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CONSTRUCTION OF TWO-BUBBLE SOLUTIONS FOR SOME ENERGY-CRITICAL WAVE EQUATIONS

JACEK JENDREJ

Abstract. We present a construction of pure two-bubbles for some energy-critical wave equations, that is solutions which in one time direction approach a superposition of two stationary states both centered at the origin, but asymptotically decoupled in scale. Our solution exists globally, with one bubble at a fixed scale and the other concentrating in infinite time, with an error tending to 0 in the energy space. We treat the cases of the power nonlinearity in space dimension 6, the radial Yang-Mills equation and the equivariant wave maps equation with equivariance class $k \geq 3$. The concentration speed of the second bubble is exponential for the first two models and a power function in the last case.

1. Introduction

This note is devoted to the study of the long-time behavior of scalar semilinear energy-critical focusing wave equations. We consider the case of the power nonlinearity in dimension $N \geq 3$:

$$\text{(NLW)} \qquad \partial_t^2 u(t,x) = \Delta u(t,x) + |u(t,x)|^{\frac{4}{N-2}} u(t,x), \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^N,$$

the equivariant wave-map equation from $\mathbb{R}^2$ to $\mathbb{S}^2$:

$$\text{(WM)} \qquad \partial_t^2 u(t,r) = \partial_r^2 u(t,r) + \frac{1}{r}\partial_r u(t,r) - \frac{k^2}{2r^2}\sin(2u(t,r)), \qquad (t,r) \in \mathbb{R} \times (0,+\infty),$$

and the critical Yang-Mills equation for radial data:

$$\text{(YM)} \qquad \begin{aligned} \partial_t^2 u(t,r) = \partial_r^2 u(t,r) + \frac{1}{r}\partial_r u(t,r) - \frac{4}{2r^2}u(t,r)(1-u(t,r))\big(1-\frac{1}{2}u(t,r)\big), \\ (t,r) \in \mathbb{R} \times (0,+\infty). \end{aligned}$$

In the case of (NLW) we always assume that the solutions are spherically symmetric in space variables: $u(t,x) = u(t,|x|)$.

Probably the first rigourous results about the dynamics of solutions of nonlinear wave equations are the works of Keller [22] and Jörgens [21]. Since that time, nonlinear wave models were examined by many authors (Morawetz, Strauss, Brenner, Grillakis, Shatah and others) from the point of view of Hamiltonian and dispersive PDEs.

About ten years ago, Kenig and Merle [23] initiated a detailed study of (NLW) in relation with the *Soliton Resolution Conjecture.* This conjecture states that “generically” a solution of a nonlinear dispersive equation should decompose as a sum of travelling waves (solitons) and a radiation term (which is a solution of the corresponding linear equation). In other words, elliptic objects should be the only manifestation of the nonlinearity after a sufficiently long time. This was proved for (NLW) in the case $N = 3$ by Duyckaerts, Kenig and Merle [12]. The only other cases where such a decomposition is known to hold are some completely integrable models, cf. Eckhaus and Schuur [14] for the KdV equation.

Whether the soliton resolution is known or just believed to hold for some model, solutions which exhibit no dispersion in one or both time directions play a distinguished role. One obvious example of such solutions are the solitons, obtained by solving appropriate elliptic equations. The situation is more complicated if we want to consider solutions decomposing into at least two solitons, the so-called (pure) *multi-solitons*. For completely integrable models such solutions can in principle be calculated explicitely and they are multi-solitons in both time directions. For non-integrable equations it is often possible to construct solutions which are multi-solitons say for large positive times, cf. Martel [27], but generally they are not expected to be multi-solitons for negative times, cf. Martel and Merle [28].

In the energy-critical setting, the role of solitons is played by the *static* solutions called *bubbles*. The problem of decomposition into bubbles was studied by several authors for the harmonic map heat flow from $\mathbb{S}^2$ to $\mathbb{S}^2$, see Topping [44] and references therein. The notion corresponding to a multi-soliton is that of a *multi-bubble*, which is a solution of the evoution equation approaching a superposition of at least two bubbles developing at a single point but at different scales.

In [19], solutions behaving as pure two-bubbles in one time direction were constructed for (NLW) in dimension $N = 6$, for (WM) in the case $k \geq 3$ and for (YM). The purpose of this note is to present these results and give the main ideas of the proofs. In Section 2, the meaning of the soliton resolution in the particular case of energy-critical wave equations is specified. The main results are stated in Section 3. Section 4 is devoted to the proof in the case of (NLW).

