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Research article

On a family of nonlinear difference equations of the fifth order solvable in closed form

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Abstract: We present some closed-form formulas for the general solution to the family of difference equations

$$x_{n+1} = \Phi^{-1}\left(\Phi(x_{n-1})\frac{\alpha\Phi(x_{n-2}) + \beta\Phi(x_{n-4})}{\gamma\Phi(x_{n-2}) + \delta\Phi(x_{n-4})}\right),$$

for $n \in \mathbb{N}_0$ where the initial values x_{-j}, $j = \overline{0,4}$ and the parameters α, β, γ and δ are real numbers satisfying the conditions $\alpha^2 + \beta^2 \neq 0$, $\gamma^2 + \delta^2 \neq 0$ and Φ is a function which is a homeomorphism of the real line such that $\Phi(0) = 0$, generalizing in a natural way some closed-form formulas to the general solutions to some very special cases of the family of difference equations which have been presented recently in the literature. Besides this, we consider in detail some of the recently formulated statements in the literature on the local and global stability of the equilibria as well as on the boundedness character of positive solutions to the special cases of the difference equation and give some comments and results related to the statements.

Keywords: difference equations; solvable equation; closed-form formula for solutions; bilinear difference equation
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1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural, whole, real and complex numbers respectively and let $\mathbb{N}_l := \{m \in \mathbb{Z} : m \geq l\}$ where $l \in \mathbb{Z}$. We use the notation $i = \overline{m,n}$ where $m, n \in \mathbb{Z}$ are such that $m \leq n$ which is the same as the expression: $m \leq i \leq n$, $i \in \mathbb{Z}$. It is also understood that $\prod_{i=m}^{m-1} c_i = 1$ for any $m \in \mathbb{Z}$ where c_i are some numbers.

Solvability of difference equations, systems of difference equations and partial difference equations and systems has been investigated for a long time. For some of the oldest sources in the topic, consult e.g., [5, 7, 15–18] where many closed-form formulas for the general solutions to the equations and systems can be found. Here, we mention a few of them which are employed in the proofs of our results.

The difference equation

$$v_{n+2} - a_1 v_{n+1} - a_0 v_n = 0, \tag{1.1}$$

for $n \in \mathbb{N}_0$ where $a_1 \in \mathbb{R}$ and $a_0 \in \mathbb{R} \setminus \{0\}$ was solved in [5, 7] where it was shown that if $a_1^2 + 4a_0 \neq 0$ the general solution to Eq (1.1) is

$$v_n = \frac{(v_1 - s_2 v_0)s_1^n - (v_1 - s_1 v_0)s_2^n}{s_1 - s_2}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where s_j, $j = 1, 2$ are the roots of the polynomial

$$Q(s) := s^2 - a_1 s - a_0$$

and if $a_1^2 + 4a_0 = 0$ the general solution to Eq (1.1) is

$$v_n = ((v_1 - s_1 v_0)n + s_1 v_0)s_1^{n-1}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

where $s_1 = \frac{a_1}{2}$.

Some methods for solving the bilinear difference equation

$$x_{n+1} = \frac{\alpha x_n + \beta}{\gamma x_n + \delta}, \quad n \in \mathbb{N}_0, \tag{1.4}$$

have been also known for a long time (see, e.g., [14–16, 19–21, 36]). For some results on the behaviour of the solutions to Eq (1.4) and related equations and topics see, e.g., [1, 6, 21, 35, 36, 42].

There has been also some recent interest in solvability and invariants of difference equations, systems of difference equations and their applications; see, e.g., [10, 23, 24, 26–29, 32, 34–42] and the related references therein.

