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ALGEBRAIC K-THEORY OF QUASI-SMOOTH BLOW-UPS AND CDH DESCENT

K-THÉORIE ALGÉBRIQUE DES ÉCLATEMENTS QUASI-LISSES ET DESCENTE POUR LA TOPOLOGIE CDH

ABSTRACT. — We construct a semi-orthogonal decomposition on the category of perfect complexes on the blow-up of a derived Artin stack in a quasi-smooth centre. This gives a generalization of Thomason's blow-up formula in algebraic K-theory to derived stacks. We also provide a new criterion for descent in Voevodsky's cdh topology, which we use to give a direct proof of Cisinski's theorem that Weibel's homotopy invariant K-theory satisfies cdh descent.

RÉSUMÉ. — Nous construisons une décomposition semi-orthogonale sur la catégorie des complexes parfaits sur l'éclaté d'un champ d'Artin dérivé le long d'un centre quasi-lisse. Ceci conduit à une généralisation de la formule d'éclatement de Thomason en K-théorie algébrique à une situation dérivée. Nous établissons aussi un nouveau critère de descente pour la topologie cdh de Voevodsky, que nous utilisons pour donner une démonstration directe d'un thèorème de Cisinski qui affirme que la K-théorie invariante par homotopie de Weibel satisfait la descente cdh.

1. Introduction

Let X be a scheme and $i : Z \to X$ a regular closed immersion. This means that Z is, Zariski-locally on X, the zero-locus of some regular sequence of functions $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$. Then the blow-up $\mathrm{Bl}_{Z/X}$ fits into a square

(1.1)
$$
\begin{array}{ccc}
\mathbf{P}\left(\mathcal{N}_{Z/X}\right) & \xrightarrow{\ i_D\ } & \mathrm{Bl}_{Z/X} \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\
Z & \xrightarrow{\ \ i\ \ } & X,
\end{array}
$$

where the exceptional divisor is the projective bundle associated to the conormal sheaf $\mathcal{N}_{Z/X}$, which under the assumptions is locally free of rank n. A result of Thomason [Tho93b] asserts that after taking algebraic K-theory, the induced square of spectra

$$
\begin{array}{ccc}
\mathrm{K}(X) & \xrightarrow{\ i^*\ } & \mathrm{K}(Z) \\
\downarrow{\scriptstyle p^*} & & \downarrow \\
\mathrm{K}\left(\mathrm{Bl}_{Z/X}\right) & \longrightarrow & \mathrm{K}\left(\mathbf{P}\left(\mathcal{N}_{Z/X}\right)\right)
\end{array}
$$

is homotopy cartesian. Here $\mathrm{K}(X)$ denotes the Bass–Thomason–Trobaugh algebraic K-theory spectrum of perfect complexes on a scheme X. We may summarize this property by saying that algebraic K-theory satisfies *descent* with respect to blow-ups in regularly immersed centres.

Now suppose that i is more generally a *quasi-smooth* closed immersion of derived schemes. This means that Z is, Zariski-locally on X, the *derived* zero-locus of some arbitrary sequence of functions $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$. (When X is a classical scheme and the sequence is regular, this is the same as the classical zero-locus, and we are in the situation discussed above.) In the derived setting there is still a conormal sheaf $\mathcal{N}_{Z/X}$ on Z, locally free of rank n, and one may still form the blow-up square (1.1), see [KR18a]. Our goal in this paper is to generalize Thomason's result above to this situation. At the same time we also allow X to be a derived *Artin stack*, and consider any *additive invariant* of stable ∞-categories (see Definition 2.6). Examples of additive invariants include algebraic K-theory K, connective algebraic K-theory K^{cn}, topological Hochschild homology THH, and topological cyclic homology TC.

THEOREM A. — *Let E be an additive invariant of stable ∞-categories. Then E satisfies descent by quasi-smooth blow-ups. That is, given a derived Artin stack X and a quasi-smooth closed immersion $i : Z \to X$ of virtual codimension $n \geqslant 1$, form the blow-up square (1.1). Then the induced commutative square*

$$
\begin{array}{ccc}
E(X) & \xrightarrow{\ i^*\ } & E(Z) \\
\downarrow{\scriptstyle p^*} & & \downarrow \\
E\left(\mathrm{Bl}_{Z/X}\right) & \longrightarrow & E\left(\mathbf{P}\left(\mathcal{N}_{Z/X}\right)\right)
\end{array}
$$

is homotopy cartesian.

We deduce Theorem A from an analysis of the categories of perfect complexes on $\mathrm{Bl}_{Z/X}$ and on the exceptional divisor $\mathbf{P}(\mathcal{N}_{Z/X})$. The relevant notion is that of a *semi-orthogonal decomposition*, see Definition 2.2.

THEOREM B. — *Let X be a derived Artin stack. For any locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank $n+1$, $n \geqslant 0$, consider the projective bundle $q : \mathbf{P}(\mathcal{E}) \to X$. Then we have:*

(i) *For each $0 \leqslant k \leqslant n$, the assignment $\mathcal{F} \mapsto q^*(\mathcal{F}) \otimes \mathcal{O}(-k)$ defines a fully faithful functor $\mathrm{Perf}(X) \to \mathrm{Perf}(\mathbf{P}(\mathcal{E}))$, whose essential image we denote $\mathbf{A}(-k)$.*

(ii) *The sequence of full subcategories $(\mathbf{A}(0), \dots, \mathbf{A}(-n))$ forms a semi-orthogonal decomposition of $\mathrm{Perf}(\mathbf{P}(\mathcal{E}))$.*

THEOREM C. — *Let X be a derived Artin stack. For any quasi-smooth closed immersion $i : Z \to X$ of virtual codimension $n \geqslant 1$, form the blow-up square (1.1). Then we have:*

(i) *The assignment $\mathcal{F} \mapsto p^*(\mathcal{F})$ defines a fully faithful functor $\mathrm{Perf}(X) \to \mathrm{Perf}(\mathrm{Bl}_{Z/X})$, whose essential image we denote $\mathbf{B}(0)$.*

(ii) *For each $1 \leqslant k \leqslant n-1$, the assignment $\mathcal{F} \mapsto (i_D)_*(q^*(\mathcal{F}) \otimes \mathcal{O}(-k))$ defines a fully faithful functor $\mathrm{Perf}(Z) \to \mathrm{Perf}(\mathrm{Bl}_{Z/X})$, whose essential image we denote $\mathbf{B}(-k)$.*

(iii) *The sequence of full subcategories $(\mathbf{B}(0), \dots, \mathbf{B}(-n+1))$ forms a semi-orthogonal decomposition of $\mathrm{Perf}(\mathrm{Bl}_{Z/X})$.*

We immediately deduce the projective bundle and blow-up formulas

$$E(\mathbf{P}(\mathcal{E})) \simeq \bigoplus_{m=0}^{n} E(X), \quad E\left(\mathrm{Bl}_{Z/X}\right) \simeq E(X) \oplus \bigoplus_{k=1}^{n-1} E(Z),$$

for any additive invariant E, see Corollaries 3.6 and 4.4, from which Theorem A immediately follows (see Subsection 4.5).

The results mentioned above admit the following interesting special cases:

(1) Suppose that X is a smooth projective variety over the field of complex numbers. This case of Theorem B was proven by Orlov in [Orl92]. He also proved Theorem C for any *smooth* subvariety $Z \hookrightarrow X$.

(2) More generally suppose that X is a quasi-compact quasi-separated classical scheme. Then the projective bundle formula (Corollary 3.6) for algebraic K-theory was proven by Thomason [Tho93a, TT90]. Similarly suppose that $i : Z \to X$ is a quasi-smooth closed immersion of quasi-compact quasi-separated classical schemes. Then it is automatically a regular closed immersion, and in this case Thomason also proved Corollary 4.4 for algebraic K-theory [Tho93b]. In fact, the papers [Tho93a] and [Tho93b] essentially contain under these assumptions proofs of Theorems B and C, respectively, even if the term "semi-orthogonal decomposition" is not used explicitly. For THH and TC, these cases of Corollaries 3.6 and 4.4 were proven by Blumberg and Mandell [BM12].

(3) More generally still, let X and Z be classical Artin stacks. These cases of Theorems B and C are proven by Bergh and Schnürer in [BS20]. However we

note that Corollaries 3.6 and 4.4 were obtained earlier by Krishna and Ravi in [KR18b], and their arguments in fact prove Theorems B and C for classical Artin stacks.

(4) Let X be a noetherian affine classical scheme, and let Z be the derived zero-locus of some functions $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$. Then the canonical morphism $i : Z \to X$ is a quasi-smooth closed immersion. In this case, Theorem A for algebraic K-theory was proven by Kerz–Strunk–Tamme [KST18] (where the blow-up $\mathrm{Bl}_{Z/X}$ was explicitly modelled as the derived fibred product $X \times_{\mathbf{A}^n} \mathrm{Bl}_{\{0\}/\mathbf{A}^n}$), as part of their proof of Weibel's conjecture on negative K-theory.