2. Preliminaries

2.1. **Scaling invariance.** Introducing a supplementary unknown $\dot{u}(t)$, equation (NLW) can be rewritten in a standard way as a first-order (in time) system:

$$\begin{cases} \partial_t u(t) = \dot{u}(t), \\ \partial_t \dot{u}(t) = \Delta u(t) + f(u(t)). \end{cases}$$

By an abuse of notation, we will say that $(u(t), \dot{u}(t))$ verifies (NLW) if it verifies the equation above. Equation (NLW) has a natural energy functional, defined for $\boldsymbol{u}_0 = (u_0, \dot{u}_0) \in \mathcal{E} := \dot{H}^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ by the formula

$$E(\boldsymbol{u}_0) := \int \frac{1}{2}|\dot{u}_0|^2 + \frac{1}{2}|\nabla u_0|^2 - F(u_0)\,\mathrm{d}x,$$

where $F(u_0) := \frac{N-2}{2N}|u_0|^{\frac{2N}{N-2}}$. Note that $E(\boldsymbol{u}_0)$ is well-defined due to the Sobolev Embedding Theorem. The differential of E is $\mathrm{D}E(\boldsymbol{u}_0) = (-\Delta u_0 - f(u_0), \dot{u}_0)$, where $f(u_0) = |u_0|^{\frac{4}{N-2}} \cdot u_0$, hence we can rewrite equation (NLW) as a Hamiltonian PDE:

$$\begin{cases} \partial_t \boldsymbol{u}(t) = J \circ \mathrm{D}E(\boldsymbol{u}(t)), \\ \boldsymbol{u}(t_0) = \boldsymbol{u}_0 \in \mathcal{E}. \end{cases}$$

Here, $J := \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$ is the natural symplectic structure.

Equation (NLW) is locally well-posed in the space $\mathcal{E}$, see for example Ginibre, Soffer and Velo [15], Shatah and Struwe [39] (the defocusing case), as well as a complete review of the Cauchy theory in Kenig and Merle [23] (for $N \in \{3, 4, 5\}$) and Bulut, Czubak, Li, Pavlović and Zhang [4] (for $N \geq 6$). By "well-posed" we mean that for any initial data $\boldsymbol{u}_0 \in \mathcal{E}$ there exists $\tau > 0$ and a unique solution in some subspace of $C([t_0 - \tau, t_0 + \tau]; \mathcal{E})$, and that this solution is continuous with respect to the inital data. By standard arguments, there exists a maximal time of existence (T_-, T_+), $-\infty \leq T_- < t_0 < T_+ \leq +\infty$, and a unique solution $\boldsymbol{u} \in C((T_-, T_+); \mathcal{E})$. If $T_+ < +\infty$, then $\boldsymbol{u}(t)$ leaves every compact subset of $\mathcal{E}$ as t approaches T_+. A crucial property of the solutions of (NLW) is that the energy E is a conservation law.

For functions $v \in \dot{H}^1_{\mathrm{rad}}(\mathbb{R}^N)$, $\dot{v} \in L^2_{\mathrm{rad}}(\mathbb{R}^N)$, $\boldsymbol{v} = (v, \dot{v}) \in \mathcal{E}$ and $\lambda > 0$, we denote

$$v_\lambda(x) := \frac{1}{\lambda^{\frac{N-2}{2}}} v\big(\frac{x}{\lambda}\big), \qquad \dot{v}_{\underline{\lambda}}(x) := \frac{1}{\lambda^{\frac{N}{2}}} \dot{v}\big(\frac{x}{\lambda}\big), \qquad \boldsymbol{v}_\lambda(x) := \big(v_\lambda, \dot{v}_{\underline{\lambda}}\big).$$

A change of variables shows that

$$E\big((\boldsymbol{u}_0)_\lambda\big) = E(\boldsymbol{u}_0).$$

Equation (NLW) is invariant under the same scaling: if $\boldsymbol{u}(t) = (u(t), \dot u(t))$ is a solution of (NLW) and $\lambda > 0$, then $t \mapsto \boldsymbol{u}\big((t-t_0)/\lambda\big)_\lambda$ is also a solution with initial data $(\boldsymbol{u}_0)_\lambda$ at time $t = 0$. This is why equation (NLW) is called *energy-critical*.

