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Research article

Semilocal convergence analysis of an eighth order iterative method for solving nonlinear systems

Xiaofeng Wang, Yufan Yang and Yuping Qin*

School of Mathematical Sciences, Bohai University, Jinzhou 121000, China

* **Correspondence:** Email: qinyuping@qymail.bhu.edu.cn.

Abstract: In this paper, the semilocal convergence of the eighth order iterative method is proved in Banach space by using the recursive relation, and the proof process does not need high order derivative. By selecting the appropriate initial point and applying the Lipschitz condition to the first order Fréchet derivative in the whole region, the existence and uniqueness domain are obtained. In addition, the theoretical results of semilocal convergence are applied to two nonlinear systems, and satisfactory results are obtained.

Keywords: nonlinear system; iterative method; recurrence relation; semilocal convergence; existence domain; unique domain
Mathematics Subject Classification: 49M15, 65J15, 65G99

1. Introduction

In mathematics and various sciences, the changes of output and input of nonlinear systems are out of proportion. Most of the systems involved in life are essentially nonlinear, so solving nonlinear problems has attracted various scientists. Scholars have proposed more efficient iterative methods for solving nonlinear systems. One of the most famous iterative methods for solving nonlinear systems is Newton's method [1],

$$x^{(k+1)} = x^{(k)} - [F'(x^{(k)})]^{-1}F(x^{(k)}), \tag{1.1}$$

for $k = 0, 1, 2, ..., x_0$ is the starting point. The Newton's method is second order convergent and effective in solving some nonlinear systems.

With the advancement of computers and numerical algebra, scholars have developed many iterative methods based on Newton's method that are more efficient than second-order Newton's method for solving nonlinear problems [2–7]. In addition, when the Jacobian matrix cannot be calculated for nonlinear systems, some effective derivative free methods can also solve nonlinear systems well (see [8–12]). We propose an eighth order iterative method with high computational efficiency, which is suitable for solving large systems of equations [13]. The specific iteration format is as follows

$$\begin{cases} y^{(k)} = x^{(k)} - \Gamma_k F(x^{(k)}), \\ w^{(k)} = y^{(k)} - [I + (I + \frac{5}{4}M^{(k)})M^{(k)}]\Gamma_k F(y^{(k)}), \\ x^{(k+1)} = w^{(k)} - [I + (I + \frac{3}{2}M^{(k)})M^{(k)}]\Gamma_k F(w^{(k)}), \end{cases} \tag{1.2}$$

where $M^{(k)} = \Gamma_k(F'(x^{(k)}) - F'(y^{(k)}))$, and $\Gamma_k = [F'(x^{(k)})]^{-1}$.

The theoretical results of local convergence and semilocal convergence of the iterative method are also important in the study. Local convergence requires the existence of the assumed solution and the initial value is close enough to the solution. Semilocal convergence does not require the existence of an assumed solution, but the selection of initial values also needs to meet certain conditions (see [14–18]). Therefore, for some systems that cannot be analyzed and solved, the results of semilocal convergence cannot only prove the convergence of iterative sequences, but also prove the existence of solutions of these systems, so as to obtain the existence domain and uniqueness domain of system solutions; for further study (see [19–22]). Based on this, we perform a semilocal convergence analysis on the method (1.2).

This paper consists of five sections. In Section 2 of the paper, the recurrence relation is explained. The semilocal convergence of the iterative method (1.2) is proved in Section 3. In Section 4, the numerical experiments of two nonlinear systems are completed. Finally, the conclusion of this paper is made.

2. Recurrence relations

In this section, let X and Y be Banach spaces and let $F : \Omega \subseteq X \to Y$ be a twice differetiable nonlinear Fréchet operator in an open Ω [23]. Let us assume that the inverse of the Jacobian matrix of the system in the iteration (1.2) is $\Gamma_0 \in \mathcal{L}(Y, X)$, which is the set of linear operation from Y to X.

Moreover, in order to obtain the semilocal convergence result for this iterative method (1.2), Kantorovich conditions are assumed:
(M_1) $\|\Gamma_0\| \leq \beta$,
(M_2) $\|\Gamma_0 F(x_0)\| \leq \eta$,
(M_3) $\|F'(x) - F'(y)\| \leq K\|x - y\|$,
where K, β, η are non-negative real numbers. For the sake of simplicity, we denote $a_0 = K\beta\eta$ and define the sequence

$$a_{k+1} = a_k f(a_k)^2 g(a_k), \tag{2.1}$$

where we use the following auxiliary functions

$$h(x) = \frac{1}{256}(256 + 256x + 384x^2 + 640x^3 + 576x^4 + 576x^5 + 528x^6 + 298x^7 + 170x^8 + 75x^9), \tag{2.2}$$

$$f(x) = \frac{1}{1 - xh(x)}, \tag{2.3}$$

and

$$g(x) = \frac{x}{131072}(196608 + 327680x + 589824x^2 + 819200x^3 + 1064960x^4 + 1351680x^5 + 1569792x^6 + 1689600x^7 + 1752576x^8 + 1693696x^9 + 1490432x^{10} + 1226752x^{11} + 913920x^{12} + 596928x^{13} + 354724x^{14} + 180520x^{15} + 73600x^{16} + 25500x^{17} + 5625x^{18}). \tag{2.4}$$

These functions play a key role in the analysis that will be performed next.

