Astérisque

HAYNES MILLER

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INTRODUCTION

Around the year 1960 the theory of surgery was developed as part of a program to classify manifolds of dimension greater than 4. Among the questions it addresses is this: Does every framed cobordism class contain a homotopy sphere?

Recall that a *framing* of a closed smooth manifold is an embedding into Euclidean space together with a trivialization of the normal bundle. A good example is given by the circle embedded in $\mathbb{R}^3$, with a framing t such that the framing normal vector fields have linking number ± 1 with the circle itself. A framed manifold is *null-bordant* if it is the boundary of a framed manifold-with-boundary, and two framed manifolds are *cobordant* if their difference is null-bordant. Cobordism classes of framed n-manifolds form an abelian group Ω_n^{fr}. The class of (S^1, t) in Ω_1^{fr} is written η.

Any closed manifold of the homotopy type of S^n admits a framing [20], and the seemingly absurdly ambitious question arises of whether every class contains a manifold of the homotopy type (and hence, by Smale, of the homeomorphism type, for $n > 4$) of a sphere. (Classes represented by some framing of the standard n-sphere form a cyclic subgroup of order given by the resolution of the Adams conjecture in the late 1960's.)

A result of Pontryagin from 1950 implied that the answer had to be "No" in general: $\eta^2 \neq 0$ in Ω_2^{fr}, while S^2 is null-bordant with any framing.

Kervaire and Milnor [20] showed that the answer is "Yes" unless $n = 4k + 2$; but that there is an obstruction, the *Kervaire invariant*

$$\kappa : \Omega_{4k+2}^{\mathrm{fr}} \to \mathbb{Z}/2\mathbb{Z},$$

which vanishes on a cobordism class if and only if the class contains a homotopy sphere. Kervaire's construction [19] of a PL 10-manifold with no smooth structure

amounted to showing that $\kappa = 0$ on $\Omega_{10}^{\mathrm{fr}}$. Pontryagin had shown that $\kappa(\eta^2) \neq 0$, and in fact the Kervaire invariant is nontrivial on the square of any element of Hopf invariant one. In [20] Kervaire and Milnor speculated that these may be the only examples with $\kappa \neq 0$.

The Pontryagin-Thom construction establishes an isomorphism between Ω_n^{fr} and the stable homotopy group $\pi_n^s(S^0)$, and in 1969 Browder [6] gave a homotopy theoretic interpretation of the Kervaire invariant (Theorem 1.1 below) which implied that $\kappa = 0$ unless $4k+2$ is of the form $2(2^j - 1)$. Homotopy theoretic calculations [27, 5, 3] soon muddied the waters by providing examples in dimensions 30 and 62. Much effort in the 1970's went into understanding the role of these classes and attempting inductive constructions of them in all dimensions.

The focus of this report is the following result.

THEOREM 0.1 (Hill, Hopkins, Ravenel, 2009, [15]). — *The Kervaire invariant $\kappa : \Omega_{4k+2}^{\mathrm{fr}} \to \mathbb{Z}/2$ is trivial unless $4k + 2 = 2, 6, 14, 30, 62$, or (possibly) 126.*

The case $4k + 2 = 126$ remains open.

In his proof that κ is trivial in dimension 10, Kervaire availed himself of the current state of the art in homotopy theory (mainly work of Serre). Kervaire and Milnor relied on contemporaneous work of Adams. Over the intervening fifty years, further developments in homotopy theory have been brought to bear on the Kervaire invariant problem. In 1964, for example, Brown and Peterson brought spin bordism into play to show that κ is trivial in dimensions $8k + 2$, $k > 0$, and Browder's work used the Adams spectral sequence. Over the past quarter century, however, essentially no further progress has been made on this problem till the present work.

Hill, Hopkins, and Ravenel (hereafter HHR) marshall three major developments in stable homotopy theory in their attack on the Kervaire invariant problem:

- The chromatic perspective based on work of Novikov and Quillen and pioneered by Landweber, Morava, Miller, Ravenel, Wilson, and many more recent workers;
- The theory of structured ring spectra, implemented by May and many others; and
- Equivariant stable homotopy theory, as developed by May and collaborators.

The specific application of equivariant stable homotopy theory was inspired by analogy with a fourth development, the motivic theory initiated by Voevodsky and Morel, and uses as a starting point the theory of "Real bordism" investigated by Landweber, Araki, Hu and Kriz. In their application of these ideas, HHR require significant extensions of the existing state of knowledge of this subject, and their paper provides an excellent account of the relevant parts of equivariant stable homotopy theory.

1. THE KERVAIRE INVARIANT

1.1. Geometry

Any compact smooth n-manifold M embeds into a Euclidean space, and in high codimension any two embeddings are isotopic. The normal bundle ν is thus well defined up to addition of a trivial bundle. A *framing* of M is a bundle isomorphism $t : \nu \to M \times \mathbb{R}^q$. The Pontryagin-Thom construction is the induced contravariant map on one-point compactifications, $S^{n+q} = \mathbb{R}^{n+q}_+ \to (M \times \mathbb{R}^q)_+ = M_+ \wedge S^q$, giving an element of $\pi_{n+q}(M_+ \wedge S^q)$.

This group becomes independent of q for $q > n$, and is termed the nth *stable homotopy group* $\pi_n^s(M_+) = \lim_{q \to \infty} \pi_{n+q}(M_+ \wedge S^q)$ of M. The Pontryagin-Thom construction provides a "stable homotopy theory fundamental class" $[M, t] \in \pi_n^s(M_+)$. Composing with the map collapsing M to a point gives an element of $\pi_n^s(S^0)$. A bordism between framed manifolds determines a homotopy between their Pontryagin-Thom collapse maps, and this construction gives an isomorphism from the framed bordism group to $\pi_n^s(S^0)$.

A framing of a manifold M of dimension $4k + 2$ determines additional structure in the cohomology of M. A cohomology class $x \in H^{2k+1}(M; \mathbb{F}_2)$ is represented by a well-defined homotopy class of maps $M_+ \to K(\mathbb{F}_2, 2k + 1)$. When we apply the stable homotopy functor we get a map $\pi_*^s(M_+) \to \pi_*^s(K(\mathbb{F}_2, 2k + 1))$. A calculation [7] shows that $\pi_{4k+2}^s(K(\mathbb{F}_2, 2k + 1)) = \mathbb{Z}/2$, so the class $[M, t]$ determines an element $q_t(x) \in \mathbb{Z}/2$.

This *Kervaire form* $q_t : H^{2k+1}(M; \mathbb{F}_2) \to \mathbb{Z}/2$ turns out to be a *quadratic refinement* of the intersection pairing $x \cdot y = \langle x \cup y, [M] \rangle$—that is to say,

$$q_t(x + y) = q_t(x) + q_t(y) + x \cdot y \,.$$

In the group (under direct sum) of isomorphism classes of finite dimensional $\mathbb{F}_2$-vector spaces with nondegenerate quadratic form, the Kervaire forms of cobordant framed manifolds are congruent modulo the subgroup generated by the *hyperbolic* quadratic space (H, q) with $H = \langle a, b \rangle$, $q(a) = q(b) = q(0) = 0$, $q(a+b) = 1$. This quotient is the *quadratic Witt group* of $\mathbb{F}_2$, and is of order 2. The element of $\mathbb{Z}/2$ corresponding to a quadratic space is given by the *Arf invariant*, namely the more popular of $\{0, 1\}$ as a value of q. The *Kervaire invariant* of (M, t) is then the Arf invariant of the quadratic space $(H^{2k+1}(M; \mathbb{F}_2), q_t)$. This defines a homomorphism

$$\kappa : \pi_{4k+2}^s(S^0) = \Omega_{4k+2}^{\mathrm{fr}} \to \mathbb{Z}/2 \,.$$

1.2. Homotopy theory

Regarding κ as defined on $\pi_{4k+2}^s(S^0)$ invites the question: What is a homotopy-theoretic interpretation of the Kervaire invariant? This was answered by Browder in a landmark paper [6], in terms of the Adams spectral sequence.

Discussion of these matters is streamlined by use of the *stable homotopy category* $h\mathcal{S}$, first described in Boardman's thesis (1964). Its objects are called *spectra* and are designed to represent cohomology theories. There are many choices of underlying categories of spectra, but they all lead the same homotopy category $h\mathcal{S}$, which is additive, indeed triangulated, and symmetric monoidal (with tensor product given by the "smash product" $\wedge$). It is an analog of the derived category of a commutative ring. There is a "stabilization" functor Σ^∞ from the homotopy category of pointed CW complexes to $h\mathcal{S}$. It sends the two point space S^0 to the "sphere spectrum" $\Sigma^\infty S^0 = \mathbb{S}$ which serves as the unit for the smash product. The suspension functor Σ is given by smashing with $\Sigma^\infty S^1$. The *homotopy* of a spectrum E is $\pi_n(E) = [\Sigma^n\mathbb{S}, E]$, so that, for a space X, $\pi_n^s(X) = \pi_n(\Sigma^\infty X_+)$. Ordinary mod 2 cohomology of a space X (which we abbreviate to $H^*(X)$) is represented by the *Eilenberg-Mac Lane spectrum* $\mathbb{H}$—$H^n(X) = [\Sigma^\infty X_+, \Sigma^n\mathbb{H}]$—and, as explained by G. Whitehead, homology is obtained as $H_n(X) = [\Sigma^n\mathbb{S}, \Sigma^\infty X_+ \wedge \mathbb{H}]$. The cup product is represented by a structure map $\mathbb{H} \wedge \mathbb{H} \to \mathbb{H}$, making $\mathbb{H}$ into a "ring-spectrum." The unit map for this ring structure, $\mathbb{S} \to \mathbb{H}$, represents a generator of $\pi_0(\mathbb{H}) = H^0(\mathbb{S})$. The graded endomorphism algebra of the object $\mathbb{H}$ is the well-known *Steenrod algebra* $\mathcal{A}$ of stable operations on mod 2 cohomology.

Evaluation of homology gives a natural transformation (generalizing the *degree*)

$$d_X : \pi_t(X) = [\Sigma^t\mathbb{S}, X] \to \mathrm{Hom}_{\mathcal{A}}(H^*(X), H^*(\Sigma^t\mathbb{S}))$$

which is an isomorphism if $X = \mathbb{H}$. This leads to the *Adams spectral sequence*

$$E_2^{s,t}(X; \mathbb{H}) = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(X), \mathbb{F}_2) = \mathrm{Ext}_{\mathcal{A}}^s(H^*(X), H^*(\Sigma^t\mathbb{S})) \Longrightarrow \pi_{t-s}(X)_2^\wedge$$

for X a spectrum such that $\pi_n(X) = 0$ for $n \ll 0$ and $H_n(X)$ is finite dimensional for all n. It converges to the 2-adic completion of the homotopy groups of X. In particular,

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_{t-s}(\mathbb{S})_2^\wedge .$$

To this day these Ext groups remain quite mysterious overall, but in [1] Adams computed them for $s \leq 2$. $E_2^{0,t}$ is of course just $\mathbb{F}_2$ in $t = 0$. The edge homomorphism

$$e : \pi_{t-1}(\mathbb{S}) \to E_2^{1,t}$$

is the "mod 2 Hopf invariant." One interpretation of this invariant is that $e(\alpha) \neq 0$ if and only if the mapping cone $S^0 \cup_\alpha e^t$ supports a nonzero Steenrod operation of positive degree. $E_2^{1,*}$ is the dual of the module of indecomposables in the Steenrod

algebra, known since Adem to be given by the Steenrod squares $Sq^1, Sq^2, Sq^4, Sq^8, \ldots$
The element dual to Sq^{2^j} is denoted by $h_j \in E_2^{1,2^j}$. There is an element of Hopf
invariant one in $\pi_{t-1}(\mathbb{S})$ if and only if there is a nonzero permanent cycle in $E_2^{1,t}$.
Adams [1] proved that elements of Hopf invariant one occur only in dimensions 0, 1,
3, and 7. This occurs because $d_2 h_j = h_0 h_{j-1}^2$ for $j > 3$ in the Adams spectral sequence
[36], a class which Adams had shown to be nontrivial.

Next, there is an edge homomorphism

$$ f : \pi_{t-2}(\mathbb{S}) \to E_2^{2,t}, \quad t - 2 \neq 2^j - 1. $$

Adams found that a basis for $E_2^{2,*}$ is given by $\{h_i^2 : i \geq 0\} \cup \{h_i h_j : j > i+1 \geq 1\}$. The
multiplicative structure of the spectral sequence together with computations [27, 25]
in $E_2^{4,*}$ imply that the only survivors to $E_3^{2,*}$ are $h_0 h_2$, $h_0 h_3$, $h_2 h_4$, $h_2 h_5$, $h_3 h_6$, and
the infinite families h_j^2 and $h_1 h_j$. This is the context of the following key result.

