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VERSAL DEFORMATION OF THE ANALYTIC SADDLE-NODE

by

Frank Loray

To Jean-Pierre Ramis for his 60th birthday

Abstract. — In the continuation of [**10**], we derive simple forms for saddle-node singular points of analytic foliations in the real or complex plane just by gluing foliated complex manifolds. We give a miniversal analytic deformation of the simplest model. We also derive a unique analytic form for those saddle-node having an analytic central manifold. By this way, we recover and generalize results earlier proved by J. Écalle by using mould theory and partially answer to some questions asked by J. Martinet and J.-P. Ramis at the end of [**11**].

***Résumé* (Déformation verselle d'un nœud-col analytique).** — Dans la continuité de [**10**], nous construisons une forme normale simple pour un feuilletage analytique au voisinage d'une singularité de type nœud-col dans le plan réel ou complexe. Nous obtenons une telle forme en recollant des variétés complexes feuilletées. Nous en déduisons une déformation analytique miniverselle dans un cas simple. Nous donnons une forme unique pour un nœud-col possédant une variété centrale analytique. Nous retrouvons ainsi géométriquement et nous généralisons des résultats obtenus par J. Écalle à l'aide de la théorie des moules. Ce travail répond partiellement à des questions ouvertes posées par J. Martinet et J.-P. Ramis à la fin de [**11**].

Introduction and results

Let X be a germ of analytic vector field at the origin of $\mathbb{C}^2$

$$X = f(x,y)\partial_x + g(x,y)\partial_y, \quad f,g \in \mathbb{R}\{x,y\} \text{ or } \mathbb{C}\{x,y\}$$

having a singularity at 0: $f(0) = g(0) = 0$. Consider $\mathcal{F}$ the germ of singular holomorphic foliation induced by the complex integral curves of X near 0. A question going back to H. Poincaré is the following:

Problem. — Find local coordinates in which the foliation is defined by a vector field having coefficients as simple as possible.

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In this problem, the vector field is considered up to analytic change of coordinates and up to multiplication by a germ of analytic function. For instance, if the vector field X has a linear part (in the matrix form)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ax+by)\partial_x + (cx+dy)\partial_y$$

having non zero eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ with eigenratio $\lambda_2/\lambda_1 \notin \mathbb{R}$, then H. Poincaré proved that the vector field X is actually linear in convenient analytic coordinates. In this situation, the eigenvalues $\{\lambda_1, \lambda_2\}$ (resp. the eigenratio λ_2/λ_1) provide a complete set of invariants for such vector fields (resp. foliations) modulo analytic change of coordinates.

In this paper, we consider *unramified saddle-nodes*, *i.e.* foliations defined by a vector field having (exactly) one zero eigenvalue and multiplicity 2. Following H. Dulac, such a foliation is defined in convenient coordinates by a vector field of the form

$$X = x^2\partial_x + y\partial_y + xf(x,y)\partial_y, \quad f \in \mathbb{C}\{x,y\}. \tag{1}$$

and one can further *formally* reduce the vector field X to a unique form

$$X_\mu = x^2\partial_x + y\partial_y + \mu xy\partial_y, \quad \mu \in \mathbb{C} \tag{2}$$

The complete *analytic* classification of those singular points has been given by J. Martinet and J.-P. Ramis in 1982 (see [**11**] or section 1), giving rise to infinitely many invariants additional to the formal one μ above. The resulting moduli space is huge and we expect that a generic saddle-node cannot be defined by a polynomial vector field in any analytic coordinates (although this is open, as far as I know). A direct application of our recent work [**10**] provides the following

Theorem 1. — *Let $\mathcal{F}$ be a germ of saddle-node foliation at the origin of $\mathbb{R}^2$ (resp. of $\mathbb{C}^2$) in the form (1) above. Then, there exist local analytic coordinates in which $\mathcal{F}$ is defined by a vector field of the form*

$$X_f = x^2\partial_x + y\partial_y + xf(y)\partial_y, \quad f \in \mathbb{C}\{y\} \tag{3}$$

where $f'(0) = \mu$.

This statement is a particular case of a general simple analytic form independantly announced by A.D. Bruno and P.M. Elizarov for all resonant saddles ($\lambda_2/\lambda_1 \in \mathbb{Q}^-$) and saddle-nodes in 1983 (see [**3, 6**]). So far, only the case of Theorem 1 with $\mu = 0$ has been proved: it is presented by J. Écalle as an application of *resurgent functions and mould theory* at the end of [**5**], p. 535. In 1994, P.M. Elizarov made an important step toward the analytic form announced by solving in [**7**] the associate cohomological equation. One can immediately deduce from his computations that the family X_f of Theorem 1 is *miniversal* at $f \equiv 0$: the coefficients of f play the role of Martinet-Ramis' invariants at the first order. This will be rigorously stated in section 1, once we have recalled the definition (and construction) of Martinet-Ramis' invariants.

It is important to notice that the form (3) is not unique. Of course, we can modify the functional coefficient f by conjugating the vector field with an homothety $y \mapsto c \cdot y$, $c \in \mathbb{C}^*$. But even if we restrict to tangent-to-the-identity conjugacies, the form (3) is perhaps locally unique at X_0 ($f \equiv 0$), but not globally for the following reason. By construction (see proof of Theorem 1), the form (3) is obtained with $f(0) \neq 0$, even if the saddle node has a *central manifold* (see below). For instance, the model X_0 has also another form (3) with $f(0) \neq 0$.

From preliminary form (1), we see that $\{x = 0\}$ is an invariant curve for the vector field that we will call *strong manifold* throughout the paper. Tangent to the zero eigendirection, there is also a unique "formal invariant curve" $\{y = \varphi(x)\}$, $\varphi \in \mathbb{R}[[x]]$ or $\mathbb{C}[[x]]$, which is generically divergent. When this curve is convergent, we call it *central manifold*. A remarquable result of Martinet-Ramis' classification is that saddle-nodes having a central manifold form an analytic submanifold of codimension one (in the unramified case). For instance, saddle-nodes in the form (3) with $f(0) = 0$ have the central manifold $\{y = 0\}$. Conversely, a natural question is:

Problem. — Given a saddle-node like in Theorem 1 having a central manifold, is it possible to put it analytically into the form (3) with $f(0) = 0$ (*i.e.* simultaneously straightening the central manifold onto $\{y = 0\}$) ?

For generic μ, the answer is yes:

Theorem 2. — *Let $\mathcal{F}$ be a germ of saddle-node foliation at the origin of $\mathbb{R}^2$ (resp. of $\mathbb{C}^2$) like in Theorem 1 with $\mu \in \mathbb{C} - \mathbb{R}^-$. If $\mathcal{F}$ has a central manifold, then there exist local analytic coordinates in which $\mathcal{F}$ is defined by*

$$X_f = x^2\partial_x + y\partial_y + xf(y)\partial_y, \quad \textit{with } f(0) = 0. \tag{4}$$

Moreover, this form is unique up to homothety $y \mapsto c \cdot y$, $c \in \mathbb{C}^*$.

In the remaining case $\mu \in \mathbb{R}^-$, we will give necessary and suffisant conditions in section 4 in terms of Martinet-Ramis' invariants (see Theorem 8), thus providing a complete answer to the question above; in the case $\mu = 0$, the condition was already given by J. Écalle in [**5**], p. 539. It turns out that these conditions are very restrictive (infinite codimension). For instance, when $\mu \in -\mathbb{N}^*$, only the saddle-nodes analytically conjugated to the formal model (2) can be normalized to the form (4). In particular, for each $\mu \in -\mathbb{N}^*$, the subfamily of those X_f satisfying $f(0) = 0$ and $f'(0) = \mu$ provides a codimension two analytically trivial deformation of the formal model (2).

Accidentally, our method to prove Theorem 2 provides in turn a simple form for saddles:

Theorem 3. — *Let $\mathcal{F}$ be a germ of saddle foliation at the origin of $\mathbb{R}^2$ (resp. of $\mathbb{C}^2$) with eigenratio $-\mu < 0$. Then there exist local analytic coordinates in which $\mathcal{F}$ is* *defined by a vector field of the form*

$$X_f = -x\partial_x + \mu(f(y)+x)y\partial_y, \quad \text{with } f(0)=1. \tag{5}$$

This latter form is not unique: for generic μ, all X_f are conjugated. For saddle-nodes having a central manifold that cannot be transformed into the form (4), it is possible to give an alternate unique form as follows.

