

Bayesian Estimation of Time to Failure Distributions Based on Skew Normal Degradation Model: An Application to GaAs Laser Degradation Data

(Anggaran Bayesian Masa untuk Taburan Kegagalan Berdasarkan Model Degradasi Normal Pencong: Aplikasi untuk Data Degradasi Laser GaAs)

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ABSTRACT

In this paper, the Bayesian method which involves informative and weakly informative priors are considered to estimate the parameters and percentiles of the time to failure distribution. The parameters of the time to failure distribution and its percentiles are determined based on linear degradation model where the degradation parameter is assumed to follow the skew normal distribution. For the prior distributions, location and scale parameters of the skew normal distribution is assumed to follow the uniform distribution while the shape parameter is assumed to follow gamma distribution. Two gamma priors are considered, either informative or weakly informative prior, depending on the assumed values of the hyper parameters. The performance of the method under the different prior assumptions is compared using a simulation study based on Markov Chain Monte Carlo method as well as a real data application. It is found that the parameter estimation based on informative prior is more precise as opposed to the weakly informative prior, especially in the case of small sample size. In addition, the skew normal degradation model is compared to the log-logistic degradation model through a simulation study and a real application of GaAs laser data. When modeling the percentiles of the time to failure distribution, results found based on the skew normal distribution is generally found to be more precise, particularly for the higher percentile values.

Keywords: Bayesian method; linear degradation model; log-logistic distribution; skew normal distribution; time to failure distribution

ABSTRAK

Dalam kertas ini, kaedah Bayesian yang melibatkan prior bermaklumat dan kurang bermaklumat dipertimbangkan untuk menganggar parameter dan persentil untuk taburan masa kegagalan. Parameter dan persentil bagi taburan masa kegagalan ditentukan berdasarkan model degradasi linear yang mana parameter degradasi diandaikan mengikuti taburan normal pencong. Untuk taburan prior, parameter skala dan lokasi bagi taburan normal pencong diandaikan mengikuti taburan seragam manakala parameter bentuk diandaikan mengikuti taburan gama. Dua prior gama yang dipertimbangkan, iaitu sama ada bermaklumat atau kurang bermaklumat, bergantung kepada nilai parameter hiper yang diandaikan. Prestasi kaedah berkenaan di bawah andaian yang berbeza dibandingkan menerusi kajian simulasi berdasarkan kaedah Rantai Markov Monte Carlo dan juga aplikasi data sebenar. Didapati bahawa penganggaran parameter berdasarkan prior bermaklumat adalah lebih persis berbanding prior kurang bermaklumat, khususnya apabila saiz sampel kecil. Seterusnya, model degradasi normal pencong dibandingkan dengan model degradasi log-logistik menerusi kajian simulasi dan aplikasi data laser GaAs. Bila memodelkan persentil bagi taburan masa kegagalan, secara amnya, hasil menunjukkan bahawa keputusan berdasarkan taburan normal pencong adalah lebih persis, khususnya untuk persentil yang bernilai tinggi.

Kata kunci: Kaedah Bayesian; model degradasi linear; taburan log-logistik; taburan masa kegagalan; taburan normal pencong

INTRODUCTION

It is desirable in life that we have products or tools which have a high quality and reliable, continue to function well after being in used for a substantial period of time. Accordingly, quality and reliability of product are always a matter of concern of both the manufacturers and customers. (Meeker & Escobar 1998) have defined reliability of a unit as the probability that the unit will perform its intended function until a specified point of time under encountered use conditions.

Problems of reliability such as degradation have always been considered in many areas of the engineering discipline and many methodologies have been developed to investigate the degradation phenomena of certain products. In particular, degradation phenomena which involves wear and tear of machinery parts, for example, have been identified as a possible cause of accident. Thus, early warning signs on possible accident that may occur due to degradation phenomena could be identified based on certain research works such as time to failure modeling.

Modeling the degradation measurements of a product over time is not straight forward since the degradation measure could be affected by the presence of more than one underlying degradation process. Many methods have been applied to estimate the parameter of time to failure distribution for the degradation models such as maximum likelihood method which has been applied by Meeker, Escobar and Lu (1999). In their paper, they have described some useful reliability models which considered degradation over time based on the physical-failure mechanisms. Also, they have presented some models which relate degradation and failure. They have used maximum likelihood method to estimate the mixed-effect accelerated degradation model parameters. In addition, the cumulative distribution function of time to failure distribution based on four different methods, which are analytical expressions, numerical evaluation, Monte Carlo evaluation and estimation, are also being considered. Confidence intervals based on bootstrap sampling for the quantities of interest are also determined. Semi-parametric method has also been applied in estimating time to failure distribution by many researchers, such as Ba Dakhn, Ebrahim and Eidous (2017), Jin (2016), and Robins and Tsiatis (1992). Ba Dakhn, Ebrahim and Eidous (2017) have applied the semi-parametric method to estimate the time-to-failure distribution and its percentiles for simple linear degradation model with no intercept. In this method, the

parametric estimator is assumed to follow either half-normal or exponential distributions. The performance of this estimator is compared with the maximum likelihood estimator and ordinary least square estimator which have been derived based on half normal distribution and exponential distribution. It is found that based on their study the performance of semi-parametric method is the best when the distribution of the random effect is unknown.

