(\Sigma_g)$ satisfies all the conditions of Theorem 3.2.2, and hence, we can introduce all the homology and cohomology groups of $SP^n(\Sigma_g)$ over $\mathbb{Z}$.

Chapter 6

Fiber Bundles and Covering Spaces

6.1 Fiber Bundles

Definition 6.1.1. Let E, B and F be topological spaces, called total space, base space and fiber respectively. A fiber bundle is a structure (E, B, q, F) with continuous surjection $q : E \to B$ satisfying the following conditions:

1. For any $b \in B$ the pre-image $q^{-1}(b)$ is homeomorphic to F and is called the fiber over b.

2. For every $b \in B$ there is an open neighbourhood $U \subseteq B$ of b such that there is a homeomorphism $\varphi : q^{-1}(U) :\to U \times F$ with subspace topology and the following diagram commutes:

$$\begin{array}{ccc} q^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ \downarrow q & \swarrow_{proj_1} & \\ U & & \end{array}$$

where $proj_1$ is the natural projection onto the first coordinate. The set of all $\{U_i, \varphi_i\}$ is called a local trivialization of the bundle.

Since projections are open maps, every fiber bundle $q : E \to B$ is an open map and hence, B has the quotient topology determined by the map q. The fiber bundle structure is determined by the projection map q, but we sometimes write a fiber bundle as a short exact sequence to indicate which space is the fiber, total space, and base space.

$$F \longrightarrow E \xrightarrow{q} B$$

Note that when the fiber is a vector space, the bundle is called a **vector bundle.**

Example 6.1.2. A fiber bundle with fiber a discrete space is a covering space. Conversely, a covering space with fibers which all have the same cardinality, such as a covering space over a connected base space, is a fiber bundle with a discrete fiber.

Example 6.1.3. Trivial Bundle.
Let $E = B \times F$ and let $q : E \to B$ be the projection onto the first coordinate. Then E is a fiber bundle over B and is called a trivial bundle.

Example 6.1.4. The $n-$dimensional real projective space $\mathbb{R}P^n$ defined by:

$$\mathbb{R}P^n := S^n/\sim$$

where $x \sim -x$ for $x \in S^n \subset \mathbb{R}^{n+1}$. Let $q : S^n \to \mathbb{R}P^n$ be the projection map, then this is a fiber bundle with fiber in the two point set and it is also a covering map.

Example 6.1.5. The $n-$dimensional complex projective space $\mathbb{C}P^n$ defined by:

$$\mathbb{C}P^n := S^{2n+1}/\sim$$

where $x \sim ux$ for $x \in S^{2n+1} \subset \mathbb{C}^n$ and $u \in S^1$. Then $q : S^{2n+1} \to \mathbb{C}P^n$ is a fiber bundle with fiber S^1.

Example 6.1.6. One of the simplest examples of a nontrivial bundle E is the Möbius band, which is a bundle over S^1 with fiber an interval.

Theorem 6.1.1. *Given a fiber bundle (E, B, q, F) and choosing a base point $e_0 \in E$; then, there is a long exact sequence of homotopy groups*

$$... \longrightarrow \pi_2(F, e_0) \longrightarrow \pi_2(E, e_0) \longrightarrow \pi_2(B, q(e_0)) \longrightarrow \pi_1(F, e_0) \longrightarrow \pi_1(E, e_0) \longrightarrow \pi_1(B, q(e_0)) \ .$$

6.1.1 Sections

Let $q : E \to M$ be a fiber bundle with the fiber $E_m = q^{-1}(m)$ for $m \in M$. A *section* is a continuous map $s : M \to E$ such that $q \circ s = id_M, i.e.,$

$$s(m) \in E_m \quad \text{for all} \quad m \in M.$$

If $E \to M$ is a vector bundle, then every fiber E_m of E is a vector space and thus has a distinguished element, the zero-vector in E_m, denoted by 0_m. It follows that every vector bundle admits the zero-section:

$$s_0(m) = (m, 0_m) \in E_m.$$

6.1.2 Pullback Bundle

Definition 6.1.7. Let $q : E \to B$ be a fiber bundle with the fiber F and let $f : B^{'} \to B$ be a continuous map. Define the pullback bundle by

$$f^*E = \{(b', e) \in B^{'} \times E \mid f(b') = q(e)\} \subseteq B^{'} \times E$$

and the projection map $q^{'} : f^*E \to B^{'}$, given by the projection onto the first coordinate and $g : f^*E \to E$, given the projection onto the second coordinate, such that the following diagram commutes:

$$\begin{array}{ccc} f^*E & \xrightarrow{g} & E \\ \downarrow{\scriptstyle q^{'}} & & \downarrow{\scriptstyle q} \\ B^{'} & \xrightarrow{f} & B \end{array} \tag{6.1.8}$$

If (U, φ) is a local trivialization of E, then $(f^{-1}(U), \psi)$ is a local trivialization of f^*E, where $\psi(b', e) = (b', proj_2(\varphi(e)))$. It then follows that f^*E is a fiber bundle over $B^{'}$ with fiber F and the bundle is called the pullback of E by f.