THEOREM 1.1 (Browder, 1969, [6]). — *The Kervaire invariant vanishes except in di-*
mensions of the form $2(2^j - 1)$, where it is detected by h_j^2.

There have been subsequent simplifications of Browder's argument. Notably,
Lannes [21] uses manifolds with boundary to express the Kervaire functional as a
certain Hopf invariant. Lannes has subsequently further simplified this approach, and
in joint work with the author he has given a proof employing characteristic numbers
of manifolds with corners. Later work [18] also revealed that it is not difficult to see,
without invoking the Adams spectral sequence, that these are the only dimensions in
which the Kervaire invariant can be nontrivial.

1.3. Implications in homotopy theory

Thus the Kervaire invariant problem is equivalent to the question of whether h_j^2 is
a permanent cycle in the Adams spectral sequence. Since h_j is permanent for $j \leq 3$,
the product structure implies that h_j^2 is too—and the squares of the framed manifolds
representing the Hopf invariant one classes are manifolds of Kervaire invariant one.
Computations [27, 5, 3] showed that h_4^2 and h_5^2 are also permanent. Much effort was
spent searching for an inductive construction: If h_j^2 survives to an element of order 2
and square zero, then h_{j+1}^2 survives [3]. Parallels were developed: Cohen, Jones, and
Mahowald [9] defined a Kervaire invariant for oriented $(4k + 2)$-manifolds immersed
in $\mathbb{R}^{4k+4}$, and showed that it does take on the value 1 in dimensions $4k+2 = 2(2^j - 1)$.

The theorem of HHR amounts to the assertion that h_j^2 supports a differential in the
Adams spectral sequence for all $j > 6$. One reason homotopy theorists were hoping
that the Kervaire classes did survive was that they had no idea what the targets of
differentials on them might be. The work of HHR sheds no light on this question—
though proving that these classes die without saying how is the genius of the paper.

Incidentally, it is known that $h_0 h_2$, $h_0 h_3$, and $h_2 h_4$ survive to homotopy classes and that $h_2 h_5$ does not. The fates of $h_3 h_6$ and of h_6^2 are at present unknown. All members of the remaining infinite family in $E_2^{2,*}$, the $h_1 h_j$, do survive, to elements denoted η_j: this is a famous theorem of Mahowald [**24**].

By the Hopf invariant one theorem, the operation $Sq^{2^{j+1}}$ must be trivial on the mapping cone of any map $S^{2^{j+1}-1} \to S^0$ if $j \geq 3$. What if we consider the mapping cone of a map $S^{2^{j+1}-1} \to S^0 \cup_2 e^1$ instead? The form of the Adams differential $d_2 h_{j+1} = h_0 h_j^2$ implies that $\mathrm{Sq}^{2^{j+1}}$ is nonzero in the mapping cone of such a map if and only if the composite $S^{2^{j+1}-1} \to S^0 \cup_2 e^1 \to S^1$ has Kervaire invariant one. The result of HHR raises the question of the minimal length [**8**] of a spectrum in which $\mathrm{Sq}^{2^{j+1}}$ is nonzero.

The Kervaire invariant question is also important unstably. The *Whitehead square* $w_n \in \pi_{2n-1}(S^n)$ is the composite $S^{2n-1} \to S^n \vee S^n \to S^n$ of the pinch map with the attaching map of the top cell in the torus $S^n \times S^n$. It is null when S^n is an H-space, i.e. when $n = 1, 3$, or 7. For n odd and not $1, 3$ or 7, it is of order 2 and generates the kernel of the suspension map $\pi_{2n-1}(S^n) \to \pi_{2n}(S^{n+1})$. Mahowald [**23**] proved that if $2\theta = w_n$ in $\pi_{2n-1}(S^n)$ then θ suspends to an element of $\pi_{2n}(S^{n+1}) \cong \pi_{n-1}(\mathbb{S})$ of order two and Kervaire invariant one. So Browder's theorem implies that, for n odd and not $1, 3$, or 7, w_n is divisible by 2 only if $n+1$ is a power of 2; computations show that it is divisible if $n = 15$ or 31; and HHR prove that it is not divisible if $n > 63$.

See [**23, 4**] for more about the roles elements of Kervaire invariant one were supposed to play in unstable homotopy theory.

2. OUTLINE

In this section we state a theorem that provides the skeleton of the HHR proof.

This begins with the Adams-Novikov spectral sequence, an analogue of the Adams spectral sequence in which ordinary mod 2 homology is replaced by complex bordism, $E_2^{*,*}(X; \mathrm{MU}) \Rightarrow \pi_*(X)$. The universal Thom class is represented by a map of spectra $\mu : \mathrm{MU} \to \mathbb{H}$ which induces a map of spectral sequences $\mu_* : E_r^{*,*}(X; \mathrm{MU}) \to E_r^{*,*}(X; \mathbb{H})$.

The central innovation in the proof is the use of G-equivariant stable homotopy theory for a finite group G (e.g. the cyclic group C_n of order n), as described in §§6–10 below. A G-spectrum X has equivariant homotopy groups $\pi_*^G(X)$, but also defines a "homotopy fixed point spectrum" X^{hG} with its own homotopy groups, and there is a natural map $\pi_*^G(X) \to \pi_*(X^{hG})$.

THEOREM 2.1 (HHR). — *There exists a C_8-spectrum $\mathbb{L}$ with the following properties.*
(a) (Detection property) $\pi_n(\mathbb{L}) = 0$ *for n odd, and there are maps of spectral sequences*

$$E_2^{*,*}(\mathbb{S};\mathbb{H}) \xleftarrow{\mu_*} E_2^{*,*}(\mathbb{S};\mathrm{MU}) \xrightarrow{i_*} H^*(C_8;\pi_*(\mathbb{L}))$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$\pi_*(\mathbb{S}) \xleftarrow{=} \pi_*(\mathbb{S}) \longrightarrow \pi_*(\mathbb{L}^{hC_8})$$

such that $\ker(i_*) \subseteq \ker(\mu_*)$ *in* $E_2^{2,2^j}$ *for $j > 6$;*
(b) (Gap property) $\pi_{-2}^{C_8}(\mathbb{L}) = 0$;
(c) (Fixed point property) $\pi_*^{C_8}(\mathbb{L}) \to \pi_*(\mathbb{L}^{hC_8})$ *is an isomorphism; and*
(d) (Periodicity property) $\pi_*(\mathbb{L}^{hC_8}) \cong \pi_{*+256}(\mathbb{L}^{hC_8})$.

Proof of Theorem 0.1. The facts that $\pi_{2^j-2}(\mathbb{L}^{hC_8}) = 0$ (by (b), (c), and (d)) and $H^0(C_8;\pi_{2^j-1}(\mathbb{L})) = 0$ (by (a)) imply that any permanent cycle in $E_2^{2,2^j}(\mathbb{S};\mathrm{MU})$ maps to zero in $H^2(C_8;\pi_{2^j}(\mathbb{L}))$, and hence (by (a)) reduces to zero in $E_2^{2,2^j}(\mathbb{S};\mathbb{H})$. But since $E_2^{1,2^j-1}(\mathbb{S};\mathrm{MU}) = 0$ and $E_2^{0,2^j-2}(\mathbb{S};\mathrm{MU}) = 0$, any class in $\pi_{2^j-2}(\mathbb{S})$ represented by an element of $E_2^{2,2^j}(\mathbb{S};\mathbb{H})$ must be the image of a permanent cycle in $E_2^{2,2^j}(\mathbb{S};\mathrm{MU})$. □

The line of attack taken by HHR is suggested by an approach used in 1978 by Ravenel [**33**] to prove an analogous result for primes $p > 3$. We take this up in §3, as it introduces the "chromatic" homotopy theory underlying the current work. In §4 the modifications required of Ravenel's method are sketched. This leads to the hope that one can take for $\mathbb{L}$ the four-fold smash product $\mathrm{MU}^{(4)}$ with a C_8 action induced from the C_2 action by complex conjugation on MU. Verifying that one can define this group action requires some technology from the modern theory of structured ring spectra and abstract homotopy theory, §5, and equivariant homotopy theory, §6.

This candidate needs to be modified if the other properties are to hold, however. The modification is to localize $\mathrm{MU}^{(4)}$ by inverting a suitable equivariant homotopy class D. Each property puts constraints on D; they are gathered in §11. The most severe constraint is imposed by the requirements of the Detection Property.

Remark. The November 2010 draft of HHR [**15**] uses $\Omega_{\mathbb{O}}$ for $\mathbb{L}$ and Ω for $\mathbb{L}^{hC_8}$. The notation used in this report more closely reflects the parallel between $\mathbb{L}$ with its C_8-action and the complex K-theory spectrum $\mathbb{K}$ with its action of C_2 by complex conjugation. The fixed point spectrum and homotopy fixed point spectrum of that action also coincide, and are given by the orthogonal K-theory spectrum $\mathbb{KO}$. So one might write $\mathbb{L}^{hC_8} = \mathbb{LO}$. In any case, the spectrum $\mathbb{L}$, or $\Omega_{\mathbb{O}}$, may in time be replaced by a more tailored construction.

3. CHROMATIC HOMOTOPY THEORY

3.1. Complex cobordism and formal groups

Much work in Algebraic Topology in the 1960s and early 1970s centered on developing the machinery of generalized homology theories, satisfying the Eilenberg-Steenrod axioms with the exception of the axiom specifying the homology of a point. These were exemplified by K-theory (created by Atiyah and Hirzebruch as an interpretation of Bott periodicity) and cobordism theory (Atiyah's interpretation of the Pontryagin-Thom theory of cobordism). Novikov [30] and Quillen [32] stressed the centrality and convenience of complex cobordism, whose coefficient ring had been shown by Milnor and Novikov to be a polynomial algebra: $MU^* = MU^*(*) = \mathbb{Z}[x_1, x_2, \dots]$, $|x_i| = -2i$. They called attention to the class of "complex oriented" multiplicative cohomology theories $E^*(-)$: those such that the cohomology of $\mathbb{C}P^\infty$ is a power series ring over the coefficients on a single 2-dimensional generator—an Euler class for complex line bundles. Many of the standard calculations of ordinary cohomology carry over to this much wider setting. See [2] and [34] for this material.

A signal complication is that the Euler class of a tensor product is no longer the sum of the Euler classes of the factors, but is rather given by a power series

$$e(\lambda_1 \otimes \lambda_2) = F(e(\lambda_1), e(\lambda_2)).$$

Standard properties of the tensor product imply that this power series obeys identities making it a *formal group*:

$$F(x,y) = x + y + \cdots \quad , \quad F(x, F(y,z)) = F(F(x,y), z) \quad , \quad F(x,y) = F(y,x).$$

More precisely it is a graded formal group (always of degree 2) over the graded ring $E^* = E^*(*) = E_{-*}(*) = E_{-*} = \pi_{-*}(\mathbb{E})$. The formal group associated with complex K-theory, for example, is the *multiplicative* formal group $G_m(x,y) = x + y - uxy$ where $u \in K^{-2}(*)$ is the Bott class, a unit by Bott periodicity. The spectrum $\mathbb{M}U$ representing complex cobordism is initial among complex oriented ring spectra, and Quillen observed that its formal group enjoys the corresponding universal property (Lazard): there is a bijection from graded ring-homomorphisms $MU_* \to R_*$ to the set of graded formal groups over R_* given by applying the homomorphism to the coefficients in the formal group defined by $\mathbb{M}U$.

The integral homology $H_*(\mathbb{M}U; \mathbb{Z})$ also admits a modular interpretation: graded ring homomorphisms from it are in bijection with formal groups $F(x,y)$ equipped with a strict isomorphism $\log(t)$ (a "logarithm") to the additive group $G_a(x,y) = x + y$: $\log(x) \equiv x \bmod \deg 2$ and $\log G(x,y) = \log(x) + \log(y)$. Thus $H_*(\mathbb{M}U; \mathbb{Z}) = \mathbb{Z}[m_1, m_2, \dots]$ where $\log(t) = \sum_i m_{i-1} t^i$, and the Hurewicz map $\pi_*(\mathbb{M}U) \to H_*(\mathbb{M}U; \mathbb{Z})$ classifies the formal group $\log^{-1}(\log(x) + \log(y))$.