To estimate the parameters of the time to failure distribution, the Bayesian approach is widely used by many researchers such as Guure and Akma (2014), Puggard, Niwitpong and Niwitpong (2022), Shafiq and Atif (2016), and Thangjai, Niwitpong and Niwitpong (2021). In addition, Hamada (2005) has applied a linear degradation model based on the Bayesian approach to estimate the parameters of the Weibull distribution. In this approach, flat priors are assumed for the parameters of interest. A comparison is carried out between the use of degradation data, lifetime data and pseudo lifetime data in computing the reliability function and percentiles. Also, Ebrahim, Alodat and Arman (2009) and Rawashdeh, Ebrahim and Momani (2018) have applied the Bayesian approach in two types of data, namely grouped and non-grouped, to estimate the parameters and percentiles of the time-to-failure distribution. While Rawashdeh, Ebrahim and Momani (2018) have assumed the degradation rate following the log-logistic distribution, Ebrahim, Alodat, and Arman (2009) have assumed the degradation rate following the exponential distribution.

In addition, the non-parametric methods have also been applied to estimate the parameter of time to failure distribution for the degradation models. Ebrahim, Eidous and Kmail (2009) have applied the nonparametric kernel density method to estimate the time to failure distribution and its percentiles based on a linear degradation model with no intercept. Eidous, Ebrahim and Ba Dakhn (2017) have estimated time-to-failure distribution and its percentiles for simple linear degradation model with no intercept based on double kernel method. Al-haj Ebrahim, Al-Momani and Eidous (2021) have presented the variable scale kernel method to estimate the time to failure distribution under linear degradation model with no intercept. In their work, the performance of the nonparametric methods is compared with the existing parametric methods such as maximum likelihood method and ordinary least square method based on different distributions such as Weibull,

exponential, half normal and log logistic. The simulation study shows that the performance of the nonparametric methods is superior than the parametric method such as maximum likelihood and ordinary least squares methods in the case when the assumption of the data distribution is violated.

The flexibility of the skew normal distribution makes it to be quite versatile in certain areas of degradation modeling. Skewed normal distribution has attracted many researchers due to its flexibility. Skew normal distribution can be reduced to other symmetric distribution based on the value of skewness parameter, say λ , such as the normal distribution if λ the skewness parameter $\lambda = 0$ and half normal distribution if approaches to infinity (Alhamidie et al. 2019; Bayes & Branco 2007; Pan, Liu & Yang 2018). Tsai and Lin (2015) have assumed that the error term under nonlinear accelerated destructive degradation test model follows the skew normal distribution. Pan, Liu, and Yang (2018) have assumed the drift parameter based on the Wiener degradation model follows the skew normal distribution for estimating the lifetime distribution. Chen et al. (2019) have presented the stochastic degradation model based on the inverse Gaussian process where the reciprocal of the degradation rate parameter is assumed to follow the skew normal distribution.

In addition, there are some authors who consider certain skewed and heavy-tailed distributions (Bryson 1974; Gómez 2005), for describing the degradation parameter of the degradation model. For example, Oliveira, Loschi, and Freitas (2018) have assumed that the degradation parameter follows the scale and log scale mixture of skew normal distribution in order to accommodate skewness and heavy-tail behavior which are present in the data. In analyzing the train wheel degradation data, particularly at the wheel position which observes heavy-tail data, they have found that their model performed better than the Weibull degradation model. It seems that the skew normal degradation model presents as an alternative model when analyzing degradation data. Accordingly, in this study the performance of skew normal degradation model is further studied based on the comparison with log-logistic linear degradation model using a simulation study and an application of GaAs laser degradation data.

Accordingly, the outline of this paper is described as the following. Next section provides an explanation on GaAs laser degradation data and usage of the data as a motivating example. In the third section, estimation of the time to failure distribution is determined based on linear

degradation model where the degradation parameter follows the skew normal distribution. This is followed by the fourth section, where the Bayesian modeling is presented. In the fifth section, the simulation study for comparing the performance of the estimated parameter using the Bayesian approach based on informative and weakly informative prior is carried out by using Markov Chain Monte-Carlo method under JAGS platform. A comparison is made between log-logistic degradation model and skew normal degradation model using simulated data in the sixth section. In the seventh section, the results found based on fitting of Bayesian models to the GaAs laser degradation data are discussed. The last section provides the conclusion of the study.

GaAs LASER DEGRADATION DATA

The Laser Degradation Data from Meeker and Escobar (1998) is considered. The dataset provides the percent increase in laser operating current for 15 GaAs laser devices which are tested at 80 °C when the output light is kept at a nearly constant reading. The data which are presented in Table 1 and Figure 1 consists of 15 units of device, and for each unit the measurements are taken at the time range from 250 to 4000 h with step equals to 250 h. The failure is assumed to occur at the critical degradation level, which is, denoted as D_f , and in this study D_f is assumed equal to 5. We believe that the choice made on this value is sensible based on the allowance of 5% increase in the operating current demarcating the start for failure of the laser device (Ebrahim, Alodat & Arman 2009). This limit is pertinent in order to allow for a prudent monitoring of the device. Based on linear interpolation carried out on the data, we found one failure for each unit. Figure 1 shows that the laser degradation data follows a linear degradation path and thus it is reasonable to apply the linear degradation model.

In conducting data analysis, the goodness of fit of a particular parametric probability distribution on the data studied is often investigated. In this subsection, the adequacy of some candidate distributions such as exponential, log-logistic, and skew normal distributions in fitting the GaAs laser degradation data is studied. In order to assess the performance of these distributions, the two criteria which are the probability plotting procedure (P-P plot) and the Akaike information criterion (AIC) are used. The probability plots of the GaAs laser degradation data based on the candidate distributions are shown in Figure 2. From the figure, it is found that the skew normal and log-logistic distributions fit the data

almost equally well and it is difficult to differentiate between them. In order to decide which model fits the data better, the AIC is computed.

