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# CM STABILITY AND THE GENERALIZED FUTAKI INVARIANT II

*by*

Sean Timothy Paul & Gang Tian

*Dedicated to Jean-Michel Bismut on the occasion of his 60$^{th}$ birthday*

***Abstract.*** — The Mabuchi K-energy map is exhibited as a singular metric on the refined CM polarization of any equivariant family $\mathbf{X} \xrightarrow{p} S$. Consequently we show that the generalized Futaki invariant is the leading term in the asymptotics of the reduced K-energy of the generic fiber of the map $p$. Properness of the K-energy implies that the generalized Futaki invariant is strictly negative.

***Résumé* (CM-stabilité et invariant de Futaki généralisé II).** — On interpréte la K-énergie de Mabuchi comme une métrique singulière sur la CM-polarisation raffinée d'une famille équivariante $\mathbf{X} \xrightarrow{p} S$. Nous montrons que l'invariant de Futaki généralisé est le terme principal de l'asymptotique de la K-énergie réduite de la fibre générique de l'application $p$. Si la K-énergie est propre, alors l'invariant de Futaki généralisé est strictement négatif.

## 1. Introduction

**1.1. Statement of results.** — Throughout this paper $\mathbf{X}$ and $S$ denote smooth, proper complex projective varieties satisfying the following conditions.

1. $\mathbf{X} \subset S \times \mathbb{P}^N$; $\mathbb{P}^N$ denotes the complex projective space of *lines* in $\mathbb{C}^{N+1}$.
2. $p := p_1 : \mathbf{X} \to S$ is flat of relative dimension $n$, degree $d$ with Hilbert polynomial $P$.
3. $L|_{\mathbf{X}_z}$ is very ample and the embedding $\mathbf{X}_z := p_1^{-1}(z) \xhookrightarrow{L} \mathbb{P}^N$ is given by a complete linear system for $z \in S$.

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4. There is an action of $G := SL(N+1, \mathbb{C})$ on the data compatible with the projection and the standard action on $\mathbb{P}^N$.

It is well known that (1) and (3) imply that

$$\mathbb{P}(p_{1*}L) \cong S \times \mathbb{P}^N. \tag{1.1}$$

Which in turn is equivalent to the existence of a line bundle $\mathcal{A}$ on $S$ such that

$$p_{1*}L \cong \underbrace{\bigoplus \mathcal{A}}_{N+1}. \tag{1.2}$$

Below $\mathrm{Chow}(\mathbf{X}/S)$ denotes the Chow form of the family $\mathbf{X}/S$, $\mu$ is the coefficient of $k^{n-1}$ in $P(k)$, and $\mathcal{M}_n$ is the coefficient of $\binom{m}{n}$ in the CGKM expansion of $\det(p_{1*}L^{\otimes m})$ for $m >> 0$. A complete discussion of these notions is given in "*CM Stability and the Generalized Futaki Invariant I*". We refer the reader to that paper for the basic definitions and constructions that are used in the present article.

We define an invertible sheaf on $S$ as follows.

***Definition 1* (The Refined CM polarization [1]).** — *We have*

$$\mathbb{L}_1(\mathbf{X}/S) := \{\mathrm{Chow}(\mathbf{X}/S) \otimes \mathcal{A}^{d(n+1)}\}^{n(n+1)+\mu} \otimes \mathcal{M}_n^{-2(n+1)} \tag{1.3}$$

With the family $p_1 : \mathbf{X} \to S$ fixed throughout, we will denote $\mathbb{L}_1(\mathbf{X}/S)$ by $\mathbb{L}_1$ in the remainder of the paper.

Our first result exhibits the Mabuchi energy as a *singular* Hermitian metric on $\mathbb{L}_1$.

***Theorem 1.*** — *Let $||\ ||$ be any smooth Hermitian metric on $\mathbb{L}_1^{-1}$.* [2] *Then there is a continuous function $\Psi_S : S \setminus \Delta \to (-\infty, c)$ such that for all $z \in S/\Delta$*

$$d(n+1)\nu_{\omega|_{\mathbf{X}_z}}(\varphi_\sigma) = \log\left(e^{(n+1)\Psi_S(\sigma z)} \frac{||\ ||^2(\sigma z)}{||\ ||^2(z)}\right). \tag{1.4}$$

*Here $c$ denotes a constant which depends only on the choice of background Kähler metrics on $S$ and $\mathbf{X}$, $\Delta$ denotes the discriminant locus of the map $p_1$, and $\omega|_{\mathbf{X}_z}$ denotes the restriction of the Fubini Study form of $\mathbb{P}^N$ to the fiber $\mathbf{X}_z$.*

***Remark 1.*** — *This should be compared with the main result in Section 8 of* [**17**]. *The principal contribution of our present work is the observation that the whole theory in Section 8 of* [**17**] *should be recast from the beginning with the sheaf $\mathbb{L}_1$.*

Let $X \hookrightarrow \mathbb{P}^N$ be an $n$ dimensional projective variety with Hilbert polynomial $P$. Let $Hilb_m(X)$ denote the *mth* Hilbert point of $X$ (see [**12**] for further information ). If $\lambda$ is a one parameter subgroup of $G$ then it is known (see [**12**] ) that the weight,

[1] We use this terminology in order to distinguish this sheaf from one introduced by the second author in ([**17**]).

[2] $\mathbb{L}_1^{-1}$ denotes the dual of $\mathbb{L}_1$.

$w_\lambda(m)$, of $Hilb_m(X)$ with respect to $\lambda$ is a *polynomial* in $m$ of degree at most $n+1$. That is,

$$w_\lambda(m) = a_{n+1}(\lambda)m^{n+1} + a_n(\lambda)m^n + \cdots$$

Then the ratio may be expanded as follows.

$$\frac{w_\lambda(m)}{mP(m)} = F_0(\lambda) + F_1(\lambda)\frac{1}{m} + \cdots + F_l(\lambda)\frac{1}{m^l} + \ldots$$

***Definition 2* (Donaldson ([5])).** — *$F_1(\lambda)$ is the generalized Futaki invariant of $X$ with respect to $\lambda$.*

In our previous paper we have shown the following.