Let KH denote homotopy invariant K-theory. Recall that this is the $\mathbf{A}^1$-localization of the presheaf $X \mapsto \mathrm{K}(X)$. That is, it is obtained by forcing the property of $\mathbf{A}^1$-homotopy invariance: for every quasi-compact quasi-separated algebraic space X, the map

$$\mathrm{KH}(X) \to \mathrm{KH}\left(X \times \mathbf{A}^1\right)$$

is invertible (see [Cis13, Wei89]). As an application of Theorem A, we give a new proof of the following theorem of Cisinski [Cis13]:

THEOREM D. — *The presheaf of spectra $S \mapsto \mathrm{KH}(S)$ satisfies cdh descent on the site of quasi-compact quasi-separated algebraic spaces.*

This was first proven by Haesemeyer [Hae04] for schemes over a field of characteristic zero, using resolution of singularities. Cisinski's proof over general bases (noetherian schemes of finite dimension) relies on Ayoub's proper base change theorem in motivic homotopy theory. A different proof of Theorem D (also in the noetherian setting) was recently given by Kerz–Strunk–Tamme [KST18, Theorem C], as an application of pro-cdh descent and their resolution of Weibel's conjecture on negative K-theory. The proof we give here is more direct and uses a new criterion for cdh descent (see Theorem 5.6 for a more precise statement):

THEOREM E. — *Let $\mathcal{F}$ be a Nisnevich sheaf of spectra on the category of quasi-compact quasi-separated algebraic spaces. Then $\mathcal{F}$ satisfies cdh descent if and only if it sends closed squares and quasi-smooth blow-up squares to cartesian squares.*

Theorem E can be compared to a similar criterion due to Haesemeyer, implicit in [Hae04], which applies to Nisnevich sheaves of spectra on the category of schemes over a field k of characteristic zero. It asserts that for such a sheaf, cdh descent is equivalent to descent for finite cdh squares and regularly immersed blow-up squares. Note that the first condition is stronger than descent for closed squares, while the second is weaker than descent for quasi-smooth blow-up squares: regularly immersed blow-up squares are precisely those quasi-smooth blow-up squares where all schemes appearing are underived. For invariants of stable ∞-categories, a similar cdh descent criterion was noticed independently by Land and Tamme [LT18, Theorem. A.2]. The h-descent criterion of Halpern-Leistner and Preygel [HLP14, Proposition 3.2.1] is also somewhat similar in spirit.

Theorem D was extended to certain nice Artin stacks recently by Hoyois and Hoyois–Krishna [Hoy16, HK19]. Our cdh descent criterion also applies in that setting (Remark 5.11 (iii)) and gives another potential approach to such results.

The organization of this paper is as follows. We begin in Section 2 with some background on derived algebraic geometry and on semi-orthogonal decompositions of stable ∞-categories. Section 3 is dedicated to the proof of Theorem B. We first show that the semi-orthogonal decomposition exists on the larger stable ∞-category $\mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$ (Theorem 3.3). Then we show that it restricts to $\mathrm{Perf}(\mathbf{P}(\mathcal{E}))$ (Subsection 3.3), and deduce the projective bundle formula (Corollary 3.6) for any additive invariant.

We follow a similar pattern in Section 4 to prove Theorem C. There is a semi-orthogonal decomposition on $\mathrm{Qcoh}(\mathrm{Bl}_{Z/X})$ (Theorem 4.3) which then restricts to $\mathrm{Perf}(\mathrm{Bl}_{Z/X})$ (Subsection 4.4). This gives both the blow-up formula (Corollary 4.4) as well as Theorem A (4.5.1) for additive invariants. As input we prove a Grothendieck duality statement for virtual Cartier divisors (Proposition 4.2) that should be of independent interest.

Section 5 contains our results on cdh descent and KH. We first give the general cdh descent criterion (Theorem 5.6). We apply this criterion to KH to give our proof of Theorem D (5.4.3).

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2. Preliminaries

Throughout the paper we work with the language of ∞-categories as in [Lur09, Lur12].

2.1. Derived algebraic geometry

This paper is set in the world of derived algebraic geometry, as in [GR17, Lur16, TV08].

2.1.1. Derived schemes and stacks

Let SCRing denote the ∞-category of simplicial commutative rings. A *derived stack* is an étale sheaf of spaces $X : \mathrm{SCRing} \to \mathrm{Spc}$. If X is corepresentable by a simplicial commutative ring A, we write $X = \mathrm{Spec}(A)$ and call X an *affine derived scheme*. A *derived scheme* is a derived stack X that admits a Zariski atlas by affine derived schemes, i.e., a jointly surjective family $(U_i \to X)_i$ of Zariski open immersions with each U_i an affine derived scheme. Allowing Nisnevich, étale or smooth atlases, respectively, gives rise to the notions of *derived algebraic space*[1]

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[1] That this agrees with the classical notion of algebraic space (at least under quasi-compactness and quasi-separatedness hypotheses) follows from [RG71, Proposition 5.7.6]. That it agrees with Lurie's definition follows from [Lur16, Example 3.7.1.5].

derived Deligne–Mumford stack, and *derived Artin stack*. The precise definition is slightly more involved, see e.g. [GR17, Volume I, Section 4.1].

Any derived stack X admits an underlying classical stack which we denote X_{cl}. If X is a derived scheme, algebraic space, Deligne–Mumford or Artin stack, then X_{cl} is a classical such. For example, $\mathrm{Spec}(A)_{\mathrm{cl}} = \mathrm{Spec}(\pi_0(A))$ for a simplicial commutative ring A.

2.1.2. Quasi-smooth morphisms

Let X be a derived scheme and let $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$ be functions classifying a morphism $f : X \to \mathbf{A}^n$ to affine space. The *derived zero-locus* of these functions is given by the derived fibred product

$$(2.1) \qquad \begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ \{0\} & \longrightarrow & \mathbf{A}^n. \end{array}$$

If X is classical, then Z is classical if and only if the sequence $(f_1, \ldots, f_n)$ is regular in the sense of [BGI71], in which case Z is regularly immersed. A closed immersion of derived schemes $i : Z \to X$ is called *quasi-smooth* (of virtual codimension n) if it is cut out Zariski-locally as the derived zero-locus of n functions on X. Equivalently, this means that i is of finite presentation and its shifted cotangent complex $\mathcal{N}_{Z/X} := \mathcal{L}_{Z/X}[-1]$ is locally free (of rank n). A closed immersion of derived Artin stacks is quasi-smooth if it satisfies this condition smooth-locally.

A morphism of derived schemes $f : Y \to X$ is quasi-smooth if it can be factored, Zariski-locally on Y, through a quasi-smooth closed immersion $i : Y \to X'$ and a smooth morphism $X' \to X$. A morphism of derived Artin stacks is quasi-smooth if it satisfies this condition smooth-locally on Y. We refer to [KR18a] for more details on quasi-smoothness.

2.1.3. Blow-ups

Important for us is the following construction from [KR18a]. Given any quasi-smooth closed immersion $i : Z \to X$ of derived Artin stacks, there is an associated *quasi-smooth blow-up square*:

$$(2.2) \qquad \begin{array}{ccc} D & \xrightarrow{\ i_D\ } & \mathrm{Bl}_{Z/X} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{\ i\ } & X. \end{array}$$

Here $\mathrm{Bl}_{Z/X}$ is the blow-up of X in Z, which is a quasi-smooth proper derived Artin stack over X, and $D = \mathbf{P}(\mathcal{N}_{Z/X})$ is the projectivized normal bundle, which is a smooth proper derived Artin stack over Z. This square is universal with the following properties:

(a) the morphism i_D is a quasi-smooth closed immersion of virtual codimension 1, i.e., a virtual effective Cartier divisor;

(b) the underlying square of classical Artin stacks is cartesian; and
(c) the canonical map $q^* \mathcal{N}_{Z/X} \to \mathcal{N}_{D/\mathrm{Bl}_{Z/X}}$ is surjective on π_0. When X is a derived scheme (resp. derived algebraic space, derived Deligne–Mumford stack), then so is $\mathrm{Bl}_{Z/X}$.

2.1.4. Quasi-coherent sheaves

Given a derived stack X, the stable ∞-category of quasi-coherent sheaves $\mathrm{Qcoh}(X)$ is the limit

$$\mathrm{Qcoh}(X) = \varprojlim_{\mathrm{Spec}(A) \to X} \mathrm{Qcoh}(\mathrm{Spec}(A))$$

taken over all morphisms $\mathrm{Spec}(A) \to X$ with $A \in \mathrm{SCRing}$. Here $\mathrm{Qcoh}(\mathrm{Spec}(A))$ is the stable ∞-category Mod_A of A-modules[2] in the sense of Lurie. Informally speaking, a quasi-coherent sheaf $\mathcal{F}$ on X is thus a collection of quasi-coherent sheaves $x^*(\mathcal{F}) \in \mathrm{Qcoh}(\mathrm{Spec}(A))$, for every simplicial commutative ring A and every A-point $x : \mathrm{Spec}(A) \to X$, together with a homotopy coherent system of compatibilities.

The full subcategory $\mathrm{Perf}(X) \subset \mathrm{Qcoh}(X)$ is similarly the limit

$$\mathrm{Perf}(X) = \varprojlim_{\mathrm{Spec}(A) \to X} \mathrm{Perf}(\mathrm{Spec}(A)),$$

where $\mathrm{Perf}(\mathrm{Spec}(A))$ is the stable ∞-category $\mathrm{Mod}_A^{\mathrm{perf}}$ of *perfect* A-modules. In other words, $\mathcal{F} \in \mathrm{Qcoh}(X)$ belongs to $\mathrm{Perf}(X)$ if and only if $x^*(\mathcal{F})$ is perfect for every simplicial commutative ring A and every morphism $x : \mathrm{Spec}(A) \to X$.

2.1.5. Functoriality

There is an inverse image functor $f^* : \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(Y)$ for any morphism of derived stacks $f : Y \to X$. It preserves perfect complexes and induces a functor $f^* : \mathrm{Perf}(X) \to \mathrm{Perf}(Y)$. Regarded as presheaves of ∞-categories, the assignments $X \mapsto \mathrm{Qcoh}(X)$ and $X \mapsto \mathrm{Perf}(X)$ satisfy *descent* for the fpqc topology ([Lur16, Corollary D.6.3.3], [GR17, Theorem 1.3.4]). This means in particular that given any fpqc covering family $(f_\alpha : X_\alpha \to X)_\alpha$, the family of inverse image functors $f_\alpha^* : \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(X_\alpha)$ is jointly conservative.