In addition, the AIC is computed for comparing the performance of several model in fitting certain dataset based on the equation given by

$$AIC = -2 \ln(\hat{L}) + 2 \omega \quad (1)$$

where (\hat{L}) is the likelihood function of the candidate distribution based on the maximum likelihood estimator and ω is the number of the unknown parameters. The distribution that has the smallest value of AIC is chosen as the best fitted model. The result of the values of AIC found are provided in Table 2.

TABLE 1. The data for GaAs Lasers Tested at 80 °C

Time (h)	Device unit number														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
250	0.47	0.71	0.71	0.36	0.27	0.36	0.36	0.46	0.51	0.41	0.44	0.39	0.30	0.44	0.51
500	0.93	1.22	1.17	0.62	0.61	1.39	0.92	1.07	0.93	1.49	1.00	0.80	0.74	0.70	0.83
750	2.11	1.90	1.73	1.36	1.11	1.95	1.21	1.42	1.57	2.38	1.57	1.35	1.52	1.05	1.29
1000	2.72	2.30	1.99	1.95	1.77	2.86	1.46	1.77	1.96	3.00	1.96	1.74	1.85	1.35	1.52
1250	3.51	2.87	2.53	2.30	2.06	3.46	1.93	2.11	2.59	3.84	2.51	2.98	2.39	1.80	1.91
1500	4.34	3.75	2.97	2.95	2.58	3.81	2.39	2.40	3.29	4.50	2.84	3.59	2.95	2.55	2.27
1750	4.91	4.42	3.30	3.39	2.99	4.53	2.68	2.78	3.61	5.25	3.47	4.03	3.51	2.83	2.78
2000	5.48	4.99	3.94	3.79	3.38	5.35	2.94	3.02	4.11	6.26	4.01	4.44	3.92	3.39	3.42
2250	5.99	5.51	4.16	4.11	4.05	5.92	3.42	3.29	4.60	7.05	4.51	4.79	5.03	3.72	3.78
2500	9.72	6.07	4.45	4.50	4.63	6.71	4.09	3.75	4.91	7.80	4.80	5.22	5.47	4.09	4.11
2750	7.13	6.64	4.89	4.72	5.24	7.70	4.58	4.16	5.34	8.32	5.20	5.48	5.84	4.83	4.38
3000	8.00	7.16	5.27	4.98	5.62	8.61	4.84	4.76	5.84	8.93	5.66	5.96	6.50	5.41	4.63
3250	8.92	7.78	5.69	5.28	6.04	9.15	5.11	5.16	6.40	9.55	6.20	6.23	6.94	5.76	5.38
3500	9.49	8.42	6.02	5.61	6.32	9.95	5.57	5.46	6.84	10.5	6.54	6.99	7.39	6.14	5.84
3750	9.87	8.91	6.45	5.95	7.10	10.5	6.11	5.81	7.20	11.3	6.96	7.37	7.85	6.51	6.16
4000	10.9	9.28	6.88	6.14	7.59	11.0	7.17	6.24	7.88	12.2	7.42	7.88	8.09	6.88	6.62