Preliminary results. In order to get the difference of the first two elements in the iterative method (1.2), we have

$$w_0 - x_0 = y_0 - x_0 - [I + (I + \frac{5}{4}\Gamma_0(F'(x_0) - F'(y_0)))\Gamma_0(F'(x_0) - F'(y_0))]\Gamma_0 F(y_0). \tag{2.5}$$

The Taylor series expansion of F around x_0 evaluated in y_0 is

$$F(y_0) = F(x_0) + F'(x_0)(y_0 - x_0) + \int_{x_0}^{y_0} (F'(x) - F'(x_0))dx, \tag{2.6}$$

where the term $F(x_0) + F'(x_0)(y_0 - x_0)$ is equal to zero, since it comes from a Newton's step. With the change $x = x_0 + t(y_0 - x_0)$, we get

$$F(y_0) = \int_0^1 (F'(x_0 + t(y_0 - x_0)) - F'(x_0))(y_0 - x_0)dt. \tag{2.7}$$

Then,

$$\begin{aligned} w_0 - x_0 &= y_0 - x_0 - (I + \Gamma_0(F'(x_0) - F'(y_0)) + \frac{5}{4}\Gamma_0(F'(x_0) - F'(y_0))\Gamma_0(F'(x_0) - F'(y_0)))\Gamma_0 F(y_0) \\ &= y_0 - x_0 - (\Gamma_0 F(y_0) + \Gamma_0(F'(x_0) - F'(y_0))\Gamma_0 F(y_0) \\ &+ \frac{5}{4}\Gamma_0(F'(x_0) - F'(y_0))\Gamma_0(F'(x_0) - F'(y_0))\Gamma_0 F(y_0)) \\ &= y_0 - x_0 - (\Gamma_0 \int_0^1 (F'(x_0 + t(y_0 - x_0)) - F'(x_0))(y_0 - x_0)dt \\ &+ \Gamma_0(F'(x_0) - F'(y_0))\Gamma_0 \int_0^1 (F'(x_0 + t(y_0 - x_0)) - F'(x_0))(y_0 - x_0)dt \\ &+ \frac{5}{4}\Gamma_0(F'(x_0) - F'(y_0))\Gamma_0(F'(x_0) - F'(y_0)) \\ &\times \Gamma_0 \int_0^1 (F'(x_0 + t(y_0 - x_0)) - F'(x_0))(y_0 - x_0)dt). \end{aligned} \tag{2.8}$$

Taking norms and applying Lipschitz condition, we get

$$\begin{aligned} \|w_0 - x_0\| \le \|y_0 - x_0\| + \frac{K}{2}\|\Gamma_0\|\,\|y_0 - x_0\|^2 + \frac{K^2}{2}\|\Gamma_0\|\,\|y_0 - x_0\|\,\|\Gamma_0\|\,\|y_0 - x_0\|^2 \\ + \frac{5K^3}{8}\,\|\Gamma_0\|\,\|y_0 - x_0\|\,\|\Gamma_0\|\,\|y_0 - x_0\|\,\|\Gamma_0\|\,\|y_0 - x_0\|^2 \end{aligned}$$

$$
\begin{aligned}
&\leq \eta + \frac{1}{2}K\beta\eta^2 + \frac{1}{2}K^2\beta^2\eta^3 + \frac{5}{8}K^3\beta^3\eta^4 \\
&= \eta(1 + \frac{1}{2}a_0 + \frac{1}{2}a_0^2 + \frac{5}{8}a_0^3),
\end{aligned} \tag{2.9}
$$

so that,

$$
||w_0 - x_0|| \leq \eta(1 + \frac{1}{2}(a_0 + a_0^2 + \frac{5}{4}a_0^3)). \tag{2.10}
$$

Using a method similar to (2.5), we get $w_0 - y_0$

$$
w_0 - y_0 = y_0 - y_0 - [I + (I + \frac{5}{4}\Gamma_0(F^{'}(x_0) - F^{'}(y_0)))\Gamma_0(F^{'}(x_0) - F^{'}(y_0))]\Gamma_0 F(y_0). \tag{2.11}
$$

So,

$$
\begin{aligned}
||w_0 - y_0|| \leq\ & ||y_0 - y_0|| + \frac{K}{2}||\Gamma_0||\,||y_0 - x_0||^2 + \frac{K^2}{2}||\Gamma_0||\,||y_0 - x_0||\,||\Gamma_0||\,||y_0 - x_0||^2 \\
&+ \frac{5K^3}{8}\,||\Gamma_0||\,||y_0 - x_0||\,||\Gamma_0||\,||y_0 - x_0||\,||\Gamma_0||\,||y_0 - x_0||^2 \\
&\leq \frac{1}{2}K\beta\eta^2 + \frac{1}{2}K^2\beta^2\eta^3 + \frac{5}{8}K^3\beta^3\eta^4 \\
&= \eta(\frac{1}{2}a_0 + \frac{1}{2}a_0^2 + \frac{5}{8}a_0^3).
\end{aligned} \tag{2.12}
$$

Next, the next step analysis

$$
x_1 - x_0 = w_0 - x_0 - [I + (I + \frac{3}{2}\Gamma_0(F^{'}(x_0) - F^{'}(y_0)))\Gamma_0(F^{'}(x_0) - F^{'}(y_0))]\Gamma_0 F(w_0). \tag{2.13}
$$