If $f : Y \to X$ is quasi-compact and *schematic*, in the sense that its fibre over any affine derived scheme is a derived scheme, then there is a direct image functor f_*, right adjoint to f^*, which commutes with colimits and satisfies a base change formula against inverse images ([Lur16, Proposition 2.5.4.5], [GR17, Volume 1, Chapter 3, Proposition 2.2.2]). If f is proper, locally of finite presentation, and of finite tor-amplitude, then f_* also preserves perfect complexes [Lur16, Theorem 6.1.3.2].

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[2] Note that if A is discrete (an ordinary commutative ring), then this is not the abelian category of discrete A-modules, but rather the derived ∞-category of this abelian category as in [Lur12, Chapter 1].

2.2. Semi-orthogonal decompositions

The following definitions were originally formulated by [BK89] in the language of triangulated categories and are standard.

DEFINITION 2.1. — *Let* $\mathbf{C}$ *be a stable ∞-category and* $\mathbf{D}$ *a stable full subcategory. An object* $x \in \mathbf{C}$ *is* left orthogonal, *resp.* right orthogonal, *to* $\mathbf{D}$ *if the mapping space* $\mathrm{Maps}_{\mathbf{C}}(x, d)$, *resp.* $\mathrm{Maps}_{\mathbf{C}}(d, x)$, *is contractible for all objects* $d \in \mathbf{D}$. *We let* $^{\perp}\mathbf{D} \subseteq \mathbf{C}$ *and* $\mathbf{D}^{\perp} \subseteq \mathbf{C}$ *denote the full subcategories of left orthogonal and right orthogonal objects, respectively.*

DEFINITION 2.2. — *Let* $\mathbf{C}$ *be a stable ∞-category and let* $\mathbf{C}(0), \ldots, \mathbf{C}(-n)$ *be full stable subcategories. Suppose that the following conditions hold:*

  (i) *For all integers* $i > j$, *there is an inclusion* $\mathbf{C}(i) \subseteq I^{\perp}\mathbf{C}(j)$.
  (ii) *The ∞-category* $\mathbf{C}$ *is generated by the subcategories* $\mathbf{C}(0), \ldots, \mathbf{C}(-n)$, *under finite limits and finite colimits.*

Then we say that the sequence $(\mathbf{C}(0), \ldots, \mathbf{C}(-n))$ *forms a semi-orthogonal decomposition of* $\mathbf{C}$.

Semi-orthogonal decompositions of length 2 come from *split short exact sequences* of stable ∞-categories, as in [BGT13].

DEFINITION 2.3. —
  (i) *A short exact sequence of small stable ∞-categories is a diagram*
$$\mathbf{C}' \xrightarrow{i} \mathbf{C} \xrightarrow{p} \mathbf{C}'',$$
  where i *and* p *are exact, the composite* $p \circ i$ *is null-homotopic,* i *is fully faithful, and* p *induces an equivalence* $(\mathbf{C}/\mathbf{C}')^{\mathrm{idem}} \simeq (\mathbf{C}'')^{\mathrm{idem}}$ *(where* $(-)^{\mathrm{idem}}$ *denotes idempotent completion).*
  (ii) *A short exact sequence of small stable ∞-categories*
$$\mathbf{C}' \xrightarrow{i} \mathbf{C} \xrightarrow{p} \mathbf{C}''$$
  is split if there exist functors $q : \mathbf{C} \to \mathbf{C}'$ *and* $j : \mathbf{C}'' \to \mathbf{C}$, *right adjoint to* i *and* p, *respectively, such that the unit* $\mathrm{id} \to q \circ i$ *and co-unit* $p \circ j \to \mathrm{id}$ *are invertible.*

Remark 2.4. — Let $\mathbf{C}$ be a small stable ∞-category, and let $(\mathbf{C}(0), \mathbf{C}(-1))$ be a semi-orthogonal decomposition. Then for any object $x \in \mathbf{C}$, there exists an exact triangle
$$x(0) \to x \to x(-1),$$
where $x(0) \in \mathbf{C}(0)$ and $x(-1) \in \mathbf{C}(-1)$. To see this, simply observe that the full subcategory spanned by objects x for which such a triangle exists, is closed under finite limits and colimits, and contains $\mathbf{C}(0)$ and $\mathbf{C}(-1)$. Moreover, the assignments $x \mapsto x(0)$ and $x \mapsto x(-1)$ determine well-defined functors $q : \mathbf{C} \to \mathbf{C}(0)$ and $p : \mathbf{C} \to \mathbf{C}(-1)$, respectively, which are right and left adjoint, respectively, to the inclusions (see e.g. [Lur16, Remark 7.2.0.2]). It follows from this that any semi-orthogonal decomposition $(\mathbf{C}(0), \mathbf{C}(-1))$ induces a split short exact sequence
$$\mathbf{C}(0) \to \mathbf{C} \xrightarrow{p} \mathbf{C}(-1).$$

Lemma 2.5. — *Let $\mathbf{C}$ be a stable ∞-category, and let $(\mathbf{C}(0), \ldots, \mathbf{C}(-n))$ be a sequence of full stable subcategories forming a semi-orthogonal decomposition of $\mathbf{C}$. For each $0 \leqslant m \leqslant n$, let $\mathbf{C}_{\leqslant -m} \subseteq \mathbf{C}$ denote the full stable subcategory generated by objects in the union $\mathbf{C}(-m) \cup \cdots \cup \mathbf{C}(-n)$, and let $\mathbf{C}_{\leqslant -n-1} \subseteq \mathbf{C}$ denote the full subcategory spanned by the zero object. Then there are split short exact sequences*

$$\mathbf{C}(-m) \hookrightarrow \mathbf{C}_{\leqslant -m} \to \mathbf{C}_{\leqslant -m-1}$$

for each $0 \leqslant m \leqslant n$.

Proof. — It follows from the definitions that for each $0 \leqslant m \leqslant n$, the sequence $(\mathbf{C}(-m), \mathbf{C}_{\leqslant -m-1})$ forms a semi-orthogonal decomposition of $\mathbf{C}_{\leqslant -m}$. Therefore the claim follows from Remark 2.4. $\qquad\square$

2.3. Additive and localizing invariants

The following definition is from [BGT13], except that we do not require commutativity with filtered colimits.

Definition 2.6. — *Let $\mathbf{A}$ be a stable presentable ∞-category. Let E be an $\mathbf{A}$-valued functor from the ∞-category of small stable ∞-categories and exact functors.*

  (i) *We say that E is an* additive invariant *if for any split short exact sequence*

$$\mathbf{C}' \xrightarrow{i} \mathbf{C} \xrightarrow{p} \mathbf{C}'',$$

   the induced map

$$E(\mathbf{C}') \oplus E(\mathbf{C}'') \xrightarrow{(i,j)} E(\mathbf{C})$$

   is invertible, where j is a right adjoint to p.
  (ii) *We say that E is a* localizing invariant *if for any short exact sequence*

$$\mathbf{C}' \xrightarrow{i} \mathbf{C} \xrightarrow{p} \mathbf{C}'',$$

   the induced diagram

$$E(\mathbf{C}') \to E(\mathbf{C}) \to E(\mathbf{C}'')$$

   is an exact triangle.

Remark 2.7. — Any localizing invariant is also additive.

Lemma 2.8. — *Let $\mathbf{C}$ be a stable ∞-category, and let $(\mathbf{C}(0), \ldots, \mathbf{C}(-n))$ be a sequence of full stable subcategories forming a semi-orthogonal decomposition of $\mathbf{C}$. Then for any additive invariant E there is a canonical isomorphism*

$$E(\mathbf{C}) \simeq \bigoplus_{m=0}^{n} E(\mathbf{C}(-m)).$$

Proof. — Follows immediately from Lemma 2.5. $\qquad\square$

3. The projective bundle formula

3.1. Projective bundles

Let X be a derived stack and $\mathcal{E}$ a locally free $\mathcal{O}_X$-module of finite rank. Recall that the *projective bundle* associated to $\mathcal{E}$ is a derived stack $\mathbf{P}(\mathcal{E})$ over X equipped with an invertible sheaf $\mathcal{O}(1)$ together with a surjection $\mathcal{E} \to \mathcal{O}(1)$. More precisely, for any derived scheme S over X, with structural morphism $x : S \to X$, the space of S-points of $\mathbf{P}(\mathcal{E})$ is the space of pairs $(\mathcal{L}, u)$, where $\mathcal{L}$ is a locally free $\mathcal{O}_S$-module of rank 1, and $u : x^*(\mathcal{E}) \to \mathcal{L}$ is surjective on π_0. We recall the standard properties of this construction:

PROPOSITION 3.1. —
   (i) *If $f : X' \to X$ is a morphism of derived stacks, then there is a canonical isomorphism $\mathbf{P}(f^*(\mathcal{E})) \to \mathbf{P}(\mathcal{E}) \times_X X'$ of derived stacks over X'.*
   (ii) *The projection $\mathbf{P}(\mathcal{E}) \to X$ is proper and schematic. In particular, if X is a derived scheme (resp. derived algebraic space, derived Deligne–Mumford stack, derived Artin stack), then the same holds for the derived stack $\mathbf{P}(\mathcal{E})$.*
   (iii) *The relative cotangent complex $\mathcal{L}_{\mathbf{P}(\mathcal{E})/X}$ is canonically isomorphic to $\mathcal{F} \otimes \mathcal{O}(-1)$, where the locally free sheaf $\mathcal{F}$ is the fibre of the canonical map $\mathcal{E} \to \mathcal{O}(1)$. In particular, the morphism $\mathbf{P}(\mathcal{E}) \to X$ is smooth of relative dimension equal to $\mathrm{rk}(\mathcal{E}) - 1$.*

PROPOSITION 3.2 (Serre). — *Let X be a derived Artin stack, and $\mathcal{E}$ a locally free sheaf of rank $n + 1$, $n \geqslant 0$. If $q : \mathbf{P}(\mathcal{E}) \to X$ denotes the associated projective bundle, then we have canonical isomorphisms*

$$q_*(\mathcal{O}(0)) \simeq \mathcal{O}_X, \quad q_*(\mathcal{O}(-m)) \simeq 0 \ (1 \leqslant m \leqslant n)$$

in $\mathrm{Qcoh}(X)$.