Proposition 6.1.2. *Let (E, B, q, F) be a trivial fiber bundle and $f : C \to B$ be a continuous map. Then the pullback of the fiber bundle along f is also a trivial fiber bundle on C with the same fiber F.*

Proof. Considering the pullback of the commutative diagram:

$$\begin{array}{ccc} q^*C & \longrightarrow & C \\ \downarrow & & \downarrow f \\ B \times F & \xrightarrow{q} & B \end{array}$$

we have

$$\begin{aligned} q^*C &= \{((b,d),c) \in B \times F \times C \,|\, q(b,d) = b = f(c)\} \\ &= \{(b,c) \in B \times C \,|\, f(c) = b\} \times F \\ &= \{(f(c),c) \,|\, c \in C\} \times F \\ &\cong C \times F. \end{aligned}$$

□

6.1.3 Covering Spaces

Definition 6.1.9. A covering space of a space X is a space $\tilde{X}$ together with a map $p : \tilde{X} \to X$ satisfying the following condition: every point $x \in X$ has an open neighbourhood $U_x \subseteq X$, such that $p^{-1}(U_x)$ is a disjoint union of open sets, each of which is mapped by p homeomorphically onto U_x.

Example 6.1.10. The map $p : \mathbb{R} \to S^1$ given by $p(t) = e^{it}$ is a covering map, wrapping the real line round and round the circle. The pre-image of a little open arc in the circle is a collection of open intervals in the real line, offset by multiples of 2π.

Example 6.1.11. Another cover of the circle is the map $p : S^1 \to S^1$ given by $p(e^{it}) = e^{int}$, where n is a positive integer. This wraps the circle around itself n times.

Example 6.1.12. The map $p : (S^1)^{2g} \to (S^1)^{2g}$ given by $p(e^{it_1}, e^{it_2}, ..., e^{it_{2g}}) = (e^{2it_1}, e^{2it_2}, ..., e^{2it_{2g}})$ is a covering map, wrapping each component two times, which makes 2^{2g} cover.

Theorem 6.1.3. *If X is a cell complex with the $n-$skeleton X_n and $\tilde{X}$ is a covering space with the covering map p, then $\tilde{X}$ is a cell complex with the $n-$skeleton $p^{-1}(X_n) = \tilde{X}_n$.*

Proof. The proof and the explanation can be found in Hatcher [7]. □

Corollary 6.1.4. *Let $p : \tilde{Y} \to Y$ be a covering map and $f : X \to Y$ be a continuous map. Pullback f_p^* of the covering map p along f is a covering map.*

$$\begin{CD} f^*(\tilde{Y}) @>\bar{f}>> \tilde{Y} \\ @VV f_p^* V @VV p V \\ X @>f>> Y \end{CD} \tag{6.1.13}$$

Proof. Take any $x \in X$ and let $f(x) = y$ in Y. Since p is a covering map, there exists an open neighbourhood $U_y \subset Y$ such that $p^{-1}(U_y) = \bigcup_{i \in I} V_i$, where each V_i is open in $\tilde{Y}$ for $i \in I$ and maps homeomorphically to U_y by p. Now, since f is continuous, $f^{-1}(U_y)$ is an open set, and let $U_x = f^{-1}(U_y)$ be the open neighbourhood of x.

Claim : U_x is covered by f_p^*. That is $(f_p^*)^{-1}(U_x) = (f_p^*)^{-1}(f^{-1}(U_y)) = \tilde{f}^{-1}(p^{-1}(U_y)) = \tilde{f}^{-1}(\bigcup_{i \in I} V_i) = \bigcup_{i \in I} \tilde{f}^{-1}(V_i)$.

So we need to check that each $\tilde{f}^{-1}(V_i)$ is mapped homeomorphically onto U_x by f_p^*. By Corollary 6.1.5 we have $\tilde{f}$ as a homeomorphism and hence we have the result.

□

Corollary 6.1.5. *Let $p : \tilde{Y} \to Y$ be a covering map and $f : X \to Y$ be a homeomorphism. If the pullback of p along f is $\tilde{X}$, and the covering map $f_p^* : \tilde{X} \to X$, then the function $\tilde{f} : \tilde{X} \to \tilde{Y}$ is a homeomorphism.*

$$\begin{CD} \tilde{X} @>\bar{f}>> \tilde{Y} \\ @VV f_p^* V @VV p V \\ X @>f>\cong> Y \end{CD} \tag{6.1.14}$$

6.1.4 Vector Bundles

An $n-$dimensional vector bundle is a structural (E, M, q) fiber bundle, such that the fibers are vector spaces isomorphic to $\mathbb{R}^n$.

Every point $m \in M$ has an open neighbourhood U along with a homeomorphism

$h : q^{-1}(U) \to U \times \mathbb{R}^n$ which takes fibers $q^{-1}(m) \to \{m\} \times \mathbb{R}^n$ so that the following diagram commutes:

$$\begin{array}{ccc} q^{-1}(U) & \xrightarrow{h} & U \times \mathbb{R}^n \\ \downarrow q & \swarrow_{proj_1} & \\ U & & \end{array}$$

A smooth vector bundle is a vector bundle (E, M, q), where E and M are smooth manifolds and $q : E \to M$ is a surjective submersion.