The difference equation of the fifth order

$$x_{n+1} = a x_{n-1} + \frac{b x_{n-1} x_{n-4}}{c x_{n-4} + d x_{n-2}}, \quad n \in \mathbb{N}_0, \tag{1.5}$$

where the parameters a, b, c, d and the initial values x_{-j}, $j = \overline{0,4}$ are real numbers has been recently investigated in [8] where some closed-form formulas for its solutions in the following four special cases: 1) $a = b = c = d = 1$; 2) $a = b = c = 1$, $d = -1$; 3) $a = c = 1$, $b = d = -1$; 4) $a = c = d = 1$, $b = -1$, are given, and where some statements on local and global stability of the solutions to Eq (1.5) as well as on their boundedness are formulated.

First, we show that a general family of nonlinear difference equations of the fifth order is solvable in closed form from which the closed-form formulas for solutions to the equations in above mentioned four special cases easily follow. To do this we find some closed-form formulas for the general solution to the family of nonlinear difference equations by applying some of the ideas and tricks in [10, 34–39]. Second, we consider in detail some of the formulated statements in [8] on the local and global stability of the equilibria of Eq (1.5) as well as on the boundedness character of positive solutions to some special cases of the equation and provide some counterexamples showing that the statements are not correct.

2. Solvability of a generalization to Eq (1.5)

Equation (1.5) is a special case of the equation

$$x_{n+1} = \Phi^{-1}\left(\Phi(x_{n-1})\frac{\alpha\Phi(x_{n-2}) + \beta\Phi(x_{n-4})}{\gamma\Phi(x_{n-2}) + \delta\Phi(x_{n-4})}\right), \quad n \in \mathbb{N}_0. \tag{2.1}$$

Indeed, for

$$\Phi(x) \equiv x, \quad \alpha = ad, \quad \beta = ac + b, \quad \gamma = d \quad \text{and} \quad \delta = c, \tag{2.2}$$

from Eq (2.1) is obtained Eq (1.5).

The following theorem shows the solvability of Eq (2.1) when Φ be a homeomorphism (for the notion and some basics see, e.g., [43]).

Theorem 1. *Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $\alpha^2 + \beta^2 \neq 0 \neq \gamma^2 + \delta^2$, Φ be a homeomorphism of $\mathbb{R}$ and $\Phi(0) = 0$. Then, Eq (2.1) is solvable in closed form.*

Proof. First, note that if there is $n_0 \in \mathbb{N}_{-1}$ such that

$$x_{n_0} = 0 \tag{2.3}$$

then if the solution is defined for all $n \in \mathbb{N}_{-4}$, we have

$$x_{n_0+2} = 0. \tag{2.4}$$

Relations (2.1), (2.3) and (2.4) imply that x_{n_0+5} is not defined which is a contradiction.

Hence, from now on we consider only solutions to Eq (2.1) such that

$$x_n \neq 0, \quad \text{for} \quad n \in \mathbb{N}_{-4}.$$

Note that for such solutions we have

$$\Phi(x_n) \neq 0, \quad \text{for} \quad n \in \mathbb{N}_{-4}.$$

Hence, the following change of variables can be used

$$y_n = \frac{\Phi(x_n)}{\Phi(x_{n-2})}, \quad n \in \mathbb{N}_{-2}, \tag{2.5}$$

from which together with the conditions posed on function Φ we have

$$y_{n+1} = \frac{\alpha y_{n-2} + \beta}{\gamma y_{n-2} + \delta}, \quad n \in \mathbb{N}_0. \tag{2.6}$$

Let

$$z_m^{(j)} = y_{3m-j}, \quad m \in \mathbb{N}_0,\ j = \overline{0,2}.$$

Then,

$$z_{m+1}^{(j)} = \frac{\alpha z_m^{(j)} + \beta}{\gamma z_m^{(j)} + \delta}, \quad m \in \mathbb{N}_0,\ j = \overline{0,2}.$$

Furthermore, let

$$z_m^{(j)} = \frac{u_{m+1}^{(j)}}{u_m^{(j)}} - \frac{\delta}{\gamma}, \quad m \in \mathbb{N}_0,\ j = \overline{0,2}, \tag{2.7}$$

then after some simple calculations it follows that

$$\gamma^2 u_{m+2}^{(j)} - \gamma(\alpha + \delta) u_{m+1}^{(j)} + (\alpha\delta - \beta\gamma) u_m^{(j)} = 0,$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$.