Proof. — There is a canonical map $\mathcal{O}_X \to q_*(\mathcal{O}(0))$, the unit of the adjunction (q^*, q_*), and there is a unique map $0 \to q_*(\mathcal{O}(-m))$ for each m. To show that these are invertible, we may use fpqc descent and base change to the case where X is affine and $\mathcal{E}$ is free. Then this is Serre's computation, as generalized to the derived setting by Lurie [Lur16, Theorem. 5.4.2.6]. □

3.2. Semi-orthogonal decomposition on $\mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$

In this subsection we will show that the stable ∞-category $\mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$ admits a canonical semi-orthogonal decomposition.

THEOREM 3.3. — *Let X be a derived Artin stack. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank $n + 1$, $n \geqslant 0$, and $q : \mathbf{P}(\mathcal{E}) \to X$ the associated projective bundle. Then we have:*
   (i) *For every integer $k \in \mathbf{Z}$, the assignment $\mathcal{F} \mapsto q^*(\mathcal{F}) \otimes \mathcal{O}(k)$ defines a fully faithful functor $\mathrm{Qcoh}(X) \to \mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$.*

(ii) *For every integer $k \in \mathbf{Z}$, let $\mathbf{C}(k) \subset \mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$ denote the essential image of the functor in (i). Then the subcategories $\mathbf{C}(k), \ldots, \mathbf{C}(k - n)$ form a semi-orthogonal decomposition of $\mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$.*

We will need the following facts (see [Lur16, Lemmas 7.2.2.2 and 5.6.2.2]):

LEMMA 3.4. — *Let R be a simplicial commutative ring and $X = \mathrm{Spec}(R)$. Denote by $\mathbf{P}_R^n = \mathbf{P}(\mathcal{O}_X^{n+1})$ the n-dimensional projective space over R. Then for every integer $m \in \mathbf{Z}$, there is a canonical isomorphism*

$$\varinjlim_{J \subsetneq [n]} \mathcal{O}(m + |J|) \xrightarrow{\sim} \mathcal{O}(m + n + 1)$$

in $\mathrm{Qcoh}(\mathbf{P}_R^n)$, where the colimit is taken over the proper subsets J of the set $[n] = \{0, 1, \ldots, n\}$, and $0 \leqslant |J| \leqslant n$ denotes the cardinality of such a subset.

LEMMA 3.5. — *Let R be a simplicial commutative ring and $X = \mathrm{Spec}(R)$. Denote by $\mathbf{P}_R^n = \mathbf{P}(\mathcal{O}_X^{n+1})$ the n-dimensional projective space over R. Then for any connective quasi-coherent sheaf $\mathcal{F} \in \mathrm{Qcoh}(\mathbf{P}_R^n)$, there exists a map*

$$\bigoplus_\alpha \mathcal{O}(d_\alpha) \to \mathcal{F},$$

with $d_\alpha \in \mathbf{Z}$, which is surjective on π_0.

Proof of Theorem 3.3. — Since the functors $- \otimes \mathcal{O}(k)$ are equivalences, it will suffice to take $k = 0$ in both claims. For claim (i) we want to show that the unit map $\mathcal{F} \to q_* q^*(\mathcal{F})$ is invertible for all $\mathcal{F} \in \mathrm{Qcoh}(X)$. By fpqc descent and base change (Subsubsection 2.1.5), we may reduce to the case where $X = \mathrm{Spec}(R)$ is affine and $\mathcal{E} = \mathcal{O}_S^{n+1}$ is free. Now both functors q^* and q_* are exact and moreover commute with arbitrary colimits (the latter by Subsubsection 2.1.5 since q is quasi-compact and schematic), and $\mathrm{Qcoh}(X) \simeq \mathrm{Mod}_R$ is generated by $\mathcal{O}_X$ under colimits and finite limits. Therefore we may assume $\mathcal{F} = \mathcal{O}_X$, in which case the claim holds by Proposition 3.2.

For claim (ii), let us first check the orthogonality condition in Definition 2.2. Thus take $\mathcal{F}, \mathcal{G} \in \mathrm{Qcoh}(X)$ and consider the mapping space

$$\mathrm{Maps}(q^*(\mathcal{F}), q^*(\mathcal{G}) \otimes \mathcal{O}(-m)) \simeq \mathrm{Maps}(\mathcal{F}, q_*(\mathcal{O}(-m)) \otimes \mathcal{G})$$

for $1 \leqslant m \leqslant n$, where the identification results from the projection formula. Since $q_*(\mathcal{O}(-m)) \simeq 0$ by Proposition 3.2, this space is contractible.

It now remains to show that every $\mathcal{F} \in \mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$ belongs to the full subcategory $\langle \mathbf{C}(0), \ldots, \mathbf{C}(-n) \rangle \subseteq \mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$ generated under finite colimits and limits by the subcategories $\mathbf{C}(0), \ldots, \mathbf{C}(-n)$. Set $\mathcal{G}_{-1} = \mathcal{F} \otimes \mathcal{O}(-1)$ and define $\mathcal{G}_m$, for $m \geqslant 0$, so that we have exact triangles

$$(3.1) \qquad q^* q_*(\mathcal{G}_{m-1} \otimes \mathcal{O}(1)) \xrightarrow{\mathrm{counit}} \mathcal{G}_{m-1} \otimes \mathcal{O}(1) \to \mathcal{G}_m.$$

For each $m \geqslant -1$, we claim that $\mathcal{G}_m$ is right orthogonal to each of the subcategories $\mathbf{C}(0), \mathbf{C}(1), \ldots, \mathbf{C}(m)$. For $m = -1$ the claim is vacuous, so take $m \geqslant 0$ and assume by induction that it holds for $m - 1$. Since $q^* q_*(\mathcal{G}_{m-1} \otimes \mathcal{O}(1))$ is contained in $\mathbf{C}(0)$, it follows that $\mathcal{G}_m$ is right orthogonal to $\mathbf{C}(0)$. To show that $\mathcal{G}_m$ is right orthogonal to $\mathbf{C}(i)$, for $1 \leqslant i \leqslant m$, it will suffice to show that the left-hand and middle terms

of the exact triangle (3.1) are both right orthogonal to $\mathbf{C}(i)$. For the left-hand term this follows from the inclusion $\mathbf{C}(0) \subset \mathbf{C}(i)^\perp$, demonstrated above. For the middle term $\mathcal{G}_{m-1} \otimes \mathcal{O}(1)$, the claim follows by the induction hypothesis.

Now we claim that $\mathcal{G}_n$ is zero. Using fpqc descent again, we may assume that $X = \mathrm{Spec}(R)$ and $\mathcal{E} = \mathcal{O}_X^{\oplus n+1}$ is free (since the sequence $(\mathcal{G}_{-1}, \mathcal{G}_0, \ldots, \mathcal{G}_n)$ is stable under base change). Using Lemma 3.5 we can build a map

$$\varphi : \bigoplus_\alpha \mathcal{O}(m_\alpha)[k_\alpha] \to \mathcal{G}_n$$

which is surjective on all homotopy groups. From Lemma 3.4 it follows that $\mathcal{G}_n$ is right orthogonal to all $\mathbf{C}(i)$, $i \in \mathbf{Z}$. Thus φ must be null-homotopic, so $\mathcal{G}_n \simeq 0$ as claimed. Working backwards, we deduce that $\mathcal{G}_{n-1} \in \mathbf{C}(-1), \ldots, \mathcal{G}_0 \in \langle \mathbf{C}(-1), \ldots, \mathbf{C}(-n) \rangle$, and then finally that $\mathcal{F} \in \langle \mathbf{C}(0), \mathbf{C}(-1), \ldots, \mathbf{C}(-n) \rangle$ as claimed. $\qquad\square$

3.3. Proof of Theorem B

We now deduce Theorem B from Theorem 3.3. First note that the fully faithful functor $\mathcal{F} \mapsto q^*(\mathcal{F}) \otimes \mathcal{O}(k)$ of Theorem 3.3 (i) restricts to a fully faithful functor $\mathrm{Perf}(X) \to \mathrm{Perf}(\mathbf{P}(\mathcal{E}))$, since q^* preserves perfect complexes. This shows Theorem B (i).

For part (ii) we argue again as in the proof of Theorem 3.3. The point is that if $\mathcal{F} \in \mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$ is perfect, then so is each $\mathcal{G}_m \in \mathrm{Qcoh}(\mathbf{P}(\mathcal{E}))$, since q^* and q_* preserve perfect complexes (the latter because q is smooth and proper).

3.4. Projective bundle formula

From Theorem B and Lemma 2.8 we deduce:

COROLLARY 3.6. — *Let X be a derived Artin stack, $\mathcal{E}$ a locally free $\mathcal{O}_X$-module of rank $n + 1$, $n \geqslant 0$, and $q : \mathbf{P}(\mathcal{E}) \to X$ the associated projective bundle. Then for any additive invariant E, there is a canonical isomorphism*

$$E(\mathbf{P}(\mathcal{E})) \simeq \bigoplus_{k=0}^n E(X)$$

induced by the functors $q^*(-) \otimes \mathcal{O}(-k) : \mathrm{Perf}(X) \to \mathrm{Perf}(\mathbf{P}(\mathcal{E}))$.

