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# On Extension Properties of Pluricomplex Green Functions [(*)]

S. Zeynep Özal Kurşungöz [(1)]

**Abstract.** — Let $\Omega^0$ be a bounded domain in $\mathbb{C}^n$ and $\mathcal{E}$ be a compact subset of $\Omega^0$ such that $\Omega := \Omega^0 \setminus \mathcal{E}$ is connected. This paper deals with the study of the extension properties of the pluricomplex Green function of $\Omega$ to strictly larger subdomains $\widetilde{\Omega}$ of $\Omega$ as a pluricomplex Green function. The problem will be studied when $\Omega^0$ is a pseudoconvex, bounded complete Reinhardt domain in $\mathbb{C}^n$ and a detailed study in unit bidisc $\Delta^2 \subset \mathbb{C}^2$ will be provided.

**Résumé.** — À condition que $\Omega^0$ est une domaine bornée dans $\mathbb{C}^n$ et $\mathcal{E}$ soit compact sous-ensemble de $\Omega^0$ en maintenant que $\Omega := \Omega^0 \setminus \mathcal{E}$ soit connexe, cet article va examiner les propriétés d'extension de la fonction de Green pluricomplexe de $\Omega$ en sous-domaines strictement plus larges $\widetilde{\Omega}$ de $\Omega$ comme une fonction de Green pluricomplexe. Le problème sera examiné quand $\Omega^0$ soit une domaine Reinhardt complète bornée pseduconvexe dans $\mathbb{C}^n$ et une étude détaillée sur unité disque $\Delta^2 \subset \mathbb{C}^2$ sera fournie.

## 1. Introduction and Statement of Results

Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ with $a \in \Omega$ and $f$ be a plurisubharmonic function in a neighborhood of $a$. If $f(z) \leqslant \log \|z-a\| + C$ for $z$ near $a$, $f$ is said to have a logarithmic pole at $a$. The extremal function

$$g_\Omega(z,a) = \sup \left\{ f(z) : \begin{array}{l} f \in \mathcal{PSH}(\Omega, [-\infty, 0)) \\ \text{and } f \text{ has a logarithmic pole at } a \end{array} \right\}$$

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is called the pluricomplex Green function of $\Omega$ with pole at $a$. This definition was given by Zakharyuta [10] and later by Klimek ([6]). It is a generalization to higher dimensions of the Green function for the Laplace operator in $\mathbb{C}$. $g_\Omega(\,\cdot\,, a)$ is negative and plurisubharmonic in $\Omega$, it has a logarithmic pole at $a$, and it is maximal in $\Omega \setminus \{a\}$. It is also decreasing under composition with holomorphic mappings, which implies that it is invariant under biholomorphic mappings.

For smooth plurisubharmonic functions, the Monge-Ampère operator is defined as the $n$th exterior power of $dd^c$, where $d = \partial + \bar{\partial}$ is the exterior derivative and $d^c = i(\bar{\partial} - \partial)$. It is proved by Demailly that for hyperconvex domains, the pluricomplex Green function is continuous and is the unique solution to the following Dirichlet problem:

$$\begin{cases} u \in C(\Omega \setminus \{a\}) \cap \mathcal{G}(\Omega, a)), \\ (dd^c u)^n = (2\pi)^n \delta_a \text{ in } \Omega, \\ u(z) \to 0 \text{ as } z \to \partial\Omega, \end{cases}$$

where $\delta_a$ is the Dirac mass at $a$.

Let $\Omega^0$ be a domain in $\mathbb{C}^n$, $n \geqslant 2$ and let $\mathcal{E}$ be a compact subset of $\Omega^0$ such that $\Omega^0 \setminus \mathcal{E}$ is connected. Hartogs' extension theorem states that every holomorphic function $f : \Omega^0 \setminus \mathcal{E} \to \mathbb{C}$ has a unique extension to $\Omega^0$. The situation in the case of plurisubharmonic functions is different.

Let $\Omega^0$ be an open subset of $\mathbb{C}^n$, $n \geqslant 2$ and let $\mathcal{E}$ be a closed subset of $\Omega^0$. Harvey and Polking [4] specified some conditions on the class of plurisubharmonic functions on $\Omega^0 \setminus \mathcal{E}$ and on the set $\mathcal{E}$ to ensure that the plurisubharmonic functions can be extended across $\mathcal{E}$.

Recall that a domain $\Omega$ is called a domain of existence for plurisubharmonic functions if there exists a plurisubharmonic function on $\Omega$ that cannot be extended to as a plurisubharmonic function to any domain $\Omega^0 \supsetneq \Omega$. Bedford and Burns [1] and Cegrell [2] showed that if a given domain $\Omega$ in $\mathbb{C}^n$, $n \geqslant 2$ satisfies certain boundary conditions, then $\Omega$ is a domain of existence for plurisubharmonic functions. In his paper [2], Cegrell also showed that if a domain $\Omega$ is contained in a domain of existence for plurisubharmonic functions $\Omega^0$, then any plurisubharmonic function on $\Omega$ is the restriction of a plurisubharmonic function on $\Omega^0$. Sadullaev [8] proved that if $\Omega$ is a closed submanifold of a Stein manifold $\Omega^0$, then every plurisubharmonic function on $\Omega$ is the restriction of a plurisubharmonic function on $\Omega^0$.

Let $\Omega$ be an open, bounded, pseudoconvex set in $\mathbb{C}^n$ for $n \geqslant 2$, $K$ be a compact subset of $\Omega$ such that $\Omega \setminus K$ is connected, and $\Omega'$ be an open subset with $K \subset \Omega' \Subset \Omega$. Cegrell [3] proved that the Monge–Ampère mass $\int_{\Omega' \setminus K} (dd^c u)^n$ for $u \in \mathcal{L}^\infty_{loc}(\Omega \setminus K)$ is finite when $K$ is a removable singularity.

A natural question is whether pluricomplex Green functions of certain domains can be extended as plurisubharmonic functions on larger domains, and in particular, as pluricomplex Green functions of larger domains. In the setting of the Hartogs extension theorem, in general the pluricomplex Green function of $\Omega^0 \setminus \mathcal{E}$ cannot be extended to $\Omega^0$, neither as a plurisubharmonic function nor as a pluricomplex Green function. However, in some cases we can extend the pluricomplex Green function of $\Omega^0 \setminus \mathcal{E}$ to a strictly larger subdomain of $\Omega^0$.

In Section 3, we show that for some Reinhardt subdomains $\Omega$ of a pseudoconvex complete Reinhardt domain $\Omega^0$ in $\mathbb{C}^n$, the pluricomplex Green function $g_\Omega(\,\cdot\,,0)$ can be extended as a pluricomplex Green function of a strictly larger Reinhardt subdomain of $\Omega^0$.

Theorem 1.1. — *Let $\Omega^0$ be a pseudoconvex, bounded complete Reinhardt domain in $\mathbb{C}^n$, and let $\Omega = \Omega^0 \setminus \mathcal{E}$, where $\mathcal{E} \not\ni 0$ is a Reinhardt compact subset of $\Omega^0$ that satisfies the following properties:*

- *$\mathcal{E}$ is strictly logarithmically convex, i.e. $\ell(\mathcal{E})$ is strictly convex,*
- *$\mathcal{E} \cap \{z_1 \dots z_n = 0\} = \emptyset$,*

*where $\ell(z_1, \dots, z_n) = (\log|z_1|, \dots, \log|z_n|)$ for $(z_1, \dots, z_n) \in \mathbb{C}^n$. Then there exists a Reinhardt domain $\widetilde{\Omega}$ such that $\Omega \subsetneq \widetilde{\Omega} \subset \Omega^0$ and*

$$g_\Omega(z,0) = g_{\widetilde{\Omega}}(z,0), \qquad \forall\, z \in \Omega.$$

In the case of $\mathbb{C}^2$, one can omit the condition $\mathcal{E} \cap \{z_1 z_2 = 0\} = \emptyset$ in Theorem 1.1.

Theorem 1.2. — *Let $\Omega^0$ be a pseudoconvex, bounded complete Reinhardt domain in $\mathbb{C}^2$, and let $\Omega = \Omega^0 \setminus \mathcal{E}$, where $\mathcal{E} \not\ni 0$ is a Reinhardt compact, strictly logarithmically convex subset of $\Omega^0$. Then there exists a Reinhardt domain $\widetilde{\Omega}$ such that $\Omega \subsetneq \widetilde{\Omega} \subset \Omega^0$ and*

$$g_\Omega(z,0) = g_{\widetilde{\Omega}}(z,0), \qquad \forall\, z \in \Omega.$$

Recall that if $g$ is a function defined on a Reinhardt domain $\Omega$ that satisfies $g(z_1, \dots, z_n) = g(|z_1|, \dots, |z_n|)$ for all $(z_1, \dots, z_n) \in \Omega$, then it is called a polyradial function. Since the pluricomplex Green functions are invariant under biholomorphic mappings, $g_\Omega(\,\cdot\,,0)$ is a polyradial function. The plurisubharmonicity of polyradial functions on Reinhardt domains can be discussed by related convex functions. Using some results on a special class of convex functions and a method introduced by Klimek [7], we will be able to find $\widetilde{\Omega}$.

In Section 4, we will discuss this problem in the unit bidisk $\Delta^2 \subset \mathbb{C}^2$. In this case, we can say more about $\widetilde{\Omega}$. When $\mathcal{E}$ is strictly logarithmically convex, we show that there exists a unique largest $\widetilde{\Omega}$. If $\mathcal{E}$ is logarithmically convex, but not strictly logarithmically convex, we show that in some cases there are infinitely many maximal subdomains of $\Delta^2$ with respect to inclusion that can be taken as $\widetilde{\Omega}$.

## 2. Preliminaries

We will first recall some basic properties of convex functions. For any two points $x^1, x^2 \in \mathbb{R}^n$, we will denote the line segment between $x^1$ and $x^2$ by $[x^1, x^2]$. Let $E$ be *any set* in $\mathbb{R}^n$ and $x^1$, $x^2 \in E$. A function $u : E \to \mathbb{R}$ is called *convex* if for any $x^1$, $x^2 \in E$ such that $[x^1, x^2] \subset E$, we have

$$u(\lambda x^1 + (1-\lambda)x^2) \leqslant \lambda u(x^1) + (1-\lambda)u(x^2), \qquad 0 \leqslant \lambda \leqslant 1.$$

For $D$ an open set in $\mathbb{R}^n$, a function $u : D \to \mathbb{R}^n$ is called *locally convex* if for any $x \in D$, there exists a ball $B_r(x) = \{y \in \mathbb{R}^n : \|x - y\| < r\} \subset D$ on which $u$ is convex.

We refer to [9, Section 2] for a discussion of several notions of convexity of functions defined on arbitrary sets. In particular, we note that our notion of convexity here corresponds to the notion of interval convexity in [9].

We are going to work with convex functions defined on domains that are not necessarily convex. In this case, proving local convexity will be enough, as is shown by the following well-known theorem (see e.g. [9, Section 2]):

Theorem 2.1. — *Let $D$ be any open set in $\mathbb{R}^n$ and $u$ be a real-valued function defined on $D$. Then, $u$ is convex if and only if $u$ is locally convex.*

The following lemma will be used several times in the proofs.

Lemma 2.2. — *If $f : (-\infty, a] \to (-\infty, 0)$ is a convex function then $f(x) \leqslant f(a)$, for any $x \in (-\infty, a)$.*

This simple fact about closed, unbounded convex sets will be used in proving some results about pluricomplex Green functions in $\mathbb{C}^2$.