Theorem 4. — *Let $\mathcal{F}$ be a germ of saddle-node foliation at the origin of $\mathbb{R}^2$ (resp. of $\mathbb{C}^2$) like in Theorem 1 having a central manifold. Let $n \in \mathbb{N}$ be such that $\mu + n \notin \mathbb{R}^-$. Then, there exist local analytic coordinates in which $\mathcal{F}$ is defined by a vector field of the form*

$$X_f = x^2\partial_x + y\partial_y + xyf(x^n y)\partial_y, \quad \text{where } f(0)=\mu. \tag{6}$$

Moreover, this form is unique up to homothety $y \mapsto c\cdot y$, $c \in \mathbb{C}^*$.

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1. Martinet-Ramis' invariants

We recall the construction of [**11**]. Consider a saddle-node in Dulac preliminary form (1)

$$X = x^2\partial_x + y\partial_y + xf(x,y)\partial_y, \quad f \in \mathbb{C}\{x,y\}.$$

The Sectorial Normalization Theorem due to Hukuhara, Kimura and Matuda reads as follows. For a sufficiently small $r, \varepsilon > 0$, there exists on each of the two sectorial domains V^+ and V^-

$$V^{\pm} := \{|x| < r,\ |y| < r,\ 0-\varepsilon < \arg(\pm x) < \pi + \varepsilon\}$$

a unique holomorphic diffeomorphism $\Phi^{\pm} : V^{\pm} \to \Phi^{\pm}(V^{\pm}) \subset \mathbb{C}^2$ of the form $\Phi(x,y) = (x, \phi(x,y))$, which is tangent to the identity at $(0,0)$ and conjugating the saddle-node above to its formal normal form (2)

$$X_\mu := x^2\partial_x + y\partial_y + \mu xy\partial_y.$$

The model X_μ admits the first integral $H_\mu(x,y) := yx^{-\mu}e^{1/x}$. Once we have fixed determinations $H_\mu^{\pm}$ of H_μ on the sectors $V^{\pm}$ coinciding over $\{-\varepsilon < \arg(x) < +\varepsilon\}$, we immediately deduce sectorial first integrals $H^{\pm} := H_\mu^{\pm} \circ \Phi^{\pm}$ for the initial saddle-node.

On the overlapping $V^+ \cap V^-$, the two first integrals H^+ and H^- factorize in the following way. Over $V^0 = \{\pi - \varepsilon < \arg(x) < \pi + \varepsilon\}$, the first integrals H^+ and H^- both identify the space of leaves with a neighborhood of $0 \in \mathbb{C}$, the size of which depending on the radius r: one can write $H^- = \varphi^0 \circ H^+$ for some germ of diffeomorphism $\varphi^0 \in \mathrm{Diff}(\mathbb{C},0)$. Over the other overlapping $V^\infty = \{-\varepsilon < \arg(x) < +\varepsilon\}$, the first integrals H^+ and H^- both identify the space of leaves with $\mathbb{C}$: one can write $H^- = \varphi^\infty \circ H^+$ for some affine automorphism φ^∞ of $\mathbb{C}$. From the asymptotics of $\Phi^{\pm}$ and the choice of the determinations $H_\mu^\pm$, one easily deduce that the linear parts of φ^0 and φ^∞ are respectively $e^{2i\pi\mu}$ and 1.

We have thus defined the *moduli map*:

$$X \longmapsto \begin{cases} \varphi^0(\zeta) \ = e^{2i\pi\mu}\zeta + \sum_{n\geqslant 2} a_n\zeta^n \ \in \mathrm{Diff}(\mathbb{C},0) \\ \varphi^\infty(\zeta) = \zeta + t \ \in \mathbb{C} \qquad\qquad \text{(a translation)} \end{cases} \tag{7}$$

The main result of [**11**] is

***Theorem* (Martinet-Ramis).** — *Any two saddle-nodes into the form (1) are conjugated by a tangent-to-the-identity diffeomorphism $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, $D\Phi(0) = I$, if, and only if, they have the same image through the moduli map above.*

Moreover, the moduli map is surjective: any pair $(\varphi^0, \varphi^\infty) \in \mathrm{Diff}(\mathbb{C}, 0) \times \mathbb{C}$ can be realized by a saddle-node of the form (1).

Two saddle-nodes X and $\widetilde{X}$ in the form (1) can be conjugated by a diffeomorphism $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ with a non trivial linear part, namely an homothety in variable y. In this case, the corresponding pairs are conjugated by an homothety:

$$(c \cdot \widetilde{\varphi}^0(\zeta), c \cdot \widetilde{\varphi}^\infty(\zeta)) = (\varphi^0(c \cdot \zeta), \varphi^\infty(c \cdot \zeta)), \quad \text{for some } c \in \mathbb{C}^*.$$

This equivalence relation on $\mathrm{Diff}(\mathbb{C}, 0) \times \mathbb{C}$ provides a complete set of invariants for saddle-nodes with multiplicity 2 with respect to the analytic conjugacy.

The classification above is a foliated version of Ecalle-Malgrange-Voronin classification of tangent-to-the-identity maps. Let us recall the Martinet-Ramis presentation in the case of multiplicity 2. Any $\varphi(x) = x + 2i\pi x^2 + \cdots \in \mathrm{Diff}(\mathbb{C}, 0)$ is conjugate by formal change of the coordinate to the 1-time map $\varphi_\mu := \exp(2i\pi \frac{x^2}{1+\mu x}\partial_x)$ for a unique $\mu \in \mathbb{C}$. On sectors $V^\pm$ like the ones above (without variable y), Leau's Theorem says that one can conjugate the dynamics of φ with that of φ_μ by tangent-to-the-identity sectorial diffeomorphisms $\Phi^\pm$. After composition with convenient determinations of the φ_μ-invariant function $H_\mu(x) := x^{-\mu}e^{1/x}$, one deduce sectorial invariant functions $H^\pm$ identifying the quotients of $V^\pm$ by the dynamics with $\mathbb{C}^*$. On V^0 (resp. V^∞) defined as before, the functions $H^\pm$ identify the set of φ-orbits with a punctured neighborhood of 0 (resp. ∞) whose size depend on the radius of the sectors $V^\pm$. Therefore, one can write $H^- = \varphi^0 \circ H^+$ (resp. $H^- = \varphi^\infty \circ H^+$) for some germ of diffeomorphism $\varphi^0 \in \mathrm{Diff}(\mathbb{C}, 0)$ (resp. $\varphi^\infty \in \mathrm{Diff}(\mathbb{C}, \infty)$). The respective linear parts of those diffeomorphisms are $e^{2i\pi\mu}$ and 1. The Ecalle-Malgrange-Voronin Theorem can be stated like Martinet-Ramis Theorem above except that φ^∞ can be any convergent power series $\zeta + \sum_{n<0} a_n\zeta^n$.

***Theorem* (Martinet-Ramis).** — *The analytic invariants $(\varphi^0, \varphi^\infty)$ of a saddle-node into the form (1) coincide with the analytic invariants of the holonomy map $\varphi(x) = x + 2i\pi x^2 + \cdots$ of the strong manifold $\{x = 0\}$.*

Therefore, any two saddle-nodes in the form (1) are analytically conjugated if and only if the holonomy maps of the corresponding strong manifolds are analytically conjugated in $\mathrm{Diff}(\mathbb{C},0)$.

Another consequence is that very few tangent-to-the-identity maps $\varphi(x) = x + 2i\pi x^2 + \cdots \in \mathrm{Diff}(\mathbb{C},0)$ are the holonomy map of the strong manifold of a saddle-node into the form (1).

***Theorem* (Martinet-Ramis).** — *A saddle-node into the form (1) admits a central manifold if and only if the translation part φ^∞ of the analytic invariants $(\varphi^0, \varphi^\infty)$ is trivial. In this case, the holonomy of the central manifold coincide with φ^0.*

When there is a central manifold, we note that the analytic class of the saddle-node is given by φ^0 up to linear conjugacy; the conjugacy class of φ^0 in $\mathrm{Diff}(\mathbb{C},0)$ does not characterize the saddle-node in general.

We also note that any germ of diffeomorphism $\varphi^0(\zeta) = e^{2i\pi\mu}\zeta + \cdots \in \mathrm{Diff}(\mathbb{C},0)$ is the holonomy map of the central manifold of a saddle-node of the form (1) with formal invariant μ.