4. The blow-up formula

4.1. Virtual Cartier divisors

Recall from [KR18a] that a *virtual (effective) Cartier divisor* on a derived Artin stack X is a quasi-smooth closed immersion $i : D \to X$ of virtual codimension 1. For any such $i : D \to X$, there is a canonical exact triangle

$$\mathcal{O}_X(-D) \to \mathcal{O}_X \to i_*(\mathcal{O}_D),$$

where $\mathcal{O}_X(-D)$ is a locally free sheaf of rank 1, equipped with a canonical isomorphism $i^*(\mathcal{O}_X(-D)) \simeq \mathcal{N}_{D/X}$ (see [KR18a, 3.2.3 and 3.2.9]).

LEMMA 4.1. — *Let X be a derived Artin stack and $i : D \to X$ a virtual Cartier divisor. Then there is a canonical isomorphism*

$$i^* i_*(\mathcal{O}_D) \simeq \mathcal{O}_D \oplus \mathcal{N}_{D/X}[1].$$

Proof. — Applying i^* to the exact triangle above (and rotating), we get the exact triangle

$$\mathcal{O}_D \to i^* i_*(\mathcal{O}_D) \to \mathcal{N}_{D/X}[1].$$

The map $\mathcal{O}_D \to i^* i_*(\mathcal{O}_D)$ is induced by the natural transformation $i^*(\eta) : i^* \to i^* i_* i^*$ (where η is the adjunction unit), so by the triangle identities it has a retraction given by the co-unit map $i^* i_*(\mathcal{O}_D) \to \mathcal{O}_D$. In other words, the triangle splits. □

4.2. Grothendieck duality

Let $i : Z \to X$ be a quasi-smooth closed immersion of derived Artin stacks. The functor i_* admits a right adjoint $i^!$, which for formal reasons can be computed by the formula

$$i^!(-) \simeq i^*(-) \otimes \omega_{D/X},$$

where $\omega_{D/X} := i^!(\mathcal{O}_X)$ is called the *relative dualizing sheaf*. See [Lur16, Corollary 6.4.2.7]. When i is a virtual Cartier divisor, $\omega_{D/X}$ can be computed as follows:

PROPOSITION 4.2 (Grothendieck duality). — *Let X be a derived Artin stack. Then for any virtual Cartier divisor $i : D \to X$, there is a canonical isomorphism*

$$\mathcal{N}_{D/X}^\vee[-1] \xrightarrow{\sim} \omega_{D/X}$$

of perfect complexes on D. In particular, there is a canonical identification $i^! \simeq i^*(-) \otimes \mathcal{N}_{D/X}^\vee[-1]$.

Proof. — Write $\mathcal{L} := \mathcal{O}_X(-D)$ and consider again the exact triangle $\mathcal{L} \to \mathcal{O}_X \to i_*(\mathcal{O}_D)$. By the projection formula, this can be refined to an exact triangle of natural transformations $\mathrm{id} \otimes \mathcal{L} \to \mathrm{id} \to i_* i^*$, or, passing to right adjoints, an exact triangle $i_* i^! \to \mathrm{id} \to \mathrm{id} \otimes \mathcal{L}^\vee$. In particular we get the exact triangle

$$(4.1) \qquad\qquad i_* i^!(\mathcal{O}_X) \to \mathcal{O}_X \to \mathcal{L}^\vee.$$

The associated map $\mathcal{L}^\vee[-1] \to i_* i^!(\mathcal{O}_X)$ gives by adjunction a canonical morphism

$$\mathcal{N}_{D/X}^\vee[-1] \simeq i^*(\mathcal{L}^\vee)[-1] \to i^!(\mathcal{O}_X),$$

which we claim is invertible. By fpqc descent and the fact that $i^!$ commutes with the operation f^*, for any morphism f [Lur16, Proposition 6.4.2.1], we may assume that X is affine. In this case the functor i_* is conservative, so it will suffice to show that the canonical map

$$i_* \left(\mathcal{N}_{D/X}^\vee \right)[-1] \to i_* i^!(\mathcal{O}_X)$$

is invertible. Considering again the triangle $\mathcal{F} \otimes \mathcal{L} \to \mathcal{F} \to i_* i^*(\mathcal{F})$ above and taking $\mathcal{F} = \mathcal{L}^\vee$, we get the exact triangle

$$\mathcal{O}_X \to \mathcal{L}^\vee \to i_* i^*(\mathcal{L}^\vee) \simeq i_* \left(\mathcal{N}_{D/X}^\vee \right),$$

since $\mathcal{L}$ is invertible. Comparing with (4.1) yields the claim. □

4.3. Semi-orthogonal decomposition on $\mathrm{Qcoh}(\mathrm{Bl}_{Z/X})$

In this subsection we prove:

THEOREM 4.3. — *Let X be a derived Artin stack and $i : Z \to X$ a quasi-smooth closed immersion of virtual codimension $n \geqslant 1$. Let $\widetilde{X} = \mathrm{Bl}_{Z/X}$ and consider the quasi-smooth blow-up square (2.2)*

$$
\begin{array}{ccc}
D & \xrightarrow{\ i_D\ } & \widetilde{X} \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\
Z & \xrightarrow{\ i\ } & X
\end{array}
$$

Then we have:

(i) *The functor $p^* : \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(\widetilde{X})$ is fully faithful. We denote its essential image by $\mathbf{D}(0) \subset \mathrm{Qcoh}(\widetilde{X})$.*

(ii) *The functor $(i_D)_*(q^*(-) \otimes \mathcal{O}(-k)) : \mathrm{Qcoh}(Z) \to \mathrm{Qcoh}(\widetilde{X})$ is fully faithful, for each $1 \leqslant k \leqslant n-1$. We denote its essential image by $\mathbf{D}(-k) \subset \mathrm{Qcoh}(\widetilde{X})$.*

(iii) *For each $1 \leqslant k \leqslant n-1$, the full stable subcategory $\mathbf{D}(-k) \subset \mathrm{Qcoh}(\widetilde{X})$ is right orthogonal to each of $\mathbf{D}(0), \ldots, \mathbf{D}(-k+1)$.*

(iv) *The stable ∞-category $\mathrm{Qcoh}(\widetilde{X})$ is generated by the full subcategories $\mathbf{D}(0)$, $\mathbf{D}(-1)$, \ldots, $\mathbf{D}(-n+1)$ under finite colimits and finite limits. In particular, the sequence $(\mathbf{D}(0), \mathbf{D}(-1), \ldots, \mathbf{D}(-n+1))$ forms a semi-orthogonal decomposition of $\mathrm{Qcoh}(\widetilde{X})$.*

4.3.1. Proof of (i)

The claim is that for any $\mathcal{F} \in \mathrm{Qcoh}(X)$, the unit map $\mathcal{F} \to p_* p^*(\mathcal{F})$ is invertible. By fpqc descent we may reduce to the case where X is affine and i fits in a cartesian square of the form (2.1). Since $\mathrm{Qcoh}(X)$ is then generated under colimits and finite limits by $\mathcal{O}_X$, and p_* commutes with colimits since p is quasi-compact and schematic (2.1.5), we may assume that $\mathcal{F} = \mathcal{O}_X$. In other words, it suffices to show that the canonical map $\mathcal{O}_X \to p_*(\mathcal{O}_{\widetilde{X}})$ is invertible.

$$
\begin{array}{ccccccc}
D & \xrightarrow{\ i_D\ } & \widetilde{X} & \qquad & \mathbf{P}^{n-1} & \longrightarrow & \mathrm{Bl}_{\{0\}/\mathbf{A}^n} \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} & & \downarrow & & \downarrow{\scriptstyle p_0} \\
Z & \xrightarrow{\ i\ } & X & \qquad & \{0\} & \xrightarrow{\ i_0\ } & \mathbf{A}^n,
\end{array}
$$

Since the left-hand square is the (derived) base change of the right-hand square along the morphism $f : X \to \mathbf{A}^n$, it follows that the map $\mathcal{O}_X \to p_*(\mathcal{O}_{\widetilde{X}})$ is the inverse image of the canonical map $\mathcal{O}_{\mathbf{A}^n} \to (p_0)_*(\mathcal{O}_{\mathrm{Bl}_{\{0\}/\mathbf{A}^n}})$. Thus we reduce to the case where i is the immersion $\{0\} \to \mathbf{A}^n$. This is well-known, see [BGI71, Example. VII].

4.3.2. Proof of (ii)

It suffices to show the unit map $\mathcal{F} \to q_*(i_D)^!(i_D)_*q^*(\mathcal{F})$ is invertible for all $\mathcal{F} \in$ Qcoh(Z). As in the previous claim we may assume X is affine and that $\mathcal{F} = \mathcal{O}_Z$. Using Proposition 4.2, the canonical identification $\mathcal{N}_{D/\widetilde{X}} \simeq \mathcal{O}_D(1)$, and Lemma 4.1, the unit map is identified with

$$\mathcal{O}_Z \to q_*\big((i_D)^*(i_D)_*(\mathcal{O}_D) \otimes \mathcal{O}_D(-1)\big)[-1]$$

Since $q : D \to Z$ is the projection of the projective bundle $\mathbf{P}(\mathcal{N}_{Z/X})$, it follows from Proposition 3.2 that we have identifications $q_*(\mathcal{O}_D(-1)) \simeq 0$ and $q_*(\mathcal{O}_D) \simeq \mathcal{O}_Z$, under which the map in question is the identity.

4.3.3. Proof of (iii)

To see that $\mathbf{D}(-k)$ is right orthogonal to $\mathbf{D}(0)$, observe that by Theorem 3.3, the mapping space

$$\mathrm{Maps}(p^*(\mathcal{F}_X), (i_D)_*(q^*(\mathcal{F}_Z) \otimes \mathcal{O}(-k))) \simeq \mathrm{Maps}(q^*i^*(\mathcal{F}_X), q^*(\mathcal{F}_Z) \otimes \mathcal{O}(-k))$$

is contractible for every $\mathcal{F}_X \in \mathrm{Qcoh}(X)$ and $\mathcal{F}_Z \in \mathrm{Qcoh}(Z)$.