Lemma 2.3. — *Let $E$ be a closed, convex, unbounded subset of $\mathbb{R}^2$ such that*

$$E \subset \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 0, m < x_2 < 0\}.$$

*Then, for any $p = (p_1, p_2) \in E$, we have $l_p \subset E$, where $l_p = \{(p_1 + t, p_2) : t \leqslant 0\}$.*

As mentioned before, the plurisubharmonicity of a polyradial function depends on the convexity of a related function. Before we discuss this, we need to give some definitions. Define functions

$$\ell(z_1,\ldots,z_n)=(\log|z_1|,\ldots,\log|z_n|),\qquad (z_1,\ldots,z_n)\in\mathbb{C}^n,$$

and

$$\mathfrak{e}(x_1,\ldots,x_n)=(e^{x_1},\ldots,e^{x_n}),\qquad (x_1,\ldots,x_n)\in[-\infty,\infty)^n,$$

where $\log 0=-\infty$ and $e^{-\infty}=0$. For a Reinhardt domain $\Omega\subset\mathbb{C}^n$, denote its logarithmic image by $\omega=\ell(\Omega)\cap\mathbb{R}^n$. For any $\omega\subset\mathbb{R}^n$ open define

$$\hat{\mathfrak{e}}(\omega)=\operatorname{int}\overline{\{(z_1,\ldots,z_n)\in\mathbb{C}^n:(|z_1|,\ldots,|z_n|)\in\mathfrak{e}(\omega)\}}\,.$$

Let $\bar{1}=(1,\ldots,1)\in\mathbb{R}^n$. A function $u:\omega\to\mathbb{R}$ is said to have *normalized growth at* $-\infty$ if, for any $a=(a_1,\ldots,a_n)\in\omega$ so that $a+t\bar{1}\in\omega$ for all $t\leqslant 0$, there exists a constant $C_a$ such that

$$u(a+t\bar{1})=u(a_1+t,\ldots,a_n+t)\leqslant t+C_a,\qquad t\leqslant 0.$$

The *convex envelope of* $\omega$ *with normalized growth at* $-\infty$ is defined by

$$u_\omega(x)=\sup\left\{u(x):\begin{array}{c}u:\omega\to(-\infty,0)\\ \text{convex with normalized growth at }-\infty\end{array}\right\}.$$

The relation between this associated convex function and the pluricomplex Green function is stated by Klimek [7]

Theorem 2.4 ([7]). — *If $\Omega\subset\mathbb{C}^n$ is a bounded Reinhardt domain, then*

$$u_\omega(x)=g_\Omega(\mathfrak{e}(x),0),\qquad x\in\omega.$$

The pluricomplex Green functions of certain domains can be found using the Minkowski functional (see [5, Proposition 4.2.21]) as follows:

Proposition 2.5. — *Let $\Omega^0$ be a bounded pseudoconvex complete Reinhardt domain with Minkowski functional $h_{\Omega^0}$. Then $g_{\Omega^0}(z,0)=\log h_{\Omega^0}(z)$, for all $z\in\Omega^0$.*

Proposition 2.5 will form a base for the discussions in proofs. We will work on a bounded, pseudoconvex complete Reinhardt domain $\Omega^0$ and on its Reinhardt subdomains. Using the pluricomplex Green function obtained from Proposition 2.5 for $\Omega^0$ and Theorem 2.4, we will find the pluricomplex Green functions of certain subdomains.

## 3. Proofs of Theorem 1.1, 1.2

The proof of Theorem 1.1 relies on analyzing the envelopes of certain classes of convex functions defined on solid hypercylinders in $\mathbb{R}^n$. These will be introduced and studied in the first section. Afterwards, proofs of Theorem 1.1 and 1.2 will be provided.

### 3.1. Special Classes of Convex Functions

Let $\beta \subset \mathbb{R}^{n-1}$ be a compact, convex set, and $q = (q_1, \dots, q_n) \in \mathbb{R}^n$ with $q_n > 0$. Consider the set

$$\mathcal{K} = \{(x, 0) + tq : x \in \beta, t \in \mathbb{R}\},$$

which is a solid hypercylinder in $\mathbb{R}^n$. The boundary of $\mathcal{K}$ is $\partial\mathcal{K} = \{(x, 0)+tq : x \in \partial\beta, t \in \mathbb{R}\}$, where $\partial\beta$ is the boundary of $\beta \subset \mathbb{R}^{n-1}$. Given a hyperplane $H$ that is not parallel to $q$, we let $D_H := \mathcal{K} \cap H$. Note that $D_H$ is a compact convex set.

Definition 3.1. — *Let $\omega \subset \mathbb{R}^n$. A function $f : \omega \to \mathbb{R}$ is called* $q$-linear *if $f(x + tq) = f(x) + t$ holds for any $x \in \omega$ and $t \in \mathbb{R}$ such that $x + tq \in \omega$.*

Let now $f : \mathcal{K} \to \mathbb{R}$ be a given convex, $q$-linear function, such that $f$ is bounded above on $D_H$, for some hyperplane $H$ not parallel to $q$. We let

$$V = V(\,\cdot\,; \mathcal{K}, f) : \mathcal{K} \to \mathbb{R}$$

be the convex envelope of $f$ defined by

$$V(x; \mathcal{K}, f) = \sup\left\{ w(x) : \begin{array}{c} w : \mathcal{K} \to \mathbb{R} \text{ is convex} \\ \text{and } w(y) \leqslant f(y) \text{ for } y \in \partial\mathcal{K} \end{array} \right\}, \quad x \in \mathcal{K}. \tag{3.1}$$

Lemma 3.2. — *The function $V$ is convex and $V(x) = f(x)$ for $x \in \partial\mathcal{K}$.*

*Proof.* — Since $f$ is $q$-linear and bounded above on $D_H$, it follows that $f$ is bounded above on each compact subset of $\mathcal{K}$. Let $x \in \mathcal{K}$ and $H_x$ be the hyperplane parallel to $H$ that contains $x$. Then $f \leqslant M$ on $D_{H_x} = H_x \cap \mathcal{K}$, for some constant $M$. If $w$ is an element of the defining family of $V$ then the restriction of $w$ to the convex set $D_{H_x}$ is convex and $w \leqslant f \leqslant M$ on the boundary of this set regarded as a subset of $H_x$. We conclude that $w(x) \leqslant M$. It follows that the functions in the defining family of $V$ are locally uniformly upper bounded on $\mathcal{K}$, hence $V$ is convex fuction on $\mathcal{K}$.

Obviously, $f$ is an element of the defining family of $V$ and therefore $f \leqslant V$ on $\mathcal{K}$ by definition. On the other hand, any element of the defining family of $V$ is dominated by $f$ on $\partial\mathcal{K}$, hence so is $V$. Therefore $V = f$ on $\partial\mathcal{K}$. □

In order to prove further properties of $V$ we need an alternate description using $q$-linear extensions of convex envelopes on slices $D_H = H \cap \mathcal{K}$, where $H$ is a hyperplane not parallel to $q$. We define

$$v(x; D_H, f) = \sup \left\{ w(x) : \begin{array}{c} w : D_H \to \mathbb{R} \text{ is convex} \\ \text{and } w(y) \leqslant f(y) \text{ on } \partial D_H \end{array} \right\}, \quad x \in D_H.$$

Here $\partial D_H$ denotes the boundary of $D_H$ seen as a subset of $H$. We denote by $v_H = v_H(\cdot\,; \mathcal{K}, f)$ the $q$-linear extension of $v(\cdot\,; D_H, f)$ to $\mathcal{K}$ defined by

$$v_H(x) = v(y; D_H, f) + t\,, \text{ where } x = y + tq\,,\ y \in D_H\,,\ t \in \mathbb{R}\,, \qquad \forall\, x \in \mathcal{K}.$$

This function is clearly $q$-linear.

Lemma 3.3. — *If $H$ is a hyperplane not parallel to $q$ then $V = v_H$ on $\mathcal{K}$. In particular, the function $V$ is $q$-linear.*

*Proof.* — Note that $v(\cdot\,; D_H, f)$ is a convex function on $D_H$, as it is the supremum of a family of uniformly upper bounded convex functions. If $H' = H + tq$ is a hyperplane parallel to $H$, for some fixed $t \in \mathbb{R}$, then $D_{H'} = D_H + tq$. For any function $w$ on $D_H$ we can define a function $w'$ on $D_{H'}$ by

$$w'(x + tq) = w(x) + t, \qquad x \in D_H.$$

Then, a simple calculation shows that $w'$ is convex on $D_{H'}$ if and only if $w$ is convex on $D_H$. Moreover, since $f$ is $q$-linear, we have that $w(x) \leqslant f(x)$ if and only if $w'(x + tq) \leqslant f(x + tq)$, where $x \in \partial D_H$. Hence

$$v(x + tq; D_{H'}, f) = v(x; D_H, f) + t\,, \qquad \forall\, x \in D_H. \tag{3.2}$$

Formula (3.2) implies that $v_{H'} = v_H$ if $H'$ is a hyperplane parallel to $H$. Also, as $f$ is a convex function in the defining family of $v(\,\cdot\,; D_H, f)$, we have that $v(x; D_H, f) = f(x)$ for $x \in \partial D_H$ and the $q$-linearity of the functions $v_H$ and $f$ implies that $v_H = f$ on $\partial\mathcal{K}$.

Next we will show that $v_H$ is also a convex function on $\mathcal{K}$. Let $x$, $x^1$, $x^2$ be points in $\mathcal{K}$ such that $x = \mu x^1 + (1-\mu)x^2$ for $0 \leqslant \mu \leqslant 1$. Now write $x = x^0 + \lambda q$, $x^j = x^{j,0} + \lambda^j q$ where $x^0, x^{j,0} \in H$ and $\lambda, \lambda^j \in \mathbb{R}$, $j = 1, 2$. We claim that $x^0 = \mu x^{1,0} + (1-\mu)x^{2,0}$ and $\lambda = \mu\lambda^1 + (1-\mu)\lambda^2$. Indeed, we have that

$$x^0 - \mu x^{1,0} - (1-\mu)x^{2,0} + (\lambda - \mu\lambda^1 - (1-\mu)\lambda^2)q = 0$$

If $x^{1,0} = x^{2,0} = x^0$ then our claim follows. Otherwise, the vectors $p := x^{2,0} - x^{1,0} \neq 0$ and $q$ are linearly independent, since $p$ is parallel to $H$ and $q$ is not parallel to $H$. As $x^0 - \mu x^{1,0} - (1-\mu)x^{2,0} = sp$ for some $s \in \mathbb{R}$, we conclude that $s = \lambda - \mu\lambda^1 - (1-\mu)\lambda^2 = 0$, which implies our claim. Using the definition of $v_H$ and the fact that the function $v(\,\cdot\,; D_H, f)$ is convex on $D_H$, we obtain

$$\begin{aligned} v_H(x) &= v(x^0; D_H, f) + \lambda \\ &\leqslant \mu\, v(x^{1,0}; D_H, f) + (1-\mu) v(x^{2,0}; D_H, f) + \mu\lambda^1 + (1-\mu)\lambda^2 \\ &= \mu\left(v(x^{1,0}; D_H, f) + \lambda^1\right) + (1-\mu)\left(v(x^{2,0}; D_H, f) + \lambda^2\right) \\ &= \mu\, v_H(x^1) + (1-\mu) v_H(x^2), \end{aligned}$$

hence $v_H$ is a convex function on $\mathcal{K}$.

Since $v_H$ is convex on $\mathcal{K}$ and $v_H = f$ on $\partial\mathcal{K}$, $v_H$ is an element of the defining family of $V$, so $v_H \leqslant V$. On the other hand, take any $x \in \mathcal{K}$. Then, there exists a hyperplane $H' \ni x$ that is parallel to $H$. Since $V|_{D_{H'}}$ is an element of the defining family of $v(\,\cdot\,; D_{H'}, f)$, we see that $V(x) \leqslant v(x; D_{H'}, f) = v_{H'}(x) = v_H(x)$, which concludes the proof. □

For the proof of Theorem 1.1 we actually need to work with envelopes of convex functions on certain subsets of $\mathcal{K}$, which we now introduce. For a fixed hyperplane $\mathcal{H}$ that is not parallel to $q$, let $D := \mathcal{K} \cap \mathcal{H}$, $C := \partial\mathcal{K} \cap \mathcal{H}$, and define

$$K := \{x + tq : x \in D,\, t < 0\}\,, \quad \partial' K := \{x + tq : x \in C,\, t < 0\}.$$

Note that $D$ is a compact convex set and $\partial K = \partial' K \cup D$ is the boundary of $K$ in $\mathbb{R}^n$.