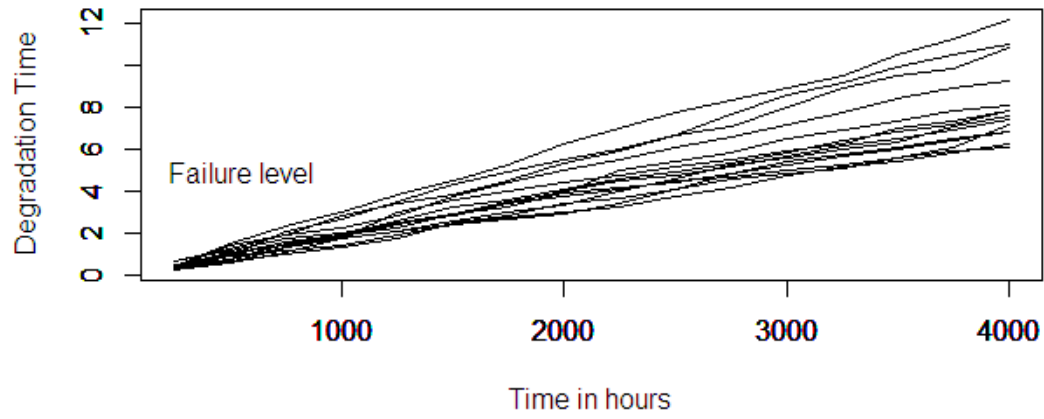


FIGURE 1. Percent increase in operating current for GaAs laser tested at 80 °C

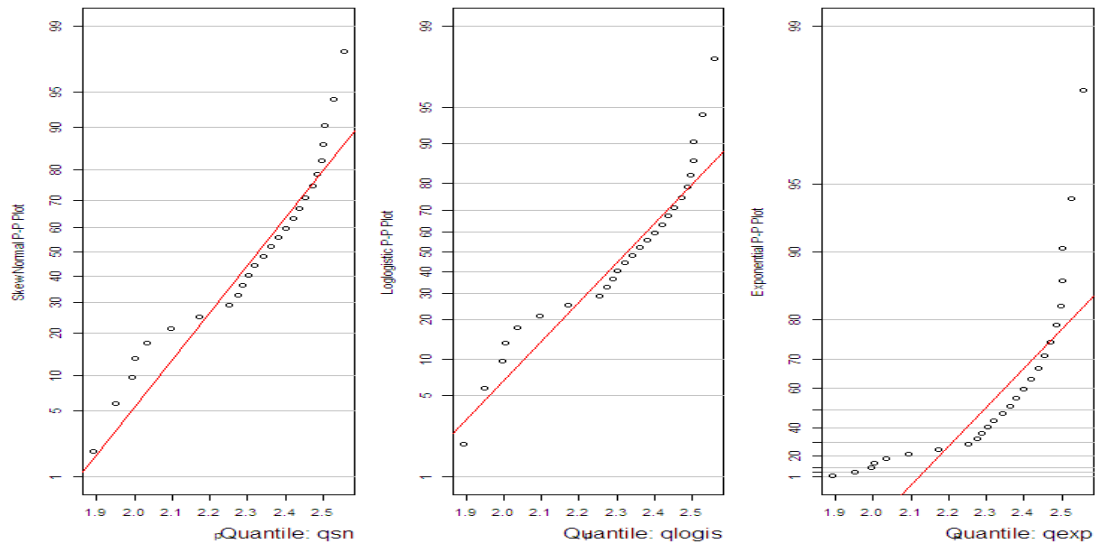


FIGURE 2. Skew normal, log-logistic and exponential probability plots for GaAs laser degradation data

TABLE 2. The AIC values found for skew normal, log-logistic and exponential linear degradation models using GaAs laser degradation data

Distributions	AIC
Skew normal	67.97
Log-logistic	68.35
exponential	101.63

From Table 2, AIC is found the smallest for skew normal distribution, slightly smaller for skew normal distribution as opposed to log logistic distribution, indicating that the most adequate model for describing the GaAs laser degradation data is the skew normal distribution among the candidate distributions considered. This finding supports the choice of skew normal distribution as a plausible candidate distribution for describing the time to failure distribution.

TIME TO FAILURE DISTRIBUTION FOR LINEAR DEGRADATION MODEL

The general degradation path model can be expressed as

$$y_{ij} = D(t_{ij}; \alpha, \beta_i) + \varepsilon_{ij}, i = 1, \dots, n; j = 1, \dots, m_i$$

where y_{ij} denotes the observed degradation measurement of the i^{th} unit at time t_{ij} , $D(t_{ij}; \alpha, \beta_i)$ is the actual path of the i^{th} unit at time t_{ij} , the term α is the vector of fixed effect parameters, $\beta_i = (\beta_{1i}, \beta_{2i}, \dots, \beta_{ki})$ is a vector of k unknown parameters for the i^{th} unit, ε_{ij} is the random error term where ε_{ij} are iid with $N(0, \sigma_\varepsilon^2)$ and σ_ε^2 is a constant, the term n is the number of units that are tested and the term m_i is the total number of inspections on unit i . It is assumed that $\{\varepsilon_{ij}\}$ and $\{\beta_i\}$ are independent and β_i 's are independent.

Failure is assumed to occur when the degradation measure exceeds the critical level of degradation. In the linear degradation path model, the actual degradation path of a particular unit, denoted as D_f , is assumed to follow a linear equation given by

$$D_f = \alpha + \frac{t}{\beta}$$

where α is a fixed effect parameter; β is a random effect parameter; and t is the time to failure. Note that the slope in the linear equation is $\frac{1}{\beta}$. Several studies such as Eidous, Ebrahim, and Ba Dakhn (2017) and Rawashdeh, Ebrahim and Momani (2018) have considered the linear degradation path model with the slope β ; however, in this study we have modified the linear degradation path model where the slope is $\frac{1}{\beta}$.

The modification involving the reciprocal of the degradation parameter is done for mathematical convenient since it is proven in the later part of this section that the pdf of the time to failure distribution is shown to be skew normal. This facilitates our data generating task because we have an identifiable

distribution to work with. Apart from the rational of mathematical convenient, we believe that in our modification, we allow for a smaller degradation rate since $\frac{1}{\beta}$ is generally smaller than β , particularly for the case when $\beta > 1$.

It can be shown that

$$F_T(t) = G_\beta \left(\frac{t}{D_f - \alpha} \right), t > 0 \quad (2)$$

where $F(\cdot)$ and $G(\cdot)$ are the cumulative distribution functions (cdfs).

In this work, the degradation parameter is assumed to follow the skew normal distribution, which is a flexible distribution that includes the skewness parameter in addition to the location and scale parameters. This skew normal distribution has been introduced by (Azzalini 1985) who defines that a random variable x follows a skew normal distribution if the probability density function (pdf) of x is given by

$$g_X(x) = 2 \phi(x) \Phi(\lambda x), x \in \mathbb{R}$$

where $\phi(\cdot)$ is the pdf and $\Phi(\cdot)$ is the cdf of standard normal distribution, $\lambda \in \mathbb{R}$ is the skewness or shape parameter.