To see that $\mathbf{D}(-k)$ is right orthogonal to $\mathbf{D}(-k')$, for $1 \leqslant k' < k$, consider the mapping space

$$\mathrm{Maps}\big((i_D)_*(q^*(\mathcal{F}_Z) \otimes \mathcal{O}(-k')), (i_D)_*(q^*(\mathcal{F}'_Z) \otimes \mathcal{O}(-k))\big),$$

for $\mathcal{F}_Z, \mathcal{F}'_Z \in \mathrm{Qcoh}(Z)$. Using fpqc descent and base change for $(i_D)_*$ against f^* for any morphism $f : U \to \widetilde{X}$, we may reduce to the case where X is affine. Since Qcoh(Z) is then generated under colimits and finite limits by $\mathcal{O}_Z$, we may assume that $\mathcal{F}_Z = \mathcal{F}'_Z = \mathcal{O}_Z$. Then we have

$$\mathrm{Maps}\big((i_D)_*(\mathcal{O}(-k')), (i_D)_*(\mathcal{O}(-k))\big) \simeq \mathrm{Maps}\big((i_D)^*(i_D)_*(\mathcal{O}(-k')), \mathcal{O}(-k)\big)$$
$$\simeq \mathrm{Maps}\big(\mathcal{O}(-k') \oplus \mathcal{O}(-k'+1)[1], \mathcal{O}(-k)\big)$$

by Lemma 4.1 and the projection formula, and this space is contractible by Theorem 3.3.

4.3.4. Proof of (iv)

Denote by $\mathbf{D}$ the full subcategory of Qcoh$(\widetilde{X})$ generated by $\mathbf{D}(0)$, $\mathbf{D}(-1)$, ..., $\mathbf{D}(-n+1)$ under finite colimits and finite limits. The claim is that the inclusion $\mathbf{D} \subseteq$ Qcoh$(\widetilde{X})$ is an equality. Note that $\mathcal{O}_{\widetilde{X}} \in \mathbf{D}(0) \subset \mathbf{D}$ and $(i_D)_*(\mathcal{O}_D(-k)) \in \mathbf{D}(-k) \subset \mathbf{D}$ for $1 \leqslant k \leqslant n - 1$. Consider the exact triangle $\mathcal{O}_{\widetilde{X}}(-D) \to \mathcal{O}_{\widetilde{X}} \to (i_D)_*(\mathcal{O}_D)$ and recall that $\mathcal{O}_{\widetilde{X}}(-D) \simeq \mathcal{O}_{\widetilde{X}}(1)$. Tensoring with $\mathcal{O}(-k)$ and using the projection formula, we get the exact triangle

$$\mathcal{O}_{\widetilde{X}}(-k+1) \to \mathcal{O}_{\widetilde{X}}(-k) \to (i_D)_*(\mathcal{O}_D(-k))$$

for each $1 \leqslant k \leqslant n - 1$. Taking $k = 1$ we deduce that $\mathcal{O}_{\widetilde{X}}(-1) \in \mathbf{D}$. Continuing recursively we find that $\mathcal{O}_{\widetilde{X}}(-k) \in \mathbf{D}$ for all $1 \leqslant k \leqslant n - 1$.

Now let $\mathcal{F} \in \text{Qcoh}(\widetilde{X})$. Denote by $\mathcal{G}_0 \in \text{Qcoh}(\widetilde{X})$ the cofibre of the co-unit map $p^* p_*(\mathcal{F}) \to \mathcal{F}$. Note that $\mathcal{G}_0$ is right orthogonal to $\mathbf{D}(0)$. For $1 \leqslant m \leqslant n-1$ define $\mathcal{G}_m$ recursively by the exact triangles

$$(i_D)_* \Big(q^* q_*((i_D)^!(\mathcal{G}_{m-1}) \otimes \mathcal{O}(m)) \otimes \mathcal{O}(-m) \Big) \xrightarrow{\text{counit}} \mathcal{G}_{m-1} \to \mathcal{G}_m.$$

Just as in the proof of Theorem 3.3, a simple induction argument shows that each $\mathcal{G}_m$ is right orthogonal to all of the subcategories $\mathbf{D}(0), \dots, \mathbf{D}(m-1)$. We now claim that $\mathcal{G}_{n-1}$ is zero; it will follow by recursion that $\mathcal{F}$ belongs to $\mathbf{D}$, as desired.

Since the objects $\mathcal{G}_k$ are stable under base change, we may use fpqc descent and base change to assume that X is affine. Moreover we may assume that $i : Z \to X$ fits in a cartesian square of the form (2.1). By [KR18a, 3.3.6], $p : \widetilde{X} \to X$ factors through a quasi-smooth closed immersion $i' : \widetilde{X} \to \mathbf{P}_X^{n-1}$. Recall from Lemma 3.4 that there is a canonical isomorphism $\varinjlim_{J \subsetneq [n-1]} \mathcal{O}(|J|) \simeq \mathcal{O}(n)$ in $\text{Qcoh}(\mathbf{P}_X^{n-1})$. Applying $(i')^*$, we get

$$\varinjlim_{J \subsetneq [n-1]} \mathcal{O}_{\widetilde{X}}(|J|) \simeq \mathcal{O}_{\widetilde{X}}(n)$$

in $\text{Qcoh}(\widetilde{X})$. In particular, every $\mathcal{O}_{\widetilde{X}}(k)$ belongs to $\mathbf{D}$ for all $k \in \mathbf{Z}$. Recall also that we may find a map $\bigoplus_\alpha \mathcal{O}(d_\alpha)[n_\alpha] \to i'_*(\mathcal{G}_{n-1})$ which is surjective on all homotopy groups (Lemma 3.5). By adjunction this corresponds to a map $\bigoplus_\alpha \mathcal{O}(d_\alpha)[n_\alpha] \to \mathcal{G}_{n-1}$ (which is also surjective on homotopy groups). But the source belongs to $\mathbf{D}$, and the target is right orthogonal to $\mathbf{D}$, so this map is null-homotopic. Thus $\mathcal{G}_{n-1}$ is zero.

4.4. Proof of Theorem C

We now deduce Theorem C from Theorem 4.3. First note that the fully faithful functor $\mathcal{F} \mapsto p^*(\mathcal{F})$ of Theorem 4.3 (i) preserves perfect complexes and therefore restricts to a fully faithful functor $\text{Perf}(X) \to \text{Perf}(\text{Bl}_{Z/X})$. This shows Theorem C (i).

Similarly, part (ii) follows from the fact that the functors q^* and $(i_D)_*$ preserve perfect complexes. For the latter, this is because i_D is quasi-smooth (and hence of finite presentation and of finite tor-amplitude).

For part (iii) we argue again as in the proof of Theorem 4.3 (iv). The point is that if $\mathcal{F} \in \text{Qcoh}(\text{Bl}_{Z/X})$ is perfect, then so is each $\mathcal{G}_m \in \text{Qcoh}(\mathbf{P}(\mathcal{E}))$, since q^*, q_*, $(i_D)_*$ and $(i_D)^!$ all preserve perfect complexes. For the latter this follows from Proposition 4.2.

4.5. Blow-up formula

By Theorem C and Lemma 2.8 we get:

COROLLARY 4.4. — *Let X be a derived Artin stack and $i : Z \to X$ a quasi-smooth closed immersion of virtual codimension $n \geqslant 1$. Then for any additive invariant E, there is a canonical isomorphism*

$$E\left(\text{Bl}_{Z/X}\right) \simeq E(X) \oplus \bigoplus_{k=1}^{n-1} E(Z).$$

4.5.1. Proof of Theorem A

Combine Corollaries 4.4 and 3.6 $\left(\text{with } \mathcal{E} = \mathcal{N}_{Z/X}\right)$.

5. The cdh topology

5.1. Definitions

The following notion was introduced by Voevodsky [Voe10b] for noetherian schemes:

DEFINITION 5.1. — *Suppose given a cartesian square Q of algebraic spaces*

(5.1)
$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{e} & X. \end{array}$$

(i) *We say that Q is a* Nisnevich square *if e is an open immersion, and p is an étale morphism inducing an isomorphism $(Y \smallsetminus B)_{\mathrm{red}} \simeq (X \smallsetminus A)_{\mathrm{red}}$.*

(ii) *We say that Q is a* proper cdh square, *or* abstract blow-up square, *if e is a closed immersion of finite presentation, and p is a proper morphism inducing an isomorphism $(Y \smallsetminus B)_{\mathrm{red}} \simeq (X \smallsetminus A)_{\mathrm{red}}$.*

(iii) *We say that Q is a* cdh square *if it is either a Nisnevich square or a proper cdh square.*

Given any class of commutative squares of algebraic spaces, we say that a presheaf satisfies *descent* for this class if it sends all such squares to homotopy cartesian squares, and the empty scheme to a terminal object. In case of the three classes considered in Definition 5.1, it follows from a theorem of Voevodsky [Voe10a, Corollary 5.10] that descent in this sense is equivalent to Čech descent with respect to the associated Grothendieck topology.

Example 5.2. — Every localizing invariant E satisfies Nisnevich descent when regarded as a presheaf on quasi-compact quasi-separated algebraic spaces with $E(X) = E(\mathrm{Perf}(X))$. This is essentially due to Thomason [TT90] and in the asserted generality is a consequence of the study of compact generation properties of the ∞-categories $\mathrm{Qcoh}(X)$ carried out by Bondal–Van den Bergh [BVdB03].

Example 5.3. — Any quasi-smooth blow-up square (2.2) induces a proper cdh square

$$\begin{array}{ccc} \mathbf{P}\left(\mathcal{N}_{Z/X}|_{Z_{\mathrm{cl}}}\right) & \longrightarrow & \left(\mathrm{Bl}_{Z/X}\right)_{\mathrm{cl}} \\ \downarrow & & \downarrow \\ Z_{\mathrm{cl}} & \longrightarrow & X_{\mathrm{cl}} \end{array}$$

on underlying classical algebraic spaces.