***Theorem* (The weight of the Refined CM polarization).** — i) *There is a natural $G$ linearization on the line bundle $\mathbb{L}_1$.*

ii) *Let $\lambda$ be a one parameter subgroup of $G$. Let $z \in \mathfrak{Hilb}^P_{\mathbb{P}^N}(\mathbb{C})$. Let $w_\lambda(z)$ denote the weight of the restricted $\mathbb{C}^*$ action (whose existence is asserted in i)) on $\mathbb{L}_1^{-1}|_{z_0}$ where $z_0 = \lambda(0)z$. Then*

$$w_\lambda(z) = F_1(\lambda). \tag{1.5}$$

The main result of the paper is the following corollary of (1.4) and (1.5).

***Corollary 1* (Algebraic asymptotics of the Mabuchi energy).** — *Let $\varphi_{\lambda(t)}$ be the Bergman potential associated to an algebraic 1psg $\lambda$ of $G$, and let $z \in S \setminus \Delta$. Then there is an asymptotic expansion*

$$d(n+1)\nu_{\omega|_{\mathbf{X}_z}}(\varphi_{\lambda(t)}) - \Psi_S(\lambda(t)) = F_1(\lambda)\log(|t|^2) + O(1) \text{ as } |t| \to 0. \tag{1.6}$$

*Moreover $\Psi_S(\lambda(t)) = \psi(\lambda)\log(|t|^2) + O(1)$ where $\psi(\lambda) \in \mathbb{Q}_{\geq 0}$. Moreover, $\psi(\lambda) \in \mathbb{Q}_+$ if and only if $\lambda(0)\mathbf{X}_z = \mathbf{X}_{\lambda(0)z}$ (the limit cycle[3] of $\mathbf{X}_z$ under $\lambda$ ) has a component of multiplicity greater than one. Here $O(1)$ denotes any quantity which is bounded as $|t| \to 0$.*

Moser iteration and a refined Sobolev inequality (see [**11**] and [**7**]) yield the following.

***Corollary 2.*** — *If $\nu_{\omega|_{\mathbf{X}_z}}$ is proper (bounded from below) then the generalized Futaki invariant of $\mathbf{X}_z$ is strictly negative (nonnegative) for all $\lambda \in G$.*

***Remark 2.*** — *We call the left hand side of (1.6) the* reduced *K-Energy along $\lambda$. We also point out that while it is certainly the case that $F_1(\lambda)$ may be defined for any subscheme of $\mathbb{P}^N$ it evidently only controls the behavior of the K-Energy when $\lambda(0)\mathbf{X}_z$ is reduced.*

[3] See [**12**] pg. 61.

***Remark 3.*** — *The precise constant $d(n+1)$ in front of $\nu_\omega$ is not really crucial, since what really matters is the sign* of $F_1(\lambda)+\psi(\lambda)$. *That $\Psi_S(\lambda(t))$ has logarithmic singularities can be deduced from* [**13**].

***Remark 4.*** — *We emphasize that we do not assume the limit cycle is smooth.*

## 2. Background and Motivation

Let $(X,\omega)$ be a compact Kähler manifold ($\omega$ not necessarily a Hodge class) and $P(X,\omega) := \{\varphi \in C^\infty(X) : \omega_\varphi := \omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi > 0\}$ the space of Kähler potentials. This is the usual description of all Kähler metrics in the same class as $\omega$ (up to translations by constants). It is not an overstatement to say that the most basic problem in Kähler geometry is the following

$$\textit{Does there exist } \varphi \in P(X,\omega) \textit{ such that } \operatorname{Scal}(\omega_\varphi) \equiv \mu? \quad (*)$$

This is a fully nonlinear *fourth order* elliptic partial differential equation for $\varphi$. $\mu$ is a constant, the average of the scalar curvature, it depends only on $c_1(X)$ and $[\omega]$. When $c_1(X) > 0$ and $\omega$ represents the *anticanonical* class a simple application of the Hodge Theory shows that $(*)$ is equivalent to the *Monge-Ampere equation.*

$$\frac{\det(g_{i\bar{j}}+\varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = e^{F-\kappa\varphi} \quad (\kappa = 1) \qquad (**)$$

where $F$ denotes the Ricci potential. When $\kappa = 0$ this is the celebrated Calabi problem solved by S.T.Yau and when $\kappa < 0$ this was solved by Aubin and Yau independently in the 70's. It is well known that $(*)$ is actually a *variational* problem. There is a natural energy on the space $P(X,\omega)$ whose critical points are those $\varphi$ such that $\omega_\varphi$ has constant scalar curvature (csc). This energy was introduced by T. Mabuchi ([**10**]) in the 1980's. It is called the *K-Energy map* (denoted by $\nu_\omega$) and is given by the following formula

$$\nu_\omega(\varphi) := -\frac{1}{V}\int_0^1 \int_X \dot{\varphi}_t(\operatorname{Scal}(\varphi_t)-\mu)\omega_t^n dt.$$

Above, $\varphi_t$ is a smooth path in $P(X,\omega)$ joining 0 with $\varphi$. The K-Energy does not depend on the path chosen. In fact there is the following well known formula for $\nu_\omega$ where $O(1)$ denotes a quantity which is bounded on $P(X,\omega)$.

$$\nu_\omega(\varphi) = \int_X \log\left(\frac{{\omega_\varphi}^n}{\omega^n}\right)\frac{{\omega_\varphi}^n}{V} - \mu(I_\omega(\varphi)-J_\omega(\varphi)) + O(1)$$

$$J_\omega(\varphi) := \frac{1}{V}\int_X \sum_{i=0}^{n-1}\frac{\sqrt{-1}}{2\pi}\frac{i+1}{n+1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^i\wedge{\omega_\varphi}^{n-i-1}$$

$$I_\omega(\varphi) := \frac{1}{V}\int_X \varphi(\omega^n - {\omega_\varphi}^n).$$

We have written down the K-energy in the case when $\omega = c_1(X)$. Observe that $\nu_\omega$ is essentially the *difference* of two positive terms. What is of interest for us is that

the problem (∗) is not only a variational problem but a *minimization* problem. With this said we have the following fundamental result.