Suppose

$$\alpha\delta \neq \beta\gamma, \quad \gamma \neq 0 \quad \text{and} \quad (\alpha + \delta)^2 \neq 4(\alpha\delta - \beta\gamma). \tag{2.8}$$

Then, the de Miovre formula (1.2) implies

$$u_m^{(j)} = \frac{(u_1^{(j)} - s_2 u_0^{(j)}) s_1^m - (u_1^{(j)} - s_1 u_0^{(j)}) s_2^m}{s_1 - s_2}, \tag{2.9}$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$ where

$$s_1 = \frac{\alpha + \delta + \sqrt{(\alpha + \delta)^2 - 4(\alpha\delta - \beta\gamma)}}{2\gamma}$$

and

$$s_2 = \frac{\alpha + \delta - \sqrt{(\alpha + \delta)^2 - 4(\alpha\delta - \beta\gamma)}}{2\gamma}.$$

From (2.7) and (2.9), we have

$$z_m^{(j)} = \frac{(z_0^{(j)} - s_2 + \frac{\delta}{\gamma}) s_1^{m+1} - (z_0^{(j)} - s_1 + \frac{\delta}{\gamma}) s_2^{m+1}}{(z_0^{(j)} - s_2 + \frac{\delta}{\gamma}) s_1^m - (z_0^{(j)} - s_1 + \frac{\delta}{\gamma}) s_2^m} - \frac{\delta}{\gamma},$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$ and consequently

$$
\begin{aligned}
y_{3m-j} =& \frac{(y_{-j} - s_2 + \frac{\delta}{\gamma})s_1^{m+1} - (y_{-j} - s_1 + \frac{\delta}{\gamma})s_2^{m+1}}{(y_{-j} - s_2 + \frac{\delta}{\gamma})s_1^{m} - (y_{-j} - s_1 + \frac{\delta}{\gamma})s_2^{m}} - \frac{\delta}{\gamma} \\
=& \frac{(\frac{\Phi(x_{-j})}{\Phi(x_{-j-2})} - s_2 + \frac{\delta}{\gamma})s_1^{m+1} - (\frac{\Phi(x_{-j})}{\Phi(x_{-j-2})} - s_1 + \frac{\delta}{\gamma})s_2^{m+1}}{(\frac{\Phi(x_{-j})}{\Phi(x_{-j-2})} - s_2 + \frac{\delta}{\gamma})s_1^{m} - (\frac{\Phi(x_{-j})}{\Phi(x_{-j-2})} - s_1 + \frac{\delta}{\gamma})s_2^{m}} - \frac{\delta}{\gamma},
\end{aligned} \tag{2.10}
$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$.

From (2.5) and (2.10) we obtain

$$\Phi(x_{6m-j}) = y_{6m-j}y_{6m-j-2}y_{6m-j-4}\Phi(x_{6(m-1)-j}), \tag{2.11}$$

for $m \in \mathbb{N}$, $j = \overline{-1,4}$, from which we get the general solution to Eq (2.1) under the assumptions in (2.8)

$$x_{6m-4} = \Phi^{-1}\left(\Phi(x_{-4})\prod_{i=1}^{m} y_{6i-4}y_{6i-6}y_{6i-8}\right), \tag{2.12}$$

$$x_{6m-3} = \Phi^{-1}\left(\Phi(x_{-3})\prod_{i=1}^{m} y_{6i-3}y_{6i-5}y_{6i-7}\right), \tag{2.13}$$

$$x_{6m-2} = \Phi^{-1}\left(\Phi(x_{-2})\prod_{i=1}^{m} y_{6i-2}y_{6i-4}y_{6i-6}\right), \tag{2.14}$$

$$x_{6m-1} = \Phi^{-1}\left(\Phi(x_{-1})\prod_{i=1}^{m} y_{6i-1}y_{6i-3}y_{6i-5}\right), \tag{2.15}$$

$$x_{6m} = \Phi^{-1}\left(\Phi(x_{0})\prod_{i=1}^{m} y_{6i}y_{6i-2}y_{6i-4}\right), \tag{2.16}$$

$$x_{6m+1} = \Phi^{-1}\left(\Phi(x_{1})\prod_{i=1}^{m} y_{6i+1}y_{6i-1}y_{6i-3}\right), \tag{2.17}$$

for $m \in \mathbb{N}_0$ where $(y_n)_{n\geq -2}$ is given by (2.10).