Definition 3.4. — *If $E$ is a compact convex subset of $\overline{K}$ so that $D \subset E$, we let $K_E = K \setminus E$ and $\partial' K_E = K_E \cap \partial' K$.*

The convex function that we will need can now be introduced. Let $f : \mathcal{K} \to \mathbb{R}$ be a given convex, $q$-linear function, such that $f$ is bounded above on $D$. For each subset $K_E$ of $K$, we let $u(\cdot\,; K_E, f) : K_E \to \mathbb{R}$ be the convex envelope of $f$ defined by

$$u(x; K_E, f) := \sup\left\{ w(x) : \begin{array}{c} w : K_E \to \mathbb{R} \text{ is convex} \\ \text{and } w(y) \leqslant f(y) \text{ for } y \in \partial' K_E \end{array} \right\}. \tag{3.3}$$

We can now state the main result of this section.

Theorem 3.5. — *For any subset $K_E$ of $K$ as above, we have that $u(\cdot\,; K_E, f) = V|_{K_E}$, where $V = V(\,\cdot\,; \mathcal{K}, f)$ is the function defined in* (3.1). *In particular,*

$$u(x; K_E, f) = u(x; K, f), \;\; \forall\, x \in K_E.$$

*Proof.* — Note that $K = K_E$ if we take $E = D$. Therefore, the second conclusion of the theorem follows at once from the first one.

A similar argument to that in the proof of Lemma 3.2 shows that the function $u(\,\cdot\,;K_E,f)$ is convex on $K_E$ and $u(x;K_E,f)=f(x)$ for $x\in\partial' K_E$. By Lemma 3.2, $V$ is a function in the defining family of $u(\,\cdot\,;K_E,f)$, so $V\leqslant u(\,\cdot\,;K_E,f)$ on $K_E$. On the other hand, let $x\in K_E$. As $E$ is convex, there exists a hyperplane $H\ni x$ such that $H\cap E=\emptyset$, so $H$ is not parallel to $q$ and $D_H\subset K_E$. Since $u(\,\cdot\,;K_E,f)|_{D_H}$ is an element of the defining family of $v(\,\cdot\,;D_H,f)$ we have by Lemma 3.3 that $u(x;K_E,f)\leqslant v(x;D_H,f)=v_H(x)=V(x)$. Hence $u(x;K_E,f)=V(x)$ for $x\in K_E$, and the theorem is proved. $\square$

### 3.2. Proof of Theorem 1.1

Let $\Omega^0$ be a pseudoconvex, bounded complete Reinhardt domain in $\mathbb{C}^n$. Let $\Omega=\Omega^0\setminus\mathcal{E}$, where $\mathcal{E}\not\ni 0$ is a Reinhardt compact subset of $\Omega^0$ that satisfies the following properties:

- $\mathcal{E}$ is strictly logarithmically convex, i.e. $\ell(\mathcal{E})$ is strictly convex,
- $\mathcal{E}\cap\{z_1\ldots z_n=0\}=\emptyset$.

For $z\in\Omega\setminus\{0\}$, let $L_z=\{\zeta z:\zeta\in\mathbb{C}\}$. $\Omega$ will be partitioned into the following sets:

$$\begin{aligned}
\Omega_1 &= \{z\in\Omega\setminus\{0\}: L_z\cap\mathcal{E}=\emptyset\},\ \text{which is an open set,}\\
\Omega_2 &= \text{the connected component of } 0 \text{ in } \Omega\setminus\Omega_1,\\
\Omega_3 &= \Omega\setminus(\Omega_1\cup\Omega_2).
\end{aligned}$$

Set $\ell(\Omega)\cap\mathbb{R}^n=\omega$, $\ell(\Omega^0)\cap\mathbb{R}^n=\omega^0$, $\ell(\Omega_i)\cap\mathbb{R}^n=\omega_i$, for $i=1,\,2,\,3$.

We study $g_\Omega$ by working with the associated convex function $u_\omega$ in logarithmic coordinates. As $\mathcal{E}\cap\{z_1,\ldots,z_n=0\}=\emptyset$, $\ell(\mathcal{E})$ is a compact set in $\mathbb{R}^n$ and we define a solid hypercylinder $\mathcal{K}$ by

$$\mathcal{K}=\bigcup_{x\in\ell(\mathcal{E})}\{x+t\bar{1}:t\in\mathbb{R}\}$$

where $\bar{1}=(1,\ldots,1)\in\mathbb{R}^n$. Let $\mathcal{H}$ be a fixed hyperplane that is not parallel to $\bar{1}$, and define $D:=\mathcal{K}\cap\mathcal{H}$, $C:=\partial\mathcal{K}\cap\mathcal{H}$,

$$K:=\{x+t\bar{1}:x\in D,\,t<0\}\,,\ \partial' K:=\{x+t\bar{1}:x\in C,\,t<0\}.$$

We assume that $\mathcal{H}$ is chosen so that $K\supset\ell(\mathcal{E})$. Note that $K$ does not necessarily lie in $\omega^0$. Let

$$E=\bigcup_{x\in\ell(\mathcal{E})}\{x+t\bar{1}:t\geqslant 0\}\cap\overline{K}.$$

Then $E$ is a compact, convex set and we set $K_E := K \setminus E$. Note that $K_E = \omega_2$.

Lemma 3.6. — *We have that $g_\Omega(z,0) = g_{\Omega^0}(z,0)$ for $z \in \Omega_1$, and $u_\omega(x) = u_{\omega^0}(x)$ for $x \in \omega_1$.*

The proof of this lemma is straightforward.

Lemma 3.7. — *The function $f := \log h_{\Omega^0} \circ \mathfrak{e}$ is a $\bar{1}$-linear convex function on $\mathbb{R}^n$. Moreover, $f = u_{\omega^0}$ on $\omega^0$.*

*Proof.* — Since the Minkowski functional is defined for all $\mathbb{C}^n$, $f$ is defined on $\mathbb{R}^n$. As $h_{\Omega^0}(\zeta z) = |\zeta| h_{\Omega^0}(z)$, for $\zeta \in \mathbb{C}$, we have that

$$\begin{aligned} f(x+t\bar{1}) = \log h_{\Omega^0}\left(\mathfrak{e}(x+t\bar{1})\right) &= \log h_{\Omega^0}\left(e^t(e^{x_1},\ldots,e^{x_n})\right) \\ &= t + \log h_{\Omega^0}\left(\mathfrak{e}(x)\right) = t + f(x), \end{aligned}$$

so $f$ is a $\bar{1}$-linear function. Now $f = \log h_{\Omega^0} \circ \mathfrak{e} = g_{\Omega^0}(\,\cdot\,,0) \circ \mathfrak{e} = u_{\omega^0}$ on $\omega^0$, which shows that $f$ is convex on $\omega^0$. Since $f$ is $\bar{1}$-linear, it follows that $f$ is convex on $\mathbb{R}^n$, using a similar argument to that in the proof of Lemma 3.3. $\square$

Lemma 3.8. — *If $f = \log h_{\Omega^0} \circ \mathfrak{e}$ then $u_\omega(x) = u(x; K_E, f)$ for $x \in \omega_2 = K_E$.*

*Proof.* — Recall that the function $u(\,\cdot\,; K_E, f)$ is defined in (3.3). By Lemmas 3.6 and 3.7, $u_\omega = u_{\omega^0} = f$ on $\omega_1$, so $u_\omega = u_{\omega^0} = f$ on $\partial' K_E$ since these functions are continuous.

As $u_\omega|_{K_E}$ is a convex function that is equal to $f$ on $\partial' K_E$, we have $u_\omega \leqslant u(\,\cdot\,; K_E, f)$ on $K_E$. For the opposite inequality, we consider the function

$$u(x) = \begin{cases} u_\omega(x), & \text{if } x \in \omega_1 \cup \omega_3, \\ u(x; K_E, f), & \text{if } x \in \omega_2 = K_E. \end{cases} \tag{3.4}$$

We will show that $u$ is an element of the defining family of $u_\omega$, hence $u \leqslant u_\omega$ on $\omega$, and in particular $u(\,\cdot\,; K_E, f) \leqslant u_\omega$ on $\omega_2$.

By Theorem 3.5, Lemma 3.2 and Lemma 3.3, the function $u(\,\cdot\,; K_E, f)$ is $\bar{1}$-linear convex on $K_E$ and $u(\,\cdot\,; K_E, f) = f$ on $\partial' K_E$. We have $u = u_\omega < 0$ on $\omega_1 \cup \omega_3$. Moreover, since $f < 0$ on $\partial' K_E$ it follows that $u(\,\cdot\,; K_E, f) < 0$ on $K_E$, so $u$ is a negative function. It has normalized growth at $-\infty$ on $\omega_1$ as $u_\omega$ has normalized growth at $-\infty$ on $\omega$. For $x \in K_E$ and $t \in \mathbb{R}$ with $x + t\bar{1} \in K_E$, the $\bar{1}$-linearity of $u(\,\cdot\,; K_E, f)$ shows that

$$u(x+t\bar{1}; K_E, f) = t + u(x; K_E, f),$$

thus $u(\,\cdot\,; K_E, f)$ has normalized growth at $-\infty$ on $\omega_2$ as well.

It remains to show that $u$ is a (locally) convex function. Note that it suffices to show that $u$ is convex in a small neighborhood of each point of $\partial' K_E$. Since $u_\omega$ is convex on $\omega$ and $u(\,\cdot\,; K_E, f)$ is convex on $K_E$, this amounts to proving the convexity inequality for points $x$, $x^1 \in \omega_1$, $x^2 \in \omega_2$, such that the segment $[x^1, x^2] \subset \omega$ and $x \in [x^1, x^2]$.

We write $x = sx^1 + (1-s)x^2$, $0 < s < 1$. Recall that $u = u(\,\cdot\,; K_E, f) = f = u_\omega$ on $\partial' K_E \subset \omega_2$ and $u_\omega \leqslant u = u(\,\cdot\,; K_E, f)$ on $K_E$. If $x \in \omega_1 \cup \partial' K_E$ then

$$u(x) = u_\omega(x) \leqslant s u_\omega(x^1) + (1-s) u_\omega(x^2) \leqslant s u(x^1) + (1-s) u(x^2),$$

since $u_\omega(x^2) \leqslant u(x^2; K_E, f) = u(x^2)$. We assume next that $x \in \omega_2$, and we let $\{y\} = [x^1, x^2] \cap \partial' K_E$, so $y = tx^1 + (1-t)x^2$ with $s \leqslant t < 1$. Then $x = \frac{s}{t}\, y + \big(1 - \frac{s}{t}\big) x^2$. As $y \in \partial' K_E$, it follows by above that $u(y) \leqslant tu(x^1) + (1-t) u(x^2)$. Since $u = u(\,\cdot\,; K_E, f)$ is convex on $\omega_2 = K_E$ we obtain

$$\begin{aligned} u(x) &\leqslant \frac{s}{t}\, u(y) + \Big(1 - \frac{s}{t}\Big)\, u(x^2) \\ &\leqslant \frac{s}{t}\big(tu(x^1) + (1-t)u(x^2)\big) + \Big(1 - \frac{s}{t}\Big)\, u(x^2) = su(x^1) + (1-s)u(x^2). \end{aligned}$$

This shows that $u$ is (locally) convex on $\omega$, hence an element of the defining family of $u_\omega$. The proof of the lemma is complete. □

Since the function $u$ defined in (3.4) is equal to $u_\omega$ on $\omega_2$ by Lemma 3.8, it should be noted that in fact we have $u_\omega = u$ on $\omega$.

Theorem 1.1 can now be proved using Lemmas 3.6, 3.7, and 3.8.

*Proof of Theorem 1.1.* — It suffices to show that there exists a domain $\widetilde{\omega}$ such that $\omega \subsetneqq \widetilde{\omega} \subset \omega^0$ and $u_\omega = u_{\widetilde{\omega}}|_\omega$. In the above setting, let $C = \ell(\mathcal{E}) \cap \partial K$, and let $\operatorname{conv}(C)$ denote the convex hull of $C$. Define

$$\widetilde{E} = \bigcup_{x \in \operatorname{conv}(C)} \{x + t\bar{1} : t \geqslant 0\} \cap \overline{K}.$$

Then $\widetilde{E}$ is a compact, convex set and we let $K_{\widetilde{E}} := K \setminus \widetilde{E}$. Since $\ell(\mathcal{E})$ is convex, $\operatorname{conv}(C) \subset \ell(\mathcal{E})$, so $\widetilde{E} \subset E$ and $K_{\widetilde{E}} \supset K_E$.

