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Jessica Guerand

Quantitative parabolic regularity à la De Giorgi

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Jessica Guerand*

Abstract

We deal with the De Giorgi Hölder regularity theory for parabolic equations with rough coefficients. We give a quantitative proof of the interior Hölder regularity of solutions of parabolic equations using De Giorgi method. More precisely, we give a quantitative proof of the last non quantitative step of the method for parabolic equations, namely *the intermediate value lemma*, one of the two main tools of the De Giorgi method sometimes called "second lemma of De Giorgi".

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1 Introduction

Let us first introduce the main results and a historical overview of the elliptic regularity theory of De Giorgi [4].

1.1 Main result

We give a quantitative version of the intermediate value lemma for parabolic equations. Let us consider the following parabolic equation with a source term

$$\partial_t f = \nabla_x \cdot (A \nabla_x f) + B \cdot \nabla_x f + s, \quad t \in (T_1, T_2), x \in \Omega, \tag{1}$$

where T_1 and T_2 are real numbers, d is a positive integer, Ω is an open set of $\mathbb{R}^d$, f is a real-valued function of (t, x), $A = A(t, x)$ a $d \times d$ bounded measurable matrix and A satisfies an ellipticity condition for two positive constants λ, Λ,

$$0 < \lambda I \leq A \leq \Lambda I, \tag{2}$$

and $s = s(t, x)$, $B = B(t, x)$ are bounded measurable coefficients, and satisfy,

$$\begin{cases} |B| \leq \Lambda, \\ |s| \leq 1. \end{cases} \tag{3}$$

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*DPMMS, University of Cambridge, United Kingdom. `jg900@cam.ac.uk`

Let $Q_1 = (T_1, T_2) \times \Omega$ and $C_c^\infty(Q_1)$ denotes the set of smooth functions compactly supported in Q_1. We say that f is a *weak subsolution* of (1) if $f \in L^\infty(T_1, T_2; L^2(\Omega))$ such that $\nabla_x f \in L^2(Q_1)$ and $\partial_t f \in L^2(T_1, T_2; H^{-1}(\Omega))$, and for all $\varphi \in C_c^\infty(Q_1)$ nonnegative we have

$$- \int_{Q_1} f \partial_t \varphi + \int_{Q_1} A \nabla_x f \cdot \nabla_x \varphi - \int_{Q_1} B \cdot \nabla_x f \varphi - \int_{Q_1} s\varphi \leq 0.$$

Let B_R be the ball centered at 0 of radius $R > 0$ in $\mathbb{R}^N$. For $X = (x_1, \ldots, x_N) \in \mathbb{R}^N$, we define $|X| := \sqrt{x_1^2 + \ldots + x_N^2}$ and for $\rho > 0$, the cylinder $Q_\rho = (T_1, T_2) \times B_\rho$. We say that a constant is universal if it depends only on R, T_1, T_2, λ, Λ, and d. Our main theorem is the following intermediate value lemma.

Theorem 1.1 (Parabolic intermediate value lemma)**.** *Let f be a subsolution of* (1) *for* $\Omega = B_{2R}$ *such that* $f \leq 1$ *on* $Q_{\frac{3}{2}R}$. *Let* $\overline{T} = \frac{T_1 + T_2}{2}$, $Q_R^- = (T_1, \overline{T}) \times B_R$ *and* $Q_R^+ = (\overline{T}, T_2) \times B_R$. *Then for all* $(k, l) \in \mathbb{R}^2$ *such that* $k < l \leq 1$, *we have*

$$C \frac{l - k}{2 - k} |\{f \leq k\} \cap Q_R^-| |\{f \geq l\} \cap Q_R^+| \leq |\{k < f < l\} \cap Q_R|^{1/20}, \tag{4}$$

where C is universal.

Remark 1.2. Theorem 1.1 is a step to obtain Hölder regularity with the De Giorgi method (see subsection 2.2). Since this theorem is true not only for solutions but also for subsolutions, it can be used to obtain a quantitative proof of Harnack inequality as in [15]. In the subsection 3.3, we will see that the intervals of time must be disjoint in the subsolution case because there exists counterexamples if they are not (see subsection 3.3.1).

1.2 Historical overview

De Giorgi [4, 5] introduced techniques in 1957 to solve 19th Hilbert problem about the analytic regularity of local minimizers of an energy functional. In fact, these minimizers are solutions of quasilinear Euler-Lagrange equations. The idea of De Giorgi was to see quasilinear elliptic equations as linear elliptic equation with merely mesurable coefficients. Thus he proved the Hölder regularity of solutions of elliptic equations with rough coefficients. In 1958, Nash [24] got the result with different techniques for both elliptic and parabolic equations. Then, Moser [23] proved in 1960 the Hölder regularity with a different approach. These methods are now called the De Giorgi-Nash-Moser techniques.

In his paper [4], De Giorgi exhibited a class of functions that satisfy energy estimates and showed that any function in this class is locally bounded and Hölder continuous. Ladyzhenskaya and Uralt'seva [20] extended his ideas to linear parabolic equations with lower order terms and to quasilinear parabolic equations. They introduced the corresponding De Giorgi classes and proved that Hölder estimate holds when $\pm u$ are both in a De Giorgi class. One can find more details in [19] and in Chapter 6 of [22].

There are extensions of the method in degenerate cases, like the $p-$Laplacian, by Ladyzhenskaya and Ural'tseva [21] in the elliptic case. Then DiBenedetto [6] covered the degenerate parabolic cases, see also DiBenedetto, Gianazza and Vespri [8, 9, 10].

Concerning nonlinear nonlocal time-dependent variational problems, Caffarelli and Vasseur [3] on the first hand and Caffarelli, Chan and Vasseur [1] on the second hand

extended the method of De Giorgi to nonlocal parabolic equations and got a Hölder regularity result for solutions of problems with translation invarient kernels. Also Caffarelli, Soria, Vázquez [2] used the De Giorgi method to prove Hölder continuity of solutions of a porous medium equation with nonlocal diffusion effects. This kind of equation has also been studied earlier by Kassmann [18] using Moser's techniques where he got local regularity results and by Kassmann and Felsinger [13] where they obtained a weak Harnack inequality .

Recently, Golse, Imbert, Mouhot, Silvestre and Vasseur proved the Hölder regularity and obtained Harnack inequalities for kinetic equations. More precisely, the Fokker-Planck kinetic equation with rough coefficients was studied by Golse, Imbert, Mouhot, Vasseur [15] and provides the results for the Landau equation. Imbert and Silvestre [17] studied a class of kinetic integro-differential equations and deduced the results for the inhomogeneous Boltzmann equation without cut-off. The quantitative versions of the intermediate value lemmas in those cases are still an open question.

1.3 Contribution of this note and comparison with existing result

The main contribution of this note is the quantitative proof of the interior Hölder regularity result with De Giorgi method for parabolic equations. It means that we can compute explicitely the Hölder exponant, at least we can give an explicit lower bound. More precisely, we give a quantitative result of one key step of the proof, which was the last non-quantitative step in the parabolic De Giorgi method. This step is sometimes called second lemma of De Giorgi or intermediate value lemma. In the elliptic case there are many quantitative versions of this lemma. De Giorgi [4, 5] obtained a quantitative version using an isoperimetric inequality argument, taken up by DiBenedetto [7] and Vasseur [25]. Recently, Hou and Niu [16] prove a quantitative version of this lemma using a Poincaré inequality. These versions are actually valid for any function in H^1. About parabolic equations, quantitative version of this lemma does not seem to exist. One can find non-quantitative versions, for example in [25], obtained by contradiction with a compactness argument. However, there exists a quantitative version of this lemma for nonlocal time-dependent integral operator [1] that does not apply for local parabolic equations.