Suppose

$$\alpha\delta \neq \beta\gamma, \quad \gamma \neq 0 \quad \text{and} \quad (\alpha+\delta)^2 = 4(\alpha\delta - \beta\gamma). \tag{2.18}$$

Then, (1.3) implies

$$u_m^{(j)} = ((u_1^{(j)} - s_1u_0^{(j)})m + s_1u_0^{(j)})s_1^{m-1}, \tag{2.19}$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$ where

$$s_1 = \frac{\alpha+\delta}{2\gamma} \neq 0.$$

Relations (2.7) and (2.19) imply

$$z_m^{(j)} = \frac{((z_0^{(j)} - s_1 + \frac{\delta}{\gamma})(m+1) + s_1)s_1}{(z_0^{(j)} - s_1 + \frac{\delta}{\gamma})m + s_1} - \frac{\delta}{\gamma},$$

and consequently

$$y_{3m-j} = \frac{((y_{-j} - s_1 + \frac{\delta}{\gamma})(m+1) + s_1)s_1}{(y_{-j} - s_1 + \frac{\delta}{\gamma})m + s_1} - \frac{\delta}{\gamma}$$
$$= \frac{(\frac{\Phi(x_{-j})}{\Phi(x_{-j-2})} - s_1 + \frac{\delta}{\gamma})(m+1) + s_1)s_1}{(\frac{\Phi(x_{-j})}{\Phi(x_{-j-2})} - s_1 + \frac{\delta}{\gamma})m + s_1} - \frac{\delta}{\gamma}, \tag{2.20}$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$.

Hence, the general solution to Eq (2.1), under the assumptions in (2.18), is given by (2.12)–(2.17) whereas $(y_n)_{n\geq -2}$ is given by (2.20).

Suppose $\gamma = 0$. Then, Eq (2.6) is

$$y_{n+1} = \frac{\alpha}{\delta} y_{n-2} + \frac{\beta}{\delta}, \quad n \in \mathbb{N}_0, \tag{2.21}$$

so that

$$z_{m+1}^{(j)} = \frac{\alpha}{\delta} z_m^{(j)} + \frac{\beta}{\delta}, \quad m \in \mathbb{N}_0, \; j = \overline{0,2}. \tag{2.22}$$

If $\alpha = \delta$ then

$$z_m^{(j)} = \frac{\beta}{\delta} m + z_0^{(j)}, \quad m \in \mathbb{N}_0, \; j = \overline{0,2},$$

that is

$$y_{3m-j} = \frac{\beta}{\delta} m + y_{-j} = \frac{\beta}{\delta} m + \frac{\Phi(x_{-j})}{\Phi(x_{-j-2})}, \quad m \in \mathbb{N}_0, \; j = \overline{0,2}. \tag{2.23}$$

Hence, the general solution to Eq (2.1) in this case is given by (2.12)–(2.17) whereas $(y_n)_{n\geq -2}$ is given by (2.23).

If $\alpha \neq \delta$ then by a Lagrange's formula [17], we have

$$z_m^{(j)} = \frac{\beta}{\alpha - \delta}\Big(\Big(\frac{\alpha}{\delta}\Big)^m - 1\Big) + \Big(\frac{\alpha}{\delta}\Big)^m z_0^{(j)},$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$, that is,

$$y_{3m-j} = \frac{\beta}{\alpha - \delta}\Big(\Big(\frac{\alpha}{\delta}\Big)^m - 1\Big) + \Big(\frac{\alpha}{\delta}\Big)^m y_{-j}$$
$$= \frac{\beta}{\alpha - \delta}\Big(\Big(\frac{\alpha}{\delta}\Big)^m - 1\Big) + \Big(\frac{\alpha}{\delta}\Big)^m \frac{\Phi(x_{-j})}{\Phi(x_{-j-2})}, \tag{2.24}$$

for $m \in \mathbb{N}_0$, $j = \overline{0,2}$.