Example 5.4. — Consider the class of proper cdh squares (5.1) where the proper morphism p is a closed immersion (with quasi-compact open complement). The associated Grothendieck topology is the same as the one generated by *closed squares*, i.e. cartesian squares as in (5.1) such that e and p are closed immersions, e is of finite presentation and p has quasi-compact open complement, and $A \sqcup Y \to X$ is surjective on underlying topological spaces.

Example 5.5. — Note that for any algebraic space X, the square

$$\begin{array}{ccc} \varnothing & \longrightarrow & X_{\mathrm{red}} \\ \downarrow & & \downarrow \\ \varnothing & \longrightarrow & X \end{array}$$

is a closed square as in Example 5.4.

5.2. A cdh descent criterion

THEOREM 5.6. — *Let $\mathcal{F}$ be a presheaf on the category $\mathbf{C}$ of algebraic spaces, with values in a stable ∞-category. Then $\mathcal{F}$ satisfies cdh descent if and only if it satisfies the following conditions:*

(i) *It sends the empty scheme to a zero object.*
(ii) *It sends Nisnevich squares to cartesian squares.*
(iii) *It sends closed squares to cartesian squares.*
(iv) *For every $X \in \mathbf{C}$ and every quasi-smooth closed immersion $Z \to X$, it sends the square (Example 5.3)*

$$\begin{array}{ccc} \mathbf{P}\left(\mathcal{N}_{Z/X}|_{Z_{\mathrm{cl}}}\right) & \longrightarrow & \left(\mathrm{Bl}_{Z/X}\right)_{\mathrm{cl}} \\ \downarrow & & \downarrow \\ Z_{\mathrm{cl}} & \longrightarrow & X \end{array}$$

to a cartesian square.

Moreover, the same holds if $\mathbf{C}$ is replaced by the full subcategory of

(a) *quasi-compact quasi-separated (qcqs) algebraic spaces,*
(b) *schemes,*
(c) *or qcqs schemes.*

Remark 5.7. — Any presheaf $\mathcal{F}$ on algebraic spaces can be trivially extended to derived algebraic spaces, by setting $\Gamma(X, \mathcal{F}) = \Gamma(X_{\mathrm{cl}}, \mathcal{F})$ for every derived algebraic space X. The condition (iv) in Theorem 5.6 is equivalent to requiring this extension to satisfy descent for quasi-smooth blow-up squares (2.2).

Example 5.8. — Let E be a localizing invariant of stable ∞-categories. Then it satisfies Nisnevich descent on qcqs algebraic spaces (Example 5.2) and quasi-smooth blow-up descent (Theorem A). Assume that E also satisfies *derived nilpotent invariance*, i.e., that the canonical map $E(X) \to E(X_{\mathrm{cl}})$ is invertible for every derived algebraic space X. Then the condition (iv) in Theorem 5.6 holds. Therefore,

E satisfies cdh descent if and only if it satisfies closed descent. Moreover, by Nisnevich descent it suffices to consider closed squares of affine schemes.

Example 5.9. — In the presence of $\mathbf{A}^1$-homotopy invariance, the Morel–Voevodsky localization theorem [MV99, Theorem 3.2.21] provides the following sufficient condition for closed descent. Let $\mathcal{F}$ be an $\mathbf{A}^1$-invariant Nisnevich sheaf on the category of algebraic spaces. Suppose that, for every algebraic space S, its restriction $\mathcal{F}_S$ to the site of *smooth* algebraic spaces over S is stable under arbitrary base change. That is, for every morphism of algebraic spaces $f : T \to S$, the canonical map $f^*(\mathcal{F}_S) \to \mathcal{F}_T$ is invertible. Then $\mathcal{F}$ satisfies closed descent. This follows immediately from the closed base change formula (cf. [Kha19, Proposition 3.3.2]).

Remark 5.10. — Let E be a localizing invariant and suppose that it is moreover *truncating* in the sense of [LT18]. That is, if R is a connective $\mathcal{E}_1$-ring spectrum and $\mathrm{Mod}_R^{\mathrm{perf}}$ denotes the stable ∞-category of left R-modules, then the canonical map $E(\mathrm{Mod}_R^{\mathrm{perf}}) \to E(\mathrm{Mod}_{\pi_0(R)}^{\mathrm{perf}})$ is invertible. Then Land–Tamme have recently proven that E has closed descent, at least if we restrict to *noetherian* algebraic spaces (see Step 1 in the proof of [LT18, Theorem. A.2]).

Remark 5.11. — There are a few variants of Theorem 5.6 with the same proof. For example:

(i) On the category of (qcqs) schemes, descent with respect to the *rh* topology (generated by Zariski squares and proper cdh squares) can be checked with the same criteria, except that Nisnevich squares are replaced by Zariski squares in condition (ii).

(ii) If we do not assume either Nisnevich or Zariski descent, descent for the proper cdh topology is still equivalent to conditions (i), (iii), and (iv), as long as we restrict to a full subcategory of algebraic spaces or schemes which satisfy Thomason's *resolution property*. For example, this holds on the category of quasi-projective schemes.

(iii) One can extend the criterion to qcqs Artin stacks as follows. The definition of Nisnevich square extends without modification (cf. [HK19, Subsection 2.3]). In the definition of proper cdh square, we add the requirement that the proper morphism p is representable (cf. op. cit.). Then the criterion of Theorem 5.6 holds for stacks *which admit the resolution property Nisnevich-locally*, see (Subsubsection 5.3.4). This condition is relatively mild. For example, many quotient stacks have the resolution property ([Tho87, Lemma 2.4], [HR17, Example 7.5]). By the Nisnevich-local structure theorem of Alper–Hall–Rydh [HK19, Theorem 2.9], any stack with linearly reductive and almost multiplicative stabilizers satisfies the resolution property Nisnevich-locally.

5.3. Proof of Theorem 5.6

Since Nisnevich squares, closed squares, and quasi-smooth blow-up squares are all cdh squares, the conditions are clearly necessary. Conversely suppose that $\mathcal{F}$ is

a presheaf satisfying the conditions and let Q be a proper cdh square of algebraic spaces of the form

(5.2)
$$\begin{array}{ccc} E & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{\ i\ } & X. \end{array}$$

It will suffice to show that the induced square $\Gamma(Q, \mathcal{F})$ is homotopy cartesian.

<div align="center">5.3.1.</div>

Assume first that Q is a blow-up square, i.e., that $Y = \mathrm{Bl}_{Z/X}$ is the blow-up of X centred in Z (and $E = \mathbf{P}(\mathcal{C}_{Z/X})$ is the projectivized normal cone). By Nisnevich descent we may assume that X satisfies the resolution property (e.g. X is affine). Since $i : Z \to X$ is of finite presentation, the ideal of definition $\mathcal{I} \subset \mathcal{O}_X$ is of finite type. Thus by the resolution property there exists a surjection $u : \mathcal{E} \to \mathcal{I}$ with $\mathcal{E}$ a locally free $\mathcal{O}_X$-module of finite rank. Denote by $V = \mathbf{V}_X(\mathcal{E}) = \mathrm{Spec}_X(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{E}))$ the associated vector bundle and $0 : X \to V$ the zero section. The $\mathcal{O}_X$-module homomorphism $u : \mathcal{E} \to \mathcal{I} \subset \mathcal{O}_X$ induces a section of V, whose derived zero-locus $\widetilde{Z}$ fits in the homotopy cartesian square

$$\begin{array}{ccc} \widetilde{Z} & \xrightarrow{\ \widetilde{i}\ } & X \\ \downarrow & & \downarrow{\scriptstyle u} \\ X & \xrightarrow{\ 0\ } & V. \end{array}$$

By construction, $\widetilde{i} : \widetilde{Z} \to X$ is a quasi-smooth closed immersion and there is a canonical morphism $Z \to \widetilde{Z}$ which induces an isomorphism $Z \simeq \widetilde{Z}_{\mathrm{cl}}$. Regarding $\mathcal{F}$ as a presheaf on derived algebraic spaces as in Remark 5.7, the square $\Gamma(Q, \mathcal{F})$ now factors as follows:

$$\begin{array}{ccc} \Gamma(X, \mathcal{F}) & \longrightarrow & \Gamma(\widetilde{Z}, \mathcal{F}) = \!\!=\!\!= \Gamma(Z, \mathcal{F}) \\ \downarrow & & \downarrow \\ \Gamma\left(\mathrm{Bl}_{\widetilde{Z}/X}, \mathcal{F}\right) & \longrightarrow & \Gamma\left(\mathbf{P}(\mathcal{N}_{\widetilde{Z}/X}), \mathcal{F}\right) \\ \downarrow & & \downarrow \\ \Gamma\left(\mathrm{Bl}_{Z/X}, \mathcal{F}\right) & \longrightarrow & \Gamma\left(\mathbf{P}\left(\mathcal{C}_{Z/X}\right), \mathcal{F}\right) \end{array}$$

The upper square is induced by a quasi-smooth blow-up square, hence is cartesian. The lower square is induced by a closed square, hence is also cartesian. Therefore it follows that the outer composite square is also cartesian. This shows that $\mathcal{F}$ satisfies descent for blow-up squares.

5.3.2.