Theorem 6.1.6. *Let $E \to M$ be a smooth vector bundle with fibers $E_m \cong \mathbb{C}^n$ and $g : X \to E$ be a smooth map on a smooth manifold X. Then, there exists a smooth section $s : M \to E$, such that g is transverse to s.*

Proof. Theorem 15.3, Chapter 02 [4] □

Corollary 6.1.7. *Let $E \to M$ be a smooth vector bundle with fibers E_m for $m \in M$. If the dimension of E is greater than twice the dimension of M, then there exists a non-vanishing section.*

Proof. Let $s_0 : M \to E$ be the zero-section, which is a smooth map on M. Now, by Theorem 6.1.6, there is a smooth section s such that s_0 is transverse to s. In fact, since the dimension of T_xM for $x \in M$ is equal to the dimension of M, whenever the dimension of E is greater than twice the dimension of M, then transversality 2.3.1 implies that there is no intersection, which is for all $m \in M$, $s_m \neq 0_m$ and hence, there exists a non-vanishing section.

□

6.1.5 Projective Bundles

Let V be a topological vector space over $\mathbb{C}$. The set of all $1 - dim$ vector subspaces of V is called the projective space $P(V)$. Topologically, it is the quotient space endowed with the quotient topology, the set of equivalence classes of $V \setminus \{0\}$ under the equivalence relation $\sim$ defined by $x \sim y$ if there is a nonzero element λ of the field such that $x = \lambda y$. If V is a finite dimension (say $n - dim$), then the dimension of $P(V)$ is $n - 1$.

Definition 6.1.15. Let $q : E \to X$ be a topological vector bundle over $\mathbb{C}$ with the base of topological space X. Then its projective bundle is the fiber bundle $P(q) : P(E) \to X$, the total space of which is a bundle of projective spaces, and the bundle projection is

$$\begin{aligned} P(E) &\longrightarrow X \\ [v] &\longmapsto q(v). \end{aligned}$$

Proposition 6.1.8. *Suppose that M is a manifold and $E \to M$ is a smooth vector bundle of rank greater than dim(M). Then the associated projective bundle $P(E)$ admits a section.*

Proof. Let $E \to M$ be a topological vector bundle with total space E and fibers E_m. By Corollary 6.1.7, there exists a non-vanishing section $s_1 : M \to E$.

For the associated projective bundle $P(E) \to M$ with fiber $P(E_m) = P(\mathbb{C}^n) \cong \mathbb{C}P^{n-1}$, we can define a section

$$s_2 : J(M) \to P(E) \quad \text{by} \quad s_2(m) = [s_1(m)]. \tag{6.1.16}$$

This implies that every non-zero section of E gives a section of $P(E)$. □

Chapter 7

The Abel-Jacobi Map

7.1 The Jacobian

Let Σ_g be a compact Riemann space of genus g. The first step is to introduce the Jacobian of Σ_g, which we will define to be the compact quotient of $\mathbb{C}^g$ by a certain lattice.

Choose smooth closed loops $\gamma_1, \gamma_2, ..., \gamma_{2g}$ representing a basis $[\gamma_1], [\gamma_2], ..., [\gamma_{2g}]$ for the homology group $H_1(\Sigma_g; \mathbb{Z}) \cong \mathbb{Z}^{2g}$.

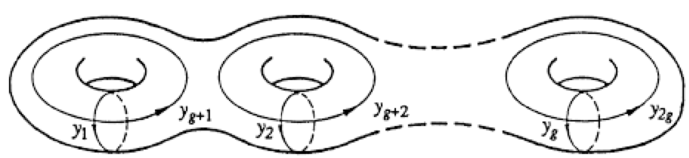


Figure 7.1: Genus-g Riemann surface with closed loops $\gamma_1, \gamma_2, ..., \gamma_{2g}$

Let $H^0(\Sigma_g; \Omega^{1,0})$ be the vector space of holomorphic $1-forms$ on Σ_g. Let $\omega_1, \omega_2, .., \omega_g$ be a basis for $H^0(\Sigma_g; \Omega^{1,0}) \cong \mathbb{C}^g$.

Now define a $2g - dimensional$ lattice $\Lambda = \{\sum_{i=1}^{2g} n_i v_i \,|\, n_i \in \mathbb{Z}\} \leq \mathbb{C}^g$, generated by the basis of $v_1, ..., v_{2g} \in \Lambda$ such that integrating each of ω_i 1−forms over γ_i,

$$v_i = (\int_{\gamma_i} \omega_1, ..., \int_{\gamma_i} \omega_g).$$

Alternatively, we can define

$$\Lambda = \{(\int_\gamma \omega_1, ..., \int_\gamma \omega_g) \,|\, [\gamma] \in H_1(X;\mathbb{Z})\}.$$

The Jacobian of the Riemann surface Σ_g, denoted by $J(\Sigma_g)$ is the compact quotient space,

$$J(\Sigma_g) = \mathbb{C}^g/\Lambda \cong \mathbb{R}^{2g}/\Lambda.$$

Since Λ is a discrete subgroup in $\mathbb{C}^g$ of maximal rank, $\mathbb{C}^g/\Lambda$ is a $(2g)-$dimensional torus which is homeomorphic to $(S^1)^{2g}$ as a topological space.