Hence, the general solution in this case is given by (2.12)–(2.17), where $(y_n)_{n\geq -2}$ is given by (2.24).

Suppose $\alpha\delta = \beta\gamma$. If $\alpha = 0$ then $\beta \neq 0$, $\gamma = 0$ and $\delta \neq 0$. Thus,

$$x_{n+1} = \Phi^{-1}\Big(\frac{\beta}{\delta}\Phi(x_{n-1})\Big), \quad n \in \mathbb{N}_0, \tag{2.25}$$

and consequently

$$x_{2m-j}=\Phi^{-1}\left(\left(\frac{\beta}{\delta}\right)^m\Phi(x_{-j})\right),\quad m\in\mathbb{N}_0,\ j=0,1. \tag{2.26}$$

If $\alpha\neq 0$ and $\beta=0$ then $\delta=0$ and $\gamma\neq 0$. So, we have

$$x_{n+1}=\Phi^{-1}\Big(\frac{\alpha}{\gamma}\Phi(x_{n-1})\Big),\quad n\in\mathbb{N}_0, \tag{2.27}$$

and consequently

$$x_{2m-j}=\Phi^{-1}\left(\left(\frac{\alpha}{\gamma}\right)^m\Phi(x_{-j})\right),\quad m\in\mathbb{N}_0,\ j=0,1. \tag{2.28}$$

If $\delta=0$ then $\gamma\neq 0$, $\beta=0$ and $\alpha\neq 0$. So, (2.27) holds which implies (2.28). If $\gamma=0$ then $\delta\neq 0$, $\alpha=0$ and $\beta\neq 0$, so (2.25) holds which implies (2.26). Finally, if $\alpha\beta\gamma\delta\neq 0$ then $\alpha=\beta\gamma/\delta$. So, (2.25), that is, (2.27) holds and consequently (2.26), that is, (2.28). □

Remark 1. The closed form formulas obtained in Theorem 1 can be employed in investigating the boundedness character, convergence, asymptotics and other properties of solutions to Eq (2.1). We will not deal with this standard problem and leave it to the interested reader as some exercises. The problem can be dealt with by employing some methods, tricks and ideas appearing in [1–4, 6, 9, 11–13, 21, 22, 25, 27, 30–34, 37, 38].

3. On some statements on local and global stability of Eq (1.5)

The local and global stability of solutions to Eq (1.5) as well as their boundedness character have been recently considered in [8]. In this section we analyse the statements therein in detail and show that practically none of them is correct.

The equilibria of Eq (1.5) were first investigated therein. If $\bar{x}$ is an equilibrium then

$$\bar{x}=a\bar{x}+\frac{b\bar{x}^2}{(c+d)\bar{x}}. \tag{3.1}$$

From this they got the relation $\bar{x}^2(1-a)(c+d)=b\bar{x}^2$ and under the assumption $(c+d)(1-a)\neq b$, concluded that $\bar{x}=0$ is a unique equilibrium point.

However, they forgot to note that (3.1) implies $\bar{x}\neq 0$. So, the statement is not true as well as Theorem 3 which states that the (wrong) equilibrium $\bar{x}=0$ is locally asymptotically stable under a condition posed to the parameters a, b, c and d.