***Theorem* (S. Bando and T. Mabuchi [1]).** — *If $\omega = c_1(X)$ admits a Kähler Einstein metric then $\nu_\omega \geq 0$. The absolute minimum is taken on the solution to (∗∗) (which is unique up to automorphisms of $X$).*

Therefore a *necessary* condition for the existence of a Kähler Einstien metric is a bound from below on $\nu_\omega$. In order to get a *sufficient* condition one requires that the K-energy *grow* at a certain rate. Precisely, it is required that the K-Energy be *proper*. This concept was introduced by the second author in [**17**].

***Definition 3.*** — *$\nu_\omega$ is* ***proper*** *if there exists a strictly increasing function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ (where $\lim_{T\longrightarrow\infty} f(T) = \infty$) such that $\nu_\omega(\varphi) \geq f(J_\omega(\varphi))$ for all $\varphi \in P(M,\omega)$.*

***Theorem* ([17]).** — *Assume that $Aut(X)$ is discrete. Then $\omega = c_1(X)$ admits a Kähler Einstein metric if and only if $\nu_\omega$ is proper.*

The next result was established by the second author and Xiuxiong Chen. It holds in an *arbitrary* Kähler class $\omega$. An alternative proof of this was given by Donaldson for polarized projective manifolds.

***Theorem* ([3]).** — *If $\omega$ admits a metric of csc then $\nu_\omega \geq 0$.*

In this paper our interest is to test for a lower bound of $\nu_\omega$ along the large but finite dimensional group $G$ of *matrices* in the polarized case. When we restrict our attention to $G$ we make the connection with Mumfords' Geometric Invariant Theory. The past couple of years have witnessed quite a bit of activity on this problem due to this connection.

To put things in historical perspective consider the various formulations of the Futaki invariant.

i) 1983 Futaki ([**6**]) introduces his invariant as a lie algebra character on a Fano manifold $X$

$$F_\omega : \eta(X) \longrightarrow \mathbb{C}.$$

ii) 1986 Mabuchi (see [**10**] ) integrates the Futaki invariant with the introduction of the K-energy map. The linearization of the K-energy along orbits of holomorphic vector fields is the real part of the Futaki invariant.

iii) 1992 Ding and Tian ([**4**]) introduced the *generalized* Futaki invariant. Here the *jumping of complex structures* is introduced. The limit of the derivative of the K-Energy map is identified with the generalized Futaki invariant of $X^{\lambda(0)}$ provided this limit has at most *normal* singularities.

iv) 1997 The CM polarization is defined (see [**17**]) for *smooth* families, as the relative canonical bundle is explicitly involved in the definition. K-Stability is defined in terms of special degenerations and the generalized Futaki invariant.

v) 1999 Yotov formulated the generalized Futaki Invariant in terms of equivariant Chow groups of a *normal* variety.

vi) 2002 For an *arbitrary scheme* Donaldson ([**5**]) defined the weight $F_1(\lambda)$. This is identified with the limit of the derivative of K-energy (by [**4**]) when the limit cycle is a *smooth* (or normal) scheme.

***Remark 5.*** — *We hope that we have clarified the role of the CM polarization. The main point is that once the CM polarization is extended to the Hilbert scheme ([**14**]) the polarization computes the precise asymptotics of the K-energy of any generic fiber of the map* $\mathbf{X} \to S$. *This extension was made possible by an application of the Knudsen Mumford expansion of the determinant of direct images of perfect complexes of sheaves (see* [**8**]*). In fact,* $\psi(\lambda)$ *already appeared in work of the second author (see* [**17**]*). Despite this, the role of* $\psi(\lambda)$ *becomes more precise in the present work.*

## 3. Algebraic potentials

In order to connect these notions to the K-Energy map we now give an account of how to associate an admissible potential $\varphi_{\lambda(t)}$ to a one parameter subgroup of $G$. In order to detect properness (conjecturally) one restricts attention to the subspace of *Bergman metrics* inside $P(M,\omega)$ since these metrics are *dense* in $P(M,\omega)$ (see [**16**], [**15**], [**19**], [**2**]). By definition these metrics are induced by the Kodaira embeddings furnished by the polarization $L$. The construction is as follows. We have an embedding

$$X \xrightarrow[L]{} \mathbb{P}(H^0(X,L)^*) = \mathbb{P}^N$$

furnished by some basis $\{S_0,\dots,S_N\}$ of $H^0(X,L)$. Observe that with the natural Hermitian metric on $H^0(X,L)$, the induced Fubini-Study metric on $\mathbb{P}^N$ is related to the curvature of the initial metric on $L$ by the formula

$$\omega_{FS}|_X = \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\sum_{i=0}^{N}||S_i||^2\right).$$

We conclude that

$$\log\left(\sum_{i=0}^{N}||S_i||^2\right) \in P(X,\omega).$$

Let $\sigma \in SL(N+1,\mathbb{C})$, then

$$\sigma^*(\omega_{FS}) = \omega_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_\sigma.$$

Where $\varphi_\sigma$ is given by the formula

$$\varphi_\sigma = \log\left(\frac{||\sigma z||^2}{||z||^2}\right).$$

We let $\{T_0, \dots T_N\}$ denote the corresponding change of basis

$$\begin{pmatrix} \sigma_{00} & \dots & \sigma_{0N} \\ \sigma_{10} & \dots & \sigma_{1N} \\ \dots & \dots & \dots \\ \sigma_{N0} & \dots & \sigma_{NN} \end{pmatrix} \begin{pmatrix} S_0 \\ \dots \\ \dots \\ S_N \end{pmatrix} = \begin{pmatrix} T_0 \\ \dots \\ \dots \\ T_N \end{pmatrix}.$$

Then we have

$$\varphi_\sigma|_X = \log\left(\frac{\sum_{i=0}^N ||T_i||^2}{\sum_{i=0}^N ||S_i||^2}\right).$$

Putting these ingredients together gives

$$\sigma^*\omega_{FS}|_X = \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\sum_{i=0}^N ||T_i||^2\right). \tag{3.1}$$