Using Taylor's expansion of $F(w_0)$ around x_0 and applying Lipschitz condition, we obtain

$$
\begin{aligned}
||x_1 - x_0|| \leq\ & ||w_0 - x_0|| + \frac{K}{2}||\Gamma_0||\,||w_0 - x_0||^2 + \frac{K^2}{2}||\Gamma_0||\,||y_0 - x_0||\,||\Gamma_0||\,||w_0 - x_0||^2 \\
&+ \frac{3K^3}{4}\,||\Gamma_0||\,||y_0 - x_0||\,||\Gamma_0||\,||y_0 - x_0||\,||\Gamma_0||\,||w_0 - x_0||^2 \\
&\leq \eta(1 + \frac{1}{2}(a_0 + a_0^2 + \frac{5}{4}a_0^3)) + \frac{K\beta\eta^2}{2}(1 + \frac{1}{2}(a_0 + a_0^2 + \frac{5}{4}a_0^3))^2 \\
&+ \frac{K^2\beta^2\eta^3}{2}(1 + \frac{1}{2}(a_0 + a_0^2 + \frac{5}{4}a_0^3))^2 + \frac{3K^3\beta^3\eta^4}{4}(1 + \frac{1}{2}(a_0 + a_0^2 + \frac{5}{4}a_0^3))^2 \\
&= \eta((1 + \frac{1}{2}(a_0 + a_0^2 + \frac{5}{4}a_0^3)) + \frac{a_0}{2}(1 + \frac{1}{2}(a_0 + a_0^2 + \frac{5}{4}a_0^3))^2 \\
&+ \frac{a_0^2}{2}(1 + \frac{1}{2}(a_0 + a_0^2 + \frac{5}{4}a_0^3))^2 + \frac{3a_0^3}{4}(1 + \frac{1}{2}(a_0 + a_0^2 + \frac{5}{4}a_0^3))^2) \\
&= \eta(\frac{1}{256}(256 + 256a_0 + 384a_0^2 + 640a_0^3 + 576a_0^4 + 576a_0^5 \\
&+ 528a_0^6 + 298a_0^7 + 170a_0^8 + 75a_0^9)).
\end{aligned} \tag{2.14}
$$

By applying Banach's lemma, one has

$$
\begin{aligned}
\|I-\Gamma_0F^{'}(x_1)\| &= \|\Gamma_0F^{'}(x_0)-\Gamma_0F^{'}(x_1)\| \\
&= \|\Gamma_0\|\,\|F^{'}(x_0)-\Gamma_0F^{'}(x_1)\| \\
&\leq K\beta\|x_1-x_0\| \\
&\leq K\beta\eta(\frac{1}{256}(256+256a_0+384a_0^2+640a_0^3+576a_0^4+576a_0^5 \\
&+528a_0^6+298a_0^7+170a_0^8+75a_0^9)) \\
&= a_0(h(a_0))<1, \qquad (2.15)
\end{aligned}
$$

where

$$h(x)=\frac{1}{256}(256+256x+384x^2+640x^3+576x^4+576x^5+528x^6+298x^7+170x^8+75x^9).$$

Then, as far as $a_0(h(a_0))<1$ (by taking $a_0<0.45807$), Banach's lemma guarantees that

$$(\Gamma_0F^{'}(x_1))^{-1}=\Gamma_1\Gamma_0^{-1}$$

exists and

$$\|\Gamma_1\|\leq\frac{1}{1-a_0(h(a_0))}\|\Gamma_0\|=f(a_0)\|\Gamma_0\|, \qquad (2.16)$$

so

$$f(x)=\frac{1}{1-\frac{1}{256}(256+256x+384x^2+640x^3+576x^4+576x^5+528x^6+298x^7+170x^8+75x^9)}.$$

Based on the above analysis, we can obtain the following theorem.

Theorem 1. For $k\geq 1$, the following conditions are valid:
$(O1_k)$ $\|\Gamma_k\|\leq f(a_{k-1})\|\Gamma_{k-1}\|$,
$(O2_k)$ $\|y_k-x_k\|=\|\Gamma_kF(x_k)\|\leq f(a_{k-1})g(a_{k-1})\|y_{k-1}-x_{k-1}\|$,
$(O3_k)$ $K\|\Gamma_k\|\,\|y_k-x_k\|\leq a_k$,
$(O4_k)$ $\|x_k-x_{k-1}\|\leq h(a_{k-1})\|y_{k-1}-x_{k-1}\|$.

Proof. The above theorem is proven through induction. Starting with $k=1$, (2.16) proved the $(O1_1)$.
$(O2_1)$: The Taylor's expansion of $F(x_1)$ around y_0, we can get

$$
\begin{aligned}
F(x_1) &= F(y_0)+F^{'}(y_0)(x_1-y_0)+\int_{y_0}^{x_1}(F^{'}(x)-F^{'}(y_0))dx \\
&= F(y_0)+(F^{'}(y_0)-F^{'}(x_0))(x_1-y_0)+F^{'}(x_0)(x_1-y_0) \qquad (2.17)\\
&+\int_0^1(F^{'}(y_0+t(x_1-y_0))-F^{'}(y_0))(x_1-y_0)dt.
\end{aligned}
$$