There are similar constructions and results for saddle-nodes

$$X = x^{k+1}\partial_x + y\partial_y + x^k f(x,y)\partial_y, \quad f \in \mathbb{C}\{x,y\}.$$

with higher multiplicity, $k \in \mathbb{N}^*$, and for tangent-to-the-identity germs $\varphi(x) = x + 2i\pi x^{k+1} + \cdots \in \mathrm{Diff}(\mathbb{C},0)$ giving rise to multiple moduli $(\varphi_l^0, \varphi_l^\infty)_{l=1,\ldots,k}$

$$X \longmapsto \begin{cases} \varphi_l^0(\zeta) \; = e^{2i\pi\mu/k}\zeta + \cdots \; \in \mathrm{Diff}(\mathbb{C},0) \\ \varphi_l^\infty(\zeta) = \zeta + \cdots \; \in \mathrm{Diff}(\mathbb{C},\infty) \end{cases} \qquad l = 1,\ldots,k \tag{8}$$

where, in the saddle-node case, all φ_l^∞ are translations. Those $2k$-uple have to be considered up to simultaneous conjugacy by an homothety and up to a cyclic permutation of the indices $\{1,\ldots,k\}$. We omit the precise statements here.

Let us now consider the following family of saddle-nodes ($\varepsilon > 0$)

$$X_\varepsilon = \frac{x^2}{1+\mu x}\partial_x + y\partial_y + \varepsilon f(x,y)y\partial_y, \quad f = \sum_{m\leqslant 0,\ n\leqslant -1} f_{m,n}x^m y^n \in \frac{1}{y}\mathbb{C}\{x,y\}$$

with $f_{0,0} = f_{0,1} = f_{1,1} = 0$, so that multiplicity is 2 and formal invariant μ, and consider its Martinet-Ramis' invariants (depending on ε)

$$\varphi^0(\zeta) = e^{2i\pi\mu}\zeta + \sum_{n>0}\varphi_n\zeta^{n+1} \quad \text{and} \quad \varphi^\infty(\zeta) = \zeta + t$$

Then, the main result of [**7**] reads

***Theorem* (Elizarov).** — *The derivative (in the sense of Gâteau) of Martinet-Ramis'moduli at $\varepsilon = 0$ is given by*

$$\frac{d\varphi_n}{\partial\varepsilon}|_{\varepsilon=0} = n^{\mu n-1}e^{-2i\pi n\mu}\sum_{m>0}\frac{m}{\Gamma(1+m+\mu n)}f_{m,n}(-n)^m$$

$$\text{and}\quad \frac{dt}{\partial\varepsilon}|_{\varepsilon=0} = (-1)^{-\mu}e^{2i\pi\mu}\sum_{m>0}\frac{m}{\Gamma(1+m-\mu)}f_{m,-1}(-n)^m$$

where Γ is the Euler's Gamma Function.

For instance, if we restrict to the family (3) of Theorem 1, we have

$$f(y) = \sum_{n\geqslant 0} a_n y^n \longmapsto \begin{cases}\varphi^0(\zeta) &= e^{2i\pi a_1}\zeta + \sum_{n\geqslant 2} a_n\zeta^n\\ \varphi^\infty(\zeta) &= \zeta + a_0\end{cases}$$

In particular, the derivative at X_0 is bijective. The theorem above motivates the following analytic form announced in [**3**]

***Conjecture* (Bruno-Elizarov).** — *Any saddle-node in the form (1) with formal invariant μ can be analytically reduced to the form*

$$x^2\partial_x + y\partial_y + x\Big(f_0 + \mu y + \sum_{(m,n)\in E_s} f_{m,n}x^m y^{n+1}\Big)\partial_y \tag{9}$$

with support in the strip $E_s = \{(m,n); n>0, \frac{n}{s}+1 \leqslant m < \frac{n}{s}+2\}$ for any slope $0 < s \leqslant +\infty$ such that E_s does not intersect the set of resonances $\{(m,n); m+\mu n \in -\mathbb{N}\}$.

For $s = +\infty$, Bruno's form (9) coincides with our (3) without restriction on μ ($E_{+\infty}$ contains resonances for $\mu \in \mathbb{Q}_*^-$).

2. Proof of Theorem 1

We repeat the geometric construction of [**10**]. Consider the germ of foliation $\mathcal{F}_0$ defined by a vector field X_0 of the form (1)

$$X_0 = x^2\partial_x + y\partial_y + xf(x,y)\partial_y, \quad f \in \mathbb{C}\{x,y\}.$$

Maybe replacing y by $x+y$, the linear part of X_0 is given by

$$\begin{pmatrix}0 & 0\\ c & 1\end{pmatrix} = (cx+y)\partial_y \quad \text{with} \quad c = f(0) \neq 0.$$

Therefore, the vector field X_0 is well-defined on the neighborhood of any small horizontal disc $\Delta_0 = \{|x| < \varepsilon\} \times \{0\}$, $\varepsilon > 0$, and transversal to Δ_0 outside the singular point. Consider inside the horizontal line $L = \overline{\mathbb{C}} \times \{0\}$ the covering given by Δ_0 and $\Delta_\infty = \{|x| > \varepsilon/2\} \times \{0\}$, and denote by $C = \Delta_0 \cap \Delta_\infty$ the intersection corona. By the flow-box Theorem, there exists a unique germ of diffeomorphism of the form

$$\Phi : (\mathbb{C}^2, C) \longrightarrow (\mathbb{C}^2, C)\ ;\ (x,y) \longmapsto (\phi(x,y), y), \quad \phi(x,0) = x$$

straightening $\mathcal{F}_0$ onto the vertical foliation $\mathcal{F}_\infty$ (defined by ∂_y) at the neighborhood of the corona C. Therefore, after gluing the germs of complex surfaces $(\overline{\mathbb{C}} \times \mathbb{C}, \Delta_0)$ and $(\overline{\mathbb{C}} \times \mathbb{C}, \Delta_\infty)$ along the corona by means of Φ, we obtain a germ of smooth complex surface S along a rational curve L equipped with a singular holomorphic foliation $\mathcal{F}$ and a (germ of) *rational fibration* $y : (S, L) \to (\mathbb{C}, 0)$ (an holomorphic fibration whose fibers are biholomorphic to $\overline{\mathbb{C}}$). Following [**8**], there exists a germ of submersion $x : (S, L) \to \overline{\mathbb{C}}$ completing y into a system of trivializing coordinates: $(x, y) : (S, L) \to \overline{\mathbb{C}} \times (\mathbb{C}, 0)$.

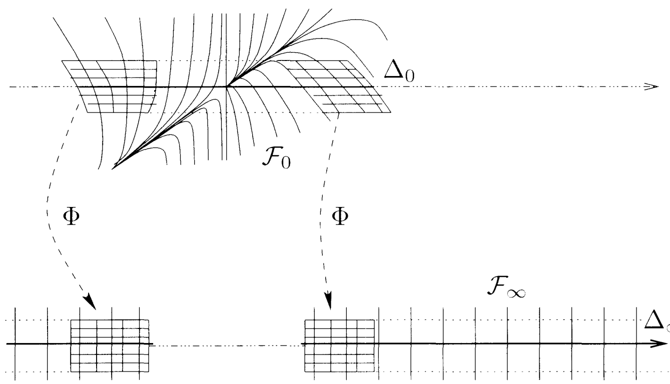

FIGURE 1. Gluing (bi)foliated surfaces

At the neighborhood of any point $p \in L$, the foliation $\mathcal{F}$ is defined by a (non unique) germ of holomorphic vector field, or equivalently by a unique germ of meromorphic vector field of the form

$$X = f(x, y)\partial_x + \partial_y$$

with f meromorphic at p. By unicity, this meromorphic vector field is actually globally defined on the neighborhood of L and is therefore rational in x, *i.e.* f is the quotient of two Weierstrass polynomials. For y fixed (close to 0), the horizontal component $f(x, y)\partial_x$ defines a meromorphic vector field on the corresponding horizontal line $\overline{\mathbb{C}} \times \{y\}$ whose zeroes and poles coincide with the tangencies between $\mathcal{F}$ and the respective vertical and horizontal fibrations. By construction, we control the number of poles: in the second chart, $\mathcal{F} = \mathcal{F}_\infty$ is transversal to y, although in the first chart, $\mathcal{F} = \mathcal{F}_0$ has exactly one simple tangency with any horizontal line. It follows that, for y fixed, the meromorphic vector field $f(x, y)\partial_x$ has exactly 1 simple pole and thus 3 zeroes (counted with multiplicity).