Now we show that $K_{\widetilde{E}} \neq K_E$. We prove in fact that $\partial K_E \setminus \partial K \subset K_{\widetilde{E}}$. Let $x \in \partial K_E \setminus \partial K$. Then $x \in (\partial \ell(\mathcal{E}) \setminus C) \cap \overline{K_E}$. Since $\ell(\mathcal{E})$ is strictly convex, there exists a hyperplane $H$ such that $H \cap \ell(\mathcal{E}) = \{x\}$, hence $H \cap C = \emptyset$. Since $C$ is compact, if $\epsilon > 0$ is small enough the hyperplane $H_\epsilon = H + \epsilon\bar{1}$ does not intersect $C$. It follows that $x$ and $\operatorname{conv}(C)$ lie on opposite sides of $H_\epsilon$, for some $\epsilon > 0$, so $x \notin \widetilde{E}$.

Let $\widetilde{\omega} = \omega_1 \cup \omega_3 \cup \widetilde{\omega_2}$, where $\widetilde{\omega_2} = K_{\widetilde{E}}$. Since $\ell(\mathcal{E}) \cap \partial K = \operatorname{conv}(C) \cap \partial K = C$ we have $\partial' K_E = \partial' K_{\widetilde{E}}$, so $\widetilde{\omega}$ is a domain contained in $\omega^0$ and $\omega \subsetneqq \widetilde{\omega}$. We have $u_{\widetilde{\omega}} \leqslant u_\omega$ on $\omega$. For the opposite inequality we define the function $\widetilde{u}$ on $\widetilde{\omega}$,

$$\widetilde{u}(x) = \begin{cases} u_\omega(x), & \text{if } x \in \omega_1 \cup \omega_3, \\ u(x; K_{\widetilde{E}}, f), & \text{if } x \in \widetilde{\omega_2} = K_{\widetilde{E}}, \end{cases}$$

where $f = \log h_{\Omega^0} \circ \mathfrak{e}$ is as in Lemma 3.7. Since $\partial' K_E = \partial' K_{\widetilde{E}}$ and, by Theorem 3.5, $u(x; K_E, f) = u(x; K_{\widetilde{E}}, f)$ for $x \in K_E$, Lemma 3.8 and its proof (see (3.4)) imply that the function $\widetilde{u}$ is an element of the defining family of $u_{\widetilde{\omega}}$. Thus $\widetilde{u} \leqslant u_{\widetilde{\omega}}$ on $\widetilde{\omega}$. As $u(\,\cdot\,; K_E, f) = u(\,\cdot\,; K_{\widetilde{E}}, f)|_{K_E}$, it follows from Lemma 3.8 that $\widetilde{u} = u_\omega$ on $\omega$. Therefore $u_\omega \leqslant u_{\widetilde{\omega}}$ on $\omega$, hence $u_\omega = u_{\widetilde{\omega}}|_\omega$. This concludes the proof of Theorem 1.1. $\square$

### 3.3. Proof of Theorem 1.2

If $\mathcal{E} \cap \{z_1 z_2 = 0\} = \emptyset$, the existence of $\widetilde{\Omega}$ follows from Theorem 1.1. So, let $\mathcal{E} \cap \{z_1 z_2 = 0\} \neq \emptyset$. This implies that $\mathcal{E}$ intersects $\{z_1 = 0\}$ or $\{z_2 = 0\}$. Since $\Omega$ is connected and Reinhardt, it can intersect only one of them. Without loss of generality, suppose that $\mathcal{E} \cap \{z_1 = 0\} \neq \emptyset$.

Following the notation in the proof of Theorem 1.1, we consider the same partition of $\Omega$.

If $\Phi(x_1, x_2) = x_2 - x_1$, the function $\Phi|_{\ell(\mathcal{E})}$ attains its minimum at a unique point $a = (a_1, a_2) \in \ell(\mathcal{E})$, as $\ell(\mathcal{E})$ is closed and strictly convex. Let

$$L = \{x \in \omega : x = a + t\bar{1},\, t \leqslant 0\}.$$

Now let

$$\widetilde{\omega_2} = \{y + (s, 0) :\ y \in L \setminus \{a\},\, s \leqslant 0\}\,,\ \widetilde{\omega} = \omega_1 \cup \widetilde{\omega_2} \cup \omega_3.$$

By the construction of $a$ and since $\ell(\mathcal{E})$ is strictly convex, it follows easily that $\widetilde{\omega}$ is a domain such that $\omega \subsetneqq \widetilde{\omega} \subset \omega^0$. Theorem 1.2 will follow if we prove that $u_\omega = u_{\widetilde{\omega}}|_\omega$. Since $\omega \subset \widetilde{\omega}$ we have $u_{\widetilde{\omega}} \leqslant u_\omega$ on $\omega$. To complete the proof, we need to show that $u_\omega \leqslant u_{\widetilde{\omega}}$ on $\omega$.

Lemma 3.6 shows that $u_\omega = u_{\widetilde{\omega}} = u_{\omega^0} = f$ on $\omega_1 \cup L$, where the function $f = \log h_{\Omega^0} \circ \mathfrak{e}$ is as in Lemma 3.7. Let us introduce the function

$$w(x_1, x_2) = x_2 - a_2 + f(a)\,,\ x = (x_1, x_2) \in \widetilde{\omega_2}.$$

We claim that $w = f$ on $L$. Indeed, $L$ has equation $x_2 = x_1 + a_2 - a_1$, $x_1 \leqslant a_1$, and by the $\bar{1}$-linearity of $f$, $w(x_1, x_1 + a_2 - a_1) = x_1 - a_1 + f(a_1, a_2) = f(x_1, x_1 + a_2 - a_1)$.

Consider now the function $u$ defined on $\widetilde{\omega}$ by

$$u(x)=\begin{cases} u_\omega(x), & \text{if } x\in\omega_1\cup\omega_3,\\ w(x), & \text{if } x\in\widetilde{\omega_2}.\end{cases}$$

Clearly $u<0$ and $u$ has normalized growth at $-\infty$. To show that $u$ is (locally) convex we proceed as in the proof of Lemma 3.8. Let $(x_1,x_2)\in\widetilde{\omega_2}$ and $x_1'$ be so that $(x_1',x_2)\in L$. Then $f(\,\cdot\,,x_2)$ is a convex function on $(-\infty,x_1']$, so by Lemma 2.2, $f(x_1,x_2)\leqslant f(x_1',x_2)=w(x_1',x_2)=w(x_1,x_2)$. Therefore we can apply the same argument as the one used in the proof of the convexity of the function defined in (3.4). We conclude that $u$ is an element of the defining family of the function $u_{\widetilde{\omega}}$, so $u\leqslant u_{\widetilde{\omega}}$ on $\widetilde{\omega}$.

We will prove that $u_\omega\leqslant w$ on $\omega_2$. This implies that $u_\omega\leqslant u\leqslant u_{\widetilde{\omega}}$ on $\omega$, which finishes the proof. To this end, we define

$$m=\inf\{x_2 : \exists\, x=(x_1,x_2)\in\ell(\mathcal{E})\}>-\infty,$$

as $0\notin\mathcal{E}$. Since $\ell(\mathcal{E})$ is strictly convex, Lemma 2.3 shows that if $(x_1',x_2')\in L$ and $x_2'\leqslant m$ then $(x_1,x_2')\in\omega_2$ for all $x_1<x_1'$. We partition $\omega_2$ as follows:

$$\omega_2'=\left\{(x_1,x_2)\in\omega_2 :\ x_2\leqslant m\right\},\ \omega_2''=\omega_2\setminus\omega_2'.$$

Let $v$ be any element of the defining family of $u_\omega$. If $(x_1,x_2)\in\omega_2'$ and $x_1'$ is so that $(x_1',x_2)\in L$ then Lemma 2.2 applied to the convex function $v(\,\cdot\,,x_2)$ on $(-\infty,x_1']$ implies that

$$v(x_1,x_2)\leqslant v(x_1',x_2)\leqslant u_\omega(x_1',x_2)=f(x_1',x_2)=w(x_1',x_2)=w(x_1,x_2).$$

If $x\in\omega_2''$, then, since $\ell(\mathcal{E})$ is convex, we can write $x=tx^1+(1-t)x^2$, $0<t<1$, with points $x^1\in\omega_2'$ and $x^2\in L$. Since $v\leqslant u_\omega=f=w$ on $L$ and $w$ is an affine function, it follows that

$$v(x)\leqslant tv(x^1)+(1-t)v(x^2)\leqslant tw(x^1)+(1-t)w(x^2)=w(x).$$

So $v\leqslant w$, and hence $u_\omega\leqslant w$, on $\omega_2$. This concludes the proof of Theorem 1.2.

## 4. Reinhardt Subdomains of the Unit Bidisk

We will now provide a detailed study when $\Omega^0=\Delta^2$ and $\mathcal{E}\not\ni 0$ is a Reinhardt compact, logarithmically convex subset of $\Delta^2$.

It should be noted that if the pluricomplex Green function of a given subdomain $\Omega\subset\Delta^2$ is identically equal to that of $\Delta^2$, the largest domain $\widetilde{\Omega}$ that satisfies the extension property given by equation (1.1) is $\Delta^2$ itself. The following theorem can be proved easily.

Theorem 4.1. — *Let $\Omega = \Delta^2 \setminus \mathcal{E}$ where $\mathcal{E} \not\ni 0$ is a Reinhardt compact, logarithmically convex subset of $\Delta^2$ such that $\operatorname{int}\mathcal{E} \cap \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2|\} = \emptyset$. Then $g_\Omega(z, 0) = g_{\Delta^2}(z, 0)$, for all $z \in \Omega$.*

*Proof.* — As in Section 3, we set

$$\omega = \ell(\Omega) \cap \mathbb{R}^2 \ , \ \ \omega^0 = \ell(\Delta^2) \cap \mathbb{R}^2 = \{(x_1, x_2) : \, x_1 < 0, \, x_2 < 0\} .$$

Without loss of generality, assume that $\mathcal{E} \subset \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leqslant |z_2|\}$. Then, $\ell(\mathcal{E})$ lies in some strip

$$S = \{(x_1, x_2) \in \mathbb{R}^2 : m \leqslant x_2 \leqslant M, \, x_1 \leqslant x_2\},$$

where $m = \inf\{x_2 : (x_1, x_2) \in \ell(\mathcal{E})\}$ and $M = \sup\{x_2 : (x_1, x_2) \in \ell(\mathcal{E})\}$. Note that $-\infty < m \leqslant M < 0$. Lemma 3.6 shows that for any $(x_1, x_2) \in \omega$ with $x_1 > x_2$, $u_\omega(x_1, x_2) = x_1$. Also, for any $(x_1, x_2) \in \omega$ with $x_1 = x_2$,

$$u_\omega(x_1, x_2) = \lim_{\substack{(y_1,y_2)\to(x_1,x_2)\\ (y_1,y_2)\in\omega, \ y_1>y_2}} u_\omega(y_1, y_2) = x_2. \tag{4.1}$$

Now, let $(x_1, x_2) \in \omega \backslash S$ with $x_1 \leqslant x_2$. Then, $\{(x_1, x_2) : \, -\infty < x_1 \leqslant x_2\} \subset \omega$ and by Lemma 2.2, $u_\omega(x_1, x_2) \leqslant x_2$. Hence we have shown that

$$u_\omega(x_1, x_2) = u_{\omega^0}(x_1, x_2) \, , \ \ \forall \ (x_1, x_2) \in (\omega \setminus S) \cup \{(x_1, x_2) \in \omega : x_1 = x_2\} .$$

Note that for any point in $\{(x_1, x_2) \in \omega : x_2 = m\}$ or $\{(x_1, x_2) \in \omega : x_2 = M\}$,using the argument in equation (4.1),

$$u_\omega(x_1, x_2) = x_2. \tag{4.2}$$

If $\mathcal{E} \cap \{(z_1, z_2) \in \mathbb{C}^2 : z_1 z_2 = 0\} \neq \emptyset$, then $\ell(\mathcal{E})$ is unbounded. Any point in $S \setminus \ell(\mathcal{E})$ lies on a segment $[P_1, P_2] \subset \omega$, where the points $P_1$, $P_2 \in \{(x_1, x_2) \in \omega : x_2 = m\} \cup \{(x_1, x_2) \in \omega : x_2 = M\} \cup \{(x_1, x_2) \in \omega : x_1 = x_2\}$. Since $u_\omega$ is (locally) convex, equations (4.1) and (4.2) yield that $u_\omega(x_1, x_2) \leqslant x_2$.