7.1.1 The Abel-Jacobi Map

Fix a point $p_0 \in \Sigma_g$. The Abel-Jacobi map is a map $AJ : \Sigma_g \to J(\Sigma_g)$. For every point $p \in \Sigma_g$, choose a curve γ from p_0 to p and define the map AJ as follows:

$$AJ(p) = (\int_{p_0}^p \omega_1, \int_{p_0}^p \omega_2, ..., \int_{p_0}^p \omega_g) + \Lambda$$

Although $\int_{p_0}^p \omega_i$ seemingly depends on the path from p_0 to p, its image in $J(\Sigma_g)$ depends only on the point p. Moreover, any two different paths γ_1, γ_2 from p_0 to p define a loop with the path concatenation in Σ_g; therefore, it become an element in $H_1(X;\mathbb{Z})$, so integration over it gives an element of Λ. That means the difference is erased to the quotient by Λ. Hence $AJ(p)$ is well-defined as a function of p and independent for choice of curve (It does however depend on the choice of the base point p_0).

In the case of general curve Σ_g, the map AJ is far from being an isomorphism unless $g = 1$. Since $J(\Sigma_g)$ is an abelian group, the Abel-Jacobi map AJ can be extended to a symmetric product,

$$AJ_n : SP^n(\Sigma_g) \to J(\Sigma_g)$$

defined by

$$AJ_n(P) := AJ(x_1) + ... + AJ(x_n)$$

where $P = (x_1, ..., x_n) \in SP^n(\Sigma_g)$.

7.2 Abel-Jacobi map as a fiber bundle

Theorem 7.2.1. *For $n > 2g-2$ the Abel-Jacobi map $AJ_n : SP^n(\Sigma_g) \to J(\Sigma_g)$ is a fiber bundle with fiber $\mathbb{C}P^{n-g}$, where $\mathbb{C}P^{n-g}$ is a complex projective space of dimension $n-g$. Moreover, $SP^n(\Sigma_g)$ is isomorphic to the associated projective bundle $P(E)$ for a vector bundle $E \to J(\Sigma_g)$.*

Proof. The proof for the first statement is Theorem 2.4 of [3] and the proof of the second part directly follows Chapter VII, Prop: 2.1 of [1] □

Since there is a natural inclusion, $SP^n(\Sigma_g) \hookrightarrow SP^{n+1}(\Sigma_g)$, and a sequence of subspaces, 5.2.1, we have a sequence of fiber bundles as follows:

$$\begin{array}{ccccc}
\mathbb{C}P^{n-g} & \longrightarrow & SP^n(\Sigma_g) & \xrightarrow{AJ_n} & J(\Sigma_g) \\
\downarrow & & \downarrow & & \downarrow \\
CP^{n+1-g} & \longrightarrow & SP^{n+1}(\Sigma_g) & \xrightarrow{AJ_{n+1}} & J(\Sigma_g) \\
\downarrow & & \downarrow & & \downarrow \\
CP^{n+2-g} & \longrightarrow & SP^{n+2}(\Sigma_g) & \xrightarrow{AJ_{n+2}} & J(\Sigma_g) \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots
\end{array}$$

Hence, taking the direct limit, we can observe that the result of a fiber bundle is

$$\mathbb{C}P^{\infty} \longrightarrow SP^{\infty}(\Sigma_g) \xrightarrow{AJ_{\infty}} J(\Sigma_g) \tag{7.2.1}$$

with fiber $\mathbb{C}P^{\infty}$. Hence, for the identity element $* \in J(\Sigma_g)$ by the definition of the fiber bundle, there is a homeomorphism

$$f : \mathbb{C}P^{\infty} \to AJ_{\infty}^{-1}(*). \tag{7.2.2}$$

Lemma 7.2.2. *If $n \geq 2g$ or $n = \infty$, then the Abel-Jacobi map $AJ_n : SP^n(\Sigma_g) \to J(\Sigma_g)$ admits a section.*

Proof. Now, by Theorem 7.2.1, we have an isomorphism h, and since $n \geq 2g$, by Proposition 6.1.8 there is a section s_2, such that

$$\begin{array}{ccc} & \overset{s_2}{\curvearrowleft} & \\ P(E) & \longrightarrow & J(\Sigma_g) \\ {\cong}\downarrow{h} & \nearrow_{AJ_n} & \\ SP^n(\Sigma_g) & & \end{array}$$

So, we can define the section $s : J(\Sigma_g) \to SP^n(\Sigma_g)$ by $s = h \circ s_2$. □

Define the map $\varphi : \mathbb{C}P^\infty \times J(\Sigma_g) \to SP^\infty(\Sigma_g)$ by $\varphi(x, y) = f(x) + s(y)$. Observe that $(AJ_\infty \circ \varphi)(x, y) = y$.