Of course, in restriction to L, the pole vanishes together with one zero at the singular point of $\mathcal{F}$. We conclude that the vector field X defining the foliation $\mathcal{F}$ takes the form

$$X = \frac{f_0(y) + f_1(y)x + f_2(y)x^2 + f_3(y)x^3}{g_0(y) + g_1(y)x}\partial_x + \partial_y \tag{10}$$

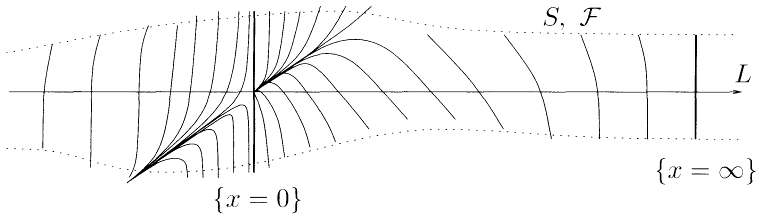

FIGURE 2. Uniformization

with $f_i, g_j \in \mathbb{C}\{y\}$. Up to a change of projective horizontal coordinate $x := \{\frac{a(y)x+b(y)}{c(y)x+d(y)}\}$ on S, one can assume that $\{x=\infty\}$ is a vertical leaf of $\mathcal{F}$, that $\{x=0\}$ is the invariant curve of the saddle-node tangent to the non zero eigendirection and that $\mathcal{F}$ has a contact of order 2 with the vertical fibration along $\{x=0\}$ (likely as in the local form (1)). Therefore, $f_0, f_1, f_3 \equiv 0$ and, reminding that $\mathcal{F}_0$ is an unramified saddle-node with 0-eigendirection transversal to L, we also have $f_2(0) \neq 0$, $g_1(0) \neq 0$, $g_0(0) = 0$ and $g_0'(0) \neq 0$. After division, $\mathcal{F}$ is actually defined by a vector field of the form

$$\widetilde{X} = x^2\partial_x + (f(y)x + yg(y))\partial_y, \quad f(0), g(0) \neq 0.$$

After change of y-coordinate, one may normalize the holomorphic vector field $yg(y)\partial_y$ to $g(0)y\partial_y$; after division by $g(0)$ and linear change of the x-coordinate, we finally obtain the form (3).

3. Gluing Lemmae

Although Theorems 2, 3 and 4 can be shortly proved by using Savelev Theorem [**15**] like in [**10**], we provide an alternate proof more "down to the earth" where we simultaneously construct the auxiliary fibration during the gluing construction. In order to do this, we need some lemmae allowing us to glue pairs of non transversal foliations.

The *order of contact* between two germs of regular holomorphic vector fields X_1 and X_2 at $0 \in \mathbb{C}^2$, or between the corresponding foliations, is by definition the order at 0 of the determinant $\det(X_1, X_2)$. For instance, X_1 and X_2 are transversal if and only if they have a contact of order $k=0$. Now, if those two foliations share a common leaf, and if moreover there is no contact between them outside this leaf, then the contact order $k \in \mathbb{N}^*$ is constant along this common leaf and classifies locally the pair of foliations:

Lemma 5. — *Let $\mathcal{F}$ be a germ of regular analytic foliation at the origin of $\mathbb{C}^2$ (or $\mathbb{R}^2$) having the horizontal axis $L_0 : \{y=0\}$ as a particular leaf and having no other contact with the horizontal fibration $\{y = constant\}$: $\mathcal{F}$ is defined by a unique function (or vector field) of the form*

$$F(x,y) = y + y^k x f(x,y) \quad \textit{with } f(0,0) \neq 0$$
$$(\textit{or } X = g(x,y)\partial_x + y^k\partial_y, \quad \textit{with } g(0,0) \neq 0)$$

where $k \in \mathbb{N}^*$ denotes the contact order between $\mathcal{F}$ and the horizontal fibration. Then, up to a change of coordinates of the form $\Phi(x,y) = (\phi(x,y),y)$, the foliation $\mathcal{F}$ is defined by the function (or vector field)

$$F_0(x,y) = y + xy^k \quad (or\ X_0 = \partial_x + y^k\partial_y).$$

The restriction of Φ to L_0 is the identity if, and only if, $f(0,w) \equiv 1$. Moreover, the normalizing coordinate Φ is unique once we have decided that it fixes the vertical axis, i.e. $\phi(x,y) = x\widetilde{\phi}(x,y)$.

Proof. — Given $\mathcal{F}$ as in the statement, choose $F(x,y)$ to be the unique function which is constant on the leaves and has restriction $F(0,y) = y$ on the vertical axis: $F(x,y) = y(1 + x\widetilde{F}(x,y))$. The assumption $dF \wedge dy = y^k u(x,y)$, $u(0,0) \neq 0$, yields $\widetilde{F}(x,y) = y^{k-1}f(x,y)$ with $f(0,0) \neq 0$, whence the form $F(x,y) = y + xy^k f(x,y)$. Now, we have

$$F = F_0 \circ \Phi_0 \quad \text{with} \quad \Phi_0(x,y) = (xf(x,y),y).$$

Thus, Φ_0 is the unique change of x-coordinate which conjugates the functions F and F_0; in particular, it conjugates the induced foliations.

Conversely, assume that $\Phi(x,y) = (\phi(x,y),y)$ is conjugating the foliations respectively induced by F and F_0: we have

$$F_0 \circ \Phi(x,y) = \varphi \circ F(x,y) \quad \text{with} \quad \varphi(y) = y + y^k\phi(0,y)$$

(the germ φ is determined by the equality restricted to $\{w = 0\}$). If we decompose $f(x,y) = u(x) + yv(x,y)$, we notice that $\varphi \circ F(x,y) = y + xy^k(u(x) + y\widetilde{v}(x,y))$, so that $\phi(x,0) = xu(x) = xf(x,0)$. Finally, if $\phi(x,y) = x\widetilde{\phi}(x,y)$, then $\varphi(y) = y$ and Φ actually conjugates the functions: we must have $\Phi = \Phi_0$ whence the unicity.

Now, if $\mathcal{F}$ is defined by $X = f(x,y)\partial_x + g(x,y)\partial_y$, assumption gives $dy(X) = g(x,y) = y^k\widetilde{g}(x,y)$ with $f(0,0), \widetilde{g}(0,0) \neq 0$. After dividing X by g, we can write $X = f(x,y)\partial x + y^k\partial_y$. We have already proved that any two such foliations (in particular those induced by X and X_0) are conjugate by a unique diffeomorphism of the form $\Phi(x,y) = (x\widetilde{\phi}(x,y),y)$. Now, if $\Phi(x,y) = (\phi(x,y),y)$ conjugates the foliations respectively induced by X and X_0, it actually conjugates these vector fields. In restriction to the trajectory L_0, we see that $\phi(x,0)$ conjugates $\widetilde{X}|_{L_0} = f(x,0)\partial_x$ to the constant vector field ∂_x. Therefore, $\phi(x,0) = \int_0^x \frac{1}{f(\zeta,0)}d\zeta$ and $\phi(x,0) \equiv x$ if, and only if, $f(x,0) \equiv 1$. □

For the next statement, denote by $\Omega \subset (\mathbb{C} \times \{0\})$ a connected open domain inside the horizontal axis.