Therefore, if we *fix* a basis of $H^0(X, L)$ we get a natural map

$$SL(N+1, \mathbb{C}) \to P(X, \omega).$$

A *one parameter subgroup* of $SL(N+1, \mathbb{C})$ is an algebraic [4] homomorphism

$$\lambda : \mathbb{C}^* \to SL(N+1, \mathbb{C}).$$

Any such $\lambda(t)$ can be diagonalised. That is, we may assume that $\lambda(t)$ takes values in the standard maximal torus $H \cong (\mathbb{C}^*)^N$ of $SL(N+1, \mathbb{C})$.

$$\lambda(t) = \begin{pmatrix} t^{m_0} & \dots & \dots & 0 \\ 0 & t^{m_1} & \dots & 0 \\ 0 & \dots & \dots & t^{m_N} \end{pmatrix}.$$

The exponents $m_i$ satisfy

$$\sum_{0 \le i \le N} m_i = 0.$$

We arrive at the following formula.

$$\varphi_{\lambda(t)}(z) := \log\left(\sum_{0 \le j \le N} |t|^{2m_j} ||S_j||^2(z)\right).$$

Now we may consider the K-energy map as a function on $SL(N+1, \mathbb{C})$.

[4] "Algebraic" means that the matrix coefficients $\lambda(t)_{i,j} \in \mathbb{C}[t, t^{-1}]$.

## 4. Singular Hermitian metrics

*Proof of Theorem 1.* — In part I of this work the authors provided the following formula for the first Chern class of $\mathbb{L}_1$.

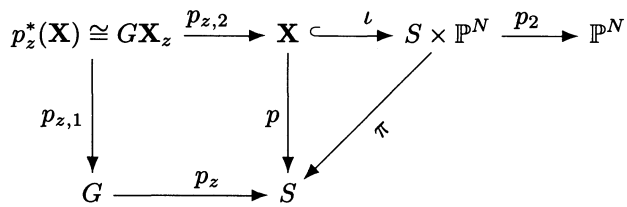
(4.1)

$$c_1(\mathbb{L}_1) = p_{1*}\left((n+1)c_1(K_{\mathbf{X}/S})c_1(L)^n + \mu c_1(L)^{n+1}\right) \quad K_{\mathbf{X}/S} := K_{\mathbf{X}} \otimes p_1^*(K_S^\vee).$$

( 4.1) allows us to exhibit the K-energy map as a *singular* metric on the CM polarization (see [**17**]). Recall that $p^{-1}(z) = \mathbf{X}_z \subset \mathbb{P}^N$, where $z \in S_\infty := S \setminus \Delta$. We define

$$G\mathbf{X}_z := \{(\sigma, y) \in G \times \mathbb{P}^N : y \in \sigma\mathbf{X}_z\}.$$

Observe that $G\mathbf{X}_z$ is biholomorphic to $G \times \mathbf{X}_z$. Then we have the following diagram, where $p_z$ denotes the evaluation map, i.e. $p_z(\sigma) := \sigma z$.




Given $z \in B \setminus \Delta$ we can consider $K_{\mathbf{X}_z}$, the canonical bundle of the fiber $\mathbf{X}_z$. These fit together holomorphically into a line bundle $K_\infty$ on $\mathbf{X} \setminus p^{-1}(\Delta)$. On the other hand, the relative canonical bundle $K_p$ of the map $p$ exists and lives on *all* of $\mathbf{X}$.

$$K_p := K_{\mathbf{X}} \otimes p^* K_S^{-1}$$

When we restrict this sheaf to $\mathbf{X} \setminus p^{-1}(\Delta)$ we have an isomorphism

$$K_p \cong K_\infty.$$

$\iota^* p_2^* \omega_{FS}$ restricts to a Kähler metric on $p^{-1}(z)$ $(z \in S_\infty)$ and hence induces a Hermitian metric on the bundle $K_\infty$. We denote its curvature by $R(\iota^* p_2^*(\omega_{FS}))$. Let $g_{\mathbf{X}}$ and $g_S$ denote two Kähler metrics on $\mathbf{X}$ and $S$ respectively. In this way we obtain a metric on the relative canonical bundle $K_p$. We let $R_f$ denote its curvature

$$R_p := R(g_{\mathbf{X}}) - p^* R(g_S).$$

In this way we obtain *two* metrics on the relative canonical bundle over the smooth locus. The crucial point is the following fact.

*The curvatures of these metrics are not the same.*

The relation between them is given in the following proposition (see [**17**] Lemma 8.5 pg. 31).

***Proposition 1* ("$\partial\overline{\partial}$ lemma along the fibers").** — *There is a smooth function* $\Psi : \mathbf{X} \setminus p^{-1}(\Delta) \to \mathbb{R}$ *such that*

*1)* $R(g_{\mathbf{X}}) - p^*R(g_S) + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\Psi = R(\iota^* p_2^*(\omega_{FS}))$;

*2)* $\Psi \leq C$, *for some constant* $C$.

***Example 1.*** — *(The universal family of hypersurfaces of degree* $d$ *in* $\mathbb{C}P^{n+1}$*)*

$$S := \mathbb{P}(H^0(\mathbb{C}P^{n+1}, \mathcal{O}(d)))$$
$$\mathbf{X} := \{([f],[z]) \in S \times \mathbb{C}P^{n+1} \mid f(z) = 0\}$$
$$p := p_1 \quad \textit{(projection onto the first factor)}.$$

*Let* $|||.|||$ *denote any norm on* $H^0(\mathbb{C}P^{n+1}, \mathcal{O}(d))$, *with associated Fubini-Study metric* $\omega_S$. *Then a computation shows that*

$$\Psi(([f],[z])) = \log\left(\frac{\sum_{i=0}^{n+1} |\frac{\partial f}{\partial z_i}(z)|^2}{|||f|||^2 ||z||^{2(d-1)}}\right).$$

The next result is a *pointwise* version of (4.1).