In this note, we derive a quantitative version of the intermediate value lemma valid for sub-solutions of parabolic equations with lower order terms and a source term.

1.4 Organization of the note

In Section 2, we recall the main steps of the De Giorgi method for the Hölder regularity of elliptic and parabolic equations in order to understand the use of the intermediate value lemma. In Section 3, we recall and simplify a proof of the intermediate value lemma in the elliptic case obtained in [16] and prove Theorem 1.1, the parabolic version.

2 De Giorgi method for elliptic and parabolic equations

We give the main steps of the De Giorgi method for elliptic and parabolic equations. In this section, a universal constant is a constant which only depends on λ, Λ and d.

2.1 Elliptic equation

Let us introduce the theorem of Hölder regularity obtained by De Giorgi [4] and recall the main tools for the proof.

2.1.1 De Giorgi Theorem

We consider the following elliptic equation

$$-\nabla.(A\nabla u) = 0 \quad \text{for } x \in B_2, \tag{5}$$

where $A = A(x)$ a $d \times d$ bounded measurable matrix satisfying (2). We say that u is a *weak solution* of (5), if $u \in H^1(B_2)$ and for all $\varphi \in H_0^1(B_2)$ we have

$$\int_{B_2} A\nabla u \cdot \nabla \varphi = 0.$$

In the following, we will use the word "solution" instead of "weak solution". De Giorgi proved the following Hölder regularity theorem.

Theorem 2.1 (Hölder continuity: elliptic case). *Let $u : B_2 \to \mathbb{R}$ be a solution of* (5). *Then $u \in C^\alpha(B_1)$ with*

$$\|u\|_{C^\alpha(B_1)} \leq C\|u\|_{L^2(B_2)},$$

where $C > 0$ and α are universal contants.

Remark 2.2. Thanks to the scaling property, Theorem 2.1 holds true for all Ω' and Ω instead of B_1 and B_2, such that $\Omega' \subset\subset \Omega$ where Ω is a bounded subset of $\mathbb{R}^d$. Indeed, let B, C, D be real constants and $x \in \Omega'$, the function $\overline{u}(y) = Bu(x + Cy) + D$ is also a solution of (5) for a matrix $\overline{A}$ which satisfies the same ellipticity condition (2) as the initial matrix A.

Let us introduce the positive part of a function $f_+ = \max(f, 0)$ and its oscillation on a set E,

$$\operatorname*{osc}_{x \in E} f(x) = \sup_{x \in E} f(x) - \inf_{x \in E} f(x).$$

We are going to give the main steps of the De Giorgi method to prove this theorem. We first explain how we can reduce Theorem 2.1 to a lemma called "lowering of the maximum". This lemma states that if a solution is below 1 and mostly below 0, then it is far from 1 in a smaller ball. To prove this lemma we need two essential results. The first result is a $L^2 - L^\infty$ estimate, sometimes called the first lemma of De Giorgi. It states that a solution bounded in L^2 is in fact bounded in L^∞ in a smaller ball. The second one is the intermediate value lemma which quantifies in measure the fact that solutions of these equations cannot make a jump between two numerical values.

Before describing the method, two following lemmas will be needed. The first lemma is a consequence of the $L^2 - L^\infty$ estimate we will present next in the De Giorgi method.

Lemma 2.3 ($L^2 - L^\infty$ bound)**.** *Let $u : B_2 \to \mathbb{R}$ be a solution of* (5). *Then we have*

$$\|u\|_{L^\infty\left(B_{3/2}\right)} \leq C\|u\|_{L^2(B_2)},$$

where C is a universal constant.

We also need the following lemma which is a consequence of an energy estimate.

Lemma 2.4 (Caccioppoli estimate)**.** *Let $u : B_2 \to \mathbb{R}$ be a solution of* (5). *Then we have*

$$\|\nabla u_+\|_{L^2(B_1)} \leq C\|u_+\|_{L^2\left(B_{3/2}\right)},$$

where C is a universal constant.

Proof of Lemma 2.4. The function u_+ is a subsolution of (5) by Stampacchia's theorem. Let $\varphi = u_+\phi^2$ where ϕ is a nonnegative C^∞ cut-off function equal to 1 in B_1 and to 0 outside $B_{\frac{3}{2}}$ such that $0 \leq \phi \leq 1$ and $|\nabla\phi| \leq 4$. Then we have

$$\int_{B_2} A\nabla u_+ \cdot \nabla(u_+)\phi^2 \leq -2 \int_{B_2} A(\phi\nabla u_+) \cdot (u_+\nabla\phi).$$

So we have using Young inequality,

$$
\begin{aligned}
\lambda \int_{B_2} |\nabla u_+|^2\phi^2 &\leq 2\Lambda \int_{B_2} |\phi\nabla u_+||u_+\nabla\phi| \\
&\leq \frac{\lambda}{2} \int_{B_2} |\nabla u_+|^2\phi^2 + \frac{2\Lambda^2}{\lambda} \int_{B_2} |\nabla\phi|^2 u_+^2.
\end{aligned}
$$

Then

$$\int_{B_2} |\nabla u_+|^2\phi^2 \leq \frac{4\Lambda^2}{\lambda^2} \int_{B_2} |\nabla\phi|^2 u_+^2,$$

and we deduce the result. $\qquad\square$

2.1.2 De Giorgi method

Let us now introduce the steps of the proof.

Preliminary step: Reduction of the problem.

We first prove that one can reduce Theorem 2.1 to the lowering of the maximum lemma (Lemma 2.7). The Hölder continuity is a consequence of the following lemma.

Lemma 2.5 (Traduction of the definition)**.** *Let $u : B_2 \to \mathbb{R}$ be a solution of* (5). *Then u satisfies*

$$\forall x_0 \in B_1, \forall r \in (0, 1/2), \quad \underset{B_r(x_0)}{\mathrm{osc}}\, u \leq Cr^\alpha\|u\|_{L^2(B_2)},$$

where C, α are universal constants.

We assume that Lemma 2.5 is true and prove Theorem 2.1.

Proof of Theorem 2.1. Let $(x, y) \in B_1^2$, $x_0 = (x + y)/2$, $x_1 = (x + x_0)/2$ and $y_1 = (x_0 + y)/2$. We choose $r = |x - x_1| = (|x - y|)/4$. Then we have

$$|f(x) - f(x_0)| \leq \operatorname*{osc}_{B_r(x_1)} u \leq \frac{C}{4^\alpha} \|u\|_{L^2(B_2)} |x - y|^\alpha.$$

We compute the same inequality with $|f(x_0) - f(y)|$ and deduce the result thanks to a triangular inequality. □

In order to get the algebraic decay of the oscillation, it is enough to prove that the rate of decay of the oscillation between the ball of radius $3/2$ and the ball of radius $1/2$ is universally bounded by a constant strictly smaller than 1.

Lemma 2.6 (Local decrease of the oscillation)**.** *Let $u : B_2 \to \mathbb{R}$ be a solution of* (5)*. Then there exists a universal constant $\theta \in (0, 1)$ such that*

$$\operatorname*{osc}_{B_{1/2}} u \leq \theta \operatorname*{osc}_{B_{3/2}} u.$$

We assume that Lemma 2.6 is true and prove Lemma 2.5.