In the case of (WM) and (YM), the energy space is given by $\mathcal{E} := \mathcal{H} \times L^2(r\,\mathrm{d}r)$, where $\mathcal{H}$ is the completion of $C_0^\infty((0,+\infty))$ for the norm

$$\|v\|_{\mathcal{H}}^2 := 2\pi \int_0^{+\infty} \big(|\partial_r v(r)|^2 + |\frac{k}{r}v(r)|^2\big)\, r\mathrm{d}r$$

(one takes $k = 2$ for the Yang-Mills equation). Note that the transformation $\widetilde v(\mathrm{e}^{i\theta}r) := \mathrm{e}^{2i\theta}v(r)$ defines an isometric embedding of $\mathcal{H}$ into $\dot H^1(\mathbb{R}^2;\mathbb{R}^2)$, whose image is given by k-equivariant functions in $\dot H^1(\mathbb{R}^2;\mathbb{R}^2)$. With this definition of $\mathcal{H}$, $u_0 \in \mathcal{H}$ forces $\lim_{r\to+\infty} u_0(r) = 0$, but we could just as well consider states of finite energy such that $\lim_{r\to+\infty} u_0(r) = l\pi$ with $l \in \mathbb{Z}$, see [7, 6] for details.

The energy functionals associated to (WM) and (YM), defined for $\boldsymbol{u}_0 = (u_0, \dot u_0) \in \mathcal{E}$, are

$$E_{\mathrm{WM}}(\boldsymbol{u}_0) := \pi \int_0^{+\infty} \big((\dot u_0)^2 + (\partial_r u_0)^2 + \frac{k^2}{r^2}(\sin(u))^2\big)\, r\mathrm{d}r$$

and

$$E_{\mathrm{YM}}(\boldsymbol{u}_0) := \pi \int_0^{+\infty} \big((\dot u_0)^2 + (\partial_r u_0)^2 + \frac{1}{r^2}(u_0(2-u_0))^2\big)\, r\mathrm{d}r$$

respectively. For $\boldsymbol{v} = (v, \dot v) \in \mathcal{E}$ we have the energy-critical scaling

$$v_\lambda(x) := v\big(\frac{x}{\lambda}\big), \qquad \dot v_{\underline{\lambda}}(x) := \frac{1}{\lambda}\dot v\big(\frac{x}{\lambda}\big), \qquad \boldsymbol{v}_\lambda(x) := \big(v_\lambda, \dot v_{\underline{\lambda}}\big).$$

The Cauchy theory in the energy space is due to Shatah and Tahvildar-Zadeh [40].

2.2. **Ground states.** A fundamental object in the study of (NLW) is the family of stationary solutions $\boldsymbol{u}(t) \equiv \pm\boldsymbol{W}_\lambda = (\pm W_\lambda, 0)$, where

$$W(x) = \Big(1 + \frac{|x|^2}{N(N-2)}\Big)^{-\frac{N-2}{2}}.$$

The functions W_λ are called *ground states* or *bubbles* (of energy). They are the only radially symmetric solutions and, up to translation, the only positive solutions of the critical elliptic problem

$$-\Delta u - f(u) = 0. \tag{2.1}$$

Let us mention that the non-radial solutions of (2.1) are not classified, see [**?**] for an overview of the problem. The ground states are the solutions of (2.1) having the least energy, which is related to their role as "mountain passes" for the potential energy $E_{\mathrm{p}}(u_0) := \int_{\mathbb{R}^N} \frac{1}{2}|\nabla u_0|^2 - F(u_0)\,\mathrm{d}x$. Aubin [1] and Talenti [42] proved the ground states achieve the optimal constant in the critical Sobolev inequality:

$$\frac{\|u\|_{L^{\frac{2N}{N-2}}}}{\|\nabla u\|_{L^2}} \le \frac{\|W\|_{L^{\frac{2N}{N-2}}}}{\|\nabla W\|_{L^2}}, \qquad \forall u \in \dot H^1(\mathbb{R}^N).$$

Payne and Sattinger [34] were the first to make a connection between variational properties of the ground state and the long-time behavior of solutions of the focusing wave equation.

In the case of the wave maps equation, the stationary states are precisely the harmonic maps. When the domain is $\mathbb{R}^2$ and the target is $\mathbb{S}^2$, they correspond to rational complex functions, hence are completely classified, even in the non-equivariant case. In the special case of k-equivariant data, we obtain stationary solutions $W^{\mathrm{WM}}(r) := 2\arctan(r^k)$ and its rescaled versions $W_\lambda^{\mathrm{WM}}(r) := 2\arctan\big(\big(\frac{r}{\lambda}\big)^k\big)$, which are the counterparts of the ground states W_λ described above. Viewed as functions from $\mathbb{R}^2$ to $\mathbb{S}^2$, W_λ^{WM} are harmonic maps of topological degree k. An elementary proof shows that W_λ^{WM} have minimal energy among k-equivariant maps joining 0 at $r = 0$ to π at $r = +\infty$. Note that $W^{\mathrm{WM}} \notin \mathcal{H}$ precisely because of the fact that $W^{\mathrm{WM}}(r) \to \pi$ as $r \to +\infty$.