Many things work just as well for $\mathbb{MU}$ as for $\mathbb{H}$. For instance, we have the Adams-Novikov spectral sequence,

$$E_2^{s,t}(X;\mathbb{MU}) \implies \pi_{t-s}(X).$$

The E_2 term is computable by homological algebra over the Hopf algebra S_* that co-represents the functor sending a commutative ring to the group (under composition) of formal powers series $f(t) = t + b_1 t^2 + b_2 t^3 + \cdots$. As an algebra, $S_* = \mathbb{Z}[b_1, b_2, \dots]$. Landweber and Novikov observed that $MU_*(X)$ is naturally a comodule for this graded Hopf algebra (in which we declare $|b_n| = 2n$). For example, the coaction $\psi : MU_* \to S_* \otimes MU_*$ (with $X = *$) is the map co-representing the action of power series on the set of formal groups by conjugation, ${}^f F(x,y) = f(F(f^{-1}(x), f^{-1}(y)))$. Then

$$E_2^{s,t}(X;\mathbb{MU}) = \operatorname{Ext}_{S_*}^{s,t}(\mathbb{Z}, MU_*(X)).$$

The universal mod 2 Thom class for complex vector bundles is represented by a map $\mu : \mathbb{MU} \to \mathbb{H}$, and this map induces a map of spectral sequences

$$\mu_* : E_r^{s,t}(X;\mathbb{MU}) \to E_r^{s,t}(X;\mathbb{H}).$$

The mod 2 Hopf invariant factors as

$$\pi_{t-1}(\mathbb{S}) \to E_2^{1,t}(\mathbb{S};\mathbb{MU}) \xrightarrow{\mu_*} E_2^{1,t}(\mathbb{S},\mathbb{H}).$$

Novikov [30] observed (see also [29]) that for $j > 3$ the class h_j is not in the image of μ. This solves the Hopf invariant one problem, but does not tell us the differential in the Adams spectral sequence that is nonzero on h_j (though Novikov [30] (see also [28]) went on to use complex bordism to recover the Adams differential as well).

3.2. Odd primary "Kervaire" classes

One may hope to address the Kervaire invariant one problem by an analogous strategy. Unfortunately, h_j^2 is always in the image of μ_*. In 1978 Ravenel [33] turned this defect to an advantage to prove that a certain analogue of h_j^2 in the classical mod p Adams spectral sequence (for p a prime larger than 3) does not survive. The HHR gambit is a variation on Ravenel's, which we therefore describe.

The elements in question are denoted b_j and they lie in $\operatorname{Ext}_{\mathcal{A}}^{2,2(p-1)p^{j+1}}(\mathbb{F}_p, \mathbb{F}_p)$. When $p = 2$, $b_j = h_{j+1}^2$. The first one, b_0, survives to an element $\beta_1 \in \pi_{2(p-1)p-2}(\mathbb{S})$. Toda proved in 1967 that for p odd, $d_{2p-1} b_1 = u h_0 b_0^p \neq 0$ in the Adams spectral sequence, where $u \in \mathbb{F}_p^\times$ and $h_0 \in \operatorname{Ext}_{\mathcal{A}}^{1,2p-2}(\mathbb{F}_p, \mathbb{F}_p)$ represents $\alpha_1 \in \pi_{2p-3}(\mathbb{S})$, the first element of order p in $\pi_*(\mathbb{S})$.

THEOREM 3.1 (Ravenel, 1978, [33]). — *For $p \geq 5$ and $j \geq 1$, the element b_j dies in the Adams spectral sequence.*

The elements h_0 and b_j are images of similarly defined elements in $E_2^{2,*}(\mathbb{S}; \mathrm{MU})$, and Toda's differential forces an analogous differential there (destroying Novikov's conjecture [30] that for odd primes it collapsed at E_2). Ravenel used Toda's calculation to ground an induction proving that in the Adams-Novikov spectral sequence

$$d_{2p-1}b_{j+1} \equiv uh_0b_j^p \pmod{\mathrm{Ann}\, b_0^{a_j}} \quad , \quad a_j = p\frac{p^j-1}{p-1} \quad , \quad u \in \mathbb{Z}_{(p)}^{\times}.$$

Because of the spacing of nonzero groups at $E_2^{*,*}$, this is the first possibly nontrivial differential, so b_{j+1} will die as long as $h_0b_j^p \neq 0$ in $E_2^{2p+1,*}$. This is well beyond the computable range in general. However, Ravenel proved the following theorem, guaranteeing that the class b_{j+1} dies in the Adams-Novikov spectral sequence:

PROPOSITION 3.2 (Ravenel, 1978, [33]). — *For any sequence $i_0, i_1, \ldots, i_k \geq 0$, the element $h_0 b_0^{i_0} b_1^{i_1} \cdots b_k^{i_k} \in E_2^{*,*}(\mathbb{S}; \mathrm{MU})$ is nonzero.*

Corresponding elements occur in $E_2^{*,*}(\mathbb{S}; \mathbb{H})$, but no such theorem holds: this is one of the advantages of MU relative to $\mathbb{H}$.

These classes are detected by a map $(f, \varphi) : (MU_*, S_*) \to (R_*, B_*)$ of rings acted on by Hopf algebras. The Hopf algebra B_* arises from a finite group G as the ring R_*^G of functions from G to R_* with Hopf algebra structure given by $\Delta(f)(g_1, g_2) = f(g_1 g_2)$, under the identification $R_*^G \otimes_{R_*} R_*^G \xrightarrow{\cong} R_*^{G \times G}$ sending $f_1 \otimes f_2$ to $(g_1, g_2) \mapsto f_1(g_1) f_2(g_2)$. A graded ring homomorphism $f : MU_* \to R_*$ classifies a graded formal group F over R_*. The map φ arises from an action of G on F as a group of strict automorphisms: formal power series $g(t) \equiv t \bmod \deg 2$ that conjugate F to itself. The map $\varphi : S_* \to R_*^G$ is determined by its composites with the evaluation maps $\mathrm{ev}_g : R_*^G \to R_*$, and $\mathrm{ev}_g \circ \varphi$ is required to classify the formal power series $g(t)$. With these definitions,

$$\mathrm{Ext}_{R_*^G}^{s,t}(R_*, R_*) \cong H^s(G; R_t).$$

3.3. Automorphisms of formal groups

One must now find a graded formal group admitting a finite group of automorphisms with rich enough cohomology. It suffices to study graded formal groups F over graded rings of the form $R_* = A[u^{\pm 1}]$, $|u| = -2$. Such a formal group must be of the form $F(x, y) = u^{-1}F_0(ux, uy)$ for a formal group F_0 over A.

Formal groups over fields are well understood. Over an algebraically closed field of characteristic p, the "height" of a formal group determines it up to isomorphism. The height is defined in terms of the self-map given by $[p](x)$, where $[k](x)$ is defined inductively by $[0](x) = 0$ and $[k](x) = F([k-1](x), x)$. A key elementary fact is that either $[p](t) = 0$ (as is the case for the additive group $G_a(x, y) = x + y$), or $[p](t) = f(t^{p^n})$ with $f'(0) \neq 0$. The *height* is the integer n (declared to be infinite if

$[p](t) = 0$). There is a formal group of any height n defined over $\mathbb{F}_p$ with the property that all endomorphisms over $\overline{\mathbb{F}}_p$ are already defined over $\mathbb{F}_{p^n}$. This endomorphism ring is well-studied (see e.g. [34]), and known to contain elements of multiplicative order p exactly when $(p-1)|n$.

Ravenel took the first possible case: F_0 is a formal group of height $n = p-1$ over $A = \mathbb{F}_q$, $q = p^n$, such that $\text{Aut}_{\mathbb{F}_q}(F_0)$ contains an element of order p. Let C_p be the subgroup generated by any such element. Then the map

$$\varphi_* : \text{Ext}_{S_*}^{*,*}(\mathbb{Z}, MU_*) \to H^*(C_p; R_*) = E[h] \otimes R_*[b] \quad , \quad |h| = 1, |b| = 2$$

sends $h_0 \mapsto u^{-n}h$ and $b_j \mapsto u^{-np^{j+1}}b$ for all j, proving Proposition 3.2.

We have not yet proved Theorem 3.1, however, since, for large j, $b_{j+1} \in E_2^{2,*}(\mathbb{S}; MU)$ is not the only class mapping to $b_{j+1} \in E_2^{2,*}(\mathbb{S}; \mathbb{H})$. The claim is that d_{2p-1} takes on the same value on all of them, modulo $\ker \varphi_*$; that is, $d_{2p-1}(\ker \mu_*) \subseteq \ker \varphi_*$. Consequently no class in $E_2^{2,*}(\mathbb{S}; MU)$ mapping to $b_{j+1} \in E_2^{2,*}(\mathbb{S}; \mathbb{H})$ survives, and so b_{j+1} does not survive either.

To see that $d_{2p-1}(\ker \mu_*) \subseteq \ker \varphi_*$, Ravenel invoked L. Smith's construction of a certain 8-cell complex $\mathbb{V}(2)$, with bottom cell $i : \mathbb{S} \to \mathbb{V}(2)$, such that the map $\varphi : MU_* \to R_*$ factors as $MU_*(\mathbb{S}) \to MU_*(\mathbb{V}(2)) \to R_*$. Thus $\ker i_* \subseteq \ker \varphi_*$. He checked that $\ker \mu_* \subseteq \ker i_*$. So if $a \in \ker \mu_*$ then $i_* d_{2p-1}a = d_{2p-1}i_*a = 0$, so $d_{2p-1}a \in \ker \varphi_*$.

Smith's complex $\mathbb{V}(2)$ exists only for $p > 3$, and this is where that assumption is required. In fact, at $p = 3$, b_1 dies but b_2 survives in the Adams spectral sequence, and the fate of the others is at present unknown. It is hoped that a modification of the HHR technique will resolve the case $p = 3$ as well.

The chromatic approach to homotopy theory rests on the following insight, which we owe to Morava: The automorphism groups of formal groups over fields control the structure of the Adams-Novikov E_2-term. Ravenel's result was the first one to depend upon this insight in an essential way.

4. THE DETECTION THEOREM

In the rest of this section, we will sketch a line of thought that leads from the ideas of §3 to a candidate for the spectrum $\mathbb{L}$ of Theorem 2.1.

4.1. Hopf algebroids and actions on rings

The starting point is finding a finite group G acting (this time nontrivially) on a graded ring R_* and a map $\bar{i}_* : E_2^{*,*}(\mathbb{S}, MU) \to H^*(G; R_*)$ such that $\ker \bar{i}_* \subseteq \ker \mu_*$ for $(*,*) = (2, 2^j)$ for large j. The construction of $\bar{i}_*$ requires a slight extension of

the notion of a Hopf algebra. A *Hopf algebroid* over a commutative ring K is a co-groupoid object in the category of commutative (perhaps graded) algebras over K. A Hopf algebroid co-represents a functor from commutative K-algebras to groupoids. For example, the functor sending a commutative graded ring R_* to the groupoid of formal groups and strict isomorphisms over R_* is corepresentable by the Hopf algebroid $(MU_*, MU_* \otimes S_*)$, with structure morphisms arising from the Hopf algebra structure of S_* and its coaction on MU_*. Comodules for Hopf algebroids and the corresponding Ext groups are defined in the evident way. A comodule for $MU_* \otimes S_*$ is the same thing as an MU_*-module over S_* and $\mathrm{Ext}^{*,*}_{MU_* \otimes S_*}(MU_*, M) = \mathrm{Ext}^{*,*}_{S_*}(\mathbb{Z}, M)$.

An action of a finite group on a graded ring determines a Hopf algebroid with object ring R_*, morphism ring R^G_*, and structure maps given as follows. The maps $\eta_L, \eta_R : R_* \to R^G_*$ are characterized by $\mathrm{ev}_g \circ \eta_L = \mathrm{id}$, $\mathrm{ev}_g \circ \eta_R = g$, $\epsilon = \mathrm{ev}_1 : R^G_* \to R_*$, and the diagonal map is defined by $\Delta(f)(g_1, g_2) = f(g_1 g_2)$ under the identification $\alpha : R^G_* \otimes_{R_*} R^G_* \to R^{G \times G}_*$ such that

$$\mathrm{ev}_{g_1, g_2} \alpha(f_1 \otimes f_2) = f_1(g_1) \cdot g_1 f(g_2).$$

A subtlety here is that (as usual in the Hopf algebroid setup) R_* must act via η_R on the left factor and via η_L on the right factor in the tensor product. Again, $\mathrm{Ext}_{R^G_*}(R_*, R_*) = H^*(G; R_*)$.