So, we must to have x_1-y_0

$$x_1-y_0=w_0-y_0-[I+(I+\frac{3}{2}\Gamma_0(F^{'}(x_0)-F^{'}(y_0)))\Gamma_0(F^{'}(x_0)-F^{'}(y_0))]\Gamma_0F(w_0). \qquad (2.18)$$

And bounding its norm, the following inequality is obtained

$$
\begin{aligned}
\|x_1 - y_0\| &\leq \|w_0 - y_0\| + \frac{K}{2}\|\Gamma_0\|\,\|w_0 - x_0\|^2 + \frac{K^2}{2}\|\Gamma_0\|\,\|y_0 - x_0\|\,\|\Gamma_0\|\,\|w_0 - x_0\|^2 \\
&\quad + \frac{3K^3}{4}\,\|\Gamma_0\|\,\|y_0 - x_0\|\,\|\Gamma_0\|\,\|y_0 - x_0\|\,\|\Gamma_0\|\,\|w_0 - x_0\|^2 \\
&\leq \eta(\frac{1}{2}a_0 + \frac{1}{2}a_0^2 + \frac{5}{8}a_0^3) + \frac{K\beta\eta^2}{2}(1 + \frac{1}{2}(a_0 + a_0^2 + \frac{5}{4}a_0^3))^2 \\
&\quad + \frac{K^2\beta^2\eta^3}{2}(1 + \frac{1}{2}(a_0 + a_0^2 + \frac{5}{4}a_0^3))^2 + \frac{3K^3\beta^3\eta^4}{4}(1 + \frac{1}{2}(a_0 + a_0^2 + \frac{5}{4}a_0^3))^2 \\
&\leq \eta(\frac{1}{256}(256 + 384a_0 + 640a_0^2 + 576a_0^3 + 576a_0^4 + 528a_0^5 + 298a_0^6 + 170a_0^7 + 75a_0^8)).
\end{aligned}
\tag{2.19}
$$

Then, using (2.17)–(2.19), the $\|F(x_1)\|$ is bounded

$$
\begin{aligned}
\|F(x_1)\| \leq\ & \frac{1}{2}K\eta^2 + K\eta^2(\frac{1}{256}(256 + 384a_0 + 640a_0^2 + 576a_0^3 \\
& + 576a_0^4 + 528a_0^5 + 298a_0^6 + 170a_0^7 + 75a_0^8)) \\
& + \frac{1}{\beta}\eta(\frac{1}{256}(256 + 384a_0 + 640a_0^2 + 576a_0^3 + 576a_0^4 \\
& + 528a_0^5 + 298a_0^6 + 170a_0^7 + 75a_0^8)) \\
& + \frac{1}{2}K\eta^2(\frac{1}{256}(256 + 384a_0 + 640a_0^2 + 576a_0^3 + 576a_0^4 \\
& + 528a_0^5 + 298a_0^6 + 170a_0^7 + 75a_0^8))^2.
\end{aligned}
\tag{2.20}
$$

Therefore, by applying $(O1_1)$, we get

$$
\begin{aligned}
\|y_1 - x_1\| &= \|\Gamma_1 F(x_1)\| = f(a_0)\|\Gamma_0\|\,\|F(x_1)\| \\
&\leq f(a_0)[\frac{1}{131072}a_0(196608 + 327680a_0 + 589824a_0^2 + 819200a_0^3 + 1064960a_0^4 \\
&\quad + 1351680a_0^5 + 1569792a_0^6 + 1689600a_0^7 + 1752576a_0^8 + 1693696a_0^9 \\
&\quad + 1490432a_0^{10} + 1226752a_0^{11} + 913920a_0^{12} + 596928a_0^{13} \\
&\quad + 354724a_0^{14} + 180520a_0^{15} + 73600a_0^{16} + 25500a_0^{17} + 5625a_0^{18})]\eta.
\end{aligned}
\tag{2.21}
$$

That is,

$$
\|y_1 - x_1\| = f(a_0)g(a_0)\eta \leq f(a_0)g(a_0)\|y_0 - x_0\|,
$$

where,

$$
\begin{aligned}
g(x) = \frac{x}{131072}&(196608 + 327680x + 589824x^2 + 819200x^3 + 1064960x^4 \\
&+ 1351680x^5 + 1569792x^6 + 1689600x^7 + 1752576x^8 + 1693696x^9 \\
&+ 1490432x^{10} + 1226752x^{11} + 913920x^{12} + 596928x^{13} \\
&+ 354724x^{14} + 180520x^{15} + 73600x^{16} + 25500x^{17} + 5625x^{18}).
\end{aligned}
$$

$(O3_1)$: Using $(O1_1)$ and $(O2_1)$,

$$
K\|\Gamma_1\|\,\|y_1 - x_1\| \leq Kf(a_0)\Gamma_0 f(a_0)g(a_0)\|y_0 - x_0\| = a_0 f(a_0)^2 g(a_0) = a_1.
$$

$(O4_1)$: For $k = 1$ it has been proven in (2.16).

The proof of $(O1_{k+1})$, $(O2_{k+1})$, $(O3_{k+1})$ and $(O4_{k+1})$ is based on the same method of proving that the inductive assumption with $(O1_k)$, $(O2_k)$, $(O3_k)$ and $(O4_k)$ as $k \geq 1$ holds true.

3. Semilocal convergence analysis

According to the convergence property of x_k sequence in Banach space, we need to prove that this sequence is a Cauchy sequence. Based on the auxiliary function, we can obtain the following results.