The stationary solutions of (YM) are $W_\lambda^{\mathrm{YM}}(r) := \frac{2r^2}{\lambda^2+r^2}$.

2.3. **Soliton resolution.** The notion of soliton resolution was described in the Introduction in a rather vague way. In the context of energy-critical equations, the commonly accepted precise meaning of a soliton resolution is provided by the following result of Duyckaerts, Kenig and Merle [12]:

Theorem 1 ([12])**.** *Let $N = 3$ and let $\boldsymbol{u}(t) : [t_0, T_+) \to \mathcal{E}$ be a radial solution of* (NLW)*. Then one of the following holds:*

- ***Type I blow-up:*** *$T_+ < \infty$ and*
$$\lim_{t\to T_+} \|\boldsymbol{u}(t)\|_{\mathcal{E}} = +\infty.$$
- ***Type II blow-up:*** *$T_+ < \infty$ and there exist $\boldsymbol{v}_0 \in \mathcal{E}$, an integer $n \in \mathbb{N}\setminus\{0\}$, and for all $j \in \{1,\ldots,n\}$, a sign $\iota_j \in \{\pm 1\}$, and a positive function $\lambda_j(t)$ defined for t close to T_+ such that*
$$\lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_n(t) \ll T_+ - t \text{ as } t \to T_+$$
$$\lim_{t\to T_+} \Big\|\boldsymbol{u}(t) - \Big(\boldsymbol{v}_0 + \sum_{j=1}^n \iota_j \boldsymbol{W}_{\lambda_j(t)}\Big)\Big\|_{\mathcal{E}} = 0.$$
- ***Global solution:*** *$T_+ = +\infty$ and there exist a solution $\boldsymbol{v}_{\mathrm{L}}$ of the linear wave equation, an integer $n \in \mathbb{N}$, and for all $j \in \{1,\ldots,n\}$, a sign $\iota_j \in \{\pm 1\}$, and a positive function $\lambda_j(t)$ defined for large t such that*
$$\lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_n(t) \ll t \text{ as } t \to +\infty$$
$$\lim_{t\to+\infty} \Big\|\boldsymbol{u}(t) - \Big(\boldsymbol{v}_{\mathrm{L}}(t) + \sum_{j=1}^n \iota_j \boldsymbol{W}_{\lambda_j(t)}\Big)\Big\|_{\mathcal{E}} = 0.$$

□

Remark 2.1. An analogous decomposition *for a sequence of times* was proved in the non-radial case by Duyckaerts, Jia, Kenig and Merle [11]. Note that non-radial multi-solitons were constructed by Martel and Merle [29] using the Lorentz transform.

Remark 2.2. Also for a sequence of times, Côte [5] proved such a decomposition for the wave maps equation for 1-equivariant data, see also Struwe [41]. In the case of higher equivariance degrees and for the Yang-Mills equation, this was settled by Jia and Kenig [20].

Remark 2.3. The decompositions in Theorem 1 are proved to hold *in the energy space*. No description of the dynamics in any stronger topology is currently available.

3. Examples of solutions with two bubbles

We are ready to state the results of [19].

Theorem 2. *There exists a solution $\boldsymbol{u} : (-\infty, T_0] \to \mathcal{E}$ of* (NLW) *such that*
$$\lim_{t\to-\infty} \|\boldsymbol{u}(t) - (\boldsymbol{W} + \boldsymbol{W}_{\frac{1}{\kappa}\mathrm{e}^{-\kappa|t|}})\|_{\mathcal{E}} = 0, \qquad \text{with } \kappa := \sqrt{\frac{5}{4}}.$$

Remark 3.1. We construct here *pure* two-bubbles, that is the solution approaches a superposition of two stationary states, with no energy transformed into radiation. By the conservation of energy and the decoupling of the two bubbles, we necessarily have $E(\boldsymbol{u}(t)) = 2E(\boldsymbol{W})$. Pure one-bubbles cannot concentrate and are completely classified, see [13].