Lemma 6. — *Let $\mathcal{F}$ and $\mathcal{F}'$ be regular holomorphic foliations defined at the neighborhood of Ω in $\mathbb{C}^2$ both having Ω as a particular leaf. Assume that the contact between each foliation with the horizontal fibration $\{y = constant\}$ reduces to Ω, with same* *order $k \in \mathbb{N}^*$. In other words, $\mathcal{F}$ and $\mathcal{F}'$ are respectively defined by vector fields*

$$X = f(x,y)\partial_x + y^k\partial_y \quad \textit{and} \quad X' = f'(x,y)\partial_x + y^k\partial_y$$

where f and f' are non vanishing functions in the neighborhood of Ω. Then, $\mathcal{F}$ and $\mathcal{F}'$ are conjugated in a neighborhood of Ω by a diffeomorphism of the form $\Phi(x,y) = (x + y\phi(x,y), \psi(y))$ (fixing Ω) if, and only if, the two following conditions hold

(1) *$f(x,0) \equiv f'(x,0)$;*

(2) *the respective holonomies φ and φ' of $\mathcal{F}$ and $\mathcal{F}'$ along Ω are analytically conjugated: $\psi \circ \varphi = \varphi' \circ \psi$.*

Proof. — Following Lemma 5, condition (1) is the necessary and sufficient condition for the existence of local conjugacies $\Phi = (y\phi(x,y), y)$ between $\mathcal{F}$ and $\mathcal{F}'$ at the neighborhood of any point $w_0 \in \Omega$. Fix one of these points and consider the respective holonomy maps φ and φ' computed on the transversal $T : \{x = x_0\}$ in the variable y. By condition (2), up to conjugate, say $\mathcal{F}'$, by a diffeomorphism of the form $(x, \psi(y))$, we may assume without loss of generality $\varphi(y) = \varphi'(y)$. We start with the local diffeomorphism $\Phi(x,y) = (y\phi(x,y), y)$ given by Lemma 5 conjugating the foliations and fixing T. Since Φ conjugates the corresponding vector fields X and X', it extends analytically along the whole of Ω by the formula $\Phi(p) := \Phi_{X'}^{-t} \circ \Phi \circ \Phi_X^t(p)$. The condition $\varphi(y) = \varphi'(y)$ implies that Φ is uniform. □

Here is a last gluing Lemma for pairs of regular foliations $\mathcal{F}$ and $\mathcal{G}$ at the neighborhood of a common leaf Ω. Again, Ω is a connected open subset of the horizontal axis $\Omega \subset (\mathbb{C} \times \{0\})$. When $\mathcal{G}$ is the horizontal fibration, the following Lemma reduces to the previous one.

Lemma 7. — *Let $\mathcal{F}$ and $\mathcal{G}$ (resp. $\mathcal{F}'$ and $\mathcal{G}'$) be regular holomorphic foliations defined at the neighborhood of Ω in $\mathbb{C}^2$ both having Ω as a regular leaf. Assume that the contact between $\mathcal{F}$ and $\mathcal{G}$ (resp. $\mathcal{F}'$ and $\mathcal{G}'$) reduces to Ω, with same order $k \in \mathbb{N}^*$. In other words, the foliations above are respectively defined by vector fields*

$$X = \partial_x + yf(x,y)\partial_y \quad \textit{and} \quad Y = X + y^k g(x,y)\partial_y,$$

$$(\textit{resp.} \quad X' = \partial_x + yf'(x,y)\partial_y \quad \textit{and} \quad Y' = X' + y^k g'(x,y)\partial_y)$$

where g and g' are non vanishing functions in the neighborhood of Ω. Then, $\mathcal{F}$ and $\mathcal{G}$ are simultaneously conjugated to $\mathcal{F}'$ and $\mathcal{G}'$ in a neighborhood of Ω by a diffeomorphism of the form $\Phi(x,y) = (x + y\phi(x,y), y\psi(x,y))$ (fixing point-wise Ω) if, and only if, the two conditions hold

(i) *for any (and for all) $x_0 \in \Omega$, we have*

$$\frac{g(x,0)}{\exp(-\int_{x_0}^x f(\zeta,0)d\zeta)} \equiv \frac{g'(x,0)}{\exp(-\int_{x_0}^x f'(\zeta,0)d\zeta)};$$

(ii) *the respective pairs of holonomies $(\varphi_\mathcal{F}, \varphi_\mathcal{G})$ and $(\varphi_{\mathcal{F}'}, \varphi_{\mathcal{G}'})$ along Ω are simultaneously analytically conjugated: $\psi \circ \varphi_\mathcal{F} = \varphi_{\mathcal{F}'} \circ \psi$ and $\psi \circ \varphi_\mathcal{G} = \varphi_{\mathcal{G}'} \circ \psi$.*

Proof. — It is similar to that of the previous Lemma. Up to a change of coordinate $y := \psi(y)$ (which does not affect neither $f(x,0)$, nor $g(x,0)$ and hence preserves equality (i)), we may assume that holonomies $(\varphi_{\mathcal{F}}, \varphi_{\mathcal{G}}) \equiv (\varphi_{\mathcal{F}'}, \varphi_{\mathcal{G}'})$ actually coincide on a transversal $T : \{x = x_0\}$. We just detail that condition (i) exactly provides the existence of local conjugacies between the given pairs of foliations fixing point-wise Ω; the unique conjugacy fixing T will extend uniformly along Ω by (ii).

At the neighborhood of any point $(x_0, 0) \in \Omega$, say $x_0 = 0$ for simplicity, we preliminary conjugate X to $X_0 = \partial_x$ by respective local changes of y-coordinate $\Psi(x,y) = (x, y\psi(x,y))$, $\psi(0,0) \neq 0$

$$\Psi_* X = X_0 = \partial_x \quad \text{and} \quad \Psi_* Y = Y_0 = z^k g_0(x,y)\partial_z + \partial_w$$

Doing the same with the pair X' and Y', we see by Lemma 5 that the corresponding pairs of foliations are conjugated by a diffeomorphism fixing point-wise $\{y = 0\}$ if, and only if, the differential form $\omega = g_0(x,0)dx$ along Ω coincide with the corresponding one $\omega' = g_0'(x,0)dx$ for $X_0' = X_0$ and $Y_0' = \partial_x + y^k g_0(x,y)\partial_y$. This 1-form ω can be redefined in the following intrinsic way: the holonomy of $\mathcal{G}$ between two transversal cross-sections T_0 and T_1 computed in any coordinate y which is $\mathcal{F}$-invariant (here $\mathcal{F}$ is defined by ∂_x) takes the form

$$\varphi(y) = y + \left(\int_{x_0}^{x_1} \omega\right) y^k + (\text{higher order terms})$$

where $(x_i, 0) := T_i \cap \Omega$, $i = 0, 1$. Since

$$\Psi^* X_0 = \partial_x - \frac{\psi_x}{\psi + y\psi_y} y\partial_y \quad \text{and} \quad \Psi^* Y_0 = \Psi^* X_0 + \frac{g_0}{\psi + y\psi_y} y^k \partial_y,$$

(ψ_x and ψ_y are partial derivatives of ψ) we derive in restriction to Ω

$$f(x,0) = -\frac{\psi_x(x,0)}{\psi(x,0)} \quad \text{and} \quad g_0(x,0) = \psi(x,0) \cdot g(x,0)$$

yielding the formula for the local invariant of our conjugacy problem

$$\omega = \frac{g(x,0)}{\exp(-\int_{x_0}^{x} f(\zeta,0)d\zeta)} dx. \qquad \square$$

4. Proof of Theorem 2

Given a saddle-node foliation $\mathcal{F}$ of the form (4), it is easy to verify that its analytic continuation at the neighborhood of the horizontal line $L = \overline{\mathbb{C}} \times \{0\}$ satisfies

(1) the line L is a global invariant curve for $\mathcal{F}$, the union of a smooth leaf together with 2 singular points;

(2) the point $x = 0$ is a saddle-node singular point with multiplicity 2, formal invariant μ and invariant curve $\{xy = 0\}$; in particular, the saddle-node has a central manifold which is contained in L;

(3) the point $x = \infty$ is a singular point with eigenratio $-\mu$ and invariant curve $\{x = \infty\} \cup \{y = 0\}$;

(4) the foliation $\mathcal{F}$ has a contact of order 2 with the vertical fibration along the invariant curve $\{x = 0\}$ (in the sense of section 3).