Lemma 1. According $h(x)$, $f(x)$ and $g(x)$, we have:
i. $f(x)$ is inceasing and $f(x) > 1$ for $x \in (0, 0.45807)$,
ii. $h(x)$ and $g(x)$ are increasing for $x \in (0, 0.45807)$.
The above lemma can be calculated from the Section 2, and the process is omitted.

Lemma 2. The $f(x)$ and $g(x)$ defined by (2.3) and (2.4). Then
i. $f(a_0)g(a_0) < 1$ for $a_0 < 0.252232$,
ii. $f(a_0)^2 g(a_0) < 1$ for $a_0 < 0.21715$,
iii. the sequence a_k is decreasing and $a_k < 0.21715$ for $k > 0$.

Proof. It is straightforword that *i*, *ii* are satisfied. As $f(a_0)^2 g(a_0) < 1$, then by construction of a_k, it is a dereasing sequence. So $a_k < a_0 \leq 0.21715$, for all $k \geq 1$.

Theorem 2. Let X, Y be Banach spaces and $F : \Omega \subseteq X \to Y$ be a nonlinear twice differentiable Fréchet operator in an open set domain Ω. Assume that $\Gamma_0 = [F'(x_0)]^{-1}$ exists in $x_0 \in \Omega$ and meet the conditions of $(M_1) - (M_3)$. Let be $a_0 = K\beta\eta$, and assume that $a_0 < 0.21$. The sequence $\{x_k\}$ defined in (2.1) and starting in x_0 converges to the solution x^* of $F(x) = 0$, if $B_e(x_0, R\eta) = x \in X : \|x - x_0\| < R\eta \subset \Omega$ where $R = \frac{h(a_0)}{1 - f(a_0)g(a_0)}$. In the case, the iterates $\{x_k\}$ and $\{y_k\}$ are contained in $B_e(x_0, R\eta)$ and $x^* \in B_e(x_0, R\eta)$. In addition, the x^* is the only solution of equation $F(x) = 0$ in $B_n(x_0, \frac{2}{K\beta} - R\eta) \cap \Omega$.

Proof. By recursively applying $(O4_k)$, we can write

$$\begin{aligned}
\|x_{k+1} - x_k\| &\leq h(a_k)\|y_k - x_k\| \\
&\leq h(a_k) f(a_{k-1}) g(a_{k-1}) \|y_{k-1} - x_{k-1}\| \\
&\leq \cdots \\
&\leq h(a_k)[\prod_{i=0}^{k-1} f(a_i)g(a_i)]\|y_0 - x_0\|.
\end{aligned} \tag{3.1}$$

Then,

$$\begin{aligned}
\|x_{k+m} - x_k\| \leq & \|x_{k+m} - x_{k+m-1}\| + \|x_{k+m-1} - x_{k+m-2}\| + \cdots + \|x_{k+1} - x_k\| \\
& \leq h(a_{k+m-1})\eta \prod_{i=0}^{k+m-2} f(a_i)g(a_i) \\
& + h(a_{k+m-2})\eta \prod_{i=0}^{k+m-3} f(a_i)g(a_i) \\
& + \cdots
\end{aligned}$$

$$+ h(a_k)\eta \prod_{i=0}^{k-1} f(a_i)g(a_i). \tag{3.2}$$

As $h(x)$ is increasing and a_k dreasing, it can be stated that

$$\|x_{k+m} - x_k\| \leq h(a_k)\eta \sum_{l=0}^{m-1}[\prod_{i=0}^{k+l-1} f(a_i)g(a_i)] \leq h(a_k)\eta \sum_{l=0}^{m-1}(f(a_0)g(a_0))^{l+k}. \tag{3.3}$$

Moreover, according Lemmas 1 and 2, by using the expression for the partial sum of a geometrical series,

$$\|x_{k+m} - x_k\| \leq h(a_k)\frac{1-(f(a_0)g(a_0))^m}{1-f(a_0)g(a_0)}(f(a_0)g(a_0))^k\eta. \tag{3.4}$$

So, the Cauchy sequence if and only if $f(a_0)g(a_0) < 1$ (Lemma 2).

For $k = 0$,

$$\begin{aligned}\|x_m - x_0\| &\leq \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + \cdots + \|x_1 - x_0\| \\ &\leq h(a_0)\|y_0 - x_0\| \sum_{r=0}^{m-1}(f(a_0)g(a_0))^r. \\ &= h(a_0)\frac{1-(f(a_0)g(a_0))^m}{1-f(a_0)g(a_0)}\eta < R\eta,\end{aligned} \tag{3.5}$$

when $m \to \infty$, we get the radius od convergence $R\eta = \frac{h(a_0)}{1-f(a_0)g(a_0)}\eta$.