If $\mathcal{E} \cap \{(z_1, z_2) \in \mathbb{C}^2 : z_1 z_2 = 0\} = \emptyset$, then $\ell(\mathcal{E})$ is bounded and $S \setminus \ell(\mathcal{E})$ has an unbounded connected component. For any point in the bounded component(s) of $S \setminus \ell(\mathcal{E})$, the previous argument shows that $u_\omega(x_1, x_2) \leqslant x_2$. Any point in the unbounded component of $S \setminus \ell(\mathcal{E})$ lies on a segment $[Q_1, Q_2] \subset \omega$, where the points $Q_1$, $Q_2 \in \{(x_1, x_2) \in \omega : x_2 = m\} \cup \{(x_1, x_2) \in \omega : x_2 = M\}$. Equation (4.2) shows that $u_\omega(x_1, x_2) \leqslant x_2$. Hence, we conclude that $u_\omega = u_{\omega^0}$ on $\omega$ and the theorem is proved. □

So, if $\Omega \subset \Delta^2$ is a domain that satisfies the requirements of Theorem 4.1, $\widetilde{\Omega} = \Delta^2$. Hence, we will focus on the domains that do not fall into this category.

Let $\mathcal{E} \not\ni 0$ be a Reinhardt compact, logarithmically convex subset of $\Delta^2$ such that $\operatorname{int}\mathcal{E} \cap \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2|\} \neq \emptyset$. We will show that there exists a unique largest subdomain in $\Delta^2$ which satisfies the extension property given by the equation (1.1) when $\mathcal{E}$ is a strictly logarithmically convex subset of $\Delta^2$. But if $\mathcal{E}$ is not strictly logarithmically convex, in some cases there exist many domains that satisfy the extension property and are maximal with respect to the inclusion.

The problem will be discussed in two cases as in Section 3. The first case will be when $\mathcal{E} \cap \{(z_1, z_2) \in \mathbb{C}^2 : z_1 z_2 = 0\} = \emptyset$. Let $A = (A_1, A_2)$, $B = (B_1, B_2)$ be points in $(0,1)^2$ with $A_1 > A_2$, $B_1 < B_2$, and define

$$\widetilde{\mathcal{E}} = \{(z_1, z_2) \in \Delta^2 : |z_1| = A_1^t B_1^{1-t},\ |z_2| = A_2^t B_2^{1-t},\ 0 \leqslant t \leqslant 1\}. \tag{4.3}$$

We will show that if $\mathcal{E}$ is strictly logarithmically convex, then $\widetilde{\Omega} = \Delta^2 \setminus \widetilde{\mathcal{E}}$ or $\widetilde{\Omega} = \Delta^2 \setminus (\widetilde{\mathcal{E}_1} \cup \widetilde{\mathcal{E}_2})$, where $\widetilde{\mathcal{E}}$, $\widetilde{\mathcal{E}_1}$, $\widetilde{\mathcal{E}_2}$ are unique and are all sets that are of the form given by equation (4.3). If $\mathcal{E}$ is not strictly logarithmically convex, $\widetilde{\Omega}$ will be of the same form, but in some cases $\widetilde{\mathcal{E}}$, $\widetilde{\mathcal{E}_1}$, $\widetilde{\mathcal{E}_2}$ will not be unique.

The second case will be when $\mathcal{E} \cap \{(z_1, z_2) \in \mathbb{C}^2 : z_1 z_2 = 0\} \neq \emptyset$. Without loss of generality, we will assume that $\mathcal{E} \cap \{(z_1, z_2) \in \mathbb{C}^2 : z_1 = 0\} \neq \emptyset$. Let $A = (A_1, A_2) \in (0,1)^2$ such that $A_1 > A_2$ and define

$$\widetilde{F} = \{(z_1, z_2) \in \Delta^2 : |z_1| = tA_1,\ |z_2| = A_2,\ 0 \leqslant t \leqslant 1\}. \tag{4.4}$$

Then, $\widetilde{\Omega} = \Delta^2 \setminus \widetilde{F}$ or $\widetilde{\Omega} = \Delta^2 \setminus (\widetilde{F_1} \cup \widetilde{F_2})$, where $\widetilde{F}$, $\widetilde{F_1}$, $\widetilde{F_2}$ are all sets of the form given by the equation (4.4) and the arguments about their uniqueness properties will be the same as in the previous case.

In order to find $\widetilde{\Omega}$ for a given $\Omega$, we will construct a basic subdomain $\Omega^B \supset \Omega$ of $\Delta^2$. This subdomain will be of the form $\widetilde{\Omega} = \Delta^2 \setminus \widetilde{\mathcal{E}}$, $\widetilde{\Omega} = \Delta^2 \setminus (\widetilde{\mathcal{E}_1} \cup \widetilde{\mathcal{E}_2})$, $\widetilde{\Omega} = \Delta^2 \setminus \widetilde{F}$, or $\widetilde{\Omega} = \Delta^2 \setminus (\widetilde{F_1} \cup \widetilde{F_2})$, as described above. We will show that $\Omega^B$ satisfies the following:

(1) The pluricomplex Green function of $\Omega^B$ can be explicitly characterized and is not identically equal to the pluricomplex Green function of $\Delta^2$.
(2) To any domain $\Omega = \Delta^2 \setminus \mathcal{E}$, one can associate a basic subdomain $\Omega^B \supset \Omega$ by a natural geometric construction.
(3) $\Omega^B$ satisfies the extension property given by equation (1.1).

This subdomain $\Omega^B$ will provide $\widetilde{\Omega}$ and will give a complete answer to our extension problem.

## 4.1. The case $\mathcal{E} \cap \{z_1 z_2 = 0\} = \emptyset$

In this section, we will first discuss the basic subdomain $\Omega^B$. The construction of $\Omega^B$ from a given $\Omega$ will be studied afterwards.

#### 4.1.1. Basic Subdomains $\Omega^B$

**The Construction and Partition of $\Omega^B$.** The construction will be discussed in logarithmic coordinates. For the simplicity of notation, we set

$$\ell(\Delta^2) := \ell(\Delta^2) \cap \mathbb{R}^2 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 0\,, x_2 < 0\}\,.$$

Let $p = (p_1, p_2)$, $q = (q_1, q_2) \in \ell(\Delta^2)$ such that $p_1 > p_2$ and $q_1 < q_2$. Then we define the following:

$$L_1 = \{tp : 0 < t \leqslant 1\},\ L_2 = \{tq : 0 < t \leqslant 1\},$$
$$L_3 = \{p + t\bar{1} : t \leqslant 0\},\ L_4 = \{q + t\bar{1} : t \leqslant 0\},$$

where $\bar{1} = (1, 1) \in \mathbb{R}^2$. Therefore, $L_1$ and $L_2$ are line segments through the origin with positive slopes $m_1 > 1$ and $m_2 < 1$, respectively, and $L_3$ and $L_4$ are rays to $-\infty$ with slope 1. We fix points

$$a = (a_1, a_2) \in L_1\,,\ b = (b_1, b_2) \in L_2\,,\ c = (c_1, c_2) \in L_3\,,\ d = (d_1, d_2) \in L_4,$$

and define following sets:

$$E_1 = \{ta + (1 - t)b : 0 \leqslant t \leqslant 1\},$$
$$E_2 = \{tc + (1 - t)d : 0 \leqslant t \leqslant 1\},$$
$$E = \{tp + (1 - t)q : 0 \leqslant t \leqslant 1\}.$$

We can now define $\omega^B$ using these sets as follows:

$$\omega^B = \begin{cases} \ell(\Delta^2) \setminus E & \text{if } a = p = c \text{ and } b = q = d\,, \\ \ell(\Delta^2) \setminus (E_1 \cup E_2) & \text{otherwise}\,. \end{cases} \tag{4.5}$$

Note that $\hat{\mathfrak{e}}(E)$, $\hat{\mathfrak{e}}(E_i)$, $i = 1, 2$ are sets of the form given by equation (4.3).

Then, the basic subdomain will be

$$\Omega^B = \hat{\mathfrak{e}}(\omega^B)\,.$$

The set $\omega^B$ will be partitioned as follows:

$$\omega^B_{1,1} = \{(x_1, x_2) \in \omega^B :\ x_2 \geqslant x_1 + d_2 - d_1\},$$
$$\omega^B_{1,2} = \left\{(x_1, x_2) \in \omega^B :\ \min\left\{\tfrac{(b_2-d_2)x_1+b_1d_2-b_2d_1}{b_1-d_1},\ \tfrac{b_2}{b_1}x_1\right\} \leqslant x_2 \leqslant x_1 + d_2 - d_1\right\},$$
$$\omega^B_1 = \omega^B_{1,1} \cup \omega^B_{1,2},$$
$$\omega^B_{2,1} = \{(x_1, x_2) \in \omega^B :\ x_2 \leqslant x_1 + c_2 - c_1\},$$
$$\omega^B_{2,2} = \left\{(x_1, x_2) \in \omega^B :\ x_1 + c_2 - c_1 \leqslant x_2 \leqslant \max\left\{\tfrac{a_2}{a_1}x_1,\ \tfrac{(a_2-c_2)x_1+a_1c_2-a_2c_1}{a_1-c_1}\right\}\right\},$$
$$\omega^B_2 = \omega^B_{2,1} \cup \omega^B_{2,2},$$
$$\omega^B_3 = \{(tx_1, tx_2) \in \omega^B :\ (x_1, x_2) \in E_1,\ t \in (0, 1)\},$$

$$\omega_4^B = \{(x_1,x_2)+t\bar{1} \in \omega^B : (x_1,x_2) \in E_2,\ t<0\},$$

$$\omega_5^B = \omega^B \setminus \bigcup_{j=1}^{4} \omega_j^B.$$

We note that in the definition of $\omega_{1,2}^B$ given above it is assumed that $b_1 \neq d_1$, i.e. $b \neq d$. If $b_1 = d_1$ then $b = d$ and we define

$$\omega_{1,2}^B = \left\{(x_1,x_2) \in \omega^B : \frac{b_2}{b_1}x_1 \leqslant x_2 \leqslant x_1 + b_2 - b_1\right\}.$$

Similarly, in the definition of $\omega_{2,2}^B$ given above it is assumed that $a_1 \neq c_1$, i.e. $a \neq c$. If $a_1 = c_1$ then $a = c$ and we define

$$\omega_{2,2}^B = \left\{(x_1,x_2) \in \omega^B : x_1 + a_2 - a_1 \leqslant x_2 \leqslant \frac{a_2}{a_1}x_1\right\}.$$

**The Pluricomplex Green Function of $\Omega^B$.** We are now ready to compute the pluricomplex Green function of $\Omega^B$ with pole at 0. This will be done in logarithmic coordinates as well. Note that

$$u_{\ell(\Delta^2)}(x) = \max\{x_1,x_2\}.$$

By Lemma 3.6,

$$u_{\omega^B}(x) = u_{\ell(\Delta^2)}(x), \qquad x \in \omega_{1,1}^B \cup \omega_{2,1}^B. \tag{4.6}$$

A simple argument using the convexity of $u_{\omega^B}$ then shows that

$$u_{\omega^B}(x) = u_{\ell(\Delta^2)}(x), \qquad x \in \omega_1^B \cup \omega_2^B.$$

Let $u_{\omega_3^B}$ be the affine mapping satisfying $u_{\omega_3^B}(0,0)=0$, $u_{\omega_3^B}(a_1,a_2)=a_1$, and $u_{\omega_3^B}(b_1,b_2)=b_2$. Note that this function's graph is a plane in $\mathbb{R}^3$. It can be written explicitly as

$$u_{\omega_3^B}(x) = \frac{b_2(a_1-a_2)x_1 + a_1(b_2-b_1)x_2}{a_1b_2 - a_2b_1}.$$