Proof of Lemma 2.5. Let us define for $n \in \mathbb{N} \setminus \{0\}$, a sequence $u_n(y) = u(x_0 + y/3^n)$ of solutions of (5) with different matrices which satisfy the ellipticity condition (2). Then using Lemma 2.6, we have

$$\operatorname*{osc}_{B_{3/2}} u_n = \operatorname*{osc}_{B_{1/2}} u_{n-1} \leq \theta \operatorname*{osc}_{B_{3/2}} u_{n-1}.$$

By induction and using Lemma 2.3 we deduce for all $n \geq 1$

$$\operatorname*{osc}_{B_{\frac{3}{2} \frac{1}{3^n}}(x_0)} u = \operatorname*{osc}_{B_{3/2}} u_n \leq \theta^{n-1} \operatorname*{osc}_{B_{3/2}} u_1 \leq \theta^{n-1} \operatorname*{osc}_{B_{3/2}} u \leq \theta^{n-1} C \|u\|_{L^2(B_2)}.$$

Assuming $\theta > 1/3$, we choose $\alpha \in (0, 1)$ such that $\theta = 1/3^\alpha$. For all $r \in (0, 1/2)$, there exists $n \in \mathbb{N} \setminus \{0\}$ such that $\frac{3}{2} \frac{1}{3^{n+1}} \leq r < \frac{3}{2} \frac{1}{3^n}$. We deduce from the previous inequality that

$$\operatorname*{osc}_{B_r(x_0)} u \leq \operatorname*{osc}_{B_{\frac{3}{2} \frac{1}{3^n}}(x_0)} u \leq \left(\frac{1}{3^n}\right)^\alpha C 3^\alpha \|u\|_{L^2(B_2)} \leq r^\alpha C 9^\alpha \|u\|_{L^2(B_2)}.$$

This achieves the proof of the lemma. □

The local decrease of the oscillation is a consequence of the following result.

Lemma 2.7 (Lowering the maximum)**.** *There exists a universal constant $\mu \in (0, 1)$ such that for any solution $v : B_2 \to \mathbb{R}$ of* (5) *satisfying*

$$\begin{cases} v \leq 1 \ in \ B_{3/2} \\ |\{v \leq 0\} \cap B_1| \geq |B_1|/2, \end{cases} \tag{6}$$

we have

$$v \leq 1 - \mu \quad in \ B_{1/2}.$$

We assume that Lemma 2.7 is true and prove Lemma 2.6.

Proof of Lemma 2.6. We set $v = \frac{2}{\operatorname*{osc}_{B_{3/2}} u} \left(u - \frac{\sup u + \inf u}{2} \right)$, where the supremum and the infimum are taken in $B_{3/2}$. Thus $-1 \leq v \leq 1$ in $B_{3/2}$ and either v or $-v$ satisfy (6). We deduce that

$$\operatorname*{osc}_{B_{1/2}} v \leq 2 - \mu,$$

and

$$\operatorname*{osc}_{B_{1/2}} u \leq (1 - \mu/2) \operatorname*{osc}_{B_{3/2}} u.$$

This implies Lemma 2.6 with $\theta = 1 - \mu/2$. □

Proof of Lemma 2.7.

Let us introduce two main results for the proof of Lemma 2.7.

Lemma 2.8 (First lemma of De Giorgi: $L^2 - L^\infty$ estimate)**.** *There exists a positive universal constant δ such that for any solution $u : B_2 \to \mathbb{R}$ of (5) the following implication holds true. If*

$$\int_{B_1} u_+^2 \leq \delta,$$

then we have

$$u_+ \leq \frac{1}{2} \quad in \ B_{1/2}.$$

Remark 2.9. The previous lemma is a consequence of energy estimates and Sobolev inequalities. Here we admit it. One can find the proof in [25]. We have the same result if B_1 and $B_{1/2}$ are replaced by B_2 and $B_{3/2}$, so applying this lemma to u and $-u$, we deduce Lemma 2.3.

We also have the following result thanks to Hou and Niu [16] and DiBenedetto [7] called sometimes the second lemma of De Giorgi.

Lemma 2.10 (Second lemma of De Giorgi: intermediate value lemma in H^1)**.** *Let $u \in H^1(B_1)$. Then we have*

$$\left| \{u \leq 0\} \cap B_1 \right| \times \left| \{u \geq 1/2\} \cap B_1 \right| \leq C \left| \{0 < u < 1/2\} \cap B_1 \right|^{1/2} \sqrt{\int_{B_1} |\nabla u_+(x)|^2 \, \mathrm{d}x},$$

where C only depends on d, λ and Λ.

Remark 2.11. One can find a slightly different version of Lemma 2.10 in [25] which uses an isoperimetric argument instead of a Poincaré inequality as in the proof of Lemma 3.1. This lemma is true not only for solutions of (5) but also for all functions in H^1.

Proof of Lemma 2.10. We apply Lemma 3.1 with $k = 0$ and $l = 1/2$. □
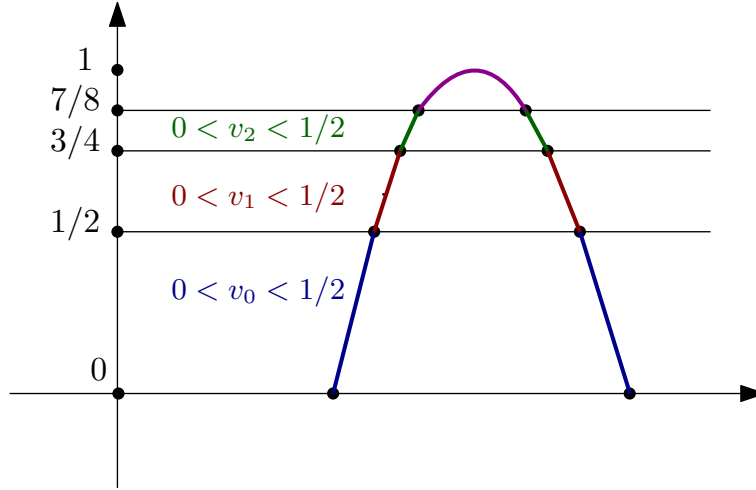
Now it remains to prove Lemma 2.7.

Figure 1: Intermediate set at each step.

Proof of Lemma 2.7. We introduce a sequence of solutions of (5),

$$\begin{cases} v_0 = v \\ v_k = 2\left(v_{k-1} - 1/2\right). \end{cases}$$

So we have $v_k = 2^k \left(v - (1 - 2^{-k})\right)$. We consider two cases. Either there exists k_0 such that $\int_{B_1} (v_{k_0})_+^2 \leq \delta$. By Lemma 2.8 we have $(v_{k_0})_+ \leq 1/2$ in $B_{1/2}$ so that $v \leq 1 - 1/2^{k_0+1}$ in $B_{1/2}$. Or for all $k \in \mathbb{N}$, we have $\int_{B_1} (v_k)_+^2 > \delta$. In this case, we deduce that

$$|\{v_k \geq 1/2\} \cap B_1| = |\{v_{k+1} \geq 0\} \cap B_1| \geq \int_{B_1} (v_{k+1})_+^2 > \delta,$$

and

$$|\{v_k \leq 0\} \cap B_1| \geq |\{v \leq 0\} \cap B_1| \geq |B_1|/2.$$

The fact that v_k is a solution of (5) and that $0 \leq (v_k)_+ \leq 1$ implies thanks to Lemma 2.4, that $\sqrt{\int_{B_1} |\nabla (v_k)_+(x)|^2 \, dx} \leq C$ where C is universal. So thanks to Lemma 2.10, there exists a universal constant $\gamma > 0$ such that

$$|\{0 < v_k < 1/2\} \cap B_1| \geq \gamma. \tag{7}$$

In Figure 1, one can see that the intermediate sets at each step are disjoints and because of (7), there cannot exists infinitely many steps. We deduce by induction that

$$\begin{aligned} |B_1| \geq |\{v_k \leq 0\} \cap B_1| &\geq |\{v_{k-1} \leq 0\} \cap B_1| + |\{0 < v_{k-1} < \tfrac{1}{2}\} \cap B_1| \\ &\geq |\{v_{k-1} \leq 0\} \cap B_1| + \gamma \\ &\geq \tfrac{|B_1|}{2} + k\gamma, \end{aligned}$$

that gives a contradiction for k big enough. So only the first case holds true and we deduce the result. Note that $k_0 \leq |B_1|/\gamma$ so $v \leq 1 - 1/2^{k_0+1} \leq 1 - 1/2^{|B_1|/\gamma+1}$ and we choose $\mu = 1/2^{|B_1|/\gamma+1}$. $\qquad\square$

2.2 Parabolic equation

Let us introduce the theorem of Hölder regularity for parabolic equations and recall the main tools for the proof.