***Proposition 2.*** — *There is a continuous Hermitian metric* $||\ ||$ *on* $\mathbb{L}_1^{-1}$ *such that, in the sense of currents we have*

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(||\ ||^2) = (n+1)p_*(R(g_{\mathbf{X}}) - p^*R(g_S))p_2^*(\omega_{FS})^n + \mu p_* p_2^*(\omega_{FS})^{n+1}.$$

*Proof.* — See Proposition 4.3 pg. 2576 of [**13**]. □

Now we pull back the curvature form of $K_\infty$ to $G\mathbf{X}_z$

$$R_{G|\mathbf{X}_z} := p_{z,2}^*(R(\pi_2^*(\omega_{FS}))).$$

Recall that for $\sigma \in G$ we define $\varphi_\sigma$ by the relation

$$\sigma^*\omega_{FS} = \omega_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi_\sigma.$$

Let $\nu_{\omega,z}(\sigma)$ denote the K energy of $(\mathbf{X}_z, \omega_{FS})$ applied to the potential $\varphi_\sigma$. With these notations in place we have the following result.

***Proposition 3* (The complex Hessian of the K-Energy map on $G$)**

*For every smooth compactly supported* $(N^2+2N-1, N^2+2N-1)$ *form* $\eta$ *on* $G$ *we have*

$$d(n+1)\int_G \nu_{\omega,z}(\varphi_\sigma)\partial\overline{\partial}\eta = \int_{G\mathbf{X}_z}((n+1)R_{G|\mathbf{X}_z} + \mu p_2^*(\omega_{FS})) \wedge p_2^*(\omega_{FS})^n \wedge p_{z,1}^*\eta.$$

The proof of Proposition 3 appears in the next section after some standard preliminaries on Bott Chern classes.

**4.1. Bott Chern secondary classes.** — Let $\phi$ be a $GL_N(\mathbb{C})$ invariant polynomial on $M_{N\times N}(\mathbb{C})$ homogeneous of degree $d$. $\phi_1$ denotes the *complete polarization* of $\phi$. Let $E$ be a holomorphic vector bundle of rank $N$ over a base $X$. Let $h_1$ and $h_0$ be two Hermitian metrics on $E$ and $\frac{\sqrt{-1}}{2\pi}R(h_i)$ the curvatures. Then we define the *Bott-Chern class* $BC(\phi, E; h_0, h_1)$ by the expression

$$BC(\phi, E; h_0, h_1) := \int_0^1 \phi_1(h_t^{-1}\dot{h}_t, \overbrace{\frac{\sqrt{-1}}{2\pi}R_t, \ldots, \frac{\sqrt{-1}}{2\pi}R_t}^{d-1})dt \in \Omega_X^{(d-1,d-1)} \tag{4.2}$$

where $h_t$ is any piecewise $C^1$ path of Hermitian metrics joining $h_0$ and $h_1$. The point of the construction is the following identity:

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}BC(\phi, E; h_0, h_1) = \left(\frac{\sqrt{-1}}{2\pi}\right)^d (\phi(R_{h_0}) - \phi(R_{h_1})).$$

Let $d = n + 1$ where, $n = dim(X)$ in this case $BC(\phi, E; h_0, h_1)$ has top dimension and we may introduce the *Donaldson Functional associated to* $\phi$.

$$D_E(h_0, h_1) := \int_X BC(\phi, E; h_0, h_1). \tag{4.3}$$

When $h_0$ is fixed, we consider it to be a functional on $\mathcal{M}_E$ (the space of hermitian metrics on $E$). In what follows we take $\phi = Ch_{n+1}$, the $n+1^{st}$ component of the chern character. We can extend the Donaldson functional to "virtual bundles" $\mathcal{E} = E - F$ by observing that a Hermitian metric $h$ on $\mathcal{E}$ is just a pair of metrics, one on $E$ and one on $F$:

$$h = (h^E, h^F).$$

We set

$$BC(\phi, \mathcal{E}; h_0, h_1) := BC(\phi, E; h_0^E, h_1^E) - BC(\phi, F; h_0^F, h_1^F). \tag{4.4}$$

Let $h : Y \to \mathcal{M}_{\mathcal{E}}$ be a smooth map, where $Y$ is a complex manifold of dimension $m$.

***Lemma 4.1.*** — *Let $\phi$ be homogeneous of degree $n + 1$ and $h_0$ a fixed metric on $\mathcal{E}$. Then for all smooth compactly supported forms $\psi$ of type $(m - 1, m - 1)$ we have the identity*

$$\frac{\sqrt{-1}}{2\pi}\int_Y D_{\mathcal{E}}(\phi; h_0, h(y))\partial_Y\overline{\partial}_Y\psi = \int_{Y\times X}\phi(R(\frac{\sqrt{-1}}{2\pi}h(y))) \wedge \pi_1^*(\psi). \tag{4.5}$$

Next we want to realize the Mabuchi K-energy as the Donaldson functional, with respect to the polynomial $\phi = Ch_{n+1}$, of a certain virtual bundle to be defined below. Then proposition (3) follows at once from the preceding lemma.

Let $X$ be a complex projective manifold (in our present application $X$ is a smooth fiber of $\mathbf{X} \xrightarrow{p} S$), and let $L$ be the restriction of $\underline{O}(1)$ to $X$. Let $\varphi$ be a kahler potential.

The two metrics $h_{FS}$ and $e^{-\varphi}h_{FS}$ induce metrics on the canonical bundle $\mathcal{K}$. We consider the virtual bundle

$$2^{n+1}\mathcal{E} := (n+1)(\mathcal{K}^{-1} - \mathcal{K})(L - L^{-1})^n - \mu(L - L^{-1})^{n+1}. \tag{4.6}$$

Here $\mu$ is the average of the scalar curvature. We need to calculate the following terms.

$$\begin{aligned} &BC(\phi; \mathcal{K}^{-1} \otimes L^{n-2j}, h_0, h_1) \\ &BC(\phi; \mathcal{K} \otimes L^{n-2j}, h_0, h_1) \\ &BC(\phi; L^{n+1-2j}, h_0, h_1). \end{aligned} \tag{4.7}$$

The path of metrics for the first two expression are given as follows.

$$\begin{aligned} h_{\mathcal{K}^{-1}\otimes L^{n-2j},t} &:= \det(g_{\alpha\overline{\beta}} + t\frac{\partial^2}{\partial z_\alpha \partial \overline{z_\beta}}\varphi)e^{-t(n-2j)\varphi}h_{FS}^{n-2j} \\ h_{\mathcal{K}\otimes L^{n-2j},t} &:= \det(g_{\alpha\overline{\beta}} + t\frac{\partial^2}{\partial z_\alpha \partial \overline{z_\beta}}\varphi)^{-1}e^{-t(n-2j)\varphi}h_{FS}^{n-2j}. \end{aligned} \tag{4.8}$$