Let's prove that x^* is the solution of $F(x) = 0$ starting from the boundary of $\|F'(x_n)\|$,

$$\begin{aligned}\|F'(x_k)\| &\leq \|F'(x_0)\| + \|F'(x_k) - F'(x_0)\| \\ &\leq \|F'(x_0)\| + K\|x_k - x_0\| \\ &\leq \|F'(x_0)\| + KR\eta.\end{aligned} \tag{3.6}$$

Then, acorrding M_2 and (3.1)

$$\begin{aligned}\|F(x_k)\| &\leq \|F'(x_k)\| \, \|y_k - x_k\| \\ &\leq \|F'(x_k)\| h(a_k)[\prod_{i=0}^{n-1} f(a_i)g(a_i)]\eta,\end{aligned} \tag{3.7}$$

as $h(x)$, $f(x)$ and $g(x)$ are increasing and a_k is the decreasing sequence,

$$\|F(x_k)\| \leq \|F'(x_k)\| h(a_k)(f(a_0)g(a_0))^k\eta. \tag{3.8}$$

Taking into account that $\|F'(x_k)\|$ is bounded and $(f(a_0)g(a_0))^k$ tends to zero when $k \to \infty$, we conclude that $\|F(x_k)\| \to 0$. As F is continuous in Ω, then $F(x^*) = 0$.

Finally, the uniqueness of x^* in $B(x_0, \frac{2}{K\beta} - R\eta) \cap \Omega$.

$$\begin{aligned}0 = F(y^*) - F(x^*) &= (F(x^*) + \int_0^1 F'(x^* + t(y^* - x^*))(y^* - x^*)dt) - (F(x^*) \\ &= (y^* - x^*)\int_0^1 F'(x^* + t(y^* - x^*))dt).\end{aligned} \tag{3.9}$$

In order to guarantee that $y^* - x^* = 0$ it is necesssary to prove that operator $\int_0^1 F'(x^* + t(y^* - x^*))dt$ is invertible. Applying hypothesis (M_3),

$$\begin{aligned}
&||\Gamma_0|| \int_0^1 ||F'(x^* + t(y^* - x^*)) - F'(x_0)||dt \\
&\leq K\beta \int_0^1 ||x^* + t(y^* - x^*) - x_0 dt|| \\
&\leq K\beta \int_0^1 ((1-t)||x^* - x_0|| + t||y^* - x_0||)dt \\
&< \frac{K\beta}{2}(R\eta + \frac{2}{K\beta} - R\eta) = 1.
\end{aligned} \tag{3.10}$$

By the Banach lemma, the intergal operator is invertible and hence $y^* = x^*$.

4. Numerical results

In this section, we provide some numerical examples to illustrate the theoretical results introduced earlier.

Example 1. Hammerstein equation is a kind of important nonlinear integral equation [24], which is given as follows:

$$x(s) = 1 + (1/5)\int_0^1 N(s,t)x(t)^3 dt, \tag{4.1}$$

where $x \in \mathbb{C}[0,1]$, $s, t \in [0,1]$, with the kernel N is

$$N(s,t) = \begin{cases} (1-s)t & t \leq s, \\ s(1-t) & s \leq t. \end{cases}$$

To solve (4.1) we transform it into a syste of nonlinear equations through a discretization process. We approximate the integral appearing in Eq (4.1) by using Gauss-Legendre quadrature,

$$\int_0^1 s(t)dt \approx \sum_{i=1}^{7} w_j s(t_j),$$

being t_j and w_j the nodes and the weights of the Gauss-Legendre polynomial. Denoting the approximation of $x(t_j)$ as $x_i, i = 1, ..., 7$, then we estimate (4.1) with the nonlinear system of equations

$$x_i - 1 - \frac{1}{5}\sum_{j=1}^{7} a_{ij}x_j^3 = 0, i = 1, ..., 7 \tag{4.2}$$

where

$$a_{ij} = \begin{cases} w_j t_j (1-t_i) & j \leq i, \\ w_j t_i (1-t_j) & i < j. \end{cases}$$

So,the system can be rewritten as

$$F(x) = x - 1 - \frac{1}{5}Av_x, \quad v_x = (x_1^3, x_2^3, ..., x_7^3)^T,$$

$$F^{'}(x) = I - \frac{3}{5}AD(x), \quad D(x) = diag(x_1^2, x_2^2, ..., x_7^2),$$

where F if a nonlinear operator in the Banach space $\mathbb{R}^L$,and $F^{'}$ is its Fréchet derivative in $\mathcal{L}(\mathbb{R}^L, \mathbb{R}^L)$.

According the method (1.2), we will use it to solve the nonlinear systems.

Taking $x_0 = (1.8, 1.8, ..., 1.8)^T, L = 7$ and the infinity norm, we get

$$||\Gamma_0|| \leq \beta, \quad \beta \approx 1.2559,$$

$$||\Gamma_0 F(x_0)|| \leq \eta, \quad \eta \approx 2.2062,$$
$$||F^{'}(x) - F^{'}(y)|| \leq k||x - y||, \quad k \approx 0.0671,$$
$$a_0 = k\beta\eta, \quad \approx 0.1860. \tag{4.3}$$

The above results satisfy the semilocal convergence condition, so this method can be applied to the system. Thus, we guarantee the existence of the solution in $B_e(x_0, 0.5646)$, and the uniqueness in $B_n(x_0, 22.4874)$. Table 1 shows the the radius of the existence domain and the radius of the unique domain under different initial values. For $x_{0i} > 1.87, i = 1, 2, ..., 7$, convergence conditions are not satisfied and, therefore, the convergence is not guaranteed.