A map $(MU_*, MU_* \otimes S_*) \to (R_*, R^G_*)$ of Hopf algebroids classifies a graded formal group F over R_* along with a strict isomorphism $\theta_g : F \to gF$, for every $g \in G$, such that

(1) $\theta_1 = \mathrm{id} \quad \text{and} \quad \theta_{g_1 g_2} = g_1 \theta_{g_2} \circ \theta_{g_1}.$

Construction of the relevant action begins with the involution on the ring spectrum $\mathbb{MU}$ arising from complex conjugation. In homology the action is given by $\overline{m}_j = (-1)^j m_j$; so as graded commutative algebras with C_2 action

(2) $$H_*(\mathbb{MU}; \mathbb{Z}) = \mathrm{Sym}\left(\bigoplus_{j>0} \sigma^{\otimes j} \right)$$

where σ is the sign representation on $\mathbb{Z}$ and Sym denotes the symmetric algebra functor. In homotopy this involution classifies the formal group

$$\overline{F}(x,y) = -F(-x, -y).$$

Note that $-[-1]_F(t)$ is a strict isomorphism $F \to \overline{F}$. If R_* is a graded ring with involution $x \mapsto \gamma x$, then the map $\pi_*(\mathbb{MU}) \to R_*$ classifying a formal group F over R_* is equivariant if and only if $\overline{F} = \gamma F$.

4.2. The norm

The action of C_2 on $\mathbb{MU}$ may be promoted to an action of a finite group containing C_2 via the *norm* construction. This construction has its origins in group cohomology [10] and is applicable in any symmetric monoidal category.

For any category $\mathcal{M}$, write $\mathcal{M}^G$ for the category of G-objects in $\mathcal{M}$, and equivariant maps. Let H be a subgroup of G of finite index. If $\mathcal{M}$ has finite coproducts, the restriction map $\mathrm{Res}_H^G : \mathcal{M}^G \to \mathcal{M}^H$ has a left adjoint Ind_H^G. The object underlying $\mathrm{Ind}_H^G X$ is the coproduct of $[G : H]$ copies of the object underlying X.

For example, let $\mathcal{M}$ be the category of commutative $\otimes$-monoids in a symmetric monoidal category $\mathcal{C}$. The category $\mathcal{M}$ has finite coproducts, given by the tensor product, so $\mathrm{Res}_H^G : \mathcal{M}^G \to \mathcal{M}^H$ has a left adjoint. A key observation of HHR is that this "multiplicative induction" descends to a functor $N_H^G : \mathcal{C}^H \to \mathcal{C}^G$, the *norm* functor.

The norm functor is transitive and multiplicative:

$$N_K^G N_H^K M \cong N_H^G M \quad , \quad N_H^G(M \otimes P) \cong N_H^G M \otimes N_H^G P \,.$$

It is also distributive over coproducts. When H is central in G this distributivity admits the following description. Let I be a set and suppose given $M_i \in \mathcal{C}$ for each $i \in I$. Pick a set F of orbit representatives for the right action of G on $\mathrm{map}(G/H, I)$. For $f \in F$ let $\mathrm{Stab}(f) = \{g \in G : fg = f\}$. Then

$$(3) \qquad N_H^G \left(\coprod_{i \in I} M_i \right) \cong \coprod_{f \in F} \mathrm{Ind}_{\mathrm{Stab}(f)}^G N_H^{\mathrm{Stab}(f)} \left(\bigotimes_{g \in G/\mathrm{Stab}(f)} M_{f(g)} \right) \,.$$

Composing the norm with the adjunction morphism $N_H^G \mathrm{Res}_H^G R \to R$ gives us a map

$$(4) \qquad \overline{N}_H^G : \mathcal{C}^H(X, \mathrm{Res}_H^G R) \to \mathcal{C}^G(N_H^G X, R)\,,$$

the *internal norm*, natural in the commutative G-monoid R and H-object X.

We can apply this construction to the stable homotopy category $h\mathscr{S}$. If H acts on a $\wedge$-monoid in $h\mathscr{S}$ such that $H_*(X; \mathbb{Z})$ is free then

$$(5) \qquad H_*(N_H^G X; \mathbb{Z}) \cong N_H^G H_*(X; \mathbb{Z})\,.$$

So we can form the C_{2n}-equivariant monoid in $h\mathscr{S}$

$$\mathbb{MU}^{(n)} = N_2^{2n} \mathbb{MU}\,,$$

where $2n$ is shorthand for the cyclic group C_{2n} of order $2n$. This is a fundamental object for HHR, where it is denoted $MU^{(C_{2n})}$. Neglecting the group action, $\mathbb{MU}^{(n)} = \mathbb{MU}^{\wedge n}$.

Its homology is easily described using (5), (2), and the fact that if V is a torsion-free $\mathbb{Z}H$-module then $N_H^G \mathrm{Sym} V \cong \mathrm{Sym}(\mathrm{Ind}_H^G V)$: As a C_{2n}-algebra,

$$(6) \qquad H_*(\mathbb{MU}^{(n)}; \mathbb{Z}) = \mathrm{Sym}\left(\bigoplus_{j>0} \mathrm{Ind}_2^{2n}\left(\sigma^{\otimes j}\right)\right).$$

The homotopy of $\mathbb{MU}^{(n)}$ is a graded ring with C_{2n} action, and an equivariant map from it to a graded C_{2n}-ring R_* classifies a graded formal group F over R_* together with a strict isomorphism $\theta : F \to \gamma F$ (where γ is a generator of C_{2n}) such that

$$(7) \qquad \gamma^n F = \overline{F} \quad \text{and} \quad \gamma^{n-1}\theta \circ \cdots \circ \gamma\theta \circ \theta = -[-1]_F .$$

Given such data, the formula $\theta_{\gamma^k} = \gamma^{k-1}\theta \circ \cdots \circ \gamma\theta \circ \theta$ satisfies (1) and so defines a map $(MU_*, MU_* \otimes S_*) \to (R_*, R_*^{C_{2n}})$. In the universal case, the ring is $MU_*^{(n)} = \pi_*(\mathbb{MU}^{(n)})$; we get maps of Hopf algebroids

$$\bar{i} : (MU_*, MU_* \otimes S_*) \to (MU_*^{(n)}, (MU_*^{(n)})^{C_{2n}}) \to (R_*, R_*^{C_{2n}}).$$

In cohomology we get

$$(8) \qquad \bar{i}_* : E_2^{*,*}(\mathbb{S}; \mathbb{MU}) \to H^*(C_{2n}; MU_*^{(n)}) \to H^*(C_{2n}; R_*).$$

4.3. Formal A-modules

Triples (R_*, F, θ) satisfying (7) arise from the theory of formal A-modules. Let A be a discrete valuation ring of characteristic zero, with quotient field K, uniformizer π, and residue field $A/(\pi) = \mathbb{F}_q$. Over K, any formal group has the form $F(x, y) = l^{-1}(l(x) + l(y))$ for a unique $l(t) \in tK[[t]]$ with $l'(0) = 1$ (the "logarithm" of F). The formal group is defined over A provided the Hazewinkel ([14], 8.3) functional equation

$$l(t) = t + \pi^{-1}l(t^q)$$

is satisfied. If this condition holds then for any $a \in A$ the power series $[a](t) = l^{-1}(al(t))$ has coefficients in A, and is an endomorphism of F. This construction defines a ring homomorphism $[-]_F : A \to \mathrm{End}_A(F)$ which splits the natural map $\mathrm{End}_A(F) \to A$ given by $f(t) \mapsto f'(0)$: F is a "formal A-module." For computations, HHR use the example ([14], 25.3.16)

$$l(t) = \sum_{i=0}^{\infty} \frac{t^{q^i}}{\pi^i}.$$

Let $2n$ be a power of 2 and write $A = \mathbb{Z}_2[\zeta]$, where ζ is a primitive $(2n)$th root of unity. The element $\pi = \zeta - 1$ serves as a uniformizer. The discrete valuation ring A is totally ramified over $\mathbb{Z}_2$, so $A/(\pi) = \mathbb{F}_2$ and $q = 2$. Let F_0 be a formal A-module,

$R_* = A[u^{\pm 1}]$ with $|u| = -2$, and $F(x,y) = u^{-1}F_0(ux,uy)$. Define an action of C_{2n} on R_* by letting a generator γ act trivially on A and by $\gamma u = \zeta u$. Define

$$\theta(t) = \zeta^{-1}u^{-1}[\zeta](ut).$$

Then $\gamma^n F = \overline{F}$, $\theta : F \to \gamma F$ is a strict isomorphism satisfying (7), and we get an equivariant ring homomorphism $MU_*^{(n)} \to R_*$.

HHR take $k = 3$ here, and check that for $j > 6$ the map $\bar{i}_*$ of (8) (with $n = 4$) satisfies the Detection Property $\ker \bar{i}_* \subseteq \ker \mu_*$ for $(*,*) = (2, 2^j)$. Smaller values of k won't do.

We will see in §6 that the C_{2n} action on the homotopy type $MU^{(n)}$ can be lifted to an action on the underlying ring-spectrum. As indicated in §9 one can then form the homotopy fixed point spectrum $(MU^{(n)})^{hC_{2n}}$, and the middle term in (8) is the E_2 term of the homotopy fixed point spectral sequence converging to $\pi_*((MU^{(n)})^{hC_{2n}})$. Unfortunately $\pi_{2^j-2}((MU^{(n)})^{hC_{2n}})$ fails to vanish (no matter what n is). To repair this, with $n = 4$, HHR define $\mathbb{L}$ to be a certain localization of $MU^{(4)}$ whose homotopy is given by $\omega^{-1}\pi_*(MU^{(4)})$ for an element $\omega \in \pi_{2k}(MU^{(4)})$ which maps to a unit in R_*. In §11 we will indicate what is required of the localization in order for such ω to exist.

The factorization (8) of $\bar{i}_*$ thus refines to a factorization

$$E_2^{*,*}(\mathbb{S}; MU) \xrightarrow{i_*} H^*(C_{2n}; \pi_*(\mathbb{L})) \to H^*(C_{2n}; R_*).$$

The fact that $\mathbb{L}$ is an MU-module-spectrum implies that i_* extends to a map of spectral sequences, and the proof of the Detection Property is complete.

5. MODEL CATEGORIES AND RING SPECTRA

Working in a homotopy category, such as the derived category of a ring or the stable homotopy category $h\mathscr{S}$, offers a valuable simplification, and the axiomatic framework of triangulated categories is very attractive. But it imposes severe restrictions on the types of construction one can make, and at least since Quillen's fundamental document [31] homotopy theory has been recognized as something more than the study of a homotopy category. Quillen wanted to say what a homotopy *theory* was, and did so by means of the theory of *model categories*. Just as a homotopy type may be represented by many non-homeomorphic spaces, so a homotopy theory may be modeled by many categorically non-equivalent (but "Quillen equivalent") model categories. A fundamental example is given by the categories of spaces and simplicial sets. Quillen equivalent model categories have equivalent homotopy categories.

There are by now many models of the homotopy theory of spectra—many Quillen equivalent model categories of "spectra," all having $h\mathscr{S}$ as homotopy category. Some

are even endowed with the structure of a symmetric monoidal category, in which the monoidal product descends to the smash product in $h\mathscr{S}$. This convenience allows one to define a "(commutative) ring spectrum" as a (commutative) monoid with respect to that operation. These form model categories in their own right.

Thom spectra, such as $\mathbb{MU}$, provide basic examples in which the structure of a commutative ring spectrum arises explicitly as part of the construction. When this does not happen, one can resort to an obstruction theory. Using this approach, Goerss, Hopkins, and the author [35, 12] built a commutative ring spectrum $E(F/k)$ associated to any formal group F of finite height over a perfect field k. The homotopy ring is $\Lambda[u^{\pm 1}]$, where Λ denotes the Lubin-Tate ring supporting the universal deformation of F. The naturality of this construction implies that any finite group G of automorphisms of F acts on the spectrum $E(F/k)$.

Basic to homotopy theory is the construction of *homotopy limits*, in particular of the homotopy fixed point object of the action of a finite group G. In the category of spaces, the homotopy fixed point space of an action of G on X is provided by the space of equivariant maps $X^{hG} = \mathrm{map}^G(EG, X)$ from a contractible CW complex EG upon which G acts freely. The map $EG \to *$ induces a map from the actual fixed point set, $X^G \to X^{hG}$. The homotopy fixed point spectrum is the "derived" fixed point object, in the sense that it is the functor best approximating $X \mapsto X^G$ that respects suitable weak equivalences. Similar constructions work in greater generality, and allow one to form, for example, the homotopy fixed point ring spectrum X^{hG} for an action of G on a ring spectrum X. This construction comes with a spectral sequence of the form

$$E_2^{s,t} = H^s(G; \pi_t(X)) \Longrightarrow \pi_{t-s}(X^{hG}).$$

Hopkins, Mahowald, and the author explored these objects in the simplest nontrivial cases, when F has height $p-1$ and G contains an element of order p. A motivating case occurs when $p = 2$ and $F(x,y)$ is the multiplicative formal group $G_m(x,y) = x + y - xy$ over $\mathbb{F}_2$. One then finds that $E(G_m/\mathbb{F}_2)^{hC_2} = KO_2^{\wedge}$, the 2-adic completion of orthogonal K-theory. Use was made in this work, as well, of the equivariant homotopy theory of representation spheres, to compute differentials in the homotopy fixed point spectral sequence.