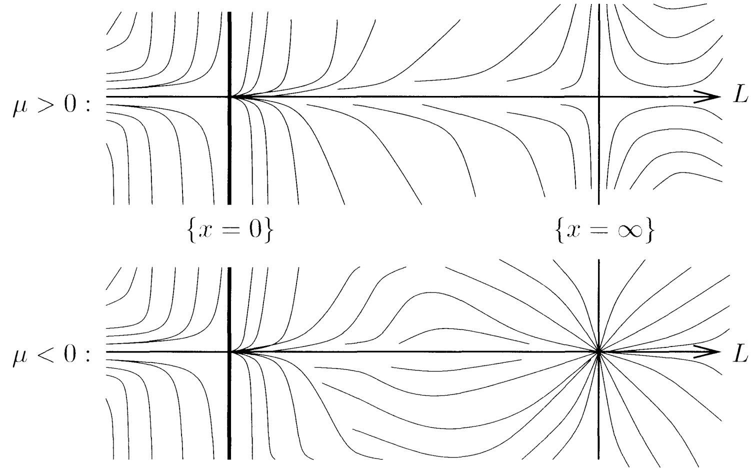

FIGURE 3. Geometry of the second normal form

Conversely, a germ of foliation $\mathcal{F}$ on $\overline{\mathbb{C}} \times (\mathbb{C}, 0)$ satisfying conditions above can easily be transformed into the form (4). Indeed, $\mathcal{F}$ is defined by a unique vector field of the form $F(x,y)\partial_x + \partial_y$ with F meromorphic at the neighborhood of the line L. In restriction to the horizontal lines, the vector field $F(x,y)\partial_x$ is rational; its zeroes and poles coincide with the points where the foliation $\mathcal{F}$ is respectively vertical and horizontal. Because we have two singular points of multiplicities 1 and 2 along L, we deduce that $F(x,y)\partial_x$ has 3 zeroes (counted with multiplicity) in restriction to each fiber; hence, it it has exactly 1 pole (the divisor of a vector field has degree 2 on $\overline{\mathbb{C}}$). From conditions (2) and (4), we actually see that the zeroes are supported by the vertical invariant curves, that L gives contribution for 1 pole and one can write $F(x,y) = \frac{x^2}{y(f(y)x+g(y))}$ for holomorphic functions $f, g \in \mathbb{C}\{y\}$. It is easy to verify that f and g do not vanish at $y = 0$ otherwise the singular points would be more degenerate. Therefore, the foliation $\mathcal{F}$ is also defined by the holomorphic vector field

$$\widetilde{X} = x^2\partial_x + (f(y)x + g(y))y\partial_y, \quad f(0), g(0) \neq 0.$$

After a change of y-coordinate, one may linearize the holomorphic vector field $yg(y)\partial_y$ to $g(0)y\partial_y$; after division by $g(0)$ and linear change of the x-coordinate, we finally obtain the form (4).

A *necessary condition* for a saddle-node to admit a form (4) is that the holonomy of the central manifold, which actually coincides with Martinet-Ramis' invariant φ^0

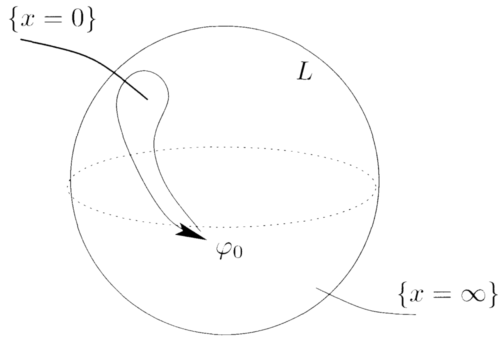

FIGURE 4. Holonomy compatibility

(see section 1), is also the anti-holonomy of the invariant curve L around the singular point $x = \infty$. This gives restriction for φ^0, and hence for the saddle-node, at least when $\mu \in \mathbb{R}^-$. In the case $\mu < 0$, the other singular point is linearizable by Poincaré's Theorem implying the linearizability of the holonomy map φ^0. Here, we use property (3) above and the fact that, in the resonant (non linearizable) case, the node has only one irreducible germ of invariant curve. In the case $\mu = 0$, the holonomy φ^0 is tangent to the identity and its inverse $(\varphi^0)^{-1}$ must be the holonomy of the *strong manifold* (the invariant curve tangent to the non zero eigendirection) of a saddle-node having a central manifold. Following section 1, this is equivalent to condition (3) of Theorem 8 below.

Theorem 8. — *Let $\mathcal{F}$ be a germ of saddle-node with multiplicity 2 at the origin of $\mathbb{R}^2$ (resp. of $\mathbb{C}^2$) having a central manifold. Then, there exist analytic coordinates in which $\mathcal{F}$ is defined by a vector field of the form (4)*

$$X_f = x^2\partial_x + y\partial_y + xf(y)\partial_y, \quad \text{with } f(0) = 0,$$

(and $\mu = f'(0)$) if, and only if, we are in one of the following cases

(1) $\mu \in \mathbb{C} - \mathbb{R}^-$,

(2) $\mu < 0$ *and φ_0 is linearizable up to conjugacy in* $\mathrm{Diff}(\mathbb{C}, 0)$,

(3) $\mu = 0$ *and Martinet-Ramis' invariants $(\widetilde{\varphi}_0^i, \widetilde{\varphi}_\infty^i)_i$ of φ_0 satisfy: all $\widetilde{\varphi}_0^i$ are linear.*

When $\mu \notin \mathbb{Q}_*^-$, the form (4) is unique up to homothety $y \mapsto c \cdot y$, $c \in \mathbb{C}^*$.

Recall that condition (2) is automatic as soon as μ is a Bruno number:

$$\mu \in \mathcal{B} \iff \sum_{n \geqslant 0} \frac{\log(q_{n+1})}{q_n} < \infty$$

(where p_n/q_n stand for successive truncatures of the continued fraction of $|\mu|$). The set $\mathcal{B}$ has full Lebesgue measure in $\mathbb{R}$. For all other values $\mu \in \mathbb{R}^- - \mathcal{B}$, condition (2) is very restrictive for φ_0, and thus for the saddle-node.

Like in Section 2, we start with a germ of saddle-node $\mathcal{F}_0$ defined on the neighborhood of some disc Δ_0 and glue it with a germ of foliation $\mathcal{F}_\infty$ along a complementary disc Δ_∞ in order to obtain a germ of 2-dimensional neighborhood S along a rational curve L equipped with a singular foliation $\mathcal{F}$. The difference with Section 2 is that we now glue $\mathcal{F}_0$ and $\mathcal{F}_\infty$ along a common invariant curve, in such a way that L becomes a global invariant curve for the foliation $\mathcal{F}$. We do it first respect to the vertical fibration; this is very easy but we need the difficult Savelev's Theorem to recover the triviality of the neighborhood (and the rational fibration). Then, we give an alternate gluing using technical (but elementary) Lemmae of section 3 in which we keep on constructing by hands the rational fibration.

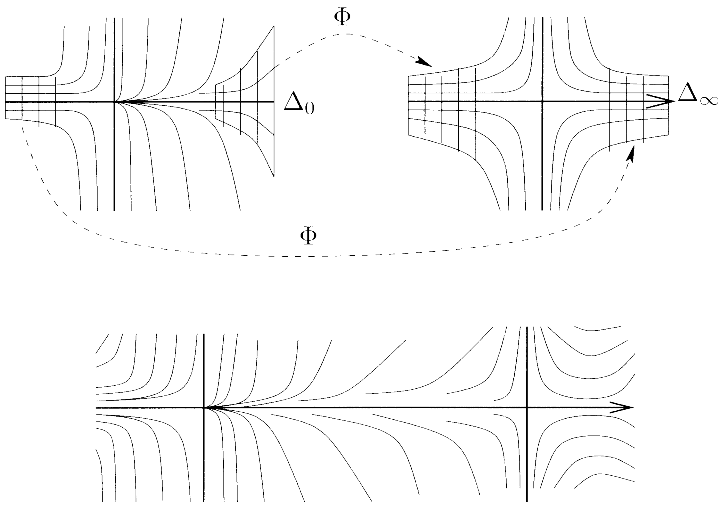

FIGURE 5. Gluing picture

We start with $\mathcal{F}_0$ into Dulac preliminary form

$$X_0 = x^2\partial_x + y\partial_y + xyf(x,y)\partial_y, \quad f \in \mathbb{C}\{x,y\},\ f(0) = \mu$$

(when the saddle-node has a central manifold, the form (1) can be achieved with the central manifold contained in $\{y = 0\}$, see [**11**]). Consider, in local coordinates $(\widetilde{x} = 1/x, y)$ at infinity, a germ of singular foliation $\mathcal{F}_\infty$ defined by

$$X_\infty = \widetilde{x}\partial_{\widetilde{x}} - (\mu + g(\widetilde{x}, y))y\partial_y, \quad g(0) = 0.$$

Up to a linear change of $\widetilde{x}$-coordinate, one may assume that $\mathcal{F}_\infty$ is actually defined on the neighborhood of Δ_∞. Obviously, there exists a germ of diffeomorphism of the form

$$\Phi : (\mathbb{C}^2, C) \longrightarrow (\mathbb{C}^2, C) \ ; \ (x,y) \longmapsto (x, \phi(x,y)), \quad \phi(x,0) = 0$$

gluing $\mathcal{F}_0$ with $\mathcal{F}_\infty$ if, and only if, the respective holonomy maps around the corona $C = \Delta_0 \cap \Delta_\infty$ are conjugated in $\mathrm{Diff}(\mathbb{C}, 0)$.