Table 1. Different initial values related parameters.

x_{0i}	β	η	k	a_0	R_e	R_n
0.2	1.0025	2.1204	0.0287	0.0610	0.0754	69.3528
0.4	1.0102	1.6005	0.0337	0.0545	0.0657	58.6428
0.6	1.0232	1.0827	0.0390	0.0433	0.0500	50.0651
0.8	1.0420	0.5637	0.0451	0.0265	0.0288	42.5422
1.0	1.0671	0.0461	0.0682	0.0034	0.0034	27.4813
1.2	1.0996	0.4949	0.0505	0.0275	0.0300	36.0019
1.4	1.1406	1.0420	0.0569	0.0676	0.0859	30.7271
1.6	1.1919	1.6098	0.0622	0.1193	0.1970	26.6603
1.7	1.2222	1.9038	0.0647	0.1505	0.3107	24.7005

Using the iterative method (1.2) to solve (4.2), the exact solution is

$$x^* = (1.003, 1.012, 1.023, 1.028, 1.023, 1.012, 1.003)^T.$$

Example 2. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}^2$ be equipped with the max-norm. Choose: $x_0 = (0.9, 0.9)^T$, $s \in [\ 0, \frac{1}{2}\ )$. Let $s = 0.49$, define function F by

$$F(x) = (x_1^3 - s, x_2^3 - s)^T, \quad x = (x_1, x_2)^T. \tag{4.4}$$

The fréchet-derivative of operator F is given by

$$F^{'}(x) = \begin{bmatrix} 3x_1^2 & 0 \\ 0 & 3x^2 \end{bmatrix}.$$

Taking $x_0 = (0.9, 0.9)^T$ and the infinity norm, we get

$$\|\Gamma_0\| \leq \beta, \quad \beta \approx 0.4115,$$

$$\|\Gamma_0 F(x_0)\| \leq \eta, \quad \eta \approx 0.1391,$$
$$\|F^{'}(x) - F^{'}(y)\| \leq k\|x - y\|, \quad k \approx 3.6113,$$
$$a_0 = k\beta\eta, \quad \approx 0.2067. \tag{4.5}$$

The convergence conditions are met and consequently the method can be applied to the system. The existence domain of the solution is $B_e(x_0, 0.9101)$, and the uniqueness domain is $B_n(x_0, 1.2192)$.

Taking $x_0 = (0.73, 0.73)^T$ and the infinity norm, then

$$\|\Gamma_0\| \leq \beta, \quad \beta \approx 0.6255,$$

$$\|\Gamma_0 F(x_0)\| \leq \eta, \quad \eta \approx 0.0893,$$
$$\|F^{'}(x) - F^{'}(y)\| \leq k\|x - y\|, \quad k \approx 3.2329,$$
$$a_0 = k\beta\eta, \quad \approx 0.1806. \tag{4.6}$$

The existence domain of the solution is $B_e(x_0, 0.5095)$, and the uniqueness domain is $B_n(x_0, 0.943534)$.

When the initial value satisfies the Kantorovich condition and the range of a_0 obtained, the initial value within that range is taken to solve the system. Iterative method (1.2) for solving nonlinear (4.4) with roots of $x^* = (0.7884, 0.7884)^T$.

Similar results can be obtained in Tables 2 and 3, that is, under the Kantorovich condition, by selecting different initial values, we can converge to a unique solution. When the initial value is closer to the root, the error estimate is lower. This semilocal convergence that can prove the existence and uniqueness of solutions under certain assumptions is very valuable.

Table 2. Numberical results of method (1.2) for nonliner equation.

x_{0i}	iter	$\|x_k - x_{k-1}\|$	$\|F(x_k)\|$
0.2	4	7.469e-336	2.149e-2021
0.4	4	2.538e-352	4.629e-2120
0.6	4	1.755e-383	8.222e-2307
0.8	4	5.848e-445	2.318e-2675
1.0	4	2.629e-701	3.000e-4096
1.2	4	2.221e-467	5.991e-2809
1.4	4	1.935e-353	8.489e-2126
1.6	4	8.010e-286	2.379e-1720
1.7	4	7.450e-259	1.285e-1558

Table 3. Numberical results of method (1.2) for nonliner equation.

x_{0i}	iter	$\|\|x_k - x_{k-1}\|\|$	$\|\|F(x_k)\|\|$	ρ
0.72	4	4.048e-331	1.878e-2640	8
0.74	4	2.665e-419	6.432e-3346	8
0.76	4	1.046e-548	3.726e-4381	8
0.78	4	2.052e-830	1.000e-6000	8
0.8	4	2.246e-767	1.000e-6000	8
0.82	4	2.005e-554	6.803e-4427	8
0.84	4	1.127e-454	6.768e-3629	8
0.86	4	9.380e-391	1.559e-3117	8
0.88	4	1.414e-344	4.163e-2748	8
0.9	4	5.796e-309	3.313e-2463	8