Given this history it was natural for HHR to hope to define $\mathbb{L}$ as $E(F/k)$ with F of height 4 and $G = C_8$ as dictated by the computations underlying the Detection Theorem. This hope foundered on the difficulty of the computations, and was replaced by the consideration of $\mathbb{MU}^{(4)}$ and the far more elaborate appeal to equivariant topology required to prove properties (b), (c), and (d) of Theorem 2.1.

The norm construction may be carried out in $\mathscr{S}$ to produce a commutative ring spectrum $N_2^{2n}\mathbb{MU} = \mathbb{MU}^{(n)}$ with an action of C_{2n}, so the homotopy fixed point set

may be formed. Proving (b)–(d) for its localization $\mathbb{L}$ requires further development of equivariant stable homotopy theory.

6. EQUIVARIANT STABLE HOMOTOPY THEORY

In order to construct the spectrum $\mathbb{L}$ and study its properties, HHR invoke a large collection of tools from equivariant stable homotopy theory. Diverse variants of this theory have been extensively developed, primarily by May and his collaborators, and HHR elect to work in the context of orthogonal spectra [26]. At many points they require details that are not fully articulated in existing references, and have provided in their paper an exhaustive account of the relevant fundamentals of this subject.

6.1. G-CW complexes

The starting point is G-equivariant *unstable* homotopy theory, where G is a finite group. A *G-CW complex* is a G-space X together with a filtration by "skeleta" such that $\mathrm{Sk}_0 X$ is a G-set and for each $n \geq 0$ there is a G-set P_n and a pushout square

$$
\begin{CD}
P_n \times S^{n-1} @>>> P_n \times D^n \\
@VVV @VVV \\
\mathrm{Sk}_{n-1} X @>>> \mathrm{Sk}_n X
\end{CD}
$$

(where D^n is the n-disk, S^{n-1} is its boundary sphere, and $S^{-1} = \varnothing$), and $X = \bigcup \mathrm{Sk}_n X$ in the weak topology. For example, the unit sphere $S(V)$ of an orthogonal representation V is a compact smooth G-manifold and hence (by a theorem of Verona) admits a finite G-CW structure. The one-point compactification S^V of V is the suspension of $S(V)$ and hence also admits a G-CW structure (with cells increased by one in dimension from those of $S(V)$).

When $G = C_{2n}$ and n is a power of 2 we can be more explicit. Decompose V into irreducibles—

$$V = a\epsilon \oplus b\sigma \oplus \lambda_1 \oplus \cdots \oplus \lambda_c$$

—where ϵ denotes the trivial one-dimensional representation, σ denotes the sign representation pulled back under the unique surjection $C_{2n} \to C_2$, and each λ_j is 2-dimensional. For each j, λ_j is a copy of $\mathbb{C}$ with a chosen generator $\gamma \in C_{2n}$ acting by a root of unity, ζ_j (which is well defined up to complex conjugation). Order the λ_j's so that if $j \leq k$ then $\langle \zeta_j \rangle \supseteq \langle \zeta_k \rangle$ as subgroups of the unit circle. Pick a basis $v_1, \ldots, v_b$ of $b\sigma$, and for each j pick a nonzero vector $w_j \in \lambda_j$. Define cones in V as follows:

$$H_a = a\epsilon \quad , \quad H_{a+i} = H_{a+i-1} \oplus \mathbb{R}\langle v_i \rangle \quad \text{for} \quad 1 \leq i \leq b,$$

and for j such that $1 \leq j \leq c$

$$H_{a+b+2j-1} = H_{a+b+2j-2} \oplus C_{2n} \cdot \mathbb{R}_+ \langle w_j \rangle \quad , \quad H_{a+b+2j} = H_{a+b+2j-2} \oplus \lambda_j .$$

Let $\mathrm{Sk}_0 S^V = \varnothing_+ = *$, and for $i > 0$ let

$$\mathrm{Sk}_i S^V = (H_i)_+ .$$

This defines a C_{2n}-CW filtration of S^V with cells in dimensions 0 and i with $a \leq i \leq \dim V$, each indexed by a transitive C_{2n}-set (with isotropy C_{2n} in dimensions 0 and a, C_n for $a+1 \leq i \leq a+b$, and $\langle \zeta_j \rangle$ in dimensions $a+b+2j-1$ and $a+b+2j$).

6.2. Orthogonal spectra

While we will suppress discussion of technicalities regarding spectra, it may be useful to spell out the definition of the variant adopted by HHR.

Let G be a finite group. Orthogonal G-spectra are defined in terms of a certain category $\mathscr{J}_G$ enriched in the category $\mathscr{T}^G$ of pointed G-spaces and equivariant maps. The objects of $\mathscr{J}_G$ are the finite-dimensional orthogonal representations of G. For $V, W \in \mathscr{J}_G$, let $O(V, W)$ be the Stiefel manifold of linear isometric embeddings from V into W. Let N be the vector bundle over $O(V, W)$ whose fiber at $i : V \hookrightarrow W$ is the orthogonal complement of the image of i in W. Then $\mathscr{J}_G(V, W)$ is defined to be the Thom space of this bundle. The group G acts on $O(V, W)$ by conjugation and N is an equivariant vector bundle, so $\mathscr{J}_G(V, W)$ receives a G-action.

Let $\mathscr{T}_G$ be the category of pointed G-spaces and all continuous pointed maps; this is enriched over $\mathscr{T}^G$. An *(orthogonal) G-spectrum* is an enriched functor $X : \mathscr{J}_G \to \mathscr{T}_G$. It assigns to every $V \in \mathscr{J}_G$ a G-space X_V, and to every pair $V, W \in \mathscr{J}_G$ a continuous equivariant map $\mathscr{J}_G(V, W) \to \mathrm{map}(X_V, X_W)$. Thus for any $i \in O(V, W)$ we receive a map $N(i)_+ \wedge X_V \to X_W$; they vary continuously and form an equivariant family. They are bonding maps for a spectrum. Morphisms in the enriched category $\mathscr{S}_G$ of G-spectra are simply the spaces of natural transformations with G acting by conjugation. Write $\mathscr{S}^G$ for the category enriched over $\mathscr{T}$ with $\mathscr{S}^G(X, Y) = \mathscr{S}_G(X, Y)^G$, and $\mathscr{S}_1 = \mathscr{S}$.

For example, the *sphere G-spectrum* $\mathbb{S}_G$ sends W to its one-point compactification S^W. The equivariant stabilization functor $\Sigma^\infty : \mathscr{T}_G \to \mathscr{S}_G$ is given on $K \in \mathscr{T}_G$ by $W \mapsto S^W \wedge K$. Any $V \in \mathscr{J}_G$ co-represents a G-spectrum, denoted by S^{-V}. Then $S^0 = S^{-0} = \mathbb{S}_G$. By Yoneda, $\mathscr{S}_G(S^{-V}, X) = X_V$. This leads to a canonical presentation of any G-spectrum as

$$(9) \qquad\qquad\qquad X = \underset{V}{\mathrm{colim}}\, S^{-V} \wedge X_V .$$

Smash products are also handled gracefully in this context. The smash product of two G-spectra X and Y is the left Kan extension of $(V, W) \mapsto X_V \wedge Y_W$ along the

functor $\mathscr{I}_G \times \mathscr{I}_G \to \mathscr{I}_G$ given by orthogonal direct sum. This makes $\mathscr{S}^G$ a closed symmetric monoidal category, and equivariant associative and commutative ring spectra are defined using this monoidal structure.

Let H be a subgroup of G. There is a restriction functor $\mathrm{Res}_H^G : \mathscr{S}^G \to \mathscr{S}^H$, which has a left adjoint

$$\mathrm{Ind}_H^G : \mathscr{S}^H \to \mathscr{S}^G.$$

The "Wirthmüller isomorphism" asserts that Ind_H^G is also right adjoint to Res_H^G.

Given subgroups H and K there are "double coset formulas"

$$(10) \qquad \mathrm{Res}_K^G \mathrm{Ind}_H^G X \cong \bigvee_{g \in H \backslash G / K} \mathrm{Ind}_{L(g)}^K \mathrm{Res}_{L(g)}^H X$$

and

$$(11) \qquad (\mathrm{Ind}_H^G X) \wedge (\mathrm{Ind}_K^G Y) \cong \bigvee_{g \in H \backslash G / K} \mathrm{Ind}_{L(g)}^G \left((\mathrm{Res}_{L(g)}^H X) \wedge (\mathrm{Res}_{L(g)}^K Y) \right)$$

where $L(g) = H \cap gKg^{-1}$. Taking $H = G$ in (11) gives the "Frobenius isomorphism"

$$(12) \qquad X \wedge \mathrm{Ind}_K^G Y \cong \mathrm{Ind}_K^G (\mathrm{Res}_K^G X \wedge Y).$$

6.3. The norm

Let $\mathscr{I}$ be the full subcategory of $\mathscr{I}_G$ consisting of the trivial representations of G. The restriction functor $\mathscr{S}_G = \mathrm{Fun}(\mathscr{I}_G, \mathscr{T}_G) \to \mathrm{Fun}(\mathscr{I}, \mathscr{T}_G) = \mathrm{Fun}(G, \mathscr{S})$ participates in an adjoint equivalence of symmetric monoidal categories enriched over $\mathscr{T}^G$: the category of G-spectra is equivalent to the category of G-objects in spectra. Homotopy theoretic aspects are not preserved by this equivalence, but the observation does allow one to easily make constructions. For example, HHR use this device to construct a *norm* functor

$$N_H^G : \mathscr{S}^H \to \mathscr{S}^G$$

associated to an inclusion $H \subseteq G$ of finite groups. This norm is compatible with the one described in §4 and with the adjoint of the restriction functor from commutative G-ring-spectra to commutative H-ring-spectra. Thus if R is a commutative G-ring-spectrum, there is a natural map of G-ring-spectra, a "power operation"

$$(13) \qquad P : N_H^G \mathrm{Res}_H^G R \to R,$$

and if R is a commutative H-ring-spectrum there is a natural "inclusion"

$$(14) \qquad i : R \to \mathrm{Res}_H^G N_H^G R$$

of G-ring-spectra. For any H-spectrum X and commutative G-ring-spectrum R we have the *internal norm* (4) in R-cohomology, studied by Greenlees and May [13]:

$$(15) \qquad \overline{N}_H^G : (\mathrm{Res}_H^G R)^*(X) \to R^*(N_H^G X).$$

The distributivity formula (3) for H central in G holds also for this norm, so for example

$$(16) \quad N_H^G \left(\bigvee_{i \geq 0} S^{iV} \right) \simeq \bigvee_{f \in F} \mathrm{Ind}_{\mathrm{Stab}(f)}^G S^{|f| \mathrm{Ind}_H^{\mathrm{Stab}(f)} V} \quad , \quad |f| = \sum_{g \in G/\mathrm{Stab}(f)} f(g) \,.$$

This norm entitles HHR to view $\mathrm{MU}^{(n)}$ as a commutative C_{2n}-ring-spectrum, rather than just a C_{2n}-object in the stable homotopy category. The action of C_{2n} on the spectrum itself yields the structure of a C_{2n}-ring-spectrum on $D^{-1}\mathrm{MU}^{(n)}$ for any equivariant map $D : S^V \to \mathrm{MU}^{(n)}$, and permits construction of the homotopy fixed point spectrum $(D^{-1}\mathrm{MU}^{(n)})^{hC_{2n}}$.

6.4. Mackey functors and Bredon cohomology

Given finite G-sets P and Q, let $M_G^+(P,Q)$ be the set of isomorphism classes of finite G-sets X equipped with equivariant maps $X \to P$, $X \to Q$. These form the morphisms in a category, in which the composition is given by pull-back. Each $M_G^+(P,Q)$ is a commutative monoid under disjoint union. Formally adjoin inverses to get an abelian group $M_G(P,Q)$. The composition is bi-additive, so it extends to the group completions and we receive a pre-additive category M_G.

A *Mackey functor* is an additive functor from M_G^{op} to the category of abelian groups. For example, the "Burnside Mackey functor" is given by $\widetilde{A} : P \mapsto M_G(P, *)$, so that $\widetilde{A}(G/K)$ is the Burnside group $A(K)$ of isomorphism classes of virtual finite K-sets.