Suppose that β belongs to a skew normal distribution. So, we denote

$$\beta \sim SN(\mu, \sigma^2, \lambda)$$

Then, the pdf of β is

$$g_\beta(\beta; \mu, \sigma^2, \lambda) = \frac{2}{\sigma} \phi\left(\frac{\beta - \mu}{\sigma}\right) \Phi\left(\lambda \left(\frac{\beta - \mu}{\sigma}\right)\right) \quad (3)$$

and the cdf of β is

$$G_\beta(\beta; \mu, \sigma^2, \lambda) = \Phi\left(\frac{\beta - \mu}{\sigma}\right) - 2T\left(\frac{\beta - \mu}{\sigma}, \lambda\right) \quad (4)$$

where $\phi(\cdot)$ is the pdf of standard normal distribution; $\Phi(\cdot)$ is the cdf of standard normal distribution; $\beta \in \mathbb{R}$, $\lambda \in \mathbb{R}$ is the skewness (shape) parameter; $\mu \in \mathbb{R}$ is the location parameter; $\sigma^2 \in \mathbb{R}$ is the scale parameter and $T(h, a)$ is Owen's T function defined as

$$T(h, a) = \frac{1}{2\pi} \int_0^a \frac{e^{-\frac{h^2(1+x^2)}{2}}}{1+x^2} dx, \quad -\infty < a, h < +\infty$$

Based on Equation (4), we modify Equation (2) to obtain the cdf of time to failure which can be given by

$$F_T(t) = \Phi\left(\frac{t - (D_f - \alpha) - \mu}{\sigma}\right) - 2T\left(\frac{t - (D_f - \alpha) - \mu}{\sigma}, \lambda\right)$$

By taking the derivative of F with respect to t , we get the pdf of time to failure distribution as follows:

$$f_T(t) = \frac{2}{(D_f - \alpha)\sigma} \phi\left(\frac{t - (D_f - \alpha)\mu}{(D_f - \alpha)\sigma}\right) \Phi\left(\lambda\left(\frac{t - (D_f - \alpha)\mu}{(D_f - \alpha)\sigma}\right)\right) \quad (5)$$

Now, it is clear that the cdf and the pdf of time to failure distribution are the cdf and the pdf of skew normal distribution with location parameter $\mu^* = (D_f - \alpha)\mu$, scale parameter $\sigma = (D_f - \alpha)\sigma$ and shape parameter λ . To compute the r th percentile of the time to failure distribution, denoted as t_r , we determine the inverse cdf of F and solved Equation (2) for t_r to obtain

$$t_r = (D_f - \alpha) G_B^{-1}(r) \quad (6)$$

Let T_1, T_2, \dots, T_n denoted a random sample of size n from the time to failure distribution with parameters λ, σ and μ . Given that $T_1 = t_1, T_2 = t_2, \dots$ and $T_n = t_n$, based on Equation (5), we obtain the likelihood function given by

$$L(\lambda, \sigma, \mu; \mathbf{t}) = \left(\frac{2}{(D_f - \alpha)\sigma}\right)^n \prod_{i=1}^n \phi\left(\frac{t_i - (D_f - \alpha)\mu}{(D_f - \alpha)\sigma}\right) \Phi\left(\lambda\left(\frac{t_i - (D_f - \alpha)\mu}{(D_f - \alpha)\sigma}\right)\right) \quad (7)$$

BAYESIAN APPROACH FOR SKEW NORMAL DEGRADATION MODEL

In this section, Bayesian approach is considered for skew normal degradation model. Both informative and non-informative priors are represented by gamma and uniform distributions, respectively, with certain known parameters. It is known that if non-informative priors are used to specify the prior distribution, the sample data would dominate the posterior distribution, particularly when the sample size is large. The posterior distribution can be given by

$$\pi(\lambda, \sigma, \mu | \mathbf{t}) = \frac{L(\lambda, \sigma, \mu; \mathbf{t})q(\lambda, \sigma, \mu)}{\int \int \int L(\lambda, \sigma, \mu; \mathbf{t})q(\lambda, \sigma, \mu) d\lambda d\sigma d\mu} \quad (8)$$

where $L(\lambda, \sigma, \mu; \mathbf{t})$ is the likelihood function of the time to failure distribution for the parameters λ, σ and μ and $q(\lambda, \sigma, \mu)$ is the joint prior distribution of λ, σ and μ . These parameters are assumed independent. So, $q(\lambda, \sigma, \mu) = h(\lambda)u(\sigma)w(\mu)$, where $h(\lambda)$, $u(\sigma)$ and $w(\mu)$ are the prior distributions.

Gamma distribution is assumed for the prior distribution of λ while the prior distribution of σ and μ are assumed uniform. These prior distributions are given as follows:

$$\text{i) Given that } \lambda \sim G(a, b), \text{ then } h(\lambda) = \begin{cases} \frac{\lambda^{a-1} e^{-b\lambda}}{\Gamma(a)b^a}, & \lambda > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{ii) Given that } \sigma \sim U(0, c), \text{ then } u(\sigma) = \begin{cases} \frac{1}{c}, & 0 < \sigma < c \\ 0, & \text{otherwise} \end{cases}$$

$$\text{iii) Given that } \mu \sim U(0, d), \text{ then } w(\mu) = \begin{cases} \frac{1}{d}, & 0 < \mu < d \\ 0, & \text{otherwise} \end{cases}$$

where a, b, c , and d are assumed known constants.

The basic mechanism in the Bayesian approach involves updating of the prior distribution for the parameter of interest using the current information based on the following relationship:

$$\text{posterior} \propto \text{prior} \times \text{likelihood}$$

Based on this relationship, involving informative and non-informative prior distributions explained earlier and the likelihood function given in Equation (7), the joint posterior density function $\pi(\lambda, \sigma, \mu | \mathbf{t})$ is proportional to

$$\frac{\lambda^{a-1} e^{-b\lambda}}{\Gamma(a) c d b^a} \left(\frac{2}{(D_f - \alpha)\sigma}\right)^n \prod_{i=1}^n \phi\left(\frac{t_i - (D_f - \alpha)\mu}{(D_f - \alpha)\sigma}\right) \Phi\left(\lambda\left(\frac{t_i - (D_f - \alpha)\mu}{(D_f - \alpha)\sigma}\right)\right) \quad (9)$$