Lemma 7.2.3. *The function $\varphi : \mathbb{C}P^\infty \times J(\Sigma_g) \to SP^\infty(\Sigma_g)$ is a homotopy equivalence.*

Proof. Consider the trivial fiber bundle $\mathbb{C}P^\infty \xrightarrow{i} \mathbb{C}P^\infty \times J(\Sigma_g) \xrightarrow{proj_2} J(\Sigma_g)$ which gives the first row of the following diagram, and since $AJ_\infty : SP^\infty(\Sigma_g) \to J(\Sigma_g)$ is a fiber bundle, the second row according to the Theorem 6.1.1. Recall that f is a homeomorphism 7.2.2.

$$\begin{array}{ccccccccccc} \ldots \to & \pi_{n+1}(J(\Sigma_g)) & \to & \pi_n(\mathbb{C}P^\infty) & \to & \pi_n(\mathbb{C}P^\infty \times J(\Sigma_g)) & \to & \pi_n(J(\Sigma_g)) & \to & \pi_{n-1}(\mathbb{C}P^\infty) & \to \ldots \\ & \downarrow{Id} & & \downarrow{f_*} & & \downarrow{\varphi_*} & & \downarrow{Id} & & \downarrow{f_*} & \\ \ldots \to & \pi_{n+1}(J(\Sigma_g)) & \to & \pi_n(AJ_\infty^{-1}(\star)) & \to & \pi_n(SP^\infty(\Sigma_g)) & \to & \pi_n(J(\Sigma_g)) & \to & \pi_{n-1}(AJ_\infty^{-1}(\star)) & \to \ldots \end{array}$$

Since all the homotopy groups here are abelian groups, by the Five-Lemma 3.3.2, the map φ_* is an isomorphism and hence, by Theorem 2.1.2, $\varphi : \mathbb{C}P^\infty \times J(\Sigma_g) \to SP^\infty(\Sigma_g)$ is a homotopy equivalence. □

Chapter 8

The Relationship Between $\widetilde{SP^n(\Sigma_g)}$ and $\widetilde{SP^\infty(\Sigma_g)}$

8.1 Covering space of $SP^n(\Sigma_g)$ as a pullback

We will construct a homomorphism $J(\Sigma_g) \to J(\Sigma_g)$ which is also a covering map. As described in Chapter 7, the Jacobian is the compact quotient space $J(\Sigma_g) = \mathbb{R}^{2g}/\Lambda$. Let $\{v_1, ..., v_{2g}\} \in \mathbb{R}^{2g}$ be a basis for Λ and $C = [v_1, ..., v_i, ..., v_{2g}]$ be the column matrix for the basis. Let A be a $2g \times 2g$ matrix with integer entries such that $\det(A) \neq 0$. Then, $CAC^{-1} = B$ is a surjective linear map $\mathbb{R}^{2g} \to \mathbb{R}^{2g}$ that sends Λ to Λ. This is determined to be a surjective homomorphism $p : J(\Sigma_g) \to J(\Sigma_g)$ by $p([x]) = [Bx]$ for $x \in \mathbb{R}^{2g}$.

Then p determines a covering map $J(\Sigma_g) \to J(\Sigma_g)$ with the number of sheets equal to $|\det(A)|$. Now we shall consider the pullback diagram of p along the Abel-Jacobi map:

$$\begin{array}{ccc} \widetilde{SP^n(\Sigma_g)} & \longrightarrow & J(\Sigma_g) \\ \Big\downarrow f_p^* & & \Big\downarrow p \\ SP^n(\Sigma_g) & \xrightarrow{A.J} & J(\Sigma_g) \end{array} \tag{8.1.1}$$

By Corollary 6.1.4, the pullback f_p^* of the covering map p is also a covering map with $|\det(A)|$ number of sheets.

8.2 Homology groups of the covering space $\widetilde{SP^\infty(\Sigma_g)}$

In this section, we will prove that a covering space of $SP^\infty(\Sigma_g)$ has the same homology as $SP^\infty(\Sigma_g)$. To begin, let us start with the following commutative diagram of continuous functions and topological spaces:

$$\begin{array}{ccccc} Z' & \xrightarrow{f'} & X' & \xleftarrow{g'} & Y' \\ \downarrow{\scriptstyle \varphi_z} & & \downarrow{\scriptstyle \varphi_x} & & \downarrow{\scriptstyle \varphi_y} \\ Z & \xrightarrow{f} & X & \xleftarrow{g} & Y \end{array}$$

Then we have, two respective pullback spaces $Z' \times_{X'} Y'$ and $Z \times_X Y$ to the diagrams $Z' \to X' \leftarrow Y'$ and $Z \to X \leftarrow Y$ such that

$$Z' \times_{X'} Y' = \{(z', y') \in Z' \times Y' \mid f'(z') = g'(y')\} \quad \text{and} \quad Z \times_X Y = \{(z, y) \in Z \times Y \mid f(z) = g(y)\}.$$

Hence, we can define a function

$$\psi : Z' \times_{X'} Y' \to Z \times_X Y \tag{8.2.1}$$

such that $\psi(z', y') = (\varphi_z(z'), \varphi_y(y'))$.