A G-module N determines a Mackey functor $\underline{N} = \underline{N}_G$ with

$$\underline{N}(P) = \mathrm{map}^G(P, N) \,.$$

The morphism $P \xleftarrow{p} X \xrightarrow{q} Q$ induces $\underline{N}(Q) \to \underline{N}(P)$ given by sending $f : Q \to N$ to

$$a \mapsto \sum_{px=a} f(qx) \,.$$

The "constant" Mackey functor $\underline{\mathbb{Z}}$, for example, arises from $\mathbb{Z}$ as a trivial G-module.

The Mackey category M_G arises in equivariant stable homotopy theory as the full subcategory of $h\mathscr{S}^G$ generated by the "discrete" spectra $\Sigma^\infty P_+$. Therefore for any equivariant cohomology theory E_G^* representable by a G-spectrum the functor $P \mapsto E_G^n(\Sigma^\infty P_+)$ defines a Mackey functor. Similarly, a G-spectrum E represents a homology theory with $E_n(X) = \pi_n^G(E \wedge X)$, which extends to a Mackey functor valued theory with $\underline{E}_n(X)(P) = E_n(\Sigma^\infty P_+ \wedge X)$. We have for example Mackey functor valued homotopy groups $\underline{\pi}_n$. If we regard G-spectra of the form $P_+ \wedge S^n$ (where P is a finite G-set) as equivariant spheres, then it is natural to extend the appellation "homotopy group" to $\underline{\pi}_n(X)(P) = [P_+ \wedge S^n, X]^G$.

Bredon cohomology with coefficients in a Mackey functor M is characterized by isomorphisms $H^0_G(\Sigma^\infty_+ A; M) = M(A)$ and $H^n_G(\Sigma^\infty_+ A; M) = 0$ for $n \neq 0$, natural in $A \in M_G$, and is represented by a G-spectrum HM. It may be computed from a G-CW structure by means of a cellular cochain complex constructed in the usual way [**22**]. It follows for example that $H^*_G(X; \underline{\mathbb{Z}}) = H^*(X/G; \mathbb{Z})$. There is of course also *Bredon homology* $H^G_*(X; M) = \pi^G_*(X \wedge HM)$.

The construction of Eilenberg-Mac Lane spectra is compatible with restriction and induction, so the Wirthmüller and Frobenius isomorphisms imply for $K \subseteq H$ that

$$(17) \qquad\qquad H^G_*(\mathrm{Ind}^G_K X; \underline{\mathbb{Z}}) \cong H^K_*(X; \underline{\mathbb{Z}}).$$

7. SLICE CELLS AND THE SLICE SPECTRAL SEQUENCE

The key to proving the Gap and Periodicity Properties is the identification of an appropriate equivariant analogue of the Postnikov system. We begin by recalling the Postnikov system in an appropriate form. See [**11, 16**] for details.

Let $\mathscr{T}^c$ be the category of path-connected pointed spaces. For $n \geq 0$ let $\mathscr{T}^c_{\leq n}$ be the class of spaces A such that $\mathrm{map}_*(S^q, A)$ is contractible for all $q > n$. (It is equivalent to require that $\pi_q(A) = 0$ for all $q > n$.) Given any $X \in \mathscr{T}^c$, there is a map $p : X \to P^n X$ such that (1) $P^n X \in \mathscr{T}^c_{\leq n}$ and (2) p is a "$\mathscr{T}^c_{\leq n}$-equivalence": $p^* : \mathrm{map}_*(P^n X, A) \xrightarrow{\simeq} \mathrm{map}_*(X, A)$ for any $A \in \mathscr{T}^c_{\leq n}$. The map $p : X \to P^n X$ is the nth *Postnikov section* of X, and is well defined up to a contractible choice. Since $\mathscr{T}^c_{\leq n-1} \subseteq \mathscr{T}^c_{\leq n}$, there is a natural factorization $X \to P^n X \to P^{n-1} X$, in which the second map may be assumed to be a fibration. The fiber $P^n_n X$ of $P^n X \to P^{n-1} X$ is a $K(\pi_n(X), n)$. The inverse limit of this *Postnikov tower* is weakly equivalent to X.

The equivariant analogue employed by HHR uses a carefully chosen set of G-spectra in place of spheres, rejecting the "spheres" $G/H_+ \wedge S^n$ used in the construction of G-CW complexes in favor of objects induced from a restricted class of representation spheres. Let $\rho = \rho_K$ denote the regular representation of K on $\mathbb{R}K$.

DEFINITION 7.1. — *A* slice cell *is a G-spectrum weakly equivalent to either* $\mathrm{Ind}^G_K S^{m\rho_K}$ *(the "regular" case) or* $S^{-1} \wedge \mathrm{Ind}^G_K S^{m\rho_K}$ *(the "irregular" case) for some subgroup K of G and some $m \in \mathbb{Z}$. The slice cell is* isotropic *if $K \neq 1$. The underlying homotopy type of a slice cell is a wedge of spheres of dimension $m|K|$ or $m|K| - 1$, and this number is declared to be its* dimension.

HHR use "$\widehat{S}$" to indicate a slice cell. The relationship between slice cells and equivariant spheres begins with the elementary observation that a slice cell of dimension

n admits the structure of a finite G-CW complex with cells in dimensions k with

(18)
$$\lfloor n/|G| \rfloor \leq k \leq n \quad \text{if} \quad n \geq 0$$
$$n \leq k \leq \lfloor n/|G| \rfloor \quad \text{if} \quad n < 0.$$

We get an important vanishing result if G is a cyclic 2-group.

PROPOSITION 7.2 (Cell Lemma). — *For any regular isotropic C_{2n}-slice cell $\widehat{S}$, where $2n$ is a power of 2, we have $H_j^{C_{2n}}(\widehat{S}; \underline{\mathbb{Z}}) = 0$ for $-3 \leq j \leq -1$.*

Proof. Suppose $\widehat{S} = \operatorname{Ind}_{2k}^{2n} S^{m\rho_{2k}}$. If $m \geq 0$ the result is clear. If $m < 0$ we use (17) to see that $H_j^{C_{2n}}(\widehat{S}; \underline{\mathbb{Z}}) = H_j^{C_{2k}}(S^{m\rho_{2k}}; \underline{\mathbb{Z}})$. An equivariant form of Spanier-Whitehead duality then implies that $H_j^{C_{2k}}(S^{m\rho_{2k}}; \underline{\mathbb{Z}}) = H_{C_{2k}}^{-j}(S^{-m\rho_{2k}}; \underline{\mathbb{Z}})$. Now the C_{2k}-CW structure on representation spheres discussed in §6.1 readily implies that these groups are zero for $-3 \leq j \leq -1$. $\qquad\square$

Slice cells restrict well: Given subgroups H and K of G, we see from (10) that $\operatorname{Res}_K^G \operatorname{Ind}_H^G S^{m\rho_H}$ is a wedge of slice cells. They induce well: if $K \subseteq H$ then

$$\operatorname{Ind}_H^G \operatorname{Ind}_K^H S^{m\rho_K} \cong \operatorname{Ind}_K^G S^{m\rho_K} \quad , \quad \operatorname{Ind}_H^G(S^{-1} \wedge \operatorname{Ind}_K^H S^{m\rho_K}) \cong S^{-1} \wedge \operatorname{Ind}_K^G S^{m\rho_K}.$$

Slice cells do not have good multiplicative properties unless G is abelian. In that case (11) shows that

(19)
$$\operatorname{Ind}_H^G S^{a\rho_H} \wedge \operatorname{Ind}_K^G S^{b\rho_K} \cong [G : H + K] \operatorname{Ind}_{H \cap K}^G S^{c\rho_{H \cap K}}$$

where $c = a[H : H \cap K] + b[K : H \cap K]$, so a smash product of regular slice cells splits as a wedge of regular slice cells. Even in the abelian case smash products of irregular slice cells are not wedges of slice cells.

We now imitate the construction of the Postnikov system. Let $\mathscr{S}_{\leq n}^G$ be the class of objects A of $\mathscr{S}^G$ such that $\mathscr{S}_G(\widehat{S}, A)$ is equivariantly contractible for all slice cells $\widehat{S}$ with $\dim \widehat{S} > n$. (It's equivalent to require that $[\widehat{S}, A]^G = 0$ for all slice cells $\widehat{S}$ with $\dim \widehat{S} > n$.) Given any $X \in \mathscr{S}_G$, there is an equivariant map $p : X \to P^n X$ such that (1) $P^n X \in \mathscr{S}_{\leq n}^G$ and (2) p is an "$\mathscr{S}_{\leq n}^G$-equivalence": $p_* : \mathscr{S}_G(P^n X, A) \to \mathscr{S}_G(X, A)$ is an equivariant weak equivalence for any $A \in \mathscr{S}_{\leq n}^G$. The map $p : X \to P^n X$ is the nth *slice section* of X. These functors assemble into a natural tower of fibrations, the *slice tower*, whose inverse limit is weakly equivalent to X. The fiber of $P^n X \to P^{n-1} X$ is written $P_n^n X$ and is called the *n-slice* of X.

The slice cells available in a given dimension vary with the dimension, but (12) implies that smashing with $S^{m\rho_G}$ induces a bijection between homotopy types of slice cells in dimension t and dimension $t + m|G|$. From this one deduces natural equivalences

(20)
$$S^{m\rho_G} \wedge P^t X \xrightarrow{\simeq} P^{t+m|G|}(S^{m\rho_G} \wedge X)$$

compatible with the projection maps. Also, since slice cells restrict to wedges of slice cells, the restriction of the slice tower for X to a subgroup $H \subseteq G$ is the slice tower of $\mathrm{Res}_H^G X$. In particular the underlying tower of spectra is the Postnikov tower.

Unfortunately, slices can be quite complicated and may not be "Eilenberg-Mac Lane objects" in any reasonable sense. This is because S^n (with trivial action) is usually not a slice cell. But S^0 and S^{-1} are (take $m = 0$ and $K = 1$ in Definition 7.1), and this implies that $P_{-1}^{-1} X = H\underline{\pi}_{-1}(X)$. Zero-slices are also of the form HM. The n-slice of an n-dimensional slice cell $\widehat{S}$ can be computed: $P_n^n \widehat{S} = H\underline{\mathbb{Z}} \wedge \widehat{S}$ if $\widehat{S}$ is regular, and $P_n^n \widehat{S} = H\widetilde{A} \wedge \widehat{S}$ if $\widehat{S}$ is irregular. In general, (18) leads to the conclusion that if X is an n-slice then $\underline{\pi}_k(X) = 0$ unless

(21)
$$\lfloor n/|G| \rfloor \leq k \leq n \quad \text{if} \quad n \geq 0$$
$$n \leq k \leq \lfloor (n+1)/|G| \rfloor \quad \text{if} \quad n < 0.$$

The *slice spectral sequence* is obtained by applying equivariant homotopy to the slice tower. HHR index it as

$$E_2^{s,t} = \pi_{t-s}^G(P_t^t X) \Longrightarrow \pi_{t-s}^G(X).$$

This indexing is in accord with that of the Atiyah-Hirzebruch spectral sequence, but HHR prefer to display it as one does the Adams spectral sequence, drawing $t - s$ horizontally and s vertically. In this display, (21) implies that the spectral sequence is nonzero only in two wedges, in the northeast and southwest quadrants, bounded by lines of slope 0 and $|G| - 1$.

8. THE SLICE THEOREM

8.1. Purity

Since slices are not generally Eilenberg-Mac Lane objects, the E_2 term of the slice spectral sequence is not generally expressible in terms of homology. However, the key observation of HHR is that the slices of the crucial C_{2n}-spectrum $\mathbb{MU}^{(n)}$ are extremely well behaved. Here is the definition.

DEFINITION 8.1. — *A G-spectrum X is* pure *if for every t there is a wedge $\widehat{W}_t$ of t-dimensional regular slice cells and an equivariant weak equivalence $P_t^t X \simeq H\underline{\mathbb{Z}} \wedge \widehat{W}_t$. A pure G-spectrum is* isotropic *if all these slice cells are isotropic.*

If X is pure and $P_t^t X \simeq H\underline{\mathbb{Z}} \wedge \widehat{W}_t$, then in the slice spectral sequence

$$E_2^{s,t} = \pi_{t-s}^G(P_t^t X) = H_{t-s}^G(\widehat{W}_t; \mathbb{Z}).$$

Pure G-spectra have a number of attractive features, among them:

LEMMA 8.2. — *If X and Y are pure and $f : X \to Y$ is such that $\mathrm{Res}_1^G f$ is a weak equivalence then f is a weak equivalence.*

The proof is an induction on the order of G.

LEMMA 8.3. — *If X is an isotropic pure C_{2n}-spectrum, where $2n$ is a power of 2, then*

$$\pi_j^{C_{2n}}(X) = 0 \quad for \quad -3 \leq j \leq -1 .$$

For the proof, just feed the Cell Lemma 7.2 into the slice spectral sequence.