SIMULATION STUDY FOR COMPARING THE PARAMETER ESTIMATION UNDER DIFFERENT PRIOR ASSUMPTIONS

The performance of Bayesian model under informative prior and weakly informative priors is studied based on simulated data which are generated using certain value of the true parameter values and the critical level of degradation D_f . The assumed true parameter values and D_f are respectively given as follows:

$$\lambda = 3; \sigma = 2; \mu = 1 \text{ and } D_f = 20$$

In the real application, we have $D_f = 5$, but in the simulation study we assumed $D_f = 20$. A much large value of D_f in the simulation study is not surprising since the simulated data depict an ideal situation, which may be related to a very reliable product, where only large values of degradation data represent cases of failure. In addition, $D_f = 20$ has been considered in the study of Rawashdeh, Ebrahim and Momani (2018). The initial value of parameter is assumed 6, while the sample sizes considered are $n = 30, 60$ and 200 .

Based on these assumed values, the data for time to failure distribution are generated using the skew normal distribution with location parameter $\mu_* = (D_f - \alpha)$, scale parameter $\sigma_* = (D_f - \alpha)\sigma$ and shape parameter λ . Since the joint posterior distribution of the parameters cannot be determined in a closed form, MCMC method is applied. In particular, samples from the joint posterior distribution of Equation (9) is generated using the JAGS (Just Another Gibbs Sampler) algorithm. The weakly informative gamma prior is represented based on the assumed values of the shape and scale hyper parameters of 0.1 and 0.01, respectively, i.e., $a = 0.1$ and $b = 0.01$, while the informative gamma prior is represented by

$a = 2$ and $b = 2$. The choice made on the true values of the shape and scale hyper parameters, although may be arbitrary, reflects the level of uncertainty on the gamma priors. The variances for the weakly informative and informative gamma priors are 1000 and 1/2, respectively. In addition, for the uniform priors, we assume the upper limits $c = 4$ and $d = 2$. We believe that this is a fair representation of the level of uncertainty that we have on the little information regarding the prior parameters.

The JAGS algorithm which has been developed by Plummer (2003) can be used to carry out MCMC simulation based on Gibbs sampling, making the posterior analysis comparatively easy. For details on the implementation of JAGS algorithm, refer to Albert (2008) and Coro (2017). For computation, the programming language R version 4.0.3 is used.

The results for the estimation of the parameters and certain percentiles based on Bayesian approach of the skew normal linear degradation model under informative and weakly informative priors for the different sample sizes are provided in Table 3. The properties of point estimates are evaluated using bias (B) and mean squared error (MSE) values of the estimators. Estimators with small bias and mean squared error are preferred.

TABLE 3. B and MSE for the parameter estimate and certain percentiles of the time to failure distribution under informative and weakly informative prior for $n = 30, 60$ and 200

Parameter values	Informative prior						Weakly Informative prior					
	$n = 30$		$n = 60$		$n = 200$		$n = 30$		$n = 60$		$n = 200$	
	B	MSE	B	MSE	B	MSE	B	MSE	B	MSE	B	MSE
$\mu = 1$	0.268	0.178	0.025	0.049	0.069	0.023	0.464	0.363	0.044	0.075	0.063	0.024
$\sigma = 2$	0.469	0.318	0.089	0.071	0.101	0.032	0.617	0.497	0.111	0.0956	0.095	0.032
$\lambda = 3$	0.314	2.808	1.332	4.870	0.554	1.027	0.970	7.535	2.937	80.400	0.523	1.062
$t_{0.05}$	1.894	14.870	1.331	7.845	0.100	2.043	0.312	19.200	1.112	7.946	0.127	2.091
$t_{0.2}$	0.719	7.819	0.410	4.590	0.336	1.578	0.042	9.978	0.104	5.479	0.324	1.594
$t_{0.5}$	1.332	9.300	0.462	5.715	0.014	1.770	1.641	10.510	0.333	6.201	0.003	1.801
$t_{0.75}$	4.027	28.240	1.071	11.290	0.656	3.393	4.664	33.630	1.131	11.800	0.641	3.384
$t_{0.9}$	7.131	75.320	1.697	23.500	1.357	7.976	8.443	95.880	1.925	26.000	1.299	7.910

The estimators are evaluated based on bias and MSE. From Table 3, we have found the following results: In most cases, B and MSE values of the estimated parameters and the associated percentiles in the Bayesian approach of skew normal linear degradation model under informative and weakly informative gamma prior decrease as n increase. The MSE of the r^{th} percentiles of the time to failure distribution increases as r increase for all sample sizes, except in the case of the low value of the r^{th} percentile. In most cases, the B values of the estimated parameters and the associated percentiles under informative gamma prior are smaller than the B values found under weakly informative gamma prior for small sample size and these results are quite close for large sample size.

Generally, in the small sample size, the performance of the Bayesian approach of skew normal linear degradation model under informative gamma prior is better than the performance of the Bayesian approach of skew normal linear degradation model under weakly informative gamma prior, while in the case of large sample size, the results found based on the two different prior assumptions are close.

COMPARISON BETWEEN LOG-LOGISTIC AND SKEW NORMAL DEGRADATION MODEL UNDER THE BAYESIAN APPROACH

In this section, the performance of Bayesian approach for modeling the time to failure distribution and its percentiles based on linear degradation model with the degradation parameter following the log-logistic distribution and skew normal distribution is compared in terms of bias and mean squared error.