The complete polarization of $\phi$ is given by

$$\phi_1(B, A \dots A) = tr(BA^n) \quad A, B \in M_k(\mathbb{C}). \tag{4.9}$$

Therefore,

(4.10)

$$\begin{aligned} BC(\mathcal{K}^{-1} \otimes L^{n-2j}, h_0, h_1) &= \int_0^1 (\Delta_{t\varphi}\varphi - (n-2j)\varphi)((n-2j)\omega_{t\varphi} + Ric_{\omega_t})^n dt \\ BC(\mathcal{K} \otimes L^{n-2j}, h_0, h_1) &= -\int_0^1 (\Delta_{t\varphi}\varphi + (n-2j)\varphi)((n-2j)\omega_{t\varphi} - Ric_{\omega_t})^n dt. \end{aligned}$$

Similarly we have

$$BC(L^{n+1-2j}, h_0, h_1) = -(n+1-2j)^{n+1}\int_0^1 \varphi\omega_t^n dt \quad \omega_t := \omega + t\partial\overline{\partial}\varphi. \tag{4.11}$$

We see that

(4.12)

$$BC((L - L^{-1})^{n+1}, h_{FS}, e^{-\varphi}h_{FS}) = -\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}(n+1-2j)^{n+1}\int_0^1 \varphi\omega_t^n dt.$$

Now we need the following numerical identity.

$$\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}(n+1-2j)^i = \begin{cases} 0 & i < n+1 \text{ or } i = n+2 \\ (n+1)!2^{n+1} & i = n+1. \end{cases} \tag{4.13}$$

It follows at once that

$$\int_X BC((L - L^{-1})^{n+1}, h_{FS}, e^{-\varphi}h_{FS}) = -(n+1)!2^{n+1}\int_0^1\int_X \varphi\omega_t^n dt. \tag{4.14}$$

It follows from (3.9) that

$$BC(\mathcal{K}^{-1}\otimes L^{n-2j}) = \int_0^1 \Delta_{t\varphi}\varphi \sum_{i=0}^{n}\binom{n}{i}(n-2j)^i Ric_t^{n-i}\omega_t^i - \int_0^1 \sum_{i=0}^{n}\binom{n}{i}(n-2j)^{i+1}\varphi Ric_t^{n-i}\omega_t^i.$$

We use the identity (4.13) to see that

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}BC(\mathcal{K}^{-1}\otimes L^{n-2j}) = n!2^n\int_0^1\left(\Delta_t\varphi\omega_t^n - \varphi n Ric_t\omega_t^{n-1}\right)dt. \tag{4.15}$$

Similarly we have the second term

$$\sum_{j=0}^{n}(-1)^{j+1}\binom{n}{j}BC(\mathcal{K}\otimes L^{n-2j}) = n!2^n\int_0^1\left(\Delta_t\varphi\omega_t^n - \varphi n Ric_t\omega_t^{n-1}\right)dt. \tag{4.16}$$

The next lemma follows at once from summing up (4.15), (4.16), and (4.14).

***Lemma 4.2.*** — *Let $D(\mathcal{E}, h_{FS}, e^{-\varphi}h_{FS})$ denote the Donaldson functional of $Ch_{n+1}$ with respect to $\mathcal{E}$. Then the following identity holds.*

$$D(\mathcal{E}, h_{FS}, e^{-\varphi}h_{FS}) = \nu_\omega(\varphi) \tag{4.17}$$

Let $\varphi = \varphi_\sigma$ and apply 4.5 to Lemma 4.2 to conclude the proof of Proposition 3. □

Next we observe that the identity

$$R_{G|\mathbf{X}_z} = p_{2,z}^*\left(R(g_{\mathbf{X}}) - p^*R(g_S) + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\Psi\right) \tag{4.18}$$

together with the previous lemmas yields the following corollary.

***Corollary 3.*** — *The function*

$$\sigma\in G\to D(\sigma) := d(n+1)\nu_{\omega,z}(\sigma) - \log\left(e^{(n+1)\Psi_S(\sigma z)}\frac{||\ ||^2(\sigma z)}{||\ ||^2(z)}\right)$$

*is pluriharmonic. Where we have defined $\Psi_S(z) := \int_{\{y\in f^{-1}(z)\}}\Psi(y)p_2^*(\omega_{FS})^n$.*

*Moreover $\Psi_S(z)\le C$ on $S\setminus\Delta$, extends continuously to the locus of reduced fibers, and $\lim_{z\to z_\infty}\Psi_S(z) = -\infty$ whenever $\mathbf{X}_{z_\infty}$ is non-reduced.*

***Remark 6.*** — *The construction of $\Psi$ and $\Psi_S$ as well as their behavior on the locus of singular fibers can be seen directly in Example 1. The general case is treated in Lemma 8.5 pg. 31 in* [**17**].

Since $\pi_1(G) = 1$ there is a (nonvanishing) entire function $\xi$ on $G$ such that

$$D(\sigma) = \log(|\xi(\sigma)|^2).$$

An analysis of the growth of this function on the standard compactification $\overline{G}$

$$\overline{G} := \{[(w_{ij}, z)]\in\mathbb{P}^{(N+1)^2} : \det(w_{ij}) = z^{N+1}\}$$

reveals that it must reduce to a *constant.*

Tying everything together establishes our main result.