2.2.1 Hölder regularity Theorem

We define the parabolic cylinder $Q_r = (-r^2, 0) \times B_r$. We study the following parabolic equation

$$\partial_t f = \nabla_x \cdot (A \nabla_x f) + B \cdot \nabla_x f + s, \quad (t,x) \in Q_2 \tag{8}$$

where $s = s(t,x)$, $B = B(t,x)$ are bounded measurable coefficients, $A = A(t,x)$ as before satisfies (2) and B, s satisfy (3). Let us state the Hölder regularity theorem in the parabolic case.

Theorem 2.12 (Hölder continuity: parabolic case)**.** *Let $u : Q_2 \to \mathbb{R}$ be a solution of* (8)*. Then $u \in C^\alpha(Q_1)$ with*

$$\|u\|_{C^\alpha(Q_1)} \le C \left(\|u\|_{L^2(Q_2)} + 1 \right),$$

where C and α are universal constants.

One can find the full proof in [15].

Remark 2.13. For the same reason as in the elliptic case (the scaling property), Theorem 2.12 holds true for all $Q' = (s,T) \times \Omega'$ and $Q = (0,T) \times \Omega$ such that $\Omega' \subset\subset \Omega$ and $0 < s < T$, instead of Q_1 and Q_2.

We have the same two lemmas as the elliptic case. Here we admit them as the proofs are very similar.

Lemma 2.14 ($L^2 - L^\infty$ bound)**.** *Let $u : Q_2 \to \mathbb{R}$ be a solution of* (8)*. Then we have*

$$\|u\|_{L^\infty(Q_{3/2})} \le C \|u\|_{L^2(Q_2)}.$$

Lemma 2.15 (Gradient L^2 bound)**.** *Let $u : Q_2 \to \mathbb{R}$ be a solution of* (8)*. Then we have*

$$\|\nabla u_+\|_{L^2(Q_1)} \le C \left(\|u_+\|_{L^2(Q_{3/2})} + 1 \right).$$

2.2.2 De Giorgi method

Let us consider the following hypothesis for the source term

$$|s| \le \beta, \tag{9}$$

where $\beta = 1/4^{|Q_1|/\gamma}$ is a universal constant smaller than 1, with $\gamma = \left(\overline{C} \delta |Q_1|/2 \right)^{20}$ where δ is the universal constant given in Lemma 2.19 and $\overline{C}$ a universal constant corresponding to $C \frac{l-k}{2-k}$ in Theorem 1.1 (here $k = 0$ and $l = 1/2$).
    Now we can introduce the steps of the proof.

Preliminary step: Reduction of the problem.

We prove step by step that one can reduce Theorem 2.12 to Lemma 2.18. Indeed, the Hölder continuity is a consequence of the following lemma.

Lemma 2.16 (Traduction of the definition). *Let $u : Q_2 \to \mathbb{R}$ be a solution of* (8) *such that the source term s satisfies* (9). *Then u satisfies*

$$\forall (t_0, x_0) \in Q_1, \forall r \in (0, 1/2), \quad \underset{Q_r(t_0, x_0)}{\mathrm{osc}}\, u \leq Cr^\alpha \left(\|u\|_{L^2(Q_2)} + 1 \right),$$

where C and α are universal constants.

We assume that Lemma 2.16 is true and prove Theorem 2.12.

Proof of Theorem 2.12. The function βu is a solution of (8) which satisfies (9). The arguments of the proof of Lemma 2.5 remains true in this case by replacing B_1 and B_2 by Q_1 and Q_2. So we get the result for βu and then for u. $\qquad\square$

The previous lemma is a consequence of the following oscillation decrease. This version of the lemma is slightly different from the elliptic case because of the source term.

Lemma 2.17 (Local decrease of the oscillation). *Let $u : Q_2 \to \mathbb{R}$ be a solution of* (8) *such that the source term s satisfies* (9). *Then there exists a universal constant $\theta \in \left(\frac{1}{2}, 1 \right)$ such that*

- *if $\underset{Q_{3/2}}{\mathrm{osc}}\, u \geq 2$,   then $\underset{Q_{1/2}}{\mathrm{osc}}\, u \leq \theta \underset{Q_{3/2}}{\mathrm{osc}}\, u$,*

- *if $\underset{Q_{3/2}}{\mathrm{osc}}\, u \leq 2$,   then $\underset{Q_{1/2}}{\mathrm{osc}}\, u \leq 2\theta$.*

We assume that Lemma 2.17 is true and prove Lemma 2.16.

Proof of Lemma 2.16. Let us define for $n \in \mathbb{N} \setminus \{0\}$ a sequence of solutions of (8) with a source term s satisfying (9) (since $1/9\theta < 1$),

$$u_n(\tau, y) = \frac{2\theta^{1-n}}{\max(2, \underset{Q_{3/2}}{\mathrm{osc}}\, u)} u\left(t_0 + \frac{\tau}{9^n}, x_0 + \frac{y}{3^n} \right).$$

By induction let us prove that for all $n \in \mathbb{N} \setminus \{0\}$,

$$\underset{Q_{1/2}}{\mathrm{osc}}\, u_n \leq 2\theta. \tag{10}$$

Indeed for $n = 1$, we have (10) thanks to Lemma 2.17. Assuming that $\underset{Q_{1/2}}{\mathrm{osc}}\, u_{n-1} \leq 2\theta$ and using Lemma 2.17, we distinguish two cases. If $\underset{Q_{3/2}}{\mathrm{osc}}\, u_n \leq 2$, we have (10). If $\underset{Q_{3/2}}{\mathrm{osc}}\, u_n \geq 2$, we have

$$\underset{Q_{1/2}}{\mathrm{osc}}\, u_n \leq \theta \underset{Q_{3/2}}{\mathrm{osc}}\, u_n = \underset{Q_{1/2}}{\mathrm{osc}}\, u_{n-1} \leq 2\theta,$$

and we deduce (10). So using (10) for $n - 1$ we have,

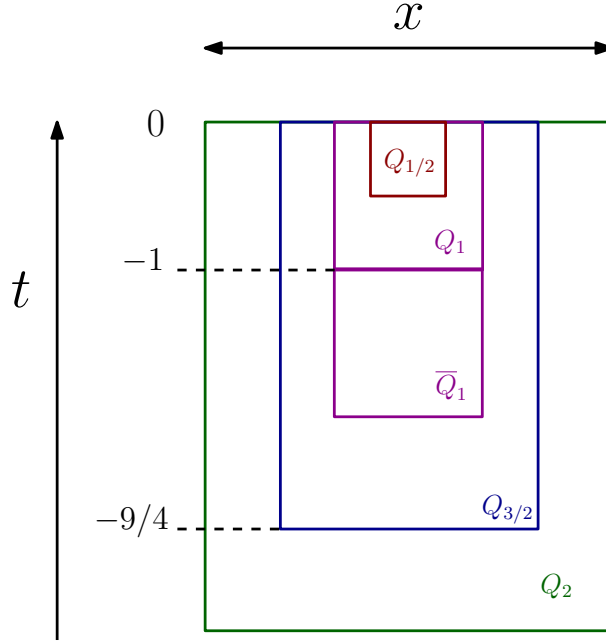$$\underset{Q_{3/2}}{\mathrm{osc}}\, u_n = \frac{1}{\theta} \underset{Q_{1/2}}{\mathrm{osc}}\, u_{n-1} \leq 2.$$

Figure 2: Parabolic cylinders.