Remark 3.2. It was proved in [18], in any dimension $N \geq 3$, that there exist no solutions $\boldsymbol{u}(t) : [t_0, T_+) \to \mathcal{E}$ of (NLW) such that $\|\boldsymbol{u}(t) - (\boldsymbol{W}_{\mu(t)} - \boldsymbol{W}_{\lambda(t)})\|_{\mathcal{E}} \to 0$ with $\lambda(t) \ll \mu(t)$ as $t \to T_+ \leq +\infty$.

Remark 3.3. In any dimension $N > 6$ one can expect an analogous result with concentration rate $\lambda(t) \sim |t|^{-\frac{4}{N-6}}$.

The corresponding result for (WM) reads:

Theorem 3. *Fix $k > 2$. There exists a solution $\boldsymbol{u} : (-\infty, T_0] \to \mathcal{E}$ of* (WM) *such that*

$$\lim_{t \to -\infty} \|\boldsymbol{u}(t) - (-\boldsymbol{W}^{\mathrm{WM}} + \boldsymbol{W}^{\mathrm{WM}}_{\frac{k-2}{\kappa}(\kappa|t|)^{-\frac{2}{k-2}}})\|_{\mathcal{E}} = 0, \qquad \textit{with } \kappa := \frac{k-2}{2}\Big(\frac{8k}{\pi}\sin\big(\frac{\pi}{k}\big)\Big)^{\frac{1}{k}}. \tag{3.1}$$

Remark 3.4. The constructed solution is a *pure* two-bubble, hence by the conservation of energy $E(\boldsymbol{u}(t)) = 2E(\boldsymbol{W}^{\mathrm{WM}})$, and it is clear that it has the homotopy degree 0. In the case of equivariant class $k = 1$, Côte, Kenig, Lawrie and Schlag [6] showed that any degree 0 initial data of energy $< 2E(\boldsymbol{W}^{\mathrm{WM}})$ leads to dispersion (the proof is expected to generalize to all equivariance classes). Theorem 4 gives the first example of a non-dispersive solution at the threshold energy.

Note that pure two-bubbles of homotopy degree $2k$ (hence of type bubble-bubble and not bubble-antibubble) do not exist because the energy of such a map has to be $> 2E(\boldsymbol{W}^{\mathrm{WM}})$. This is similar to the case of opposite signs for (NLW), see Remark 3.2.

Remark 3.5. The fact that the signs of the bubbles in (3.1) are opposite comes from a slightly different structure of the corresponding elliptic problem. As a result, the case of opposite signs for (NLW) resembles the case of the same sign for (WM).

Remark 3.6. It is conceivable that the proof could be adapted to deal with a more general equation $\partial_t^2 u = \partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}(gg')(u)$ with g satisfying the assumptions of [7] and $g'(0) \in \{3, 4, 5, \ldots\}$.

We have a very similar result for (YM):

Theorem 4. *There exists a solution $\boldsymbol{u} : (-\infty, T_0] \to \mathcal{E}$ of* (YM) *such that*

$$\lim_{t \to -\infty} \|\boldsymbol{u}(t) - (-\boldsymbol{W}^{\mathrm{YM}} + \boldsymbol{W}^{\mathrm{YM}}_{\frac{1}{\kappa}\mathrm{e}^{-\kappa|t|}})\|_{\mathcal{E}} = 0, \qquad \textit{with } \kappa := 2\sqrt{3}.$$

Remark 3.7. The case of wave maps in the equivariance class $k = 2$ should be almost the same.

Remark 3.8. In all the three cases, we expect the constructed solutions to be smooth and very unstable.

We see that the common feature of all these theorems is that one bubble is stationary and the other one concentrates. It is clear that the control of the concentrating bubble is going to be the delicate part of the proof. The possibility of concentration of a harmonic map at the origin for the wave maps flow was observed numerically by Bizoń, Chmaj and Tabor [3]. First mathematical results in this direction were obtained by Krieger, Schlag and Tataru [25, 26], both for the power nonlinearity case in dimension $N = 3$ and for the wave maps equation. These seminal works were followed by [10], [24], [9], [33] and [35].

Stable (in an appropriate topology) blow-up solutions for the wave maps and Yang-Mills equations were obtained in [38] and [37]. A similar result for (NLW) in dimension $N = 4$ is proved in [16].

A still different approach to controlling the concentration of one bubble, which is adopted here, was developped in [17].

Let us stress that an important difference between our results and other constructions of multi-solitons lies in the *strong interaction* of the two bubbles for long times. By this, we mean that the interaction has an influence on the dynamics – it induces the concentration of the second bubble. To the author's knowledge, the only result where a strong interaction of solitons is observed for a Hamiltonian equation is the recent work of Martel and Raphaël [31].