We can define the pullback space for the diagram $\mathbb{C}P^\infty \times J(\Sigma_g) \xrightarrow{proj_2} J(\Sigma_g) \xleftarrow{p} J(\Sigma_g)$, as a covering space $\widetilde{\mathbb{C}P^\infty \times J(\Sigma_g)}$ of $\mathbb{C}P^\infty \times J(\Sigma_g)$. Similarly, the pullback space for the diagram $SP^\infty(\Sigma_g) \xrightarrow{AJ_\infty} J(\Sigma_g) \xleftarrow{p} J(\Sigma_g)$ is $\widetilde{SP^\infty(\Sigma_g)}$.

Following 8.2.1, this determines the function $\psi : \widetilde{SP^\infty(\Sigma_g)} \to \widetilde{\mathbb{C}P^\infty \times J(\Sigma_g)}$.

Note that the pullback of a trivial fiber bundle is a trivial fiber bundle with the same fiber. Thus $\widetilde{\mathbb{C}P^\infty \times J(\Sigma_g)} \to J(\Sigma_g)$ is a trivial bundle. Moreover, since we consider the pullback as the covering space, by Proposition 6.1.2,

$$\widetilde{\mathbb{C}P^\infty \times J(\Sigma_g)} \cong \mathbb{C}P^\infty \times J(\Sigma_g). \tag{8.2.2}$$

Lemma 8.2.1. *$\psi : \widetilde{SP^\infty(\Sigma_g)} \to \widetilde{\mathbb{C}P^\infty \times J(\Sigma_g)}$ is a homotopy equivalence.*

Proof. Since we have pullback diagrams of a trivial fiber bundle and fiber bundle, we can say that $\mathbb{C}P^{\infty} \to \widetilde{\mathbb{C}P^{\infty} \times J(\Sigma_g)} \to J(\Sigma_g)$ is a trivial bundle and $\mathbb{C}P^{\infty} \to \widetilde{SP^{\infty}(\Sigma_g)} \to J(\Sigma_g)$ is a fiber bundle. Now, by Theorem 6.1.1, we have long exact sequence of homotopy groups, which make the following commutative diagram:

$$\begin{array}{ccccccccccccc}
\ldots \to & \pi_{n+1}(J(\Sigma_g)) & \longrightarrow & \pi_n(\mathbb{C}P^{\infty}) & \longrightarrow & \pi_n(\widetilde{\mathbb{C}P^{\infty} \times J(\Sigma_g)}) & \to & \pi_n(J(\Sigma_g)) & \longrightarrow & \pi_{n-1}(\mathbb{C}P^{\infty}) & \longrightarrow \ldots \\
& \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle f_*} & \\
\ldots \to & \pi_{n+1}(J(\Sigma_g)) & \to & \pi_n(AJ_{\infty}^{-1}(\star)) & \longrightarrow & \pi_n(\widetilde{SP^{\infty}(\Sigma_g)}) & \longrightarrow & \pi_n(J(\Sigma_g)) & \to & \pi_{n-1}(AJ_{\infty}^{-1}(\star)) & \to \ldots
\end{array}$$

Since all the homotopy groups here are abelian, by the Five-Lemma 3.3.2 the map, ψ_* is an isomorphism. Hence, by Theorem, 2.1.2, ψ is a homotopy equivalence. □

Corollary 8.2.2. *Let Σ_g be a genus g compact Riemannian space and $\mathbb{F}$ be a field. Then*

$$H_k(\widetilde{SP^{\infty}(\Sigma_g)}, \mathbb{F}) = H_k(SP^{\infty}(\Sigma_g), \mathbb{F}).$$

Proof. By Lemma 8.2.1, we have the homotopy equivalences,

$$\begin{aligned}
\widetilde{SP^{\infty}(\Sigma_g)} &\cong \widetilde{\mathbb{C}P^{\infty} \times J(\Sigma_g)} \\
&\cong \mathbb{C}P^{\infty} \times J(\Sigma_g) \\
&\cong SP^{\infty}(\Sigma_g).
\end{aligned}$$

□

As a consequence of Corollary 8.2.2 we have

$$H_k(\widetilde{SP^{\infty}(\Sigma_g)}, \mathbb{F}) = \begin{cases} \mathbb{F}^{\sum_{i=0}^{k/2} \binom{2g}{2i}} & \text{if } k = 0, \textit{even} \\ \mathbb{F}^{\sum_{i=0}^{(k-1)/2} \binom{2g}{2i+1}} & \text{if } k = \textit{odd}. \end{cases} \tag{8.2.3}$$

Now let us consider the relationship between the covering spaces. Let $p : \tilde{X} \to X$ be a covering map and let X be a cell-complex with the $n-$skeleton X_n. Then $\tilde{X}$ is a cell-complex with $\tilde{X}_n = p^{-1}(X_n)$ representing the $n-$skeleton of $\tilde{X}$.