Thus we deduce by induction and using Lemma 2.14 that for all $n \geq 1$,

$$\underset{Q_{\frac{3}{2}\frac{1}{3^n}}(x_0)}{\mathrm{osc}}\, u = \frac{\max(2,\, \underset{Q_{3/2}}{\mathrm{osc}}\, u)\theta^{n-1}}{2} \underset{Q_{\frac{3}{2}}}{\mathrm{osc}}\, u_n \leq \theta^{n-1}\max(2,\, \underset{Q_{\frac{3}{2}}}{\mathrm{osc}}\, u) \leq \theta^{n-1}C\left(\|u\|_{L^2(Q_2)}+1\right).$$

We choose $\alpha \in (0,1)$ such that $\theta = 1/3^\alpha$. Let $r \in (0,1/2)$. In particular there exists $n \in \mathbb{N} \setminus \{0\}$ such that $\frac{3}{2}\frac{1}{3^{n+1}} \leq r < \frac{3}{2}\frac{1}{3^n}$. So we deduce that

$$\underset{Q_r(x_0)}{\mathrm{osc}}\, u \leq \underset{Q_{\frac{3}{2}\frac{1}{3^n}}(x_0)}{\mathrm{osc}}\, u \leq \left(\frac{1}{3^n}\right)^\alpha C3^\alpha\left(\|u\|_{L^2(Q_2)}+1\right) \leq r^\alpha C9^\alpha\left(\|u\|_{L^2(Q_2)}+1\right).$$

$\square$

We define $\overline{Q}_r = (-2r^2, -r^2] \times B_r$. The local decrease of the oscillation is a consequence of the following result.

Lemma 2.18 (Lowering the maximum). *There exists a universal constant $\mu \in (0,1)$ such that for any solution $v : Q_2 \to \mathbb{R}$ of (8) with a source term s satisfying (9), if v verifies*

$$\begin{cases} v \leq 1 \ in \ Q_{3/2} \\ |\{v \leq 0\} \cap \overline{Q}_1| \geq |\overline{Q}_1|/2, \end{cases} \tag{11}$$

then

$$v \leq 1 - \mu \quad in \ Q_{\frac{1}{2}}.$$

These cylinders are represented in Figure 2. We assume that Lemma 2.18 is true and prove Lemma 2.17.

Proof of Lemma 2.17. We distinguish two cases: either $\underset{Q_{3/2}}{\operatorname{osc}} u \geq 2$ or $\underset{Q_{3/2}}{\operatorname{osc}} u \leq 2$. In the

first case, we set $v = \frac{2}{\underset{Q_{3/2}}{\operatorname{osc}} u} \left(u - \frac{\sup u + \inf u}{2} \right)$, where the supremum and the infimum are

taken in $Q_{3/2}$. So v is a solution of (8) with $\overline{A}, \overline{B}$ and $\overline{s}$ satisfying (2),(3) and (9). And following the same steps as the proof of Lemma 2.6, we deduce the result. In the second case, we set $v = u - \frac{\sup u + \inf u}{2}$. The functions v and $-v$ are solutions of (8) with a source term s satisfying (9). And either v or $-v$ satisfies (11). So we have in both cases

$$\underset{Q_{1/2}}{\operatorname{osc}} u = \underset{Q_{1/2}}{\operatorname{osc}} v \leq 2 - \mu.$$

We deduce the result taking $\theta = \max(1 - \mu/2)$. □

Proof of Lemma 2.18.

Let us introduce two main results for the proof of Lemma 2.18 where we consider solutions such that the source term s satisfies $|s| \leq 1$ so it does not depend on the universal constant β.

Lemma 2.19 (First lemma of De Giorgi: $L^2 - L^\infty$ estimate)**.** *There exists a positive universal constant δ such that for any solution $u : Q_2 \to \mathbb{R}$ of (8) the following implication holds true. If*

$$\int_{Q_1} u_+^2 \leq \delta,$$

then we have

$$u_+ \leq \frac{1}{2} \quad in \ Q_{1/2}.$$

Remark 2.20. The previous lemma is a consequence of energy estimates and Sobolev inequalities. Here we admit it. One can find the proof in [15, Theorem 12]. We have the same result for Q_2 and $Q_{3/2}$, so applying this lemma to u and $-u$, we deduce Lemma 2.14.

Here, the second result is the parabolic intermediate value lemma, Theorem 1.1, sometimes called second lemma of De Giorgi.

Now let us prove Lemma 2.18.

Proof of Lemma 2.18. We introduce a sequence of solutions of (8),

$$\begin{cases} v_0 = v \\ v_k = 2\left(v_{k-1} - 1/2 \right). \end{cases}$$

Here v is a solution of (8) where the source term satisfies (9) and the functions v_k are solutions of (8) where the source term satisfies $|s| \leq 1$ (it will be explain at the end of the proof). More precisely, we have $v_k = 2^k \left(v - (1 - 2^{-k}) \right)$. We consider two cases. Either there exists k_0 such that $\int_{Q_1} (v_{k_0})_+^2 \leq \delta$. By Lemma 2.19 we have $(v_{k_0})_+ \leq 1/2$ in $Q_{1/2}$ so that $v \leq 1 - 1/2^{k_0+1}$ in $Q_{1/2}$. Or for all $k \in \mathbb{N}$, we have $\int_{Q_1} (v_k)_+^2 > \delta$. In this case, we deduce that

$$|\{v_k \geq 1/2\} \cap Q_1| = |\{v_{k+1} \geq 0\} \cap Q_1| \geq \int_{Q_1} (v_{k+1})_+^2 > \delta,$$

and
$$|\{v_k \le 0\} \cap \overline{Q}_1| \ge |\{v \le 0\} \cap \overline{Q}_1| \ge \frac{|\overline{Q}_1|}{2}.$$

We define $\tilde{Q} = Q_1 \cup \overline{Q}_1$. So thanks to Theorem 1.1, there exists $\gamma > 0$ only depending on λ, Λ and d such that
$$|\{0 < v_k < 1/2\} \cap \tilde{Q}| \ge \gamma.$$

Then we deduce recursively that

$$
\begin{aligned}
|\tilde{Q}| \ge |\{v_k \le 0\} \cap \tilde{Q}| &\ge |\{v_{k-1} \le 0\} \cap \tilde{Q}| + |\{0 < v_{k-1} < \tfrac{1}{2}\} \cap \tilde{Q}| \\
&\ge |\{v_{k-1} \le 0\} \cap \tilde{Q}| + \gamma \\
&\ge \frac{|Q_1|}{2} + k\gamma,
\end{aligned}
$$

which gives a contradiction for k big enough. So only the first case holds true and we deduce the result. Note that $k_0 \le 2|Q_1|/\gamma$ so $v \le 1 - 1/2^{k_0+1} \le 1 - 1/2^{(2|Q_1|/\gamma)+1}$ and we choose $\mu = 1/2^{(2|Q_1|/\gamma)+1}$. And the universal constant $\beta = 1/2^{2|Q_1|/\gamma}$ implies that v_k are solutions of (8) with $|s| \le 1$. $\qquad\square$

3 Intermediate value lemmas

In this section, we deal with intermediate value lemmas for elliptic and parabolic equations. We first recall the lemma in the elliptic case since we use it in the proof of the parabolic case. Then we give the proof of Theorem 1.1, an intermediate value lemma for solutions of parabolic equations with lower order terms.

3.1 Functions in H^1

We give a simpler proof of [16, Theorem 2.9], about an intermediate value lemma for functions which are bounded in the Sobolev space H^1. This lemma in an alternative version of the De Giorgi isoperimetric inequality [25, Lemma 10]. As we previously saw, it is a crucial tool in the De Giorgi proof of the Hölder regularity for solutions of elliptic equations.