In the context of the harmonic map heat flow, Topping [43] proved the existence of towers of bubbles for a well chosen target manifold. Towers of bubbles for the Yamabe flow were constructed in [8] by using a change of variables which reduces the problem to a construction of spatially decoupled bubbles.

4. Elements of the proof

In this section the main ideas of the proof of Theorem 2 are presented. The proofs of Theorems 3 and 4 follow a similar scheme.

4.1. **Formal computation.** The usual method of performing a formal analysis of blow-up solutions is to search a series expansion with respect to a small scalar parameter depending on time and converging to 0 at blow-up. In our case the blow-up time is $-\infty$.

Recall that $N = 6$, hence for $v \in \dot H^1_{\mathrm{rad}}(\mathbb{R}^6)$, and $\dot v \in L^2_{\mathrm{rad}}(\mathbb{R}^6)$ we have

$$v_\lambda(x) := \frac{1}{\lambda^2} v\Big(\frac{x}{\lambda}\Big), \qquad \dot v_{\underline{\lambda}}(x) := \frac{1}{\lambda^3}\dot v\Big(\frac{x}{\lambda}\Big).$$

We denote $\Lambda v := -\frac{\partial}{\partial\lambda} v_\lambda|_{\lambda=1} = (2 + x\cdot\nabla)\dot v$ and $\Lambda_0 \dot v := -\frac{\partial}{\partial\lambda}\dot v_{\underline{\lambda}}|_{\lambda=1} = (3 + x\cdot\nabla)\dot v$. For $\boldsymbol{v} = (v, \dot v)$ we denote $\Lambda\boldsymbol{v} := (\Lambda v, \Lambda_0 \dot v)$.

If $u(t) \simeq W + W_{\lambda(t)}$, then $\partial_t u(t) \simeq -\lambda'(t)\Lambda W_{\underline{\lambda(t)}}$, hence

$$\boldsymbol{u}(t) \simeq (W + W_{\lambda(t)}, 0) - \lambda'(t)\cdot(0, \Lambda W_{\underline{\lambda(t)}}) = \boldsymbol{W} + \boldsymbol{U}^{(0)}_{\lambda(t)} + b(t)\cdot\boldsymbol{U}^{(1)}_{\lambda(t)},$$

with $b(t) := \lambda'(t)$, $\boldsymbol{U}^{(0)} := (W, 0)$ and $\boldsymbol{U}^{(1)} := (0, -\Lambda W)$. This suggests considering $b(t) = \lambda'(t)$ as the small parameter with respect to which the formal expansion should be sought. Hence, we make the ansatz

$$\boldsymbol{u}(t) = \boldsymbol{W} + \boldsymbol{U}^{(0)}_{\lambda(t)} + b(t)\cdot\boldsymbol{U}^{(1)}_{\lambda(t)} + b(t)^2\cdot\boldsymbol{U}^{(2)}_{\lambda(t)},$$

and try to find the conditions under which a satisfactory candidate for $\boldsymbol{U}^{(2)} = (U^{(2)}, \dot U^{(2)})$ can be proposed. Neglecting irrelevant terms and replacing $\lambda'(t)$ by $b(t)$, we compute

$$\partial_t^2 u(t) = -b'(t)(\Lambda W)_{\underline{\lambda(t)}} + \frac{b(t)^2}{\lambda(t)}(\Lambda_0\Lambda W)_{\underline{\lambda(t)}} + \mathrm{lot}.$$

On the other hand, using the fact that $f(W + W_\lambda) = f(W) + f(W_\lambda) + f'(W_\lambda)W \simeq f(W) + f(W_\lambda) + f'(W_\lambda)$ for $\lambda \ll 1$ and $f'(W_\lambda) = \lambda f'(W)_{\underline{\lambda}}$, we get

$$\Delta u(t) + f(u(t)) = -\frac{b(t)^2}{\lambda(t)}(LU^{(2)})_{\underline{\lambda(t)}} + \lambda(t) f'(W)_{\underline{\lambda(t)}} + \mathrm{lot},$$

where $L := -\Delta - f'(W)$ is the linearization of $-\Delta - f(u)$ near $u = W$. We discover that, formally at least, we should have

$$LU^{(2)} = -\Lambda_0\Lambda W + \frac{\lambda(t)}{b(t)^2}\big(b'(t)\cdot\Lambda W + \lambda(t)\cdot f'(W)\big). \tag{4.1}$$