When $\mu \notin \mathbb{R}$, the holonomy map φ^0 of $\mathcal{F}_0$ around C (or $x = 0$) is hyperbolic and hence linearizable by Kœnigs' Theorem. It is therefore enough to choose X_∞ linear. When $\mu > 0$, then the holonomy map φ^0 can be realized as the holonomy of a saddle $\mathcal{F}_\infty$ like above following [**12, 14**]. When $\mu < 0$ and φ^0 is linearizable, we obviously realize it with X_∞ linear. Finally, when $\mu = 0$, condition (3) of Theorem 2 is exactly the one to realize $(\varphi^0)^{-1}$ as the holonomy of the strong manifold of a saddle node $\mathcal{F}_\infty$ having a central manifold. After gluing $\mathcal{F}_0$ and $\mathcal{F}_\infty$ along C, we obtain a germ of surface S containing a rational curve L which, by Camacho-Sad's Formula (see [**4**]), has 0 self-intersection in S. Following Savelev's Theorem [**15**], there exists a system of trivializing coordinates: $(x, y) : (S, L) \to \overline{\mathbb{C}} \times (\mathbb{C}, 0)$. Up to a change of trivializing coordinates $x := \{\frac{a(y)x+b(y)}{c(y)x+d(y)}\}$ and $y = \varphi(y)$ on S, one may assume properties (1), (2), (3) and (4) of the begining of the section all satisfied. Therefore, $\mathcal{F}$ is defined by a vector field of the form (4). The existence part is proved. □

Let us now show how to avoid with Savelev Theorem by using section 3. We first choose germs of foliations $\mathcal{F}_0$ and $\mathcal{F}_\infty$ with compatible holonomy as in the previous proof. Instead of X_0, we define the foliation $\mathcal{F}_0$ by the meromorphic vector field

$$\widetilde{X}_0 = \frac{x^2}{1 + xf(x, y)}\partial_x + y\partial_y, \quad f(0) = \mu.$$

After a local change of the x-coordinate, we may assume that the restriction $\widetilde{X}_0|_L = \frac{x^2}{1+xf(x,0)}\partial_x$ to $L = \{y = 0\}$ coincides with the global meromorphic vector field $\frac{x^2}{1+\mu x}\partial_x$. By the same way, the alternate meromorphic vector field

$$\widetilde{X}_\infty = \frac{x}{\mu + g(1/x, y)}\partial_x + y\partial_y, \quad g(0) = 0.$$

defines $\mathcal{F}_\infty$ at the neighborhood of $x = \infty$ and its restriction $\frac{x}{\mu+g(1/x,0)}\partial_x$ coincide with $\frac{x^2}{1+\mu x}\partial_x$ after a local change of x-coordinate at infinity (they are both conjugated to $\frac{1}{\mu}x\partial_x$ at $x = \infty$).

Assume first that $\widetilde{X}_0$ and $\widetilde{X}_\infty$ are defined at the neighborhood of some horizontal discs Δ_0 and Δ_∞ covering L. Maybe restricting to slightly smaller discs, one may assume that the intersecting corona $C = \Delta_0 \cap \Delta_\infty$ does not contain $-1/\mu$ (the pole of $\frac{1}{\mu}x\partial_x$): therefore, the vector fields $\widetilde{X}_0$ and $\widetilde{X}_\infty$ are both holomorphic on the neighborhood of C and can be glued by means of Lemma 6. By this way, we construct a surface S equipped with a global foliation $\mathcal{F}$ and a rational fibration $y : S \to (\mathbb{C}, 0)$. By Fisher-Grauert [**8**], S is a germ of trivial $\overline{\mathbb{C}}$-bundle and we can end the proof as before.

The problem is that $\widetilde{X}_0$ and $\widetilde{X}_\infty$ are *a priori* defined on small respective neighborhoods Ω_0 and Ω_∞ of $x = 0$ and $x = \infty$. We would like to apply a change of coordinate in variable x in order to enlarge Ω_∞, for instance. We cannot do this with an homothety anymore because we need to preserve the restrition of $\widetilde{X}_\infty$ to L in order to apply Lemma 6. We can only use the changes of coordinates which commute with $\frac{x^2}{1+\mu x}\partial_x$, *i.e.* those ones given by an element of the flow $\exp(t\frac{x^2}{1+\mu x}\partial_x)$, $t \in \mathbb{C}$. By this way, we will not be able to cover the complement of Ω_0 with the new domain Ω_∞, but at least, we may assume that Ω_0 and Ω_∞ intersect. Then, we can complete a covering of L by adding a third open set Ω_1 in such a way that the intersections $\Omega_0 \cap \Omega_1$, $\Omega_0 \cap \Omega_\infty$ and $\Omega_1 \cap \Omega_\infty$ do not contain neither 0, $-1/\mu$ nor ∞. We refer to [**1**] for a complete description of the flow $\exp(t\frac{x^2}{1+\mu x}\partial_x)$ in function of μ. Finally, consider the third foliation defined on the neighborhood of Ω_1 by the rational vector field

$$\widetilde{X}_\mu = \frac{x^2}{1+\mu x}\partial_x + y\partial_y.$$

By means of Lemma 6, we can glue the 3 foliations together on the neighborhood of L, simultaneously preserving the y-coordinate. This finishes the second proof of the construction of form (4). □

It remains to prove the unicity (up to homothety) of form (4) in case μ is not rational negative. Assume that $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are of the form (4) and are analytically conjugated on a neighborhood of $(x,y) = 0$. Following [**11**], they are also conjugated by a germ of diffeomorphism of the form

$$\Phi_0 : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)\ ;\ (x,y) \longmapsto (x, \phi_0(x,y))$$

which must preserve the central manifold: $\phi_0(x,0) = 0$. One can extend analytically Φ_0 on a neighborhood of $L - \{x = \infty\}$ in the obvious way, by lifting-path-property. We *claim* that Φ_0 extends until the other singular point $x = \infty$. Before proving this, let us show how to conclude the proof. Therefore, we obtain a global diffeomorphism Φ along L conjugating $\mathcal{F}$ and $\widetilde{\mathcal{F}}$. By Blanchard's argument, Φ permutes the horizontal lines: for any line L' close to L, the restriction of y along the image $\Phi(L')$ is an holomorphic map from a compact manifold into a bounded domain; therefore, $y|_{\Phi(L')}$ is constant and $\Phi(L')$ is actually a fiber of y. Therefore, one can write $\Phi(x,y) = (x, \phi(y))$ and due to the form (4), ϕ has to commute with $y\partial_y$ and must be linear. This concludes the proof of Theorem 8. □

We first prove the *claim* in case $\mathcal{F}_\infty$ is in the Poincaré domain ($\mu \in \mathbb{C} - \mathbb{R}^-$). Recall that property (3) implies that $\mathcal{F}_\infty$ is non resonant and hence linearizable by a local change of coordinates of the form $(\tilde{x}, y) \mapsto (\tilde{x}, \phi_\infty(\tilde{x}, y))$. Therefore, we can assume that $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are defined by

$$X_\infty = \tilde{x}\partial_{\tilde{x}} - \mu y\partial_y, \quad \mu \not\geq 0.$$

and that $\Phi_0(\tilde{x}, y) = (\tilde{x}, \phi(\tilde{x}, y))$ is a self-conjugacy of $\mathcal{F}_\infty$ at the neighborhood of the punctured disc $\Delta^* := \Delta_\infty - \{\tilde{x} = 0\}$. The question is, when does Φ_0 coincide with a symetry of $\mathcal{F}_\infty$

$$\Phi_\infty(\tilde{x}, y) = (\tilde{x}, c \cdot y), \quad c \in \mathbb{C}^*.$$

Of course, this is the case if, and only if, $\phi(\widetilde{x}, y)$ is linear in y. In fact, for x fixed, $\phi(\widetilde{x}, y)$ commutes with the holonomy $y \mapsto e^{-2i\pi\mu}y$ of $\mathcal{F}_\infty$ and is therefore linear as soon as μ is not rational.