Lemma 3.1 (Intermediate value lemma in H^1). *Let $u \in H^1(B_R)$. Then for all $(k,l) \in \mathbb{R}^2$ such that $k \le l$, we have*

$$(l-k)\big|\{u \le k\} \cap B_R\big| \times \big|\{u \ge l\} \cap B_R\big| \le C\big|\{k < u < l\} \cap B_R\big|^{1/2}\sqrt{\int_{B_R} |\nabla(u-k)_+(x)|^2\,\mathrm{d}x}, \tag{12}$$

where C is a universal constant.

We will use the shorthand notations $|u \le k|$, $|u \ge l|$ and $|k < u < l|$ for the mesures of the sets $\{x \in B_R, u(x) \le k\}$, $\{x \in B_R, u(x) \ge l\}$ and $\{x \in B_R, k < u(x) < l\}$.

Proof. We define the following truncated function

$$v(x) = \begin{cases} 0 & \text{if} \quad u(x) \le k, \\ u(x) - k & \text{if} \quad k < u(x) < l, \\ l - k & \text{if} \quad u(x) \ge l. \end{cases} \tag{13}$$

By Stampacchia theorem in [14, Theorem 7.8] or [11], we have $v \in H^1(B_R)$. By Poincaré inequality since $v \in W^{1,1}(B_R)$, see for example [12, Theorem 2, p.293], we have

$$\int_{B_R} |v(x) - \bar{v}| \, \mathrm{d}x \leq CR \int_{B_R} |\nabla v(x)| \, \mathrm{d}x, \tag{14}$$

where $\bar{v} = \frac{1}{|B_R|} \int_{B_R} v(x) \, \mathrm{d}x$. The sets $\{x \in B_R, v(x) = 0\}$, and $\{x \in B_R, v(x) = l - k\}$ are respectively denoted by $\{v = 0\}$ and $\{v = l - k\}$ and their mesures by $|v = 0|$ and $|v = l - k|$. We have the following inequalities

$$\frac{(l-k)}{|B_R|}|v = 0||v = l - k| \leq \int_{\{v=0\}} \bar{v} \, \mathrm{d}x \leq \int_{\{v=0\}} |v(x) - \bar{v}| \, \mathrm{d}x \leq \int_{B_R} |v(x) - \bar{v}| \, \mathrm{d}x, \tag{15}$$

and by Cauchy-Schwarz inequality

$$\int_{B_R} |\nabla v(x)| \, \mathrm{d}x = \int_{\{k<u<l\}} |\nabla v(x)| \, \mathrm{d}x \leq \sqrt{\int_{B_R} |\nabla(u-k)_+(x)|^2 \, \mathrm{d}x} |k < u < l|^{1/2}. \tag{16}$$

Using (14), (15) and (16) and the equalities $|v = 0| = |u \leq k|$ and $|v = l - k| = |u \geq l|$, we deduce (12). $\qquad\square$

3.2 Subsolutions of parabolic equations

In this section, we prove Theorem 1.1. We first introduce two lemmas. The first lemma is a Cacciopoli inequality already obtained in [12, 25, 15].

Lemma 3.2 (Caccioppoli inequality). *Let $u : Q_{2R} \to \mathbb{R}$ be a non-negative subsolution of (1) bounded by a constant $M > 0$ on $Q_{\frac{3}{2}R}$. Then there exists a universal constant $C > 0$ such that*

$$\int_{T_1}^{T_2} \int_{B_R} |\nabla_x u(\tau, x)|^2 \, \mathrm{d}x \mathrm{d}\tau \leq CM^2.$$

Proof of Lemma 3.2. Let ϕ be a smooth cut-off function such that $0 \leq \phi \leq 1$, $|\nabla_x \phi| \leq 2/R$ and

$$\phi(x) = \begin{cases} 1 & \text{in } B_R \\ 0 & \text{outside } B_{\frac{3}{2}R}. \end{cases}$$

We multiply the inequation by $u\phi^2$ which is non negative, and integrate over $(t_1, t_2) \times B_{2R}$ with $T_1 \leq t_1 \leq t_2 \leq T_2$. We get for almost every t_1, t_2

$$\int_{B_{\frac{3}{2}R}} u^2(t_2, x)\phi^2(x) \, \mathrm{d}x$$

$$\leq \int_{B_{\frac{3}{2}R}} u^2(t_1, x)\phi^2(x) \, \mathrm{d}x - \int_{t_1}^{t_2} \int_{B_{\frac{3}{2}R}} A(\tau, x)\nabla_x u(\tau, x) \cdot \nabla_x(u\phi^2)(\tau, x) \, \mathrm{d}x \mathrm{d}\tau$$

$$+ \int_{t_1}^{t_2} \int_{B_{\frac{3}{2}R}} B(\tau, x) \cdot \nabla_x u(\tau, x)\phi^2(x)u(\tau, x) \, \mathrm{d}x \mathrm{d}\tau + \int_{t_1}^{t_2} \int_{B_{\frac{3}{2}R}} |s(\tau, x)|u^2(\tau, x) \, \mathrm{d}x \mathrm{d}\tau.$$

So using the bounds on A, B and s we have,

$$\lambda \int_{t_1}^{t_2} \int_{B_{\frac{3}{2}R}} |\nabla_x u(\tau, x)|^2 \phi^2(x) \, \mathrm{d}x\mathrm{d}\tau \leq \int_{B_{\frac{3}{2}R}} u^2(t_1, x)\phi^2(x) \, \mathrm{d}x$$

$$+ 2\Lambda \int_{t_1}^{t_2} \int_{B_{\frac{3}{2}R}} \phi(x)|\nabla_x u(\tau, x)| \times |\nabla_x \phi(\tau, x)|u(\tau, x) \, \mathrm{d}x\mathrm{d}\tau$$

$$+ \Lambda \int_{t_1}^{t_2} \int_{B_{\frac{3}{2}R}} |\nabla_x u(\tau, x)|\phi^2(x)u(\tau, x) \, \mathrm{d}x\mathrm{d}\tau + \int_{t_1}^{t_2} \int_{B_{\frac{3}{2}R}} u^2(\tau, x) \, \mathrm{d}x\mathrm{d}\tau.$$

Then using that $0 \leq u \leq M$ on Q_{2R} and a Young inequality on the second and the third terms of the right hand side, we get

$$\lambda \int_{t_1}^{t_2} \int_{B_{\frac{3}{2}R}} |\nabla_x u(\tau, x)|^2 \phi^2(x) \, \mathrm{d}x\mathrm{d}\tau \leq CM^2 + \frac{\lambda}{4} \int_{t_1}^{t_2} \int_{B_{\frac{3}{2}R}} |\nabla_x u(\tau, x)|^2 \phi^2(x) \, \mathrm{d}x\mathrm{d}\tau$$

$$+ \frac{4\Lambda}{\lambda^2} \int_{t_1}^{t_2} \int_{B_{\frac{3}{2}R}} |\nabla_x \phi(\tau, x)|^2 u^2(\tau, x) \, \mathrm{d}x\mathrm{d}\tau$$

$$+ \frac{\lambda}{4} \int_{t_1}^{t_2} \int_{B_{\frac{3}{2}R}} |\nabla_x u(\tau, x)|^2 \phi^2(x) \, \mathrm{d}x\mathrm{d}\tau$$

$$+ \frac{2\Lambda}{\lambda^2} \int_{t_1}^{t_2} \int_{B_{\frac{3}{2}R}} |\phi(\tau, x)|^2 u^2(\tau, x) \, \mathrm{d}x\mathrm{d}\tau,$$

where C is a universal constant. Thus we deduce the desired result. $\qquad \square$

The second lemma is a first step for the proof of Theorem 1.1. It gives "almost" an intermediate value lemma with an error which is small for close times. The mesures of the following sets $\{(t, x) \in (\tau_1, \tau_2) \times B_R, f(t, x) < l\}$, $\{(t, x) \in (\tau_1, \tau_2) \times B_R, f(t, x) \leq l\}$ and $\{(t, x) \in (\tau_1, \tau_2) \times B_R, k < f(t, x) < l\}$ are respectively denoted by $|f < l, (\tau_1, \tau_2)|$, $|f \leq l, (\tau_1, \tau_2)|$ and $|k < f < l, (\tau_1, \tau_2)|$.