The next statement (Theorem 4 in [8]) is the following:

Statement 1. *If $c(1-a)\neq b$ then the unique equilibrium point of Eq (1.5) is globally asymptotically stable.*

As we have shown the relation $\bar{x}\neq 0$ must hold, so $\bar{x}=0$ is not an equilibrium of Eq (1.5) implying that the statement is not well formulated. It can happen that some solutions converge to something which is not an equilibrium. However, the second problem with the statement is that the conclusion is not correct. This we show by giving an example of Eq (1.5) possessing solutions which are even unbounded and consequently cannot converge to any finite real number.

Example 1. Let $a = c = 1/2$, $b = d = 1$. Then, Eq (1.5) becomes

$$x_{n+1} = x_{n-1}\frac{2x_{n-2} + 5x_{n-4}}{4x_{n-2} + 2x_{n-4}}, \quad n \in \mathbb{N}_0, \tag{3.2}$$

and the condition $c(1 - a) \neq b$ is satisfied.

We can apply Theorem 1 but for the benefit of the reader we repeat some of the steps in the proof of the theorem.

Using the change of variables

$$y_n = \frac{x_n}{x_{n-2}}, \quad n \geq -2, \tag{3.3}$$

we get

$$y_{n+1} = \frac{2y_{n-2} + 5}{4y_{n-2} + 2}, \quad n \in \mathbb{N}_0, \tag{3.4}$$

so the sequences $y_m^{(j)} = y_{3m-j}$, $m \in \mathbb{N}_0$, $j = \overline{0, 2}$, satisfy the equation

$$u_{m+1} = \frac{2u_m + 5}{4u_m + 2}, \quad m \in \mathbb{N}_0. \tag{3.5}$$

Let

$$y_m^{(j)} = \frac{z_{m+1}^{(j)}}{z_m^{(j)}} - \frac{1}{2}, \quad m \in \mathbb{N}_0, \ j = \overline{0, 2}. \tag{3.6}$$

Then,

$$z_{m+2}^{(j)} - z_{m+1}^{(j)} - z_m^{(j)} = 0, \quad j = \overline{0, 2}, \tag{3.7}$$

and by the de Moivre formula we have

$$z_m^{(j)} = \frac{(z_1^{(j)} - s_2 z_0^{(j)})s_1^m - (z_1^{(j)} - s_1 z_0^{(j)})s_2^m}{s_1 - s_2}, \tag{3.8}$$

for $m \in \mathbb{N}_0$, $j = \overline{0, 2}$ where $s_1 = \frac{1+\sqrt{5}}{2}$ and $s_2 = \frac{1-\sqrt{5}}{2}$ and consequently

$$y_m^{(j)} = \frac{(y_0^{(j)} - s_2 + \frac{1}{2})s_1^{m+1} - (y_0^{(j)} - s_1 + \frac{1}{2})s_2^{m+1}}{(y_0^{(j)} - s_2 + \frac{1}{2})s_1^m - (y_0^{(j)} - s_1 + \frac{1}{2})s_2^m} - \frac{1}{2}, \tag{3.9}$$

for $m \in \mathbb{N}_0$, $j = \overline{0, 2}$.

Assume that the initial values x_{-j}, $j = \overline{0, 4}$ are chosen such that

$$y_0^{(j)} \neq s_2 - \frac{1}{2} = -\frac{\sqrt{5}}{2}, \quad j = \overline{0, 2},$$

that is,

$$\frac{x_{-2}}{x_{-4}} \neq -\frac{\sqrt{5}}{2} \neq \frac{x_{-1}}{x_{-3}}, \quad \frac{x_0}{x_{-1}} \neq -\frac{\sqrt{5}}{2}.$$

For instance, this is possible if the initial values are chosen to be some rational numbers different from zero.

Taking the limit in (3.9) we get

$$\lim_{m\to+\infty} y_m^{(j)} = \lim_{m\to+\infty} y_{3m-j} = s_1 - \frac{1}{2} = \frac{\sqrt{5}}{2}, \quad j = \overline{0, 2}. \tag{3.10}$$

From (3.3) we have

$$x_{6m-j} = x_{-j} \prod_{i=1}^{m} y_{6i-j} y_{6i-j-2} y_{6i-j-4}, \tag{3.11}$$

for $m \in \mathbb{N}$, $j = \overline{-1, 4}$.