Finally, in the remaining case $\mu \geqslant 0$, the fact that Φ_0 extends at the singular point at infinity is due to J.-F. Mattei and R. Moussu ([**13**], p. 484-485 or [**12**], p. 595-596) in the case $\mu > 0$ and to M. Berthier, R. Meziani and P. Sad ([**2**], Theorem 1.1) in the case $\mu = 0$. Actually, in both cases, it is proved that any conjugacy between the holonomy maps of two saddles ($\mu > 0$) or strong manifolds of two saddle-nodes with a central manifold ($\mu = 0$) extends as a conjugacy of the respective foliations of the form $\Phi_\infty(x, y) = (x, \phi_\infty(x, y))$; this Φ_∞ will automatically coincide with Φ_0 and extend it at the singular point $x = \infty$. The claim is proved. □

Remark 9. — In the case $\mu \in \mathbb{Q}_*^-$, it is easy to construct examples of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ like above that are not globally conjugated and giving rise to non unique form (4).

Proof of Theorem 3. — It is the same with $\mu > 0$, except that we start with the saddle $\mathcal{F}_\infty$ at $x = \infty$. Following Martinet-Ramis (see section 1), the anti-holonomy $\varphi^0(y) = e^{2i\pi\mu}y + \cdots$ of $\mathcal{F}_\infty$ can be realized as the holonomy of the central manifold of a saddle-node $\mathcal{F}_0$. Like above, we can glue those two foliations and obtain normal form (4). We deduce the normal form (5) for the saddle $\mathcal{F}_\infty$ by setting $\widetilde{x} = 1/x$ in the form (4). □

5. Proof of Theorem 4

Let us start by blowing-up a saddle-node of the form (4)

$$X_f = x^2\partial_x + y\partial_y + xyf(y)\partial_y, \quad f(0) = \mu.$$

Along the exceptional divisor, we have one saddle with eigenratio -1 and a saddle-node, given in the chart (x, t), $y = tx$, by

$$\widetilde{X}_f = x^2\partial_x + t\partial_t + xt(f(xt) - 1)\partial_t.$$

In particular, $\widetilde{X}_f$ takes the form (6) of Theorem 4 with $n = 1$ and has formal invariant $\widetilde{\mu} = \mu - 1$.

After n successive blow-ups of the saddle-nodes, we obtain an exceptional divisor like in the picture below where the new saddle-node takes the form (6) of Theorem 4 with formal invariant $\widetilde{\mu} = \mu - n$. All other singular points are saddles with -1 eigenratio.

The rough idea to put a given saddle-node $\widetilde{\mathcal{F}}$ into the form (6) is to realize it as the n^{th} blowing-up of a saddle-node $\mathcal{F}$, then apply Theorem 2 to put $\mathcal{F}$ into the form (4). We first detail the case $n = 1$.

Since the holonomy map φ of the strong manifold of $\widetilde{\mathcal{F}}$ is tangent-to-the-identity, it can be realized as the holonomy map of a saddle with -1 eigenratio following

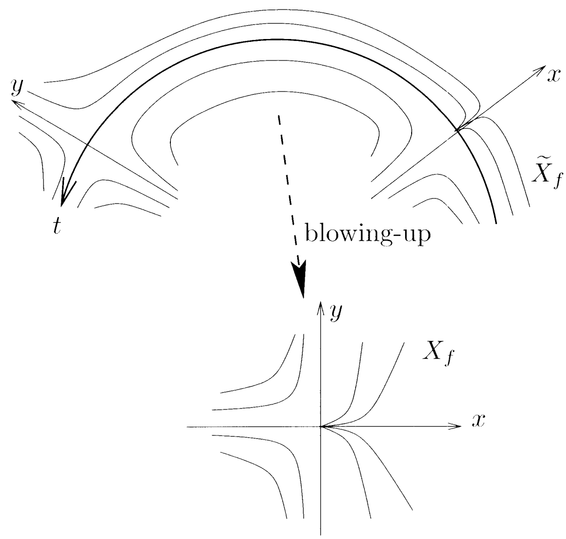

FIGURE 6. Blowing-up a saddle-node

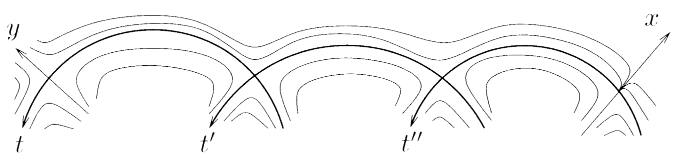

FIGURE 7. After 3 blowing-ups

Martinet-Ramis (see [**12**]). Therefore, one can glue those two foliations along their invariant curve like we did in section 4 to prove Theorem 2 (first gluing construction). By this way, we obtain a germ of surface S around a rational curve L having self-intersection -1 by Camacho-Sad index Theorem [**4**]. Following Grauert (see [**9**]), *the neighborhood of a smooth rational curve with negative self-intersection in a surface is rigid*: maybe replacing S by a smaller neighborhood of L, S is biholomorphic to the neighborhood of the exceptional divisor after blowing-up the origin of $\mathbb{C}^2$ (-1 self-intersection). After making this identification, the global foliation $\widetilde{\mathcal{F}}$ on S becomes the germ of a saddle-node $\mathcal{F}$ at the origin of $\mathbb{C}^2$. The corresponding formal invariants are related by $\mu = \tilde{\mu} - 1$ so that if $\widetilde{\mathcal{F}}$ satisfies the assumptions of Theorem 4 with $n = 1$, then one can apply Theorem 2 to $\mathcal{F}$. Once $\mathcal{F}$ is in the form (4), we obtain the form (6) for $\widetilde{\mathcal{F}}$. Here, we implicitely use the known fact that one can blow up a diffeomorphism: the conjugacy from $\mathcal{F}$ to its normal form (4) induces after blowing up a conjugacy from $\widetilde{\mathcal{F}}$ to its normal form (6). This proves the existence part. □

The unicity also follows from that of form (4) proved in Section 4. Indeed, if two such foliations $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$ are locally conjugated, then the corresponding holonomies along the exceptional divisor L are conjugated. By Mattei-Moussu [**13**], this implies that the -1 saddles are conjugated; therefore, the holonomies of the saddles along the other invariant curve $\{t = \infty\}$ are conjugated as well. This latter means that after blowing down, the holonomies of the strong manifold of the corresponding saddle-nodes $\mathcal{F}$ and $\mathcal{F}'$ are conjugated. We can apply unicity of Theorem 2. □

The general case $n \in \mathbb{N}^*$ is proved by the same way. Starting from a saddle-node $\widetilde{\mathcal{F}}$ with formal invariant $\mu > -n$ (or $\mu \notin \mathbb{R}$), we glue it successively with -1 saddles in order to construct a n-blow-up configuration as in the picture; then, Grauert's Theorem permits to blow down successively all irreducible components of the divisor: at each step, the component which contains the saddle-node has again self-intersection -1 by Camacho-Sad. After blowing down the whole divisor, we can apply Theorem 2 to the resulting saddle-node.

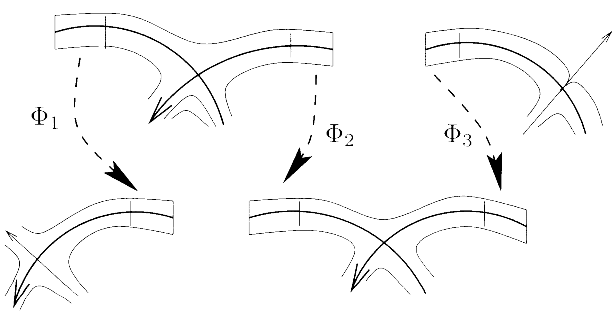

FIGURE 8. Gluing foliations along an exceptional divisor

For the unicity, given a conjugacy between two saddle-nodes $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$ like above, we successively deduce by Mattei-Moussu the conjugacy of all respective -1 saddles and finally of the resulting saddle-nodes $\mathcal{F}$ and $\mathcal{F}'$ after blowing down. The unicity follows again from that of Theorem 2.

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