Lemma 3.3 (A key inequality for close times). *Let $f : Q_{2R} \to \mathbb{R}$ be a subsolution of* (1) *such that $f \leq 1$ on $Q_{\frac{3}{2}R}$. Then for all $(k, l) \in \mathbb{R}^2$ such that $k < l \leq 1$ and for all $(t_1, t_2, \tau) \in (T_1, T_2)^3$ such that $T_1 < t_1 < \tau < t_2 < T_2$, we have*

$$C\frac{l-k}{2-k}|f < l, (t_1, \tau)||f \geq l, (\tau, t_2)| \leq (t_2 - t_1)|k < f < l, (t_1, \tau)|^{1/4} + (t_2 - t_1)^{9/4}.$$

Proof of Lemma 3.3. Let $u = (f - k)_+$. The function u is a subsolution of the following equation

$$\partial_t u = \nabla_x \cdot (A\nabla_x u) + B \cdot \nabla_x u + \mathbb{1}_{\{u>0\}}s.$$

Let $\varphi(x) = \left(1 - \frac{|x|^2}{R^2}\right)^{3/4}$. We multiply the inequation by φ which is non-negative on B_R, and integrate over $(s, t) \times B_R$ where $T_1 \leq s < t \leq T_2$. We get for almost every s, t

$$\int_{B_R} u(t, x)\varphi(x) \, \mathrm{d}x \leq \int_{B_R} u(s, x)\varphi(x) \, \mathrm{d}x - \int_s^t \int_{B_R} A(\tau, x)\nabla_x u(\tau, x) \cdot \nabla_x \varphi(x) \, \mathrm{d}x\mathrm{d}\tau$$

$$+ \int_s^t \int_{B_R} B(\tau, x) \cdot \nabla_x u(\tau, x)\varphi(x) \, \mathrm{d}x\mathrm{d}\tau + \int_s^t \int_{B_R} |s(\tau, x)| \, \mathrm{d}x\mathrm{d}\tau. \tag{17}$$

By using a reverse Hölder inequality, we find a lower bound for the left hand side of (17),

$$
\begin{aligned}
\int_{B_R} u(t,x)\varphi(x)\,\mathrm{d}x &\geq (l-k)\int_{\{f(t,x)\geq l\}} \varphi(x)\,\mathrm{d}x \\
&\geq (l-k)\left(\int_{B_R} \mathbb{1}_{\{f(t)\geq l\}}\,\mathrm{d}x\right)^2 \left(\int_{B_R}\frac{1}{\varphi(x)}\,\mathrm{d}x\right)^{-1} \\
&\geq C(l-k)|f(t)\geq l|^2,
\end{aligned}
\tag{18}
$$

where C is a constant which depends only on R, $\{f(t)\geq l\}$ denotes the set of points $x \in B_R$ which satisfy $f(t,x) \geq l$ and $|f(t) \geq l|$ its measure. Also $|k < f(s) < l|$ and $|f(s) \leq k|$ will denote respectively the mesures of the set of points $x \in B_R$ which satisfy $k < f(s,x) < l$ and $f(s,x) \leq k$. For the first term of the right hand side of (17), we have

$$
\begin{aligned}
\int_{B_R} u(s,x)\varphi(x)\,\mathrm{d}x &\leq (1-k)\left(\int_{B_R}\mathbb{1}_{\{k<f(s)<l\}} + \mathbb{1}_{\{f(s)\geq l\}}\,\mathrm{d}x\right) \\
&\leq (1-k)\left(|k < f(s) < l| + |f(s) \geq l|\right).
\end{aligned}
\tag{19}
$$

For the second term of the right hand side of (17), using Cauchy-Schwarz inequality and Lemma 3.2, we get

$$
\begin{aligned}
-\int_s^t \int_{B_R} A(\tau,x)\nabla_x u(\tau,x)\cdot\nabla_x\varphi(x)\,\mathrm{d}x\mathrm{d}\tau &\leq \Lambda\int_s^t\int_{B_R}|\nabla_x u(\tau,x)|\times|\nabla_x\varphi(x)|\,\mathrm{d}x\mathrm{d}\tau \\
&\leq \Lambda\sqrt{\int_s^t\int_{B_R}|\nabla_x u|^2}\sqrt{\int_s^t\int_{B_R}|\nabla_x\varphi|^2} \\
&\leq C(1-k)\sqrt{t-s}.
\end{aligned}
\tag{20}
$$

With the same arguments, the third term is bounded by $C(1-k)\sqrt{t-s}$. The last term in (17) is bounded by $C(t-s)$. By combining (18), (19) and (20), we deduce,

$$
C(l-k)|f(t)\geq l|^2 \leq (1-k)\left(|k<f(s)<l|+|u(s)\geq l-k|\right)+C(1-k)\sqrt{t-s}+C(t-s).
\tag{21}
$$

For almost every $s \in [T_1, t)$, $f(s) \in H^1(B_R)$ so multiplying (21) by $(l-k)|f(s) \leq k| = (l-k)|u(s) \leq 0|$ and applying Lemma 3.1 to u between 0 and $l-k$, we get

$$
\begin{aligned}
C(l-k)^2|f(s)\leq k||f(t)\geq l|^2 &\leq (1-k)^2|\{k<f(s)<l\}| \\
&\quad + (1-k)|k<f(s)<l|^{1/2}\sqrt{\int_{B_R}|\nabla u(s,x)|^2\,\mathrm{d}x}+(1-k)^2\sqrt{t-s}.
\end{aligned}
\tag{22}
$$

We integrate (22) in $s \in [t_1, \tau]$ and $t \in [\tau, t_2]$ with $T_1 \leq t_1 < \tau < t_2 \leq T_2$. By Jensen inequality, we have

$$
\int_\tau^{t_2}|f(t)\geq l|^2\,\mathrm{d}t \geq \frac{1}{t_2-\tau}\left(\int_\tau^{t_2}|f(t)\geq l|\,\mathrm{d}t\right)^2 = \frac{1}{t_2-\tau}|f\geq l,\,(\tau,t_2)|^2.
\tag{23}
$$

By Cauchy-Schwarz inequality and Lemma 3.2, we get

$$
\begin{aligned}
\int_{t_1}^\tau &\left(|k<f(s)<l|^{1/2}\sqrt{\int_{B_R}|\nabla u(s,x)|^2\,\mathrm{d}x}\right)\mathrm{d}s \\
&\leq |k<f<l,\,(t_1,\tau)|^{1/2}\sqrt{\int_{t_1}^\tau\int_{B_R}|\nabla u(s,x)|^2\,\mathrm{d}x\mathrm{d}s} \\
&\leq C|k<f<l,\,(t_1,\tau)|^{1/2}.
\end{aligned}
\tag{24}
$$

Using (22), (23), (24), we deduce from (21) that

$$C(l-k)^2|f \leq k, \, (t_1, \tau)||f \geq l, \, (\tau, t_2)|^2$$
$$\leq (1 + (1-k)^2)\left((t_2 - t_1)|k < f < l, \, (t_1, \tau)|^{1/2} + (t_2 - t_1)^{7/2}\right).$$