Rawashdeh, Ebrahim and Momani (2018) have estimated the time to failure distribution based on linear degradation path $D = +X$ with random degradation rate X which follows log-logistic distribution. The cdf of time to failure distribution is given as

$$F_{T_L}(t_L; \alpha_L, \beta_L, \emptyset) = \frac{1}{1 + \left(\frac{t_L}{\alpha_*}\right)^{-\beta_L}}, \alpha_* = \frac{D - \emptyset}{\alpha_L}$$

where \emptyset is the degradation level at at $t_L = 0$; α_L and β_L are the scale and the shape parameters of log-logistic distribution, respectively, and D is the critical degradation level at which failure is declared. It has been shown that the time to failure distribution is also log-logistic with scale parameter α_* and the shape parameter β_L . The percentiles of the time to failure distribution based on Equation (10) can be shown as

$$t_{L-r} = \frac{D - \emptyset}{\alpha_L} \left(\frac{r}{1-r}\right)^{\frac{1}{\beta_L}} \quad (11)$$

The Bayesian method has been applied for grouped and non-grouped data and the results found are compared. The prior distributions of parameters and are assumed to be non-informative prior. The non-informative prior that has been considered by them are uniform and Jeffery's priors.

In the conclusion of this study, they have stated that the performance of the Bayesian model for the non-grouped method is better due to the smaller value of mean squared error. They have shown that the joint posterior distribution of non-grouped data is given as

$$\pi(\alpha_L, \beta_L, \emptyset | t_L, D) \propto \frac{\beta_L^n}{D} \left(\prod_{i=1}^n t_{L_i}\right)^{\beta_L-1} \alpha_L^{n\beta_L-1} (D - \emptyset)^{-n\beta_L} \frac{1}{\prod_{i=1}^n \left(1 + \left(\frac{\alpha_L t_{L_i}}{D - \emptyset}\right)^{\beta_L}\right)^2} \quad (12)$$

In this work, a slight modification is done in the work by Rawashdeh, Ebrahim and Momani (2018) where we consider the linear degradation path model $D = \emptyset + \frac{t}{X}$ which involves reciprocal of the degradation rate instead of the degradation rate. Here X is assumed to follow log-logistic distribution. Thus, the cdf of time to failure distribution can be shown as

$$F_{T_L}(t_L; \alpha_L, \beta_L) = \frac{1}{1 + \left(\frac{t_L}{(D - \emptyset)\alpha_L}\right)^{-\beta_L}} \quad (13)$$

Then, the pdf of time to failure distribution is given as

$$f_{T_L}(t_L; \alpha_L, \beta_L) = \frac{\beta_L}{(D - \emptyset)\alpha_L} \left(\frac{t_L}{(D - \emptyset)\alpha_L}\right)^{\beta_L-1} \left(1 + \left(\frac{t_L}{(D - \emptyset)\alpha_L}\right)^{\beta_L}\right)^{-2} \quad (14)$$

Note that the time to failure distribution is also log-logistic distribution with scale parameter $(D - \emptyset)\alpha_L$ and the shape parameter β_L . Based on Equation (13), it can be shown that the percentiles of the time to failure distribution is

$$t_{L-r} = (D - \emptyset)\alpha_L \left(\frac{r}{1-r}\right)^{\frac{1}{\beta_L}} \quad (15)$$

To make the skew normal and log-logistic degradation models comparable, both the prior distributions for the scale and the shape parameters for each model are assumed to be the same as in the previous section where the values of the hyper parameters of the informative gamma prior for both models are equal, i.e., $a = b = 2$ and $d = s = 2$. Note that the location parameter, which is another parameter for the skew normal distribution, is assumed fixed and it is not estimated. In addition, for both models, the initial and the critical degradation values are also assumed the same as previous section. Then, it can be shown that the joint posterior distribution $\pi_L(\alpha_L, \beta_L; t_L)$ is proportional to

$$\frac{\beta_L^{d-1} e^{-\frac{\beta_L}{s}}}{\Gamma(d) d^s b} \left(\frac{\beta_L}{(D - \phi)\alpha_L}\right)^n \prod_{i=1}^n \left(\frac{t_{L_i}}{(D - \phi)\alpha_L}\right)^{\beta_L-1} \left(1 + \left(\frac{t_{L_i}}{(D - \phi)\alpha_L}\right)^{\beta_L}\right)^{-2} \tag{16}$$

The results of the simulation study for comparison between the models are provided in Table 4.

The estimators are evaluated based on bias and MSE. From Table 4, we have found the following results: In most cases, the MSE of the estimated percentiles in the Bayesian approach of skew normal and log-logistic linear degradation model decreases as n increase. The MSE of the r^{th} percentile positions increases as r increases for all sample sizes except in the lower percentile position where $n = 30$ and 60 for skew normal linear degradation model. In most cases for all sample sizes, the B values of the estimated percentiles in both models increase as r increase. In most cases, for small sample size such as $n = 30$ and 60 , for lower percentile position of time to failure distribution, the B values for skew normal linear degradation model are larger than the B values for log-logistic linear degradation model and vice versa for the case of larger percentile position.

Generally, the performance of the Bayesian approach of skew normal linear degradation model is found better than the performance of the Bayesian approach of log-logistic linear degradation model, particularly in the case of higher percentile position of time to failure distribution.