Relations (3.10), (3.11) and the fact $\frac{\sqrt{5}}{2} > 1$, imply

$$\lim_{n\to+\infty} |x_n| = +\infty, \tag{3.12}$$

which shows the existence of unbounded solutions to Eq (3.2).

Remark 2. If the initial values x_{-j}, $j = \overline{0, 4}$ are chosen to be positive numbers then a simple inductive argument shows that such solutions of Eq (3.2) are positive. From this and (3.12) for such solutions we have

$$\lim_{n\to+\infty} x_n = +\infty,$$

that is, we have solutions which diverge to $+\infty$.

The last statement in [8] on the long-term behaviour of solutions to Eq (1.5) is the following:

Statement 2. *All solutions of Eq* (1.5) *are bounded when* $a + \frac{b}{c} < 1$.

If a solution to Eq (1.5) as well as the parameters a, b, c and d are positive then the theorem trivially follows from the obvious estimate

$$x_{n+1} \leq a x_{n-1} + \frac{b x_{n-1} x_{n-4}}{c x_{n-4}} = x_{n-1} \left(a + \frac{b}{c} \right). \tag{3.13}$$

Moreover, estimate (3.13) shows that each positive solution to Eq (1.5) in this case converge to zero. Indeed, from (3.13) we have

$$0 < x_{2m-j} \leq x_{-j} \left(a + \frac{b}{c} \right)^m \to 0,$$

as $m \to +\infty$ for $j = 0, 1$, as claimed.

Remark 3. This is one of the simplest ways for proving the boundedness of positive solutions to difference equations. For some more complex tricks and methods see, e.g., [2, 3, 22, 24, 25, 29–33].

Remark 4. It should be pointed out that the condition on positivity of the parameters a, b, c and d was not posed in [8]. Moreover, in the introduction they said that the parameters are real numbers so the proof therein cannot be regarded as complete one.

If the positivity condition is not posed, then the statement is not true which is shown in the example which follows.

Example 2. Let

$$d = 0 \quad \text{and} \quad c = 1. \tag{3.14}$$

Then, Eq (1.5) becomes

$$x_{n+1} = (a + b)x_{n-1}, \quad n \in \mathbb{N}_0,$$

from which it follows that

$$x_{2m-j} = (a + b)^m x_{-j}, \tag{3.15}$$

for $m \in \mathbb{N}_0$, $j = 0, 1$.

If the real parameters a and b are chosen such that the following condition also holds

$$a + b < -1, \tag{3.16}$$

then from (3.14) and (3.16) we see that the condition $a + \frac{b}{c} < 1$ obviously holds.

However, from (3.16) we have

$$|a + b| > 1. \tag{3.17}$$

From (3.15) and (3.17) it follows that for the initial values such that

$$x_{-1} \neq 0 \neq x_0, \tag{3.18}$$

we have

$$\lim_{m\to\infty} |x_{2m-j}| = +\infty$$

for $j = 0, 1$, that is, such solutions are unbounded.

Remark 5. Note that instead of condition (3.18) we can choose the initial values x_{-1} and x_0 such that one of them is different from zero to obtain unbounded solutions to Eq (3.2).

Remark 6. The closed form formulas obtained in Theorem 1 can be also employed for getting all the closed form formulas in [8]. We leave the verification of the facts to the reader.

4. Conclusions

Many recent papers are devoted to concrete nonlinear difference equations and systems. Some of them study their solvability but a big part of them do not present almost any theory. Here, we show that a general nonlinear difference equation of the fifth order is solvable in closed form from which the closed-form formulas for solutions to some very special cases in the literature easily follow. We also consider in detail some of the recently formulated statements in the literature on the local and global stability of the equilibria as well as on the boundedness of solutions to some special cases of the general equation and give several comments. We present some ideas and tricks which can be employed in studying related difference equations in a proper way to avoid some problems which appear in the literature from time to time.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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