So we have,

$$C\frac{(l-k)^2}{1 + (1-k)^2}|f \leq k, \, (t_1, \tau)||f \geq l, \, (\tau, t_2)|^2 \leq (t_2 - t_1)|k < f < l, \, (t_1, \tau)|^{1/2} + (t_2 - t_1)^{7/2},$$

which gives also, taking the square root of the last inequality and using $|f \leq k, \, (t_1, \tau)| \leq |B_R|^{1/2}(\tau - t_1)^{1/2}|f \leq k, \, (t_1, \tau)|^{1/2}$,

$$C\frac{l-k}{2-k}|f \leq k, \, (t_1, \tau)||f \geq l, \, (\tau, t_2)| \leq (t_2 - t_1)|k < f < l, \, (t_1, \tau)|^{1/4} + (t_2 - t_1)^{9/4}. \quad (25)$$

Using the equality $|f \leq k, \, (t_1, \tau)| = |f < l, \, (t_1, \tau)| - |k < f < l, \, (t_1, \tau)|$, we deduce from (25) the desired result. $\qquad\square$

Now let us prove Theorem 1.1. The idea of the proof is to understand that the "error" term $(t_2 - t_1)^{9/4}$ in Lemma 3.3 is negligible compared to the other terms when $t_2 - t_1$ is small and when the intervals are well-chosen.

Proof of Theorem 1.1. Let $n \in \mathbb{N}\backslash\{0\}$, $\alpha_n = \frac{T_2 - T_1}{2n}$, $\overline{T} = \frac{T_2 + T_1}{2}$ and $t_k = k\alpha_n$. Necessarily, there exists $i \in [1, n]$ such that

$$|f < l, \, (t_{i-1}, t_i)| \geq \frac{|f \leq k, \, (T_1, \overline{T})|}{2n}, \quad (26)$$

and there exists $j \in [n, 2n - 1]$ such that

$$|f \geq l, \, (t_j, t_{j+1})| \geq \frac{|f \geq l, \, (\overline{T}, T_2)|}{2n}. \quad (27)$$

We distinguish two cases, either there exists $m \in [i, 2n - 1]$ such that $m + 1$ does not satisfy (26) (i.e., (26) is false for $i = m + 1$), or for all $m \in [i, 2n - 1]$, $m + 1$ does satisfy (26). In the first case, letting p be the first integer m satisfying "$m + 1$ does not satisfy (26)", we have

$$|f < l, \, (t_p, t_{p+1})| < \frac{|f \leq k, \, (T_1, \overline{T})|}{2n},$$

so

$$|f \geq l, \, (t_p, t_{p+1})| \geq |B_R|\alpha_n - \frac{|f \leq k, \, (T_1, \overline{T})|}{2n} \geq \frac{|f \geq l, \, (\overline{T}, T_2)|}{2n}$$

and

$$|f < l, \, (t_{p-1}, t_p)| \geq \frac{|f \leq k, \, (T_1, \overline{T})|}{2n}.$$

In the second case, let $p = j$. Then in all cases, using Lemma 3.3 we have,

$$C\frac{l-k}{2-k}\frac{|f \leq k, \, (T_1, \overline{T})|}{2n}\frac{|f \geq l, \, (\overline{T}, T_2)|}{2n} \leq C\frac{l-k}{2-k}|f < l, \, (t_{p-1}, t_p)||f \geq l, \, (t_p, t_{p+1})|$$
$$\leq \frac{|k < f < l, \, (T_1, T_2)|^{1/4}}{n} + n^{-9/4}.$$

Thus, we have

$$C\frac{l-k}{2-k}|f \leq k, (T_1, \overline{T})||f \geq l, (\overline{T}, T_2)| \leq n|k < f < l, (T_1, T_2)|^{1/4} + n^{-1/4}.$$

So necessarily $|k < f < l, (T_1, T_2)| > 0$. And taking n such that $n^{-1/4} \leq \frac{n|k<f<l, (T_1,T_2)|^{1/4}}{2}$, for example $n = \left\lfloor \frac{2}{|k<f<l, (T_1,T_2)|^{1/5}} \right\rfloor + 1$, we get

$$C\frac{l-k}{2-k}|f \leq k, (T_1, \overline{T})||f \geq l, (\overline{T}, T_2)| \leq |k < f < l, (T_1, T_2)|^{1/20}.$$

This achieves the proof of the theorem. $\qquad\square$

3.3 Remarks and counterexamples

In this subsection we exhibit some counterexamples which help us to understand some issues.

3.3.1 Parabolic intermediate value lemma

Theorem 1.1 is false for subsolutions if we replace Q_R^- and Q_R^+ by Q_R. For example, for $T_1 = 0$ and $T_2 = 1$ the function

$$f(t,x) = \begin{cases} 1 & \text{for } t \in \left(0, \frac{1}{2}\right] \\ 0 & \text{for } t \in \left(\frac{1}{2}, 1\right), \end{cases}$$

is a subsolution of (1) but does not satisfy Lemma 1.1 with Q_R instead of Q_R^- and Q_R^+. That is why we have a weaker version of the intermediate value lemma for subsolutions which takes into account the disjoint intervals of time in a specific order.

3.3.2 Extension to kinetic equations?

Let us consider the following kinetic Fokker-Planck equation of [15],

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (A\nabla_v f) + B \cdot \nabla_v f + s, \quad (t, v, x) \in Q_{2R}, \tag{28}$$

where $Q_R = (-R^2, 0) \times B_R \times B_{R^3}$. We define $Q_R^+ = (-R^2, 0) \times B_R \times B_{R^3}$ and $Q_R^- = (-2R^2, -R^2) \times B_R \times B_{R^3}$. In [15], the authors prove a non-quantitative version of an intermediate value lemma. Trying to extend Theorem 1.1 to subsolutions of (28), some problems are encountered. First, subsolutions of (28) are not H^1 in the variable x. Second, the following relevant generalization of the parabolic intermediate value lemma to the kinetic one,

$$|\{f \leq k\} \cap Q_R^-|^\alpha|\{f \geq l\} \cap Q_R^+|^\beta \leq C|\{k < f < l\} \cap Q_R|^\gamma, \tag{29}$$

for some universal constants α, β, γ and C which depends also on k, l, is not true since there exists counterexamples. For example, let $d = 1$, the function

$$f(t,x,v) = \begin{cases} 1 & \text{for } x + 2Rt < -2R^3 \\ 0 & \text{for } x + 2Rt \geq -2R^3, \end{cases}$$

is a subsolution of (28) but does not satisfy (29). In fact, for some parameters $c > R$ (to have a subsolution) and $a \in \mathbb{R}$,

$$f_{a,c}(t,x,v) = \begin{cases} 1 & \text{for } x + ct < a \\ 0 & \text{for } x + ct \geq a, \end{cases}$$

is also a subsolution of (28). Drawing many lines of discontinuity $x + ct = a$, we notice that to find a valid intermediate value inequality, we must consider two cylinders which cannot be both crossed by the same line of discontinuity $x + ct = a$. More precisely, we must have a "gap" in time between the two cylinders of the same size (or at least not smaller) than the two cylinders. Let us define $\overline{Q}_R = (-3R^2, -2R^2) \times B_R \times B_{R^3}$. The two domains Q_R and $\overline{Q}_R$ are never both crossed by any possible line of discontinuity $x + ct = a$. That is why this intermediate value inequality seems to be more accurate,

$$|\{f \leq k\} \cap \overline{Q}_R|^\alpha |\{f \geq l\} \cap Q_R|^\beta \leq C|\{k < f < l\} \cap Q_R|^\gamma.$$

So it seems necessary to add new arguments which take into account a " good gap" between the two cylinders to find a quantitative version of the intermediate value lemma.

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