The operator L is self-adjoint. Because of the scaling invariance, $\Lambda W \in \ker(L)$, which yields a natural solvability (or Fredholm) condition

$$\int \Lambda W \cdot \Big(-\Lambda_0 \Lambda W + \frac{\lambda(t)}{b(t)^2}\big(b'(t)\cdot \Lambda W + \lambda(t)\cdot f'(W)\big)\Big)\,\mathrm{d}x = 0.$$

Since $\langle \Lambda W, \Lambda_0 \Lambda W\rangle_{L^2} = 0$, we get equivalently

$$\int \Lambda W \cdot \big(b'(t)\cdot \Lambda W + \lambda(t)\cdot f'(W)\big)\,\mathrm{d}x = 0 \Leftrightarrow b'(t) = \frac{5}{4}\lambda(t) = \kappa^2\lambda(t).$$

It turns out that if this condition is satisfied, then equation (4.1) has indeed a decaying regular solution $U^{(2)} = Q + \frac{\lambda(t)^2}{b(t)^2}P$, where P and Q are some explicit profiles. The *formal parameter equations*

$$\lambda'(t) = b(t), \qquad b'(t) = \kappa^2\lambda(t) \tag{4.2}$$

have a solution

$$(\lambda_{\mathrm{app}}(t), b_{\mathrm{app}}(t)) = \big(\frac{1}{\kappa}\mathrm{e}^{-\kappa|t|}, \mathrm{e}^{-\kappa|t|}\big), \qquad t \le T_0 < 0.$$

In any space dimension N, ignoring the problems related to the slow decay of W, a similar analysis would yield $b'(t) = \kappa^2\lambda(t)^{\frac{N-4}{2}}$. For $N < 6$ this leads to a finite time blow-up, which was studied in [17] for $N = 5$. For $N > 6$, we obtain a global solution $\lambda(t) \sim |t|^{-\frac{4}{N-6}}$, see Remark 3.3.

4.2. **Control of the error.** The computation in the last paragraph leads us to define the following approximate solution:

$$\boldsymbol{\varphi}(\mu,\lambda,b) := \boldsymbol{W}_\mu + \boldsymbol{U}_\lambda^{(0)} + b\cdot \boldsymbol{U}_\lambda^{(1)} + b^2\cdot \boldsymbol{U}_\lambda^{(2)}.$$

We search a solution which is close in the energy space to $\boldsymbol{\varphi}(1,\lambda_{\mathrm{app}}(t), b_{\mathrm{app}}(t))$ as $t\to-\infty$.

The key idea is to obtain a *uniformly controlled* sequence of solutions of (NLW), that is solutions $\boldsymbol{u}_n(t) : [T_n, T_0] \to \mathcal{E}$ with $T_n \to -\infty$ such that

$$\|\boldsymbol{u}_n(t) - \boldsymbol{\varphi}(1,\lambda_{\mathrm{app}}(t), b_{\mathrm{app}}(t))\|_{\mathcal{E}} \le C\mathrm{e}^{-\frac{1}{2}\kappa|t|} \qquad \text{for } t\in[T_n,T_0], \tag{4.3}$$

with C independent of n.

Since the bound (4.3) is the most restricitve at time $t = T_n$, a natural idea is to impose the initial condition at this time, which will guarantee that (4.3) will hold at least in a neighborhood of $t = T_n$. We consider thus the sequence of solutions $\boldsymbol{u}_n(t)$ of (NLW) with the initial data

$$\boldsymbol{u}_n(T_n) = \boldsymbol{\varphi}(1,\lambda_{\mathrm{app}}(T_n), b_{\mathrm{app}}(T_n)),$$

where T_n is any decreasing sequence tending to $-\infty$. In fact, an adjustment has to be made because of the exponential instability of the flow near $\boldsymbol{W}$, but this is not a major difficulty and will not be discussed here.

Suppose for a moment that we have proved (4.3) and let us show how to finish the proof of Theorem 2. The sequence $\boldsymbol{u}_n(T_0)$ being bounded in $\mathcal{E}$, it contains a subsequence (still denoted $\boldsymbol{u}_n$) converging weakly to a certain $\boldsymbol{u}_0 \in \mathcal{E}$. Let $\boldsymbol{u}(t)$ be the solution of (NLW) with the initial data $\boldsymbol{u}(T_0) = \boldsymbol{u}_0$. Using the profile decomposition of Bahouri and Gérard [2], one can prove that $\boldsymbol{u}(t)$ is defined for $t\in(-\infty,T_0]$ and for all $t\in(-\infty,T_0]$ we have $\boldsymbol{u}_n(t) \rightharpoonup \boldsymbol{u}(t)$. By the Fatou property of the weak limit, we obtain

$$\|\boldsymbol{u}(t) - \boldsymbol{\varphi}(1,\lambda_{\mathrm{app}}(t), b_{\mathrm{app}}(t))\|_{\mathcal{E}} \le C\mathrm{e}^{-\frac{1}{2}\kappa|t|} \qquad \text{for } t\in(-\infty,T_0],$$

which finishes the proof.