***Theorem 1* (The K-Energy as a singular metric on $\mathbb{L}_1^{-1}$).** — *We have*

$$d(n+1)\nu_{\omega,z}(\sigma) = \log\left(e^{(n+1)\Psi_S(\sigma z)}\frac{||\ ||^2(\sigma z)}{||\ ||^2(z)}\right). \tag{4.19}$$

We proceed to the proof of Corollary 1. First substitute $\sigma = \lambda(t)$ in (4.19). Then we have the string of identities.

$$\begin{aligned} d(n+1)\nu_{\omega,z}(\lambda(t)) &= \log\left(e^{(n+1)\Psi_S(\lambda(t)z)}\frac{||\ ||^2(\lambda(t)z)}{||\ ||^2(z)}\right) \\ &= (n+1)\Psi_S(\lambda(t)z) + \log\left(\frac{||\ ||^2(\lambda(t)z)}{||\ ||^2(z)}\right) \\ &= (n+1)\Psi_S(\lambda(t)z) + \log\left(\frac{||\ ||^2(t^{w_\lambda(z)-w_\lambda(z)}\lambda(t)z)}{||\ ||^2(z)}\right) \\ &= (n+1)\Psi_S(\lambda(t)z) + w_\lambda(z)\log(|t|^2) + O(1) \\ &= F_1(\lambda)\log(|t|^2) + (n+1)\Psi_S(\lambda(t)z) + O(1). \end{aligned}$$

The passage from line 3 to 4 follows from the defining property of the weight (see the introduction to [**14**]). The passage from line 4 to 5 is the statement of (1.5).

Rationality of the contribution from $\Psi_S(\lambda(t)z)$ follows easily from [**13**] Theorem 3.5 pg. 2564 and Zhiqin Lu's explicit computation of the asymptotics of the K-Energy on hypersurfaces (see [**9**]). This completes the proof of Corollary 1. □

**4.2. Properness Implies that $F_1(\lambda) < 0$.** — Let $X := \mathbf{X}_z$ a smooth fiber of $p$. Recall that the algebraic potential associated to a one parameter subgroup $\lambda$ is given by

$$\varphi_t := \varphi_{\lambda(t)} = \log(\sum_{i=0}^{N} t^{2q_i}||S_i||^2).$$

Then, as we have seen, $\varphi_t \in P(X,\omega)$. Following Yau [**18**], our plan is to use the standard Moser iteration to control $Osc(\varphi_t)$ by $I_\omega(\varphi_t)$. Define

$$\varphi_- := \mathrm{Max}\{-\varphi_t, 1\} \geq 1.$$

Let $p \in \mathbb{Z}_+$. Then we have the (obvious ) inequality

$$\varphi_-^p \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi \wedge \omega_\varphi^{n-1} \leq \varphi_-^p \omega_\varphi^n.$$

Trivially this implies

$$\int_X \varphi_-^p \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi \wedge \omega_\varphi^{n-1} \leq \int_X \varphi_-^p \omega_\varphi^n \leq \int_X \varphi_-^{p+1}\omega_\varphi^n.$$

Next integrate by parts on the leftmost side of this inequality

$$\begin{aligned}\int_X \varphi_-^p \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi \wedge \omega_\varphi^{n-1} &= -\int_X \frac{\sqrt{-1}}{2\pi}\partial\varphi_-^p \wedge \overline{\partial}\varphi\, \omega_\varphi^{n-1} \\ &= \int_X \frac{\sqrt{-1}}{2\pi}\partial\varphi_-^p \wedge \overline{\partial}\varphi_- \omega_\varphi^{n-1} \\ &= \frac{4p}{(p+1)^2}\frac{\sqrt{-1}}{2\pi}\int_X \partial\varphi_-^{\frac{p+1}{2}} \wedge \overline{\partial}\varphi_-^{\frac{p+1}{2}} \wedge \omega_\varphi^{n-1}.\end{aligned}$$

Since $\varphi_- \geq 1$ we deduce the gradient estimate

$$\frac{4p}{(p+1)^2}\frac{\sqrt{-1}}{2\pi}\int_X \partial\varphi_-^{\frac{p+1}{2}} \wedge \overline{\partial}\varphi_-^{\frac{p+1}{2}} \wedge \omega_\varphi^{n-1} \leq \int_X \varphi_-^p \omega_\varphi^n \leq \int_X \varphi_-^{p+1}\omega_\varphi^n.$$

We concentrate on the outermost inequality

$$\frac{4p}{n(p+1)^2}\int_X ||\nabla_{\varphi_t}\varphi_-^{\frac{p+1}{2}}||^2_{\varphi_t}\omega_{\varphi_t}^n \leq \int_X \varphi_-^{p+1}\omega_\varphi^n.$$

Now we invoke the Sobolev inequality

$$\left(\int_X \varphi_-^{\frac{(p+1)n}{n-1}}\frac{\omega_\varphi^n}{V}\right)^{\frac{n-1}{n}} \leq \mathcal{C}(\varphi_t)\left(\int_X ||\nabla_{\varphi_t}\varphi_-^{\frac{p+1}{2}}||^2_{\varphi_t}\frac{\omega_{\varphi_t}^n}{V} + \int_X \varphi_-^{p+1}\frac{\omega_\varphi^n}{V}\right).$$

$\mathcal{C}(\varphi_t)$ is the Sobolev constant of the metric $\omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi_t$. Concerning this constant we have the crucial

***Proposition 4*** **([11], [7]).** — *There is a positive constant $\delta = \delta(n)$ such that for all $\sigma \in SL(N+1,\mathbb{C})$ we have*

$$\mathcal{C}(\varphi_\sigma) < \delta.$$

This follows from the fact the complex projective subvarieties are *minimal* as Riemannian submanifolds of $\mathbb{P}^N$ and hence have vanishing mean curvature.

Therefore,

$$\left(\int_X \varphi_-^{\frac{(p+1)n}{n-1}}\frac{\omega_\varphi^n}{V}\right)^{\frac{n-1}{n}} \leq n(p+1)\delta\int_X \varphi_-^{p+1}\frac{\omega_\varphi^n}{V}.$$

Now extract the $p+1$st root of both sides to get

$$\left(\int_X \varphi_-^{\frac{(p+1)n}{n-1}}\frac{\omega_\varphi^n}{V}\right)^{\frac{n-1}{n(p+1)}} \leq (n(p+1)\delta)^{\frac{1}{p+1}}\left(\int_X \varphi_-^{p+1}\frac{\omega_\varphi^n}{V}\right)^{\frac{1}{p+1}}.$$