Proposition 8.2.3. *Let Σ_g be a compact Riemann surface of genus g. Then,*

$$H_k(\widetilde{SP^n(\Sigma_g)}, \mathbb{F}) \cong H_k(\widetilde{SP^\infty(\Sigma_g)}, \mathbb{F}) \qquad for \quad k = 0, 1, .., n-1. \tag{8.2.4}$$

Proof. Consider the following pullback diagram:

$$\begin{array}{ccc} \widetilde{SP^n(\Sigma_g)}_n & \xrightarrow{\tilde{f}} & \widetilde{SP^\infty(\Sigma_g)}_n \\ \Big\downarrow{\scriptstyle f_p^*} & & \Big\downarrow{\scriptstyle p} \\ SP^n(\Sigma_g)_n & \xrightarrow{f} & SP^\infty(\Sigma_g)_n \end{array}$$

Recall that we have a homeomorphism of $n-$skeletons $SP^n(\Sigma_g)_n \cong SP^\infty(\Sigma_g)_n$ for up to the $n-th$ skeleton, and by Corollary 6.1.5, we have $\widetilde{SP^n(\Sigma_g)}_n \cong \widetilde{SP^\infty(\Sigma_g)}_n$ for up to the $n-th$ skeleton. Now Lemma 3.2.1 gives the proof. ☐

Thus, as a result

$$H_k(\widetilde{SP^n(\Sigma_g)}, \mathbb{F}) = \begin{cases} \mathbb{F}^{\sum_{i=0}^{k/2} \binom{2g}{2i}} & \text{if } k = even, \\ \mathbb{F}^{\sum_{i=0}^{(k-1)/2} \binom{2g}{2i+1}} & \text{if } k = odd. \end{cases} \tag{8.2.5}$$

for $k = 0, 1, 2, ..., n-1$.

Since $SP^n(\Sigma_g)$ is a closed manifold, its covering space is also a closed manifold. By the Poincaré duality theorem

$$H_{2n-k}(\widetilde{SP^n(\Sigma_g)}, \mathbb{F}) = H_k(\widetilde{SP^n(\Sigma_g)}, \mathbb{F}) \tag{8.2.6}$$

for $k = 1, 2, ..., n-1$ and it remains to determine the $n-th$ Betti number of the covering space.

8.3 Euler Characteristics

The Euler characteristic $\chi(X)$ can be defined purely in terms of homology and hence depends only on the homotopy type of X. In particular, $\chi(X)$ is independent of the choice of the cell complex structure on X.

Definition 8.3.1. Euler characteristic.
Let X be a finite cell complex, and c_i be the number of $i-cells$ of X. Then, the Euler characteristic $\chi(X)$ is defined as:

$$\chi(X) = \sum_i (-1)^i c_i.$$

There is an alternative definition for the Euler characteristics connected with the Betti numbers. We are going to use that definition for the calculations.

Proposition 8.3.1. *For a finite cell complex the Euler characteristic is equal to the alternating sum of Betti numbers:*

$$\chi(X) = \sum_i (-1)^i b_i.$$

Theorem 8.3.2. *Let X be a finite cell complex and $p : \tilde{X} \to X$ be an $m-$fold covering map for the covering space X. Then,*

$$\chi(\tilde{X}) = m\chi(X).$$

Proof. Since p is an $m-$fold covering map, we have the number of $i - cells$ of $\tilde{X}$,

$$c_i(\tilde{X}) = mc_i(X).$$

Now, by the Definition 8.3.1,

$$\begin{aligned}\chi(\tilde{X}) &= \sum_i (-1)^i c_i(\tilde{X}) \\ &= \sum_i (-1)^i m c_i(X) \\ &= m\chi(X).\end{aligned}$$

□

8.3.1 The n-th Betti number of $\widetilde{SP^n(\Sigma_g)}$

Let b_i and $\tilde{b}_i$ be the $i-th$ Betti number of $SP^n(\Sigma_g)$ and its covering space, respectively. Then we have $b_i = \tilde{b}_i$ for all i except $i = n$, and from section 8.1 we have the covering map $f_p^\star : \widetilde{SP^n(\Sigma_g)} \to SP^n(\Sigma_g)$ with $|\det(A)|$ number of sheets.