Let us sketch the proof of (4.3). Using suitable orthogonality conditions in order to determine $\mu(t)$ and $\lambda(t)$, we decompose $\boldsymbol{u}_n(t) = \boldsymbol{\varphi}(\mu(t), \lambda(t), b(t)) + \boldsymbol{g}(t)$. The parameter $b(t)$ is defined as follows:

$$b(t) := b_{\mathrm{app}}(T_n) + \kappa^2 \int_{T_n}^{t} \lambda(\tau)\,\mathrm{d}\tau,$$

because that is what equation (4.2) indicates. Of course $\mu(t)$, $\lambda(t)$, $b(t)$ and $\boldsymbol{g}(t)$ depend on n. The goal is to control the size of $\boldsymbol{g}(t)$, uniformly in n, on a time interval $[T_n, T_0]$.

To this end, we introduce a *mixed energy-virial functional* $H(t)$, which is a correction of the energy functional $E(\boldsymbol{\varphi}(t)+\boldsymbol{g}(t)) - E(\boldsymbol{\varphi}(t)) - \langle \mathrm{D}E(\boldsymbol{\varphi}(t)), \boldsymbol{g}(t)\rangle$ by a small localized virial term. This functional has the following coercivity property:

$$\|\boldsymbol{g}(t)\|^2_{\dot H^1\times L^2} \lesssim H(t) \qquad \text{(modulo the unstable modes).}$$

Moreover, the fact that $\boldsymbol{\varphi}(t)$ is a refined ansatz can be used to show that for some large constant C_0 we have

$$\|\boldsymbol{g}(t)\|_{\mathcal{E}} \leq C_0 \mathrm{e}^{-\frac{3}{2}\kappa|t|} \text{ for } t\in[T_n, T] \quad \Rightarrow \quad H'(t) \leq c\cdot C_0^2 \cdot \mathrm{e}^{-3\kappa|t|} \text{ for } t \in [T_n, T], \tag{4.4}$$

with a small constant c_0. A classical continuity argument yields the uniform control

$$\|\boldsymbol{g}\|_{\mathcal{E}} \leq C_0 \mathrm{e}^{-\frac{3}{2}\kappa|t|}. \tag{4.5}$$

While proving (4.4), one shows in particular that the bound $\|\boldsymbol{g}(t)\|_{\mathcal{E}} \leq C_0\mathrm{e}^{-\frac{3}{2}\kappa|t|}$ implies the following bounds for $t \in [T_n, T_0]$:

$$\begin{aligned} |\mu(t)/\mu_{\mathrm{app}}(t) - 1| &\lesssim C_0 \mathrm{e}^{-\frac{1}{2}\kappa|t|}, \\ |\lambda(t)/\lambda_{\mathrm{app}}(t) - 1| &\lesssim C_0 \mathrm{e}^{-\frac{1}{2}\kappa|t|}. \end{aligned} \tag{4.6}$$

This is proved by solving the modulation equations (modulation inequalities, to be precise). Now (4.6) and (4.5) lead to (4.3).

The idea of constructing a uniformly controlled sequence of solutions converging to a singular solution was introduced by Merle [32]. Combining this technique with energy estimates was an idea of Martel [27]. This is a typical scheme for proofs of existence of “special” objects like multi-solitons or minimal blow-up solutions. Such a proof provides no information on the stability of the constructed solutions. This being said, we do not expect any finite-codimensional stability of our solutions, hence the method seems well adapted to the problem at hand.

Raphaël and Szeftel [36] used a virial correction of the energy functional in a similar context in their study of minimal mass blow-up solutions for the non-homogeneous nonlinear Schrödinger equation. The first step of the proof (the formal computation) is also inspired by the work of Martel, Merle and Raphaël [30] on exotic blow-up for the L^2-critical generalized Korteweg-de Vries equation. They observed that the blow-up rate is directly related to the size of interaction of the bubble with the “background”, which is the heart of our construction of the approximate solution.

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