Also, observe that

$$u_{\omega_3^B}(x) = x_1 = u_{\ell(\Delta^2)}(x), \qquad x \in \partial\omega_2^B \cap \partial\omega_3^B \cap \omega,$$

and

$$u_{\omega_3^B}(x) = x_2 = u_{\ell(\Delta^2)}(x), \qquad x \in \partial\omega_1^B \cap \partial\omega_3^B \cap \omega.$$

Since $u_{\omega_3^B}$ is affine, it is the maximal convex function on $\omega_3^B$ that is equal to $u_{\ell(\Delta^2)}$ on $\partial\omega_1^B \cap \partial\omega_3^B \cap \omega$ and $\partial\omega_2^B \cap \partial\omega_3^B \cap \omega$. This shows that

$$u_{\omega^B}(x) \leqslant u_{\omega_3^B}(x), \qquad x \in \omega_3^B.$$

Let $u_{\omega_4^B}$ be the affine mapping defined by

$$u_{\omega_4^B}(x) = x_2 = u_{\ell(\Delta^2)}(x), \qquad x \in \partial\omega_1^B \cap \partial\omega_4^B \cap \omega,$$

and

$$u_{\omega_4^B}(x) = x_1 = u_{\ell(\Delta^2)}(x), \qquad x \in \partial\omega_2^B \cap \partial\omega_4^B \cap \omega.$$

Therefore the graph of this function is a plane through parallel lines in $\mathbb{R}^3$. Explicitly,

$$u_{\omega_4^B}(x) = \frac{(c_1 - c_2)x_1 + (d_2 - d_1)x_2 + (c_1 - c_2)(d_2 - d_1)}{c_1 - c_2 - d_1 + d_2}.$$

Using the same argument as in the case of $u_{\omega_3^B}$, we conclude that

$$u_{\omega^B}(x) \leqslant u_{\omega_4^B}(x), \qquad x \in \omega_4^B.$$

Assume now that $a \neq p \neq c$ and $b \neq q \neq d$. To simplify notation we let

$$f(x) = u_{\ell(\Delta^2)}(x) \text{ for } x \in \partial'\omega_5^B\,, \ \text{ where } \partial'\omega_5^B = \partial\omega_5^B \cap \omega^B\,.$$

Then $f < -\varepsilon$ on $\partial'\omega_5^B$, for some $\varepsilon > 0$. Define

$$u_{\omega_5^B}(x) = \sup\left\{v(x) :\ v : \omega_5^B \cup \partial'\omega_5^B \to \mathbb{R} \text{ convex},\ v \leqslant f \text{ on } \partial'\omega_5^B\right\}.$$

Then $u_{\omega_5^B}$ is a convex function which satisfies $u_{\omega_5^B} \leqslant -\varepsilon$ on $\omega_5^B \cup \partial'\omega_5^B$. Also, as $u_{\ell(\Delta^2)}$ is an element of the defining family of $u_{\omega_5^B}$, we have

$$u_{\omega_5^B}(x) = u_{\ell(\Delta^2)}(x), \qquad x \in \partial'\omega_5^B.$$

Now define a function $u : \omega^B \to \mathbb{R}$ as

$$u(x) = \begin{cases} u_{\ell(\Delta^2)}(x) & \text{if } x \in \omega_1^B \cup \omega_2^B, \\ u_{\omega_3^B}(x) & \text{if } x \in \omega_3^B, \\ u_{\omega_4^B}(x) & \text{if } x \in \omega_4^B, \\ u_{\omega_5^B}(x) & \text{if } x \in \omega_5^B. \end{cases} \tag{4.7}$$

Proposition 4.2. — *With the above notation, if $a \neq p \neq c$ and $b \neq q \neq d$, then $u = u_{\omega^B}$.*

*Proof.* — We will show that $u$ is an element of the defining family of $u_{\omega^B}$, that is, $u$ is a negative locally convex function with normalized growth at $-\infty$.

The negativity of $u$ is trivial. As each function in the definition of $u$ is convex, it is enough to check the (local) convexity in a small neighborhood of each point of $\partial\omega_k^B \cap \partial\omega_j^B \cap \omega^B$, $k \neq j$. But $u_{\omega_j^B}$, $j = 3, 4$, are affine functions and

$$u(x) = \max\{u_{\ell(\Delta^2)}(x), u_{\omega_3^B}(x), u_{\omega_4^B}(x)\}, \qquad x \in \omega^B \setminus \omega_5^B.$$

Therefore it remains to check the convexity of $u$ in neighborhoods of points of $\partial'\omega_5^B$. This follows by the same steps as in the proof of the convexity of the function given by equation (3.4), since the convex function $u_{\ell(\Delta^2)} \leqslant u_{\omega_5^B}$ on $\omega_5^B \cup \partial'\omega_5^B$ and $u_{\ell(\Delta^2)} = u_{\omega_5^B}$ on $\partial'\omega_5^B$. We conclude that $u$ is convex on $\omega^B$.

Note that the condition of having normalized growth at $-\infty$ should be checked for rays in $\omega_1^B \cup \omega_2^B \cup \omega_4^B$. As $u$ is equal to $u_{\ell(\Delta^2)}$ on $\omega_1^B \cup \omega_2^B$, it suffices to check this on $\omega_4^B$. But this follows easily as $u_{\omega_4^B}$ is $\bar{1}$-linear. Therefore, $u$ is an element of the defining family of $u_{\omega^B}$ and $u \leqslant u_{\omega^B}$.

On the other hand, we have already shown that $u_{\omega^B} \leqslant u$ on $\omega^B \setminus \omega_5^B$. As the restriction of $u_{\omega^B}$ to $\omega_5^B \cup \partial'\omega_5^B$ is an element of the defining family of $u_{\omega_5^B}$, $u_{\omega^B} \leqslant u$ on $\omega_5^B$ as well. Hence $u = u_{\omega^B}$. □

If $a = p = c$ and $b = q = d$ then $\omega_5^B = \emptyset$ and $u_{\omega^B} = u$ where $u$ is the function given in (4.7). This gives an explicit formula for the pluricomplex Green function of $\Omega^B$ with pole at the origin, which was previously obtained by Klimek [7, Example 5.8]. Note that $g_{\Omega^B}(\,\cdot\,, 0)$ extends continuously to $\Delta^2$, but the extended function is not plurisubharmonic on $\Delta^2$.

Proposition 4.3. — *With the above notation, if $a = p = c$ and $|\{b, q, d\}| > 1$, or if $b = q = d$ and $|\{a, p, c\}| > 1$, then there is an explicit formula for $u_{\omega^B}$.*

*Proof.* — In logarithmic coordinates, $\omega_5^B$ is a triangular region in $\mathbb{R}^2$. Without loss of generality, let $a = p = c$. Then $u_{\omega^B}$ can be calculated in $\omega^B \setminus \omega_5^B$ as in Proposition 4.2. If a segment $[P_1, P_2]$ lies in $\omega^B$ and $P_1, P_2 \notin \omega_5^B$ then $[P_1, P_2] \cap \omega_5^B = \emptyset$. This implies that $u_{\omega^B}|_{\omega_5^B}$ is the largest negative convex function on $\omega_5^B$ that is equal to $u_{\ell(\Delta^2)}$ on $\partial\omega_5^B \cap \omega^B$. Hence $u_{\omega_5^B}$ is the affine mapping such that $u_{\omega_5^B}(b_1, b_2) = b_2$, $u_{\omega_5^B}(d_1, d_2) = d_2$, and $u_{\omega_5^B}(a_1, a_2) = 0$. It is given by the equation

$$u_{\omega_5^B} = \frac{a_2(d_2 - b_2)x_1 + (d_2(b_1 - a_1) - b_2(d_1 - a_1))x_2 - a_2(b_1 d_2 - b_2 d_1)}{a_2(d_1 - b_1) + a_1(b_2 - d_2) + b_1 d_2 - b_2 d_1}\,.$$

A simple calculation shows that the function $u$ defined as in the equation (4.7) using this function $u_{\omega_5^B}$ is equal to $u_{\omega^B}$, and the result follows. □

### 4.1.2. Construction of $\widetilde{\Omega}$

Let $\Omega = \Delta^2 \setminus \mathcal{E}$, where $\mathcal{E} \not\ni 0$ is a Reinhardt compact, logarithmically convex subset of $\Delta^2$ such that $\mathcal{E} \cap \{(z_1, z_2) \in \mathbb{C}^2 : z_1 z_2 = 0\} = \emptyset$. We will show that given such $\Omega$, the subdomain $\widetilde{\Omega}$ of $\Delta^2$ that satisfies the extension property given by (1.1) will be $\Delta^2$ itself or a basic subdomain $\Omega^B$. To this end, we show how to find points that are required to construct $\Omega^B$ from a given $\Omega$, by using two functions in logarithmic coordinates.

The first of these functions is $\Psi(x_1, x_2) = \arctan(x_2/x_1)$. Its level sets are lines through the origin, and $\Psi|_{\ell(\mathcal{E})}$ attains its maximum $\theta_1$ and minimum $\theta_2$ as $\ell(\mathcal{E})$ is compact. Note that $\theta_1$, $\theta_2 \geqslant \pi/4$ or $\theta_1$, $\theta_2 \leqslant \pi/4$ if and only if $\operatorname{int}\mathcal{E} \cap \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2|\} = \emptyset$. In this case Theorem 4.1 shows that $\widetilde{\Omega} = \Delta^2$.

We assume next that $\theta_1 > \pi/4 > \theta_2$. The second function is $\Phi(x_1, x_2) = x_2 - x_1$. The level sets of this function are lines with slope 1. Like the previous function, $\Phi|_{\ell(\mathcal{E})}$ also attains its minimum $m$ and maximum $M$ as $\ell(\mathcal{E})$ is compact. Note that $\Psi(y) > \pi/4$ for any $y \in \ell(\mathcal{E})$ with $\Phi(y) = m$, and $\Psi(y) < \pi/4$ for any $y \in \ell(\mathcal{E})$ with $\Phi(y) = M$. We define the points $p$, $q$ as intersections of lines as follows:

$$\begin{aligned}\{p\} &= \{(p_1, p_2)\} = \{x_2 = x_1 \tan\theta_1\} \cap \{x_2 - x_1 = m\},\\ \{q\} &= \{(q_1, q_2)\} = \{x_2 = x_1 \tan\theta_2\} \cap \{x_2 - x_1 = M\}.\end{aligned}$$

Note that $p_1 > p_2$ and $q_1 < q_2$. We can now use the construction from Section 4.1.1 and define $L_1$, $L_2$, $L_3$, and $L_4$ as in that section.

**Case I : $\mathcal{E}$ is strictly logarithmically convex.** Let $\ell(\mathcal{E})$ be strictly convex. This implies that the extrema of the functions $\Psi|_{\ell(\mathcal{E})}$ and $\Phi|_{\ell(\mathcal{E})}$ are attained at unique points. Suppose that $\Psi|_{\ell(\mathcal{E})}$ attains its maximum at $a$ and its minimum at $b$, while $\Phi|_{\ell(\mathcal{E})}$ attains its minimum at $c$ and its maximum at $d$.

It is easily seen that as $\ell(\mathcal{E})$ is strictly convex, $\Psi(a) \neq \Psi(c)$ and $\Phi(a) \neq \Phi(c)$ unless $a = c$, and $\Psi(b) \neq \Psi(d)$ and $\Phi(b) \neq \Phi(d)$ unless $b = d$. Then, one can construct a basic subdomain $\Omega^B$ using the points $a$, $b$, $c$, and $d$ as in equation (4.5). Note that as $\ell(\mathcal{E})$ is convex and $a$, $b$, $c$, $d$ are in it, so is $E_1$ and $E_2$. Therefore, $\omega \subset \omega^B$ and $\omega \neq \omega^B$.

The following proposition easily follows:

Proposition 4.4. — *Let $\Omega = \Delta^2 \setminus \mathcal{E}$, where $\mathcal{E} \not\ni 0$ is a Reinhardt compact, strictly logarithmically convex subset of $\Delta^2$ such that $\mathcal{E} \cap \{(z_1, z_2) \in \mathbb{C}^2 : z_1 z_2 = 0\} = \emptyset$. Using the above notation, suppose that $\theta_1 > \pi/4 > \theta_2$. Then $\widetilde{\Omega} = \Omega^B$.*

Hence in this case the pluricomplex Green function of $\Omega$ extends to the pluricomplex Green function of $\widetilde{\Omega} = \Omega^B$, and $\Omega^B$ is the unique largest domain with this property.