For $m = |\det(A)|$, by Theorem 8.3.2 and (5.4.4) we have:

$$\tilde{b}_n = b_n + (m-1)\chi(SP^n(\Sigma_g)) \tag{8.3.2}$$

$$= b_n + (m-1)(-1)^n \binom{2g-2}{n}, \tag{8.3.3}$$

where

$$b_n = \begin{cases} \sum_{k=0}^{n/2} \binom{2g}{2k} & \text{if } n = 0, even, \\ \sum_{k=0}^{(n-1)/2} \binom{2g}{2k+1} & \text{if } n = odd. \end{cases}$$

So the results are:

- Euler characteristic by using (5.4.4):

$$\chi(\widetilde{SP^n(\Sigma_g)}) = m\chi(SP^n(\Sigma_g)) = m(-1)^n \binom{2g-2}{n} \tag{8.3.4}$$

- The Poincaré polynomial

$$P_t(\widetilde{SP^n(\Sigma_g)}) = P_t(SP^n(\Sigma_g)) + (m-1)\binom{2g-2}{n} t^n \tag{8.3.5}$$

- Zeta function

$$\widetilde{\zeta(u,t)} = \sum_{n=0}^{\infty} P_t(\widetilde{SP^n(\Sigma_g)})u^n = \frac{(1+ut)^{2g}}{(1-u)(1-t^2u)} + (m-1)(1+tu)^{2g-2} \tag{8.3.6}$$

Finally, note that the Betti numbers are independent of the characteristic of $\mathbb{F}$. It follows by Theorem 3.2.2 that

$$H_n(\widetilde{SP^n(\Sigma_g)}, \mathbb{Z}) = \mathbb{Z}^{\tilde{b}_n} \tag{8.3.7}$$

where $\tilde{b}_n$ is given by the equation 8.3.2.

Chapter 9

Results

1. Homology and Cohomology of $SP^{\infty}(\Sigma_g)$:

$$H_k(SP^{\infty}(\Sigma_g),\mathbb{Z}) = H^k(SP^{\infty}(\Sigma_g),\mathbb{Z}) = \begin{cases} \mathbb{Z}^{\sum_{i=0}^{k/2}\binom{2g}{2i}} & \text{if } k = 0, even, \\ \mathbb{Z}^{\sum_{i=0}^{(k-1)/2}\binom{2g}{2i+1}} & \text{if } k = odd. \end{cases}$$

2. Homology and Cohomology of $\widetilde{SP^{\infty}(\Sigma_g)}$:

$$H_k(\widetilde{SP^{\infty}(\Sigma_g)},\mathbb{Z}) = H^k(\widetilde{SP^{\infty}(\Sigma_g)},\mathbb{Z}) = \begin{cases} \mathbb{Z}^{\sum_{i=0}^{k/2}\binom{2g}{2i}} & \text{if } k = 0, even, \\ \mathbb{Z}^{\sum_{i=0}^{(k-1)/2}\binom{2g}{2i+1}} & \text{if } k = odd. \end{cases}$$

3. Homology and Cohomology of $SP^{n}(\Sigma_g)$:

$$H_k(SP^{n}(\Sigma_g),\mathbb{Z}) = H^k(SP^{n}(\Sigma_g),\mathbb{Z}) = \begin{cases} \mathbb{Z}^{\sum_{i=0}^{k/2}\binom{2g}{2i}} & \text{if } k = 0, even, \\ \mathbb{Z}^{\sum_{i=0}^{(k-1)/2}\binom{2g}{2i+1}} & \text{if } k = odd \end{cases}$$

   for $k = 0, 1, 2, ..., n$ and $H_k \cong H^k \cong H_{2n-k} \cong H^{2n-k}$.

4. If the zeta function is $\zeta(u,t) = \sum_{n=0}^{\infty} P_t(SP^n(\Sigma_g))u^n$, then

$$\zeta(u,t) = \frac{(1+ut)^{2g}}{(1-u)(1-t^2u)}.$$

5. Homology and cohomology of $\widetilde{SP^n(\Sigma_g)}$:

$$H_k(\widetilde{SP^n(\Sigma_g)}, \mathbb{Z}) = H^k(\widetilde{SP^n(\Sigma_g)}, \mathbb{Z}) = \begin{cases} \mathbb{Z}^{\sum_{i=0}^{k/2} \binom{2g}{2i}} & \text{if } k = 0, even, \\ \mathbb{Z}^{\sum_{i=0}^{(k-1)/2} \binom{2g}{2i+1}} & \text{if } k = odd \end{cases}$$

for $k = 0, 1, ..., n-1$ and $H_k \cong H^k \cong H_{2n-k} \cong H^{2n-k}$ for all k except n.

For $k = n$,

$$H_n(\widetilde{SP^n(\Sigma_g)}, \mathbb{Z}) = H^n(\widetilde{SP^n(\Sigma_g)}, \mathbb{Z}) = \mathbb{Z}^{\tilde{b}_n},$$

where

$$\tilde{b}_n = b_n + (m-1)(-1)^n \binom{2g-2}{n}$$

and

$$b_n = \begin{cases} \sum_{k=0}^{n/2} \binom{2g}{2k} & \text{if } n = 0, even, \\ \sum_{k=0}^{(n-1)/2} \binom{2g}{2k+1} & \text{if } n = odd. \end{cases}$$

6. If $\widetilde{\zeta(u,t)} = \sum_{n=0}^{\infty} P_t(\widetilde{SP^n(\Sigma_g)})u^n$, then

$$\widetilde{\zeta(u,t)} = \frac{(1+ut)^{2g}}{(1-u)(1-t^2u)} + (m-1)(1+tu)^{2g-2}.$$

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