The slice filtration in general does not have good multiplicative properties, but for pure spectra it does:

PROPOSITION 8.4. — *If X and Y are pure G-spectra, there is a pairing of slice spectral sequences $E_r(X) \otimes E_r(Y) \to E_r(X \wedge Y)$ compatible with the pairing in homotopy.*

The mainspring of the HHR approach is contained in the following theorem.

THEOREM 8.5 (Slice Theorem). — *If $2n$ is a power of 2, then the C_{2n}-spectrum $\mathbb{MU}^{(n)}$ is pure and isotropic.*

Notice that (20) implies that if X is pure and isotropic then so is any suspension $S^{l\rho_G} \wedge X$. This motivates the type of localization used to construct $\mathbb{L}$ from $\mathbb{MU}^{(n)}$. If R is a G-ring-spectrum we may use an equivariant map $D : S^{l\rho_G} \to R$ to form a telescope

$$(22) \qquad\qquad R \to S^{-l\rho_G} \wedge R \to S^{-2l\rho_G} \wedge R \to \cdots .$$

The direct limit is written $D^{-1}R$. When G is a cyclic 2-group and R is pure and isotropic, the Cell Lemma 7.2 implies that $\pi_j^G(D^{-1}R) = 0$ for $-3 \leq j \leq -1$. So the Slice Theorem has the following corollary.

COROLLARY 8.6 (Gap Theorem). — *Let the number $2n$ be a power of 2 and let $D : S^{l\rho_{2n}} \to \mathbb{MU}^{(n)}$ be any equivariant map. Then $\pi_j^{(2n)}(D^{-1}\mathbb{MU}^{(n)}) = 0$ for $-3 \leq j \leq -1$.*

The spectrum $\mathbb{L}$ will be of the form $D^{-1}\mathbb{MU}^{(n)}$ with $n = 4$ and D a certain map chosen to make the other parts of Theorem 2.1 work out.

8.2. Refinements—proving the Slice Theorem

The proof of the Slice Therem is the heart of this work. HHR begin by approximating $\mathbb{MU}^{(n)}$ by a wedge of slice cells, by means of a "refinement."

A *refinement of the homotopy* of a G-spectrum X is a wedge $\widehat{W}$ of slice cells together with an equivariant map $\alpha : \widehat{W} \to X$ for which there exists a map of graded abelian groups $H_*(\widehat{W}; \mathbb{Z}) \to \pi_*(\widehat{W})$ such that

$$
\begin{array}{ccc}
 & H_*(\widehat{W}; \mathbb{Z}) & \\
{\scriptstyle\cong}\swarrow & \downarrow & \searrow{\scriptstyle\mathrm{id}} \\
\pi_*(X) \xleftarrow[\alpha_*]{} & \pi_*(\widehat{W}) \xrightarrow[h]{} & H_*(\widehat{W}; \mathbb{Z})
\end{array}
$$

commutes. Here h denotes the Hurewicz map. If X is a G-ring-spectrum, the refinement is *multiplicative* if $\widehat{W}$ is equipped with a G-ring-spectrum structure for which α is multiplicative.

For example, algebra generator of $\pi_*(\mathbb{MU})$ in dimension $2j$ can be chosen to be restriction to the trivial group of a C_2-equivariant map $S^{j\rho_2} \to \mathbb{MU}$. It follows that

$$
\alpha_1 : \widehat{W}(1) = \bigwedge_{j>0} \bigvee_{i\geq 0} S^{ij\rho_2} \to \mathbb{MU}
$$

(where the infinite smash product is the "weak smash," the homotopy colimit over finite sub smash products) is a multiplicative refinement of homotopy. Note that $\widehat{W}(1)$ is regular and isotropic.

HHR use the theory of formal groups to define certain equivariant homotopy classes

$$
(23) \qquad\qquad \bar{r}_j^{(2n)} : S^{j\rho_2} \to \mathrm{Res}_2^{2n}\mathbb{MU}^{(n)}
$$

with a variety of properties making them central actors in this work. Using the internal norm (13) they lead to classes

$$
(24) \qquad g_j = \overline{N}_2^{2n}\bar{r}_j^{(2n)} : S^{j\rho_{2n}} \simeq N_2^{2n}S^{j\rho_2} \xrightarrow{N\bar{r}_j} N_2^{2n}\mathrm{Res}_2^{2n}\mathbb{MU}^{(n)} \xrightarrow{P} \mathbb{MU}^{(n)}
$$

which play an important role. When $j = 2^k - 1$ this composite is written $\Delta_k^{(2n)}$ or Δ_k.

Using the C_{2n}-ring-spectrum structure of $\mathrm{Res}_2^{2n}\mathbb{MU}^{(n)}$, the classes $\bar{r}_j$ give a C_{2n}-ring-spectrum map

$$
\bigwedge_{j>0} \bigvee_{i\geq 0} S^{ij\rho_2} \to \mathrm{Res}_2^{2n}\mathbb{MU}^{(n)}.
$$

Apply the norm to this map, and compose with the adjunction (13) to get a map

$$
\alpha_n : \widehat{W}(n) = N_2^{2n}\left(\bigwedge_{j>0} \bigvee_{i\geq 0} S^{ij\rho_2} \right) \to \mathbb{MU}^{(n)}.
$$

The distributivity formula (16) implies

$$N_2^{2n} \left(\bigvee_{i \geq 0} S^{i\rho_2} \right) \cong \bigvee_{f \in F} \mathrm{Ind}_{\mathrm{Stab}(f)}^{G} S^{|f|\rho_{\mathrm{Stab}(f)}}$$

and this is a wedge of regular slice cells, isotropic since C_2 is normal in C_{2n}. Thus by (19) and multiplicativity of the norm, $\widehat{W}(n)$ is a wedge of regular isotropic slice cells.

HHR prove, starting from (6), that α_n is a multiplicative refinement of homotopy.

They develop a "method of polynomial algebras" filling out the following argument. Write A for the ring spectrum $\widehat{W}(n)$ and I^t for the wedge of all the slice cells in A of dimension at least t. We have a tower under A

$$\cdots \to A/I^3 \to A/I^2 \to A/I^1 = S^0.$$

The fiber $\widehat{W}_t = I^t/I^{t+1}$ in this tower is a wedge of slice cells of dimension t if t is even, and is contractible if t is odd. The tower maps to the slice tower of $R = \mathbb{MU}^{(n)}$, and so extends to a map of towers of R-module-spectra

$$
\begin{array}{ccccccc}
\cdots & \longrightarrow & R \wedge_A A/I^3 & \longrightarrow & R \wedge_A A/I^2 & \longrightarrow & R \wedge_A A/I^1 \\
& & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & P^2 R & \longrightarrow & P^1 R & \longrightarrow & P^0 R \,.
\end{array}
$$

A characterization of the slice tower implies that this is an equivalence of towers. On the fibers, the maps are

$$R \wedge_A \widehat{W}_t = R \wedge_A I^t/I^{t+1} \xrightarrow{\simeq} P_t^t R\,.$$

The action of A on the quotient I^t/I^{t+1} factors though the augmentation $A \to S^0$, so

$$(R \wedge_A S^0) \wedge \widehat{W}_t \xrightarrow{\simeq} P_t^t R\,.$$

We can now deduce the Slice Theorem 8.5 from the following key fact.

THEOREM 8.7 (Reduction Theorem). — *Let $A \to R$ be the multiplicative refinement of homotopy of $R = \mathbb{MU}^{(n)}$ constructed above, and suppose that n is a power of 2. Then $R \wedge_A S^0 \simeq H\underline{\mathbb{Z}}$.*

The proof of the Reduction Theorem is a fairly elaborate induction, which we will not attempt to summarize in this report. The case $n = 1$ is due to Hu and Kriz [17], and motivic analogues were known to Hopkins and Morel.

9. THE FIXED POINT THEOREM

The Detection Property pertains to a *homotopy* fixed point set while the Gap Property deals with certain equivariant homotopy groups and hence pertain to an actual fixed point set. The task of unraveling the relationship between these two constructions in various settings has been a common one in modern homotopy theory (Thomason, Carlsson, . . .), and this task plays a role here too.

Let $\mathcal{F}$ be a *family* of subgroups of G, i.e. a set of subgroups closed under conjugation and passage to subgroups. An elementary construction yields a G-CW complex $E\mathcal{F}$ characterized up to equivariant homotopy equivalence by

$$(E\mathcal{F})^H \simeq * \quad \text{if} \quad H \in \mathcal{F} \quad , \quad (E\mathcal{F})^H = \varnothing \quad \text{if} \quad H \notin \mathcal{F}.$$

Denote the suspension G-spectra of these G-spaces with disjoint basepoint adjoined by $E\mathcal{F}_+$. We have the important cofibration sequences

$$E\mathcal{F}_+ \to S^0 \to \widetilde{E}\mathcal{F}.$$

Two families are of special importance: the family $\{1\}$, and the family of proper subgroups. Write $E = E_G$ and $E' = E'_G$ for the corresponding G-spaces. Thus E_G is the usual free contractible G-CW complex.

For a G-spectrum X and a subgroup $H \subseteq G$ one has the categorically defined fixed point sub-object

$$X^H = \text{map}^G(G/H, X) = \text{map}^H(*, \text{Res}_H^G X).$$

The *homotopy fixed point* spectrum is X^{hG}, where $X^h = \text{map}(E_G, X)$. The question arises of when the map $i : X \to X^h$ induced by $E_G \to *$ is a weak G-equivalence. The tool for attacking this question is the map of cofiber sequences

$$
\begin{array}{ccccc}
E_+ \wedge X & \longrightarrow & X & \longrightarrow & \widetilde{E} \wedge X \\
\downarrow & & \downarrow{\scriptstyle i} & & \downarrow \\
E_+ \wedge X^h & \longrightarrow & X^h & \longrightarrow & \widetilde{E} \wedge X^h.
\end{array}
$$

The left vertical map is automatically an equivalence, since $\text{Res}_1^G i$ is a weak equivalence and E is entirely built from free G-cells. It remains to analyze the map $\widetilde{E} \wedge i$.

This analysis is carried out using the "*geometric fixed point construction*"

$$\Phi^H X = (E'_H \wedge \text{Res}_H^G X)^H.$$

In many ways Φ^H behaves the way one expects a fixed point construction to behave. It preserves weak G-equivalences. If X is a G-spectrum and T a G-space then $\Phi^G(X \wedge T) \simeq \Phi^G(X) \wedge T^G$. In particular $\Phi^G S^W \simeq S^{W^G}$, so, for $H \subseteq G$, $\Phi^H S^{m\rho_G} \simeq$

$S^{m[G:H]}$. It interacts well with the norm, as well (up to cofibrancy issues): for a G-spectrum X and subgroup $H \subseteq G$, $\Phi^G N_H^G X \simeq \Phi^H X$.

An easy induction implies that if X is *geometrically free*—that is, $\Phi^H X \simeq *$ for all nontrivial subgroups $H \subseteq G$—then $\widetilde{E} \wedge X \simeq *$.

To obtain a condition under which the "Tate spectrum" $\widetilde{E} \wedge X^h$ is contractible, note that if R is a G-ring-spectrum and M an R-module-spectrum, then $\widetilde{E} \wedge R \simeq *$ implies $\widetilde{E} \wedge M \simeq *$. Since the R-module structure on M induces one on M^h, we can conclude:

LEMMA 9.1. — *If the G-ring-spectrum R is geometrically free, then $i : M \to M^h$ is a weak G-equivalence for any R-module-spectrum M.*

HHR wish apply this to the G-ring-spectrum $\mathbb{L} = D^{-1}\mathbb{MU}^{(n)}$ (with $G = C_{2n}$, $2n$ a power of 2). This puts a constraint on $D : S^{m\rho_{2n}} \to \mathbb{MU}^{(n)}$. To express it succinctly it is convenient to regard $\pi_0(\mathscr{S}^G(S^V, X))$ as a homotopy group of the G-spectrum X in "dimension" V: $\pi_V^G(X)$. With care one can allow V to be a virtual representation. If X is a G-ring-spectrum then we obtain a graded ring, graded by the real representation ring $RO(G)$.

One has $\Phi^{(2)}\mathbb{MU} \simeq \mathbb{MO}$, the unoriented Thom spectrum; so $\Phi^{(2n)}\mathbb{MU}^{(n)} \simeq \mathbb{MO}$. Among the properties of the generators described in (23) is this: if we apply $\Phi^{(2n)}$ to the composite (24) we get classes in $\pi_j(\mathbb{MO})$ which are zero for $j = 2^k - 1$ and indecomposable otherwise. So $\Phi^{(2m)}\Delta_k^{(2m)} = 0$.