TABLE 4. Bias (B) and MSE of the percentiles for the skew normal and log-logistic degradation models for the simulated data for $n = 30, 60$ and 200

Parameter values	Skew normal degradation model						Log-logistic degradation model					
	$n = 30$		$n = 60$		$n = 200$		$n = 30$		$n = 60$		$n = 200$	
	B	MSE	B	MSE	B	MSE	B	MSE	B	MSE	B	MSE
$t_{0.05}$	0.787	17.805	0.466	7.825	0.613	2.900	0.027	2.420	1.226	3.194	0.672	0.806
$t_{0.2}$	0.679	6.845	1.080	4.090	0.668	1.347	0.656	6.099	1.659	6.295	1.594	3.346
$t_{0.5}$	2.686	17.146	1.997	7.333	2.246	5.943	2.093	21.337	1.760	12.834	3.372	13.649
$t_{0.75}$	4.790	48.523	3.048	17.183	3.779	16.386	4.380	84.856	1.172	38.467	6.043	45.353
$t_{0.9}$	6.927	99.588	4.252	33.804	5.375	33.108	8.412	395.311	0.851	177.867	10.739	159.160

APPLICATION OF REAL DEGRADATION DATA

In the second section, the GaAs laser degradation data is described and applied to investigate the adequacy of the skew normal, log-logistic and exponential degradation models based on probability plotting and the computed values of AIC of all models. In this section, the GaAs laser

degradation data is applied to compare the performance of the weakly informative and informative prior of the skew normal degradation model. The comparison is made in terms of point estimate (PE) and standard error (SE) of the percentiles of the time to failure distribution. The results found are provided in Table (5).

TABLE 5. PE and SE for certain percentiles of the skew normal linear degradation model under informative and weakly informative prior based on the GaAs laser degradation data

parameters	Informative prior		Weakly informative prior	
	PE	SE	PE	SE
$t_{0.05}$	4.449	0.168	5.891	0.148
$t_{0.2}$	6.635	0.098	7.113	0.097
$t_{0.5}$	9.068	0.200	9.431	0.204
$t_{0.75}$	11.86	0.339	12.001	0.347
$t_{0.9}$	14.433	0.485	14.667	0.495

According to Table 5, referring to the estimated parameters and standard errors, when estimating the percentiles of the time to failure distribution under the application of laser degradation data, the performance of the skew normal degradation model based on informative prior is generally better than the performance of the skew normal degradation model based on weakly informative prior.

As mentioned in the second section, the AIC found for fitted skew normal and log-logistic models are found close to each other. In order to compare the performance of those models, the point estimate (PE) and standard error (SE) of the percentiles of the time to failure distribution found based on skew normal and log-logistic distributions that are indicated in Equations (5) and (14), respectively, are computed. Also, deviance

information criterion (DIC) is determined for selecting the best fitted model to the real data. Spiegelhalter, Best and Carlin (2002) define the DIC for the vector of parameters interest, say θ , as the following:

$$DIC = \bar{D}(\theta) + P_D$$

where D is the deviance which is defined by $-2\log(l)$; l is the likelihood function; \bar{D} is the posterior deviance mean which is given based on the values of the estimated parameters of the posterior distribution; and P_D is the effective number of parameters of the model which is defined as $\bar{D}(\theta) - \bar{D}(\bar{\theta})$ where $\bar{D}(\bar{\theta})$ is the deviance which is found by finding the mean of each value of the estimated parameters in the posterior distribution. Results of the comparison involving the parameter estimates, standard error and DIC are provided in Table 6.

TABLE 6. PE and SE of the percentiles and DIC for the skew normal and log-logistic degradation models for laser degradation data

parameters	Interpolated values	Skew normal degradation model		Log-logistic degradation model	
		PE	SE	PE	SE
$t_{0.05}$	7.128	6.650	0.237	5.393	0.234
$t_{0.2}$	8.125	8.332	0.176	7.478	0.219
$t_{0.5}$	10.581	10.154	0.153	10.050	0.233
$t_{0.75}$	11.671	11.678	0.179	12.749	0.350
$t_{0.9}$	12.212	13.100	0.234	16.226	0.627
DIC		67.550		68.530	

Comparison of the skew normal and the log-logistic degradation model in Table 6 indicates the following results: In most cases, the standard error for the estimated percentiles of skew normal degradation model is smaller than the standard error for the estimated percentiles of log-logistic degradation model. The point estimate of the percentiles of skew normal degradation model is found closer to the observed value of the percentiles reported in the laser degradation dataset than those values found for the log-logistic degradation model. This finding can be proven by using the linear interpolation of the observed values. For example, consider the interpolated value of the 75th percentile of the time to failure values in Table 1, i.e., 11.671. Corresponding to the estimated values in the Table 6, we have for the skew normal degradation model while for the log-logistic degradation model, and the standard errors are 0.18 and 0.35, respectively. This indicates that the estimated value of the parameter for the skew normal degradation model is more precise and hence it is more appropriate to use this result as compared to that found based on log-logistic degradation model. The deviance information criterion for skew normal degradation model is slightly smaller than the deviance information criterion for log-logistic degradation model, indicating a better fitting for skew normal degradation model.

CONCLUSIONS

In this study, we present the Bayesian approach of skew normal linear degradation model based on the informative and weakly informative prior. From the simulation results, we conclude that the Bayesian approach which considers informative prior outperformed the Bayesian model found under weakly informative prior, especially in the case of small sample size while for large sample size both results are found to be close. In addition, comparison of the skew normal and log-logistic linear degradation models is presented in terms of point estimate, standard error, Akaike information criterion and deviance information criterion. It is found that the linear degradation model with the reciprocal of the degradation parameter as the slope in the degradation path equation where the degradation parameter follows the skew normal distribution presents as a better alternative model than the log-logistic linear degradation model for describing the degradation data and these results are found more apparent in the real data application of laser degradation data in terms of point estimate, standard error and deviance information criteria.

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