**Case II : $\mathcal{E}$ is logarithmically convex but not strictly logarithmically convex.** Let $\ell(\mathcal{E})$ be a convex set that is not strictly convex. In this case, the functions $\Psi|_{\ell(\mathcal{E})}$ and $\Phi|_{\ell(\mathcal{E})}$ do not necessarily attain their extrema at unique points. We will show that in this case, one can pick any point from each of the sets of points where these functions attain their extrema and use them to construct basic subdomains $\Omega^B$ that satisfy the extension property given by (1.1). In some cases, we will also prove that among these basic subdomains, there exists one which has the largest pluricomplex Green function, hence is the most natural to choose.

Observe that if the extrema are attained at unique points, the proof of Proposition 4.4 can be used to find $\widetilde{\Omega}$ and this will be unique. Now, suppose that they are attained at more than one point. We are going to prove the results in the "generic" case, when $\widetilde{\omega} = \omega^B$ is constructed using four distinct points $a, b, c, d$. Notice that this implies $p,\ q \notin \ell(\mathcal{E})$. The other cases are similar to this one.

We let $a, a' \in L_1 \cap \ell(\mathcal{E})$, $b, b' \in L_2 \cap \ell(\mathcal{E})$, $c, c' \in L_3 \cap \ell(\mathcal{E})$, and $d, d' \in L_4 \cap \ell(\mathcal{E})$, where $\|\nu - p\| < \|\nu' - p\|$ for $\nu = a, c$, and $\|\nu - q\| < \|\nu' - q\|$ for $\nu = b, d$. Denote the basic subdomains constructed by these points by

$$\omega^B = \ell(\Delta^2) \setminus ([a,b] \cup [c,d])\ ,\ \ \omega^{B'} = \ell(\Delta^2) \setminus ([a',b'] \cup [c',d']).$$

Proposition 4.5. — *Using the above notation,* $u_\omega(x) = u_{\omega^B}(x) = u_{\omega^{B'}}(x)$ *for* $x \in \omega$.

Notice that the subdomains given in Proposition 4.5 are not comparable to each other by inclusion. Therefore we have infinitely many subdomains in $\Delta^2$ that satisfy the extension property given by equation (1.1) and are maximal with respect to inclusion.

We now let $a$, $b$, $c$, $d$ satisfy

$$\|a - p\| = \min_{\nu \in L_1 \cap \ell(\mathcal{E})} \|\nu - p\|\ , \qquad \|b - q\| = \min_{\nu \in L_2 \cap \ell(\mathcal{E})} \|\nu - q\|, \tag{4.8}$$

$$\|c - p\| = \min_{\nu \in L_3 \cap \ell(\mathcal{E})} \|\nu - p\|\ , \qquad \|d - q\| = \min_{\nu \in L_4 \cap \ell(\mathcal{E})} \|\nu - q\|. \tag{4.9}$$

We also let $(u_{\omega^B})^*$ and $(u_{\omega^{B'}})^*$ denote the upper semicontinuous regularizations of $u_{\omega^B}$ and $u_{\omega^{B'}}$ on $\ell(\Delta^2)$, respectively. Note that this says that for $(u_{\omega^B})^*$,

$$(u_{\omega^B})^*(x) = \lim_{\substack{y \to x \\ y \in \omega_3^B}} u_{\omega^B}(y), \qquad x \in [a,b]$$

and

$$(u_{\omega^B})^*(x) = \lim_{\substack{y \to x \\ y \in \omega_4^B}} u_{\omega^B}(y), \qquad x \in [c,d].$$

Analogous results also hold for $(u_{\omega^{B'}})^*$.

Proposition 4.6. — *With the above notations, $(u_{\omega^B})^* \geqslant (u_{\omega^{B'}})^*$ on $\ell(\Delta^2)$.*

*Proof.* — First of all, note that $\omega_i^B \subset \omega_i^{B'}$ for $i = 1, 2$ and $\omega_i^B \supset \omega_i^{B'}$ for $i = 3, 4$. Then, the results of Section 4.1.1 show that $(u_{\omega^B})^* = (u_{\omega^{B'}})^*$ on $\omega_1^B \cup \omega_2^B \cup \omega_3^{B'} \cup \omega_4^{B'}$. Since the function $(u_{\omega^{B'}})^*$ is convex on $\omega_3^B \setminus \omega_3^{B'}$, it is bounded above there by the affine function that is equal to $x_1$ on $L_1 \cap (\omega_3^B \setminus \omega_3^{B'})$ and $x_2$ on $L_2 \cap (\omega_3^B \setminus \omega_3^{B'})$. Since $(u_{\omega^B})^*$ is equal to that affine function there, we conclude that $(u_{\omega^B})^* \geqslant (u_{\omega^{B'}})^*$ on $\omega_3^B \setminus \omega_3^{B'}$, hence on $\omega_3^B$. A similar argument shows that $(u_{\omega^B})^* \geqslant (u_{\omega^{B'}})^*$ on $\omega_4^B$ as well. Lastly, recall from Section 4.1.1 the definition of $u_{\omega^B} = u_{\omega_5^B}$ on $\omega_5^B \cup \partial' \omega_5^B$ as the supremum of a class of convex functions with a given boundary condition. We note that the restriction of $(u_{\omega^{B'}})^*$ to $\omega_5^B \cup \partial' \omega_5^B$ is an element of the defining family of the function $u_{\omega_5^B}$. Thus $(u_{\omega^B})^* \geqslant (u_{\omega^{B'}})^*$ on $\omega_5^B$, and the result follows. □

Proposition 4.6 shows that if we use the points that satisfy equations (4.8) and (4.9) to construct $\omega^B$, then the associated basic subdomain $\Omega^B$ will have the largest pluricomplex Green function. Therefore in this case we let $\widetilde{\Omega}$ be given by this basic subdomain $\Omega^B$.

## 4.2. The case $\mathcal{E} \cap \{z_1 z_2 = 0\} \neq \emptyset$

The structure of this section will closely follow that of Section 4.2. The basic subdomain $\Omega^B$ will be introduced and investigated before discussing its construction from a given subdomain $\Omega \subset \Delta^2$. Without loss of generality, we assume that $\mathcal{E} \cap \{z_1 = 0\} \neq \emptyset$.

### 4.2.1. Basic Subdomains $\Omega^B$

**Construction and Partition of $\Omega^B$.** The construction will be discussed in logarithmic coordinates. Let $p = (p_1, p_2) \in \ell(\Delta^2)$ such that $p_1 > p_2$. We then define

$$L_1 = \{tp : 0 < t \leqslant 1\},\ L_2 = \{p + t\bar{1} : t \leqslant 0\}$$

where $\bar{1} = (1, 1) \in \mathbb{R}^2$. Hence, $L_1$ is a line segment thorough the origin with slope $m > 1$ and $L_2$ is a ray to $-\infty$ with slope 1. We fix points $a \in L_1$ and $b \in L_2$, and define the following sets:

$$E_1 = \{(ta_1, a_2) : t \geqslant 1\}, \tag{4.10}$$

$$E_2 = \{(tb_1, b_2) : t \geqslant 1\}, \tag{4.11}$$

$$E = \{(tp_1, p_2) : t \geqslant 1\}. \tag{4.12}$$

The set $\omega^B$ can now be defined using these sets as follows:

$$\omega^B = \begin{cases} \ell(\Delta^2) \setminus E & \text{if } a = p \text{ or } b = p, \\ \ell(\Delta^2) \setminus (E_1 \cup E_2) & \text{otherwise.} \end{cases} \tag{4.13}$$

Note that $\hat{\mathfrak{e}}(E)$, $\hat{\mathfrak{e}}(E_i)$, $i = 1, 2$, are sets of type given by equation (4.4). The basic subdomain will be $\Omega^B = \hat{\mathfrak{e}}(\omega^B)$.

In the case $a \neq p \neq b$, the subdomain $\omega^B$ will be partitioned into the following sets:

$$\begin{aligned}
\omega^B_{1,1} &= \{(x_1, x_2) \in \omega^B : x_2 \leqslant x_1 - b_1 + b_2\}, \\
\omega^B_{1,2} &= \left\{(x_1, x_2) \in \omega^B : x_1 - b_1 + b_2 \leqslant x_2 \leqslant \max\left\{\tfrac{a_2}{a_1} x_1, \tfrac{(a_2 - b_2)x_1 + a_1 b_2 - a_2 b_1}{a_1 - b_1}\right\}\right\}, \\
\omega^B_1 &= \omega^B_{1,1} \cup \omega^B_{1,2}, \\
\omega^B_2 &= \left\{(x_1, x_2) \in \omega^B : x_2 > a_2,\ x_1 \leqslant \frac{a_1}{a_2} x_2\right\}, \\
\omega^B_3 &= \{(x_1, x_2) \in \omega^B : x_1 - b_1 + b_2 \leqslant x_2 < b_2\}, \\
\omega^B_4 &= \omega^B \setminus \bigcup_{i=1}^{3} \omega^B_i.
\end{aligned}$$

In the case $a = p$ or $b = p$, we construct the partition of $\omega^B = \ell(\Delta^2) \setminus E$ by taking $a = b = p$ in the above formulas. We obtain:

$$\begin{aligned}
\omega^B_{1,1} &= \{(x_1, x_2) \in \omega^B : x_2 \leqslant x_1 - p_1 + p_2\}, \\
\omega^B_{1,2} &= \left\{(x_1, x_2) \in \omega^B : x_1 - p_1 + p_2 \leqslant x_2 \leqslant \frac{p_2}{p_1} x_1\right\}, \\
\omega^B_1 &= \omega^B_{1,1} \cup \omega^B_{1,2}, \\
\omega^B_2 &= \left\{(x_1, x_2) \in \omega^B : x_2 > p_2,\ x_1 \leqslant \frac{p_1}{p_2} x_2\right\}, \\
\omega^B_3 &= \{(x_1, x_2) \in \omega^B : x_1 - p_1 + p_2 \leqslant x_2 < p_2\}.
\end{aligned}$$

**The Pluricomplex Green Function of $\Omega^B$.** We can now compute the pluricomplex Green function of $\Omega^B$ with pole at 0. This will be done in logarithmic coordinates as well. Recall that

$$u_{\ell(\Delta^2)}(x) = \max\{x_1, x_2\}.$$

By Lemma 3.6,

$$u_{\omega^B}(x) = u_{\ell(\Delta^2)}(x) = \max\{x_1, x_2\} = x_1, \qquad x \in \omega^B_{1,1}.$$

A simple argument using the convexity of $u_{\omega^B}$ then shows that

$$u_{\omega^B}(x) = u_{\ell(\Delta^2)}(x) = x_1, \ x \in \omega_1^B.$$

We now let $\mathcal{B}_j = \partial\omega_1^B \cap \partial\omega_j^B \cap \omega^B$, $j = 2, 3, 4$. Note that $\mathcal{B}_j$ is a line segment for $j = 2, 4$ and $\mathcal{B}_3$ is a ray to $-\infty$. It is easily seen that

$$\begin{aligned}
\mathcal{B}_2 &= \left\{(x_1, x_2) \in \omega^B : a_2 < x_2 < 0, \ x_1 = \frac{a_1}{a_2}x_2\right\}, \\
\mathcal{B}_3 &= \{(x_1, x_2) \in \omega^B : x_2 < b_2, \ x_1 = x_2 + b_1 - b_2\}, \\
\mathcal{B}_4 &= \left\{(x_1, x_2) \in \omega^B : b_2 < x_2 < a_2, \ x_1 = \frac{(a_1 - b_1)x_2 - a_1 b_2 + a_2 b_1}{a_2 - b_2}\right\}.
\end{aligned}$$

We define the affine functions $u_{\omega_j^B}$, $j = 2, 3, 4$ as follows:

$$\begin{aligned}
u_{\omega_2^B}(x_1, x_2) &= \frac{a_1}{a_2}x_2, \\
u_{\omega_3^B}(x_1, x_2) &= x_2 + b_1 - b_2, \\
u_{\omega_4^B}(x_1, x_2) &= \frac{(a_1 - b_1)x_2 - a_1 b_2 + a_2 b_1}{a_2 - b_2}.
\end{aligned}$$

Let $x = (x_1, x_2) \in \mathcal{B}_j$. Then $(x_1', x_2) \in \omega_j^B$ for all $x_1' \leqslant x_1$, and these functions verify

$$u_{\omega_j^B}(x_1', x_2) = u_{\omega_j^B}(x_1, x_2) = x_1 = u_{\ell(\Delta^2)}(x_1, x_2) = u_{\omega^B}(x_1, x_2).$$

Applying Lemma 2.2 to the convex function $u_{\omega^B}(\,\cdot\,, x_2)$ on $(-\infty, x_1]$ we see that

$$u_{\omega^B}(x_1', x_2) \leqslant u_{\omega^B}(x_1, x_2) = u_{\omega_j^B}(x_1, x_2) = u_{\omega_j^B}(x_1', x_2)\,, \ j = 2, 3, 4.$$

Therefore $u_{\omega^B} \leqslant u_{\omega_j^B}$ on $\omega_j^B$ for $j = 2, 3, 4$.