5. Conclusions

In this paper, the semilocal convergence of the eighth order iterative method (1.2) is studied. By analyzing the behavior of the iterative method under the Kantorovich condition, the Lipschitz condition is applied to the first derivative, and the theory of semilocal convergence of the iterative method is obtained by using the recurrence relation. The existence and uniqueness domain of the solution of the nonlinear system is obtained. In the experimental part, a classical Hammerstein nonlinear integral equation and a matrix function are solved. The experimental results are consistent with expectations, and the high-precision approximation of the system solution also proves the effectiveness of the method numerically.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. J. M. Ortega, W. C. Rheinboldt, *Iterative solution of nonlinear equations in several variables*, New York: Academic Press, 1970. https://doi.org/10.1016/C2013-0-11263-9
2. R. Behl, S. Bhalla, Á. A. Magrenán, S. Kumar, An efficient high order iterative scheme for large nonlinear systems with dynamics, *J. Comput. Appl. Math.*, **404** (2022), 113249. http://dx.doi.org/10.1016/j.cam.2020.113249
3. C. Chun, B. Neta, Developing high order methods for the solution of systems of nonlinear equations, *Appl. Math. Comput.*, **342** (2019), 178–190. http://dx.doi.org/10.1016/j.amc.2018.09.032
4. X. Wang, Y. Cao, A numerically stable high-order Chebyshev-Halley type multipoint iterative method for calculating matrix sign function, *AIMS Math.*, **8** (2023), 12456–12471. http://dx.doi.org/10.3934/math.2023625
5. X. Wang, W. Li, Stability analysis of simple root seeker for nonlinear equation, *Axioms*, **12** (2023), 215. https://doi.org/10.3390/axioms12020215
6. T. Zhanlav, K. Otgondorj, Higher order Jarratt-like iterations for solving systems of nonlinear equations, *Appl. Math. Comput.*, **395** (2021), 125849. https://dx.doi.org/10.1016/j.amc.2020.125849
7. S. Regmi, *Optimized iterative methods with applications in diverse disciplines*, New York: Nova Science Publishers, Inc., 2021.
8. B. Neta, A new derivative-free method to solve nonlinear equations, *Mathematics*, **9** (2021), 583. http://dx.doi.org/10.3390/math9060583
9. C. Chun, B. Neta, An efficient derivative-free method for the solution of systems of equations, *Numer. Func. Anal. Opt.*, **42** (2021), 834–848. http://dx.doi.org/10.1080/01630563.2021.1931313
10. R. Behl, A. Cordero, J. R. Torregrosa, A new higher-order optimal derivative free scheme for multiple roots, *J. Comput. Appl. Math.*, **404** (2022), 113773. http://dx.doi.org/10.1016/j.cam.2021.113773
11. M. Kansal, A. S. Alshomrani, S. Bhalla, R. Behl, M. Salimi, One parameter optimal derivative-free family to find the multiple roots of algebraic nonlinear equations, *Mathematics*, **8** (2020), 2223. http://dx.doi.org/10.3390/math8122223
12. J. R. Sharma, S. Kumar, L. Jantschi, On derivative free multiple-root finders with optimal fourth order convergence, *Mathematics*, **8** (2020), 1091. http://dx.doi.org/10.3390/math8071091
13. X. Wang, Fixed-point iterative method with eighth-order constructed by undetermined parameter technique for solving nonlinear systems, *Symmetry*, **13** (2021), 863. http://dx.doi.org/10.3390/sym13050863
14. I. K. Argyros, S. George, On the complexity of extending the convergence region for Traub's method, *J. Complexity*, **56** (2020), 101423. http://dx.doi.org/10.1016/j.jco.2019.101423
15. I. K. Argyros, S. George, Ball comparison between four fourth convergence order methods under the same set of hypotheses for solving equations, *Int. J. Appl. Comput. Math.*, **7** (2021), 9. http://dx.doi.org/10.1007/S40819-020-00946-8

16. I. K. Argyros, A new convergence theorem for the Steffenssen method in Banach space and applications, *Rev. Anal. Numér. Théor. Approx*, **29** (2000), 119–127.
17. A. Cordero, E. G. Villalba, J. R. Torregrosa, P. Triguero-Navarro, Convergence and stability of a parametric class of iterative schemes for solving nonlinear systems, *Mathematics*, **9** (2021), 86. http://dx.doi.org/10.3390/math9010086
18. I. K. Argyros, S. George, S. Shakhno, H. Yarmola, Perturbed Newton methods for solving nonlinear equations with applications, *Symmetry*, **14** (2022), 2206. http://dx.doi.org/10.3390/sym14102206
19. A. Cordero, J. G. Maimó, E. Martinez, J. R. Torregrosa, Semilocal convergence of the extension of Chun's method, *Axioms*, **10** (2021), 161. http://dx.doi.org/10.3390/axioms10030161
20. S. Amat, M. A. Hernández, N. Romero, Semilocal convergence of a sixth order iterative method for quadratic equations, *Appl. Numer. Math.*, **62** (2012), 833–841. http://dx.doi.org/10.1016/j.apnum.2012.03.001
21. A. Cordero, M. A. Hernández-Verón, N. Romero, J. R. Torregrosa, Semilocal convergence by using recurrence relations for a fifth-order method in Banach spaces, *J. Comput. Appl. Math.*, **273** (2015), 205–213. http://dx.doi.org/10.1016/j.cam.2014.06.008
22. M. A. Hernández-Verón, E. Martinez, C. Teruel, Semilocal convergence of a k-step iterative process and its application for solving a special kind of conservative problems, *Numer. Algor.*, **76** (2017), 309–331. http://dx.doi.org/10.1007/s11075-016-0255-z
23. V. Candela, A. Marquina, Recurrence relations for rational cubic models II: the Chebyshev method, *Computing*, **45** (1990), 355–367. http://dx.doi.org/10.1007/BF02238803
24. J. A. Ezquerro, M. A. Hernández-Verón, Halley's method for operators with unbounded second derivative, *Appl. Numer. Math.*, **57** (2007), 354–360. http://dx.doi.org/10.1016/j.apnum.2006.05.001

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