Slightly more generally, suppose that $Y = \mathrm{Bl}_{Z'/X}$ is a blow-up centred in some closed immersion $Z' \to X$ with $|Z'| \subseteq |Z|$ on underlying topological spaces, and let $E' \to Y$ denote the exceptional divisor. Since $\mathcal{F}$ is invariant under nilpotent extensions (Example 5.5) we may assume that $i' : Z' \to X$ actually factors through a closed immersion $Z' \to Z$ (see Example 5.5). Applying descent for blow-up squares (Subsubsection 5.3.1), it will suffice to show that $\mathcal{F}$ satisfies descent for the square

$$
\begin{array}{ccc}
E' & \longrightarrow & E \\
\downarrow & & \downarrow \\
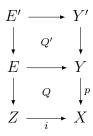Z' & \longrightarrow & Z.
\end{array}
$$

Note that the blow-up $\mathrm{Bl}_{Z'/Z}$ is equipped with a canonical closed immersion into E so that $E' \to E$ and $\mathrm{Bl}_{Z'/Z} \to Z$ form a closed covering. Applying closed descent and descent for the blow-up square associated to $Z' \to Z$ (Subsubsection 5.3.1), we conclude.

5.3.3.

For the case of an arbitrary proper cdh square, we recall the standard argument using Raynaud–Gruson's technique of *platification par éclatements* [RG71, I, Corollary 5.7.12] to reduce to the case considered above (see e.g. [KST18, Subsection 5.2]).

CONSTRUCTION 5.12. — *Let Q be a proper cdh square of the form (5.2). Assume that X is quasi-compact and quasi-separated and that the open subspace $X \smallsetminus Z$ is quasi-compact and dense in X. Then there exists a proper cdh square Q' sitting above Q such that the composite $Q' \circ Q$*

$$
\begin{array}{ccc}
E' & \longrightarrow & Y' \\
\downarrow & \quad Q' & \downarrow \\
E & \longrightarrow & Y \\
\downarrow & \quad Q & \downarrow{\scriptstyle p} \\
Z & \xrightarrow{\;i\;} & X
\end{array}
$$

is of the form considered in (Subsubsection 5.3.2). That is, Y' is a blow-up of X centred in some closed immersion $Z' \to X$ with $|Z'| \subseteq |Z|$. This follows from Raynaud–Gruson just as in the proof of [HK19, Corollary 2.4].

Let Q be a proper cdh square of the form (5.2). Since $\mathcal{F}$ is a Nisnevich sheaf, we may assume that X is quasi-compact and quasi-separated. Since $i : Z \to X$ is of finite presentation, its open complement $U = X \smallsetminus Z$ is quasi-compact. Using closed descent, we can ensure that U is dense in X. Now apply the construction above to get a proper cdh square Q' such that $Q' \circ Q$ is of the form considered in (Subsubsection 5.3.2). Applying the construction again, this time to Q', we end up with a third square Q'' such that the composite $Q'' \circ Q'$ is also of the form considered

in (Subsubsection 5.3.2). Then we know that $\Gamma(Q' \circ Q, \mathcal{F})$ and $\Gamma(Q'' \circ Q', \mathcal{F})$ are both homotopy cartesian. It follows that the square $\Gamma(Q', \mathcal{F})$ is also homotopy cartesian (since $\mathcal{F}$ takes values in a stable ∞-category, it suffices to check that the induced map on homotopy fibres is invertible), and hence so is $\Gamma(Q, \mathcal{F})$.

5.3.4.

We now discuss the extension to stacks mentioned in Remark 5.11 (iii). The precise statement is as follows. Let $\mathbf{C}$ be a category of qcqs Artin stacks such that

(a) every stack $X \in \mathbf{C}$ admits a Nisnevich atlas by stacks with the resolution property;
(b) for every stack $X \in \mathbf{C}$ and every blow-up $Y \to X$, the qcqs Artin stack Y also belongs to $\mathbf{C}$.

Then the statement of Theorem 5.6 holds for presheaves on $\mathbf{C}$.

The proof for the case of a blow-up square (Subsubsection 5.3.1) has been presented in such a way that it holds *mutatis mutandis* under the above assumptions. The argument of [KST18, Claim 5.3] also goes through, using descent for closed squares and blow-up squares, to deal with the slightly more general case where $Y = \mathrm{Bl}_{Z'/X}$ is a blow-up centred in some closed immersion $Z' \to X$ that factors through Z. To reduce a general proper cdh square to that case, we use Rydh's extension of Raynaud–Gruson [HK19, Theorem 2.2]. First, closed descent allows us to assume that $X \smallsetminus Z$ is dense in X. Then we apply Rydh–Raynaud–Gruson just as in the proof of [HK19, Corollary 2.4]. The only difference with the case of schemes or algebraic spaces is that in general we get a *sequence* of $(X \smallsetminus Z)$-admissible blow-ups $\widetilde{X} \to X$ which factors through $p : Y \to X$. The addition of a simple induction is then the only modification required to run the same argument.

5.4. Homotopy invariant K-theory

5.4.1.

For any qcqs algebraic space X, its homotopy invariant K-theory spectrum is given by the formula

$$(5.3) \qquad \Gamma(X, \mathrm{KH}) = \varinjlim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \mathrm{K}(X \times \mathbf{A}^n).$$

In other words, $\Gamma(X, \mathrm{KH})$ is the geometric realization of the simplicial diagram $\mathrm{K}(X \times \mathbf{A}^\bullet)$, where $\mathbf{A}^\bullet$ is regarded as a cosimplicial scheme in the usual way (see e.g. [MV99, p. 45]). This extends the usual definition [Wei89, TT90], and is a way to formally impose the property of $\mathbf{A}^1$-homotopy invariance: for any qcqs algebraic space X, the projection $p : X \times \mathbf{A}^1 \to X$ induces an isomorphism of spectra

$$p^* : \Gamma(X, \mathrm{KH}) \to \Gamma(X \times \mathbf{A}^1, \mathrm{KH}).$$

5.4.2.

As the previous paragraph makes sense when X is derived, we may regard KH as a presheaf on qcqs derived algebraic spaces. Given a Nisnevich square of the form (5.1), Nisnevich descent for K-theory (Example 5.2) yields homotopy cartesian squares of spectra

$$
\begin{array}{ccc}
\mathrm{K}(X \times \mathbf{A}^n) & \longrightarrow & \mathrm{K}(A \times \mathbf{A}^n) \\
\downarrow & & \downarrow \\
\mathrm{K}(Y \times \mathbf{A}^n) & \longrightarrow & \mathrm{K}(B \times \mathbf{A}^n)
\end{array}
$$

for every $[n] \in \boldsymbol{\Delta}^{\mathrm{op}}$. Passing to the colimit over n, we deduce that KH also satisfies Nisnevich descent. We have:

THEOREM 5.13. — *For every qcqs derived algebraic space S, the canonical morphism of spectra*

$$
\Gamma(S, \mathrm{KH}) \to \Gamma(S_{\mathrm{cl}}, \mathrm{KH})
$$

is invertible.

Proof. — By Nisnevich descent, we may as well assume S is an affine derived scheme. Let KH_S denote the restriction of KH to the site of affine derived schemes that are smooth and of finite presentation over S. This is still an $\mathbf{A}^1$-homotopy invariant Nisnevich sheaf, and it is equipped with a canonical morphism

$$
\mathrm{K}_S^{\mathrm{cn}} \to \mathrm{K}_S \to \mathrm{KH}_S,
$$

where $\mathrm{K}_S^{\mathrm{cn}}$ and K_S are the respective restrictions of connective and non-connective K-theory. By Cisinski, this morphism exhibits KH_S as the *Bott periodization* of the $\mathbf{A}^1$-localization of $\mathrm{K}_S^{\mathrm{cn}}$, i.e., the periodization with respect to the Bott element $b \in \mathrm{K}_1(\mathbf{G}_{m,S})$ (the proof is the same as in the case where S is classical [Cis13, Corollary 2.12]). It follows from this description that for any morphism of affine derived schemes $f : T \to S$, there is a canonical isomorphism $f^*(\mathrm{KH}_S) \simeq \mathrm{KH}_T$, where f^* denotes the functor of inverse image of $\mathbf{A}^1$-invariant Nisnevich sheaves. Indeed, we reduce to checking the same property for $\mathrm{K}_S^{\mathrm{cn}}$, which is clear as this is identified up to Zariski localization with the group completion of the presheaf $\coprod_n \mathrm{BGL}_{n,S}$.

In particular, we get a canonical isomorphism $i^*(\mathrm{KH}_S) \simeq \mathrm{KH}_{S_{\mathrm{cl}}}$, where $i : S_{\mathrm{cl}} \to S$ is the inclusion of the underlying classical scheme. Moreover, i^* induces an equivalence between the ∞-categories of $\mathbf{A}^1$-invariant Nisnevich sheaves on S and S_{cl}, respectively, by [Kha16, Corollary 1.3.5]. We deduce that the canonical morphism

$$
\Gamma(S, \mathrm{KH}) \simeq \Gamma(S, \mathrm{KH}_S) \to \Gamma(S_{\mathrm{cl}}, \mathrm{KH}_{S_{\mathrm{cl}}}) \simeq \Gamma(S_{\mathrm{cl}}, \mathrm{KH})
$$

is invertible. □

5.4.3. Proof of Theorem D

We use the criterion of Theorem 5.6. Condition (i) is obvious. Nisnevich descent (condition (ii)) was verified above (Subsubsection 5.4.2). For condition (iv), it will

suffice by Theorem 5.13 and Remark 5.7 to show that KH sends quasi-smooth blow-up squares of derived algebraic spaces to homotopy cartesian squares. This follows from the same property for K-theory (Theorem A) using the formula (5.3) (just as in the proof of Nisnevich descent). For closed descent (condition (iii)), we may restrict our attention to closed squares of *affine* schemes (by Nisnevich descent). This is classical, see [TT90, Exer. 9.11 (f)] or [Wei89, Corollary 4.10]. Alternatively, it follows from the criterion of Example 5.9.

Remark 5.14. — By continuity for KH (e.g. [HK19, Theorem. 4.9 (5)]), once we have descent for proper cdh squares as in Definition 5.1 (ii), we can immediately drop the finite presentation hypothesis on e.

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