The adjunction morphism (14) provides us with a C_{2m}-ring-spectrum map

$$(25) \qquad \mathbb{MU}^{(m)} \to \mathrm{Res}_{2m}^{2n} N_{2m}^{2n} \mathbb{MU}^{(m)} \cong \mathrm{Res}_{2m}^{2n} \mathbb{MU}^{(n)} .$$

Using it we may ask that $\mathrm{Res}_{2m}^{2n} D \in \pi_{(ln/m)\rho_{2m}}^{(2m)}(\mathrm{Res}_{2m}^{2n} \mathbb{MU}^{(n)})$ be divisible by (the image of) $\Delta_k^{(2m)} \in \pi_{(2^k-1)\rho_{2m}}^{(2m)}(\mathbb{MU}^{(m)})$. Note that the adjunction implies that $\Delta_k^{(2m)}$ divides $\mathrm{Res}_{2m}^{2n} N_{2m}^{2n} \Delta_k^{(2m)} \in \pi_{(2^k-1)\rho_{2n}}^{(2n)}(\mathbb{MU}^{(n)})$ in this sense.

Lemma 9.1 implies the following theorem (in which $\mathbb{L} = D^{-1}\mathbb{MU}^{(n)}$).

THEOREM 9.2 (Fixed Point Theorem). — *Let $2n$ be a power of 2, and assume that $D \in \pi_{l\rho_{2n}}^{(2n)}(\mathbb{MU}^{(n)})$ has the following property: For every divisor m of n, $\mathrm{Res}_{2m}^{2n} D$ is divisible by $\Delta_k^{(2m)}$ for some k. Then $\mathbb{L} \to \mathbb{L}^h$ is a weak C_{2n}-equivalence, and $\pi_*^{C_{2n}}(\mathbb{L}) \to \pi_*(\mathbb{L}^{hC_{2n}})$ is an isomorphism.*

10. THE PERIODICITY THEOREM

Our candidate for $\mathbb{L}$ is $\mathbb{L} = D^{-1}\mathrm{MU}^{(n)}$ for a suitable choice of n (a power of 2) and $D \in \pi_{l\rho_{2n}}(\mathrm{MU}^{(n)})$. The Periodicity Property will follow from the construction of an element of $\omega \in \pi_{2nj}^{(2n)}(\mathbb{L})$ (for $2n = 8$ and $j = 32$) that restricts to a unit in $\pi_{2nj}(\mathrm{Res}_1^{(2n)}\mathbb{L})$. For: Such an element ω determines an equivariant map $\overline{\omega} : S^{2nj} \wedge \mathbb{L} \to \mathbb{L}$ which is a weak equivalence of underlying spectra and hence induces a weak equivalence $S^{2nj} \wedge \mathbb{L}^{hC_{2n}} \to \mathbb{L}^{hC_{2n}}$; this is the Periodicity Property. (By the Slice Theorem 8.5, $\mathbb{L}$ is pure, so by Lemma 8.2 the map $\overline{\omega}$ is actually an equivariant weak equivalence.)

Since we are inverting D, any divisor of D becomes a unit in $RO(C_{2n})$-graded equivariant homotopy. The problem is to find a unit with integral grading. HHR start with the class $\Delta_1^{(2n)} = \overline{N}_2^{2n}\overline{r}_1^{(2n)} \in \pi_{\rho_{2n}}^{(2n)}(\mathrm{MU}^{(n)})$, and arrange D to be divisible by it. They then work hard (and impose further restrictions on D) to find a class $v \in \pi_{j(2n-\rho_{2n})}(D^{-1}\mathrm{MU}^{(n)})$, for some j, that restricts to $1 \in \pi_0(\mathrm{Res}_1^{(2n)}D^{-1}\mathrm{MU}^{(n)})$. Then $\omega = (\Delta_1^{(2n)})^j v \in \pi_{2nj}(\mathbb{L})$ will serve the purpose. The size of j determines how many Kervaire invariant one classes are allowed to survive.

The construction of the class v begins with the "orientation class"

$$u_V \in H_d^G(S^V; \mathbb{Z}) = \pi_d^G(S^V \wedge H\mathbb{Z}) = \pi_{d-V}^G(H\mathbb{Z})$$

of an oriented G-representation V of dimension d. This class satisfies the identities

$$(26) \quad u_{U \oplus V} = u_U \cdot u_V \quad , \quad u_{\mathrm{Res}_H^G V} = \mathrm{Res}_H^G u_V \quad , \quad u_{\mathrm{Ind}_H^G W} = u_{\mathrm{Ind}_H^G \dim W} \cdot \overline{N}_H^G u_W$$

in which $\overline{N}_H^G$ refers to the internal norm described in (4). In particular, $u_V \in \pi_{d-V}^G(H\mathbb{Z})$ restricts to $1 \in \pi_0(H\mathbb{Z})$. Note that twice any representation admits an orientation.

First consider the "sign" representation $\sigma = \sigma_{2n}$ of C_{2n}, the pullback of the usual sign representation under the unique surjection $C_{2n} \to C_2$. We have the orientation class $u_{2\sigma} \in \pi_2^{(2n)}(S^{2\sigma} \wedge H\mathbb{Z})$, and we wonder first whether it, or some power of it, lifts to a class in $\pi_2^{(2n)}(S^{2\sigma} \wedge \mathrm{MU}^{(n)})$.

The slice spectral sequence provides potential obstructions to lifting a class from $H\mathbb{Z} = P^0\mathrm{MU}^{(n)}$ to $\mathrm{MU}^{(n)}$. Since the class of interest is not an integral grading we have to apply $RO(G)$-graded homotopy to the slice tower. There results family of spectral sequences indexed by V,

$$E_2^{s,t+V} = \pi_{t-s+V}^G(P_{d+t}^{d+t}X) \Longrightarrow \pi_{t-s+V}^G(X).$$

If the spectrum X is pure, the multiplicative structure ties them all together.

Thus we can regard $u_{2\sigma}$ as an element of $E_2^{0,2-2\sigma}$. HHR need some other classes to express differentials on this class. For any G-representation V, the inclusion of

the trivial subspace defines a G-map $S^0 \to S^V$. If $V^G \neq 0$ then this G-map is null-homotopic, but in general it is not; it defines an "Euler class"

$$a_V \in \pi_0^G(S^V) = \pi_{-V}^G(S^0).$$

Write a for the Hurewicz image of a_σ: $a \in \pi_0^{(2n)}(S^\sigma \wedge H\underline{\mathbb{Z}}) = \pi_{-\sigma}^{(2n)}(H\underline{\mathbb{Z}})$; so $a \in E_2^{1,1-\sigma}$. It survives to the $\mathbb{MU}^{(n)}$ Hurewicz image of a_σ in $\pi_{-\sigma}^{(2n)}(\mathbb{MU}^{(n)})$.

Finally, let $b = 1 \wedge a_{\overline{\rho}} : S^1 \to S^{\rho_{2n}}$, where $\overline{\rho}$ denotes the reduced regular representation of C_{2n}. Composing b^j with the class g_j defined in (24) gives $g_j b^j \in \pi_j^{(2n)}(\mathbb{MU}^{(n)})$, represented by a class $f_j \in E_2^{(2n-1)j,2nj}$ (because $S^{j\rho_{2n}}$ is a slice cell of dimension $2nj$).

No power of $u_{2\sigma}$ survives the slice spectral sequence, by virtue of the following computation (due in case $n = 1$ to Araki (unpublished) and Hu and Kriz [17]).

PROPOSITION 10.1. — *Let $2n$ be a power of 2. In the $RO(C_{2n})$-graded slice spectral sequence for $\mathbb{MU}^{(n)}$, $u_{2\sigma}^{2^{k-1}}$ survives to E_r with $r = 1 + 2n(2^k - 1)$, and*

$$d_r u_{2\sigma}^{2^{k-1}} = a^{2^k} f_{2^k-1} \neq 0.$$

These very differentials imply the survival of certain multiples of these classes, however. The element $\Delta_k = \overline{N}_2^{2n}\overline{r}_{2^k-1}^{(2n)}$ (cf (24)) defines a map $S^{(2^k-1)\rho_{2n}} \wedge \mathbb{MU}^{(n)} \to \mathbb{MU}^{(n)}$, via the G-ring-spectrum structure of $\mathbb{MU}^{(n)}$, which is compatible with the slice tower and gives a map of slice spectral sequences. A simple identity shows that the target of the differential on $u_{2\sigma}^{2^k}$ is killed by Δ_k. This is the last differential that can be nonzero on $\Delta_k u_{2\sigma}^{2^k}$, so that class survives in the slice spectral sequence. This implies that the image of $u_{2\sigma}^{2^k}$ survives in the localization of the slice spectral sequence obtained by inverting Δ_k, and so $u_{2\sigma}^{2^{k'}}$ does too for $k' \geq k$.

The class $\Delta_1^{(2n)}$ involves the regular representation, not the sign representation. Fortunately, the internal norm (4) provides a relation between their orientation classes. Since $\rho_2 = 1 \oplus \sigma_2$, $u_{2\rho_2} = u_{2\sigma_2}$. Using the equations $\mathrm{Ind}_H^G \rho_H = \rho_G$ and $\mathrm{Ind}_n^{2n} 1 = 1 \oplus \sigma_{2n}$, one sees that

$$u_{2\rho_4} = u_{2\sigma_4}^4 \cdot \overline{N}_2^4 u_{2\rho_2} \quad , \quad u_{2\rho_8} = u_{2\sigma_8}^8 \cdot \overline{N}_4^8 u_{2\rho_4}$$

and so on. Thus

$$u_{2\rho_8}^{2^k} = u_{2\sigma_8}^{2^{k+3}} \cdot \overline{N}_4^8 \left(u_{2\sigma_4}^{2^{k+2}} \cdot \overline{N}_2^4 u_{2\sigma_2}^{2^k} \right)$$

will survive in the localized slice spectral sequence provided that

(27)
$$\begin{aligned}
\Delta_{k_3}^{(8)} \quad &\text{divides} \quad \mathrm{Res}_8^8 D \quad \text{for some} \quad k_3 \leq k+3 \\
\Delta_{k_2}^{(4)} \quad &\text{divides} \quad \mathrm{Res}_4^8 D \quad \text{for some} \quad k_2 \leq k+2 \\
\Delta_{k_1}^{(2)} \quad &\text{divides} \quad \mathrm{Res}_2^8 D \quad \text{for some} \quad k_1 \leq k.
\end{aligned}$$

Any element $v \in \pi_{2^{k+1}(8-\rho_8)}^{(8)}(D^{-1}\mathbb{MU}^{(4)})$ represented by $u_{2\rho_8}^{2^k}$ will restrict to 1 in the group $\pi_0(\mathrm{Res}_1^{(8)} D^{-1}\mathbb{MU}^{(4)})$. Taking

$$\omega = (\Delta_1^{(8)})^{2^{k+1}} v \in \pi_{2^{k+4}}(\mathbb{L})$$

will then give us the Periodicity Property.

11. WRAP-UP

Proof of Theorem 2.1. The spectrum $\mathbb{L}$ is $D^{-1}\mathbb{MU}^{(n)}$. We have now accumulated several requirements on $2n$ (a power of 2) and D:

- The Gap Property requires that $D \in \pi_{l\rho_{2n}}^{(2n)}(\mathbb{MU}^{(n)})$ for some l.
- The Fixed Point Property requires that for each divisor m of n there exists k such that $\Delta_k^{(2m)}$ divides $\mathrm{Res}_{2m}^{2n} D$.
- The Periodicity Property requires, for $n = 4$, that D satisfies (27) and that $\Delta_1^{(8)}$ divides D.
- The Detection Property requires that $n \geq 4$ and for $n = 4$ that D satisfies (27) for certain values of k_1, k_2, k_3.

A calculation verifies that this last requirement is met by $k_1 = 4$, $k_2 = 2$, $k_3 = 1$ and no smaller values. Recalling that $\Delta_k^{(2m)}$ divides $\mathrm{Res}_{2m}^{2n} N_{2m}^{2n} \Delta_k^{(2m)}$, we can take $k = 4$ and

$$D = N_2^8 \Delta_4^{(2)} \cdot N_4^8 \Delta_2^{(4)} \cdot \Delta_1^{(8)} \in \pi_{19\rho_8}^{(8)}(\mathbb{MU}^{(4)}).$$

Then $v \in \pi_{2^{k+1}(8-\rho_8)}^{(8)}(D^{-1}\mathbb{MU}^{(4)})$ with $2^{k+1} = 32$, so $\omega = (\Delta_1^{(8)})^{32} v \in \pi_{256}(\mathbb{L})$. $\qquad\square$

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Haynes MILLER

Mass. Institute of Technology
Department of Mathematics
77 Massachusetts Avenue
Cambridge, MA 02139–4307 – U.S.A.
E-mail : hrm@math.mit.edu