Now we start the standard iteration: Let $p_0 := 1$ and $p_{j+1}+1 := \frac{n}{n-1}(p_j+1)$. Then we have that

$$\begin{aligned}||\varphi_-||_{p_{j+1}+1} &\leq C^{\frac{1}{p_j+1}}(p_j+1)^{\frac{1}{p_j+1}}||\varphi_-||_{p_j+1} \leq \cdots \\ &\leq C^{\sum_{i=0}^{\frac{1}{p_i+1}}}\prod_{i=0}^{j}(p_i+1)^{\frac{1}{p_j+1}}||\varphi_-||_2.\end{aligned}$$

That is to say

$$||\varphi_-||_{p_j+1+1} \leq C^{\sum_{i=0}^{\frac{1}{p_i+1}}} \prod_{i=0}^{j} (p_i+1)^{\frac{1}{p_j+1}} ||\varphi_-||_2.$$

It is not hard to check that the infinite product converges. Taking limits as $j \longrightarrow \infty$ gives

$$||\varphi_-||_\infty \leq C \left( \int_X \varphi_-^2 \frac{\omega_\varphi^n}{V} \right)^{\frac{1}{2}} \leq ||\varphi_-||_\infty^{\frac{1}{2}} C \left( \int_X \varphi_- \frac{\omega_\varphi^n}{V} \right)^{\frac{1}{2}}.$$

Which implies

$$||\varphi_-||_\infty \leq C^2 \left( \int_X \varphi_- \frac{\omega_\varphi^n}{V} \right).$$

Since $\varphi_t \leq C$ as $t \to 0$ we have

$$-\inf_X \varphi_t \leq C_1 \int_X (-\varphi_t) \frac{\omega_\varphi^n}{V} + C_2.$$

Now, by the Green identity we deduce

$$\mathrm{Osc}_X(\varphi_t) := \mathrm{Sup}_X(\varphi_t) - \mathrm{Inf}_X(\varphi_t) \leq C_1 \left( \int_X \varphi_t \omega^n - \int_X \varphi_t \omega_\varphi^n \right) + C_2.$$

Using the properness assumption gives:

$$f(\mathrm{Osc}_X(\varphi_t)) \leq \nu_\omega(\varphi_t).$$

Now we are prepared to complete the proof of the corollary.

*Case 1:* Assume that $X^{\lambda(0)} \neq X$ and moreover that $X^{\lambda(0)}$ is reduced, then by the same argument as in [**17**] we have

$$\lim_{t \to 0} \mathrm{Osc}_X(\varphi_t) \to \infty.$$

Consequently we deduce that

$$\lim_{t \to 0} \nu_\omega(\varphi_t) \to \infty.$$

Corollary 1 yields the precise asymptotics [(5)]

$$\nu_\omega(\varphi_{\lambda(t)}) = F_1(\lambda) \log(t^2) + O(1).$$

This forces the desired sign $F_1(\lambda) < 0$.

*Case 2:* If $X^{\lambda(0)}$ is nonreduced, then $\Psi(\lambda(t)) \to -\infty$, however, under the properness assumption the K-Energy is bounded from below, and we again have that $F_1(\lambda) < 0$. This completes the proof of Corollary 2. □

(5) Recall that when $X^{\lambda(0)}$ is multiplicity free $\Psi(\lambda(t)) = O(1)$.

## References


[1] S. BANDO & T. MABUCHI – "Uniqueness of Einstein Kähler metrics modulo connected group actions", in *Algebraic geometry, Sendai, 1985*, Adv. Stud. Pure Math., vol. 10, North-Holland, 1987, p. 11–40.

[2] D. CATLIN – "The Bergman kernel and a theorem of Tian", in *Analysis and geometry in several complex variables (Katata, 1997)*, Trends Math., Birkhäuser, 1999, p. 1–23.

[3] X. CHEN & G. TIAN – "Geometry of kahler metrics and foliations by holomorphic discs", preprint arXiv:math.DG/0507148, 2005.

[4] W. Y. DING & G. TIAN – "Kähler-Einstein metrics and the generalized Futaki invariant", *Invent. Math.* **110** (1992), p. 315–335.

[5] S. K. DONALDSON – "Scalar curvature and stability of toric varieties", *J. Differential Geom.* **62** (2002), p. 289–349.

[6] A. FUTAKI – "An obstruction to the existence of Einstein Kähler metrics", *Invent. Math.* **73** (1983), p. 437–443.

[7] D. HOFFMAN & J. SPRUCK – "Sobolev and isoperimetric inequalities for Riemannian submanifolds", *Comm. Pure Appl. Math.* **27** (1974), p. 715–727.

[8] F. F. KNUDSEN & D. MUMFORD – "The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div"", *Math. Scand.* **39** (1976), p. 19–55.

[9] Z. LU – "$K$ energy and $K$ stability on hypersurfaces", *Comm. Anal. Geom.* **12** (2004), p. 601–630.

[10] T. MABUCHI – "$K$-energy maps integrating Futaki invariants", *Tohoku Math. J.* **38** (1986), p. 575–593.

[11] J. H. MICHAEL & L. M. SIMON – "Sobolev and mean-value inequalities on generalized submanifolds of $R^n$", *Comm. Pure Appl. Math.* **26** (1973), p. 361–379.

[12] D. MUMFORD – "Stability of projective varieties", *Enseignement Math.* **23** (1977), p. 39–110.

[13] S. T. PAUL & G. TIAN – "Analysis of geometric stability", *Int. Math. Res. Not.* **2004** (2004), p. 2555–2591.

[14] ———, "CM Stability and the generalised Futaki invariant I", preprint arXiv:math.AG/0605278, 2006.

[15] W.-D. RUAN – "Canonical coordinates and Bergmann [Bergman] metrics", *Comm. Anal. Geom.* **6** (1998), p. 589–631.

[16] G. TIAN – "On a set of polarized Kähler metrics on algebraic manifolds", *J. Differential Geom.* **32** (1990), p. 99–130.

[17] ———, "Kähler-Einstein metrics with positive scalar curvature", *Invent. Math.* **130** (1997), p. 1–37.

[18] S. T. YAU – "On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I", *Comm. Pure Appl. Math.* **31** (1978), p. 339–411.

[19] S. ZELDITCH – "Szegő kernels and a theorem of Tian", *Int. Math. Res. Not.* **1998** (1998), p. 317–331.



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