Now we define a function $u : \omega^B \to \mathbb{R}$ as

$$u(x) = \begin{cases} u_{\ell(\Delta^2)}(x) & \text{if } x \in \omega_1^B, \\ u_{\omega_2^B}(x) & \text{if } x \in \omega_2^B, \\ u_{\omega_3^B}(x) & \text{if } x \in \omega_3^B, \\ u_{\omega_4^B}(x) & \text{if } x \in \omega_4^B. \end{cases} \tag{4.14}$$

Proposition 4.7. — *If $a \neq p \neq b$, then $u_{\omega^B} = u$. Moreover, the function $u_{\omega^B}$ extends continuously to $\ell(\Delta^2)$.*

*Proof.* — It is obvious that the function $u$ extends continuously to $\ell(\Delta^2)$. We have already shown that $u_{\omega^B} \leqslant u$ on $\omega^B$ and $u_{\omega^B} = u$ on $\omega_1^B$. We will prove that $u$ is an element of the defining family of $u_{\omega^B}$, that is, $u$ is a negative (locally) convex function with normalized growth at $-\infty$. This will imply that $u \leqslant u_{\omega^B}$.

Clearly, $u < 0$. As each function in the definition of $u$ is convex, it is enough to check the convexity of $u$ in a small neighborhood of each point $x \in \mathcal{B}_j$, $j = 2, 3, 4$. This follows easily since in a small neighborhood of such $x$ we have that $u = \max\{u_{\omega_j^B}, u_{\ell(\Delta^2)}\}$. Finally, we check that $u$ has normalized growth at $-\infty$ on $\omega_1^B \cup \omega_3^B$. Indeed, $u$ is equal to $u_{\ell(\Delta^2)}$ on $\omega_1^B$ and is $\bar{1}$-linear on $\omega_3^B$. $\square$

We remark that the continuous extension of $u_{\omega^B}$ to $\ell(\Delta^2)$ is not convex on $\ell(\Delta^2)$. If $\omega^B = \ell(\Delta^2) \setminus E$, then $\omega_4^B = \emptyset$ and $u_{\omega^B} = u$, where $u$ is obtained by replacing $a = p$ and $b = p$ in formula (4.14). This explicit formula for the pluricomplex Green function of $\Omega^B$ with pole at 0 was obtained by Klimek [7, Example 5.9].

### 4.2.2. Construction of $\widetilde{\Omega}$

Let $\Omega = \Delta^2 \setminus \mathcal{E}$, where $\mathcal{E} \not\ni 0$ is a Reinhardt compact, logarithmically convex subset of $\Delta^2$ such that $\mathcal{E} \cap \{(z_1, z_2) \in \mathbb{C}^2 : z_1 = 0\} \neq \emptyset$. We will show that given such $\Omega$, the subdomain $\widetilde{\Omega}$ of $\Delta^2$ that satisfies the extension property given by equation (1.1) will be $\Delta^2$ itself or a basic subdomain $\Omega^B$. We will use the functions $\Psi$ and $\Phi$ that were defined in Section 4.1.2 to find points that are required to construct the basic subdomain $\Omega^B$.

Recall that $\Psi(x_1, x_2) = \arctan(x_2/x_1)$. Since $\ell(\mathcal{E})$ is closed, $\Psi|_{\ell(\mathcal{E})}$ attains its maximum $\theta$. Observe that $\theta \leqslant \pi/4$ if and only if $\operatorname{int} \mathcal{E} \cap \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2|\} = \emptyset$ and Theorem 4.1 shows that in this case $\widetilde{\Omega} = \Delta^2$.

We assume that $\theta > \pi/4$. Recall that $\Phi(x_1, x_2) = x_2 - x_1$. $\Phi|_{\ell(\mathcal{E})}$ attains its minimum $m$ as $\ell(\mathcal{E})$ is closed. Observe that $m < 0$ and $\Psi(y) > \pi/4$ for any $y \in \ell(\mathcal{E})$ with $\Phi(y) = m$. We define the point $p$ as intersection of lines as follows:

$$\{p\} = \{(p_1, p_2)\} = \{x_2 = x_1 \tan\theta\} \cap \{x_2 - x_1 = m\}.$$

Note that $p_1 > p_2$. We can use the construction from Section 4.2.1 and define $L_1$ and $L_2$ as in that section.

**Case I: $\mathcal{E}$ strictly logarithmically convex.** Let $\ell(\mathcal{E})$ be strictly convex. Therefore, the maximum of $\Psi|_{\ell(\mathcal{E})}$ and the minimum of $\Phi|_{\ell(\mathcal{E})}$ are attained at unique points $a$ and $b$ of $\ell(\mathcal{E})$, respectively.

We observe that as $\ell(\mathcal{E})$ is strictly convex, we have that $a = p$ if and only if $b = p$, so $\Psi(a) \neq \Psi(b)$ and $\Phi(a) \neq \Phi(b)$ unless $a = b = p$. Then, one can construct a basic subdomain $\Omega^B$ using the points $a$ and $b$ as in equation (4.13). Since $a, b \in \ell(\mathcal{E})$ and $\ell(\mathcal{E})$ is a closed, unbounded convex set which is contained in a horizontal strip, we have $E_1, E_2 \subset \ell(\mathcal{E})$ by Lemma 2.3. So, $\omega \subset \omega^B$ and $\omega \neq \omega^B$.

The following proposition can be proved easily.

Proposition 4.8. — *Let $\Omega = \Delta^2 \setminus \mathcal{E}$, where $\mathcal{E} \not\ni 0$ is a Reinhardt compact, strictly logarithmically convex subset of $\Delta^2$ such that $\mathcal{E} \cap \{(z_1, z_2) \in \mathbb{C}^2 : z_1 = 0\} \neq \emptyset$. Using the above notation, suppose that $\theta > \pi/4$. Then $\widetilde{\Omega} = \Omega^B$.*

Hence when $\mathcal{E}$ is strictly logarithmically convex, the pluricomplex Green function of $\Omega$ extends to the pluricomplex Green function of $\widetilde{\Omega} = \Omega^B$, and $\Omega^B$ is the unique largest domain with this property.

**Case II: $\mathcal{E}$ is logarithmically convex but not strictly logarithmically convex.** Let $\ell(\mathcal{E})$ be a convex set that is not strictly convex. In this case, the maximum $\theta$ of $\Psi|_{\ell(\mathcal{E})}$ and the minimum $m$ of $\Phi|_{\ell(\mathcal{E})}$ are not necessarily attained at unique points. We will show that we can pick any point at which $\Psi|_{\ell(\mathcal{E})}$ attains its maximum and any other point at which $\Phi|_{\ell(\mathcal{E})}$ attains its minimum to construct the basic subdomain $\Omega^B$ that satisfies the extension property given by (1.1). We will prove that among these $\Omega^B$, there exists one which has the largest pluricomplex Green function.

Note that if the maximum of $\Psi|_{\ell(\mathcal{E})}$ and the minimum of $\Phi|_{\ell(\mathcal{E})}$ are attained at unique points, the proof of Proposition 4.8 can be used to find $\widetilde{\Omega} = \Omega^B$ and this will be unique. We will consider here the case when they are both attained at more than one point. The other cases can be treated in a similar manner.

Let $a = (a_1, a_2), a' = (a'_1, a'_2) \in L_1 \cap \ell(\mathcal{E})$ and $b = (b_1, b_2), b' = (b'_1, b'_2) \in L_2 \cap \ell(\mathcal{E})$, where $\|\nu - p\| < \|\nu' - p\|$ for $\nu = a, b$. Let $E_1, E_2$ be defined by the equations (4.10) and (4.11), respectively, and let

$$E'_1 = \{(ta'_1, a'_2) : t \geqslant 1\},\ E'_2 = \{(tb'_1, b'_2) : t \geqslant 1\}.$$

We denote the basic subdomains constructed as follows:

$$\omega^B = \ell(\Delta^2) \setminus (E_1 \cup E_2),\ \omega^{B'} = \ell(\Delta^2) \setminus (E'_1 \cup E'_2).$$

Proposition 4.9. — *Using the above notation, $u_\omega(x) = u_{\omega^B}(x) = u_{\omega^{B'}}(x)$ for $x \in \omega$.*

The subdomains given in Proposition 4.9 are not comparable to each other by inclusion. As in Section 4.1.2, we have infinitely many subdomains that satisfy the extension property given by (1.1) and are maximal with respect to inclusion.

To show that there exists a basic subdomain with the largest pluricomplex Green function, we let $a$ and $b$ satisfy

$$\|a-p\| = \min_{\nu\in L_1\cap\ell(\mathcal{E})} \|\nu - p\|, \ \|b-p\| = \min_{\nu\in L_2\cap\ell(\mathcal{E})} \|\nu - p\|. \tag{4.15}$$

Let $(u_{\omega^B})^*$ and $(u_{\omega^{B'}})^*$ denote the upper semicontinuous regularizations of $u_{\omega^B}$ and $u_{\omega^{B'}}$, respectively. Note that they are simply equal to the continuous extensions of these functions to $\ell(\Delta^2)$ (see Proposition 4.7).

PROPOSITION 4.10. — *With the above notation,* $(u_{\omega^B})^* \geqslant (u_{\omega^{B'}})^*$ *on* $\ell(\Delta^2)$.

*Proof.* — Note that $\omega_1^B \subset \omega_1^{B'}$ and $\omega_i^B \supset \omega_i^{B'}$ for $i = 2, 3$. The results of Section 4.2.1 show that $(u_{\omega^B})^* = (u_{\omega^{B'}})^*$ on $\omega_1^B \cup \omega_2^{B'} \cup \omega_3^{B'}$. If $(x_1, x_2) \in \omega_2^B \setminus \omega_2^{B'}$, or $(x_1, x_2) \in \omega_3^B \setminus \omega_3^{B'}$, or $(x_1, x_2) \in \omega_4^B$, we consider the point $(x_1', x_2)$ such that $(x_1', x_2) \in [a, a']$, or $(x_1', x_2) \in [b, b']$, or $(x_1', x_2) \in [a, b]$, respectively. The function $(u_{\omega^{B'}})^*(\,\cdot\,, x_2)$ is convex on $(-\infty, x_1']$, so Lemma 2.2 implies that $(u_{\omega^{B'}})^*(x_1, x_2) \leqslant (u_{\omega^{B'}})^*(x_1', x_2) = (u_{\omega^B})^*(x_1', x_2) = (u_{\omega^B})^*(x_1, x_2)$. Thus $(u_{\omega^B})^* \geqslant (u_{\omega^{B'}})^*$ on $(\omega_2^B \setminus \omega_2^{B'}) \cup (\omega_3^B \setminus \omega_3^{B'}) \cup \omega_4^B$, and the result follows. ☐

Proposition 4.10 shows that if we use the points that satisfy formulas (4.15) to construct $\omega^B$, the associated basic subdomain $\Omega^B$ will have the largest pluricomplex Green function with pole at 0. So, in this case $\widetilde{\Omega}$ will be given by this basic subdomain $\Omega^B$.


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