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AN EXPLICIT PROOF OF THE GENERALIZED GAUSS-BONNET FORMULA

by

Henri Gillet & Fatih M. Ünlü

To Jean-Michel Bismut on the occasion of his 60^th birthday

Abstract. — In this paper we construct an explicit representative for the Grothendieck fundamental class $[Z] \in \mathrm{Ext}^r(\mathcal{O}_Z, \Omega^r_X)$ of a complex submanifold Z of a complex manifold X when Z is the zero locus of a real analytic section of a holomorphic vector bundle E of rank r on X. To this data we associate a super-connection A on $\bigwedge^* E^\vee$, which gives a "twisted resolution" T^* of $\mathcal{O}_Z$ such that the "generalized super-trace" of $\frac{1}{r!}A^{2r}$, which is a map of complexes from T^* to the Dolbeault complex $\mathcal{A}^{r,*}_X$, represents $[Z]$. One may then read off the Gauss-Bonnet formula from this map of complexes.

Résumé **(Une démonstration explicite de la formule de Gauss-Bonnet généralisée)**
Dans cet article nous construisons un représentant explicite de la classe fondamentale de Grothendieck $[Z] \in \mathrm{Ext}^r(\mathcal{O}_Z, \Omega^r_X)$ d'une sous-variété Z dans une variété lisse complexe X quand Z est le lieu des zéros d'une section réelle analytique d'un fibré vectoriel holomorphe E de rang r sur X. Nous associons à cette donnée une super-connection A sur $\bigwedge^* E^\vee$, qui fournit une « résolution tordue » T^* de $\mathcal{O}_Z$ telle que la « super-trace généralisée » de $\frac{1}{r!}A^{2r}$, qui est un morphisme de complexes de T^* vers le complexe de Dolbeault $\mathcal{A}^{r,*}_X$, représente $[Z]$. On peut alors lire la formule de Gauss-Bonnet à partir de cette application entre complexes.

Introduction

If X is a complex manifold, and τ is a holomorphic section, transverse to the zero section, of the dual $E^\vee$ of a rank r holomorphic vector bundle, it is well known that

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the fundamental class of the locus Z of zeros of τ is equal to the top Chern class of the bundle $E^\vee$:

$$[Z] = c_r(E^\vee) = (-1)^r c_r(E)$$

For Hodge cohomology, this is the fact that the image of the Grothendieck fundamental class

$$[Z] \in \mathrm{Ext}^r(\mathcal{O}_Z, \Omega^r_X)$$

under the map

$$\mathrm{Ext}^r(\mathcal{O}_Z, \Omega^r_X) \to \mathrm{Ext}^r(\mathcal{O}_X, \Omega^r_X) = \mathrm{H}^r(X, \Omega^r_X)$$

coincides with the top Chern class of $E^\vee$. Proofs of this result tend to be indirect, i.e. they depend on the axioms for cycle classes and Chern classes, and comparison with "standard" cases.

However, one may observe that the section τ gives rise to an explicit global Koszul resolution

$$K^*(\tau) = (\bigwedge{}^* E^\vee, \iota_\tau) \to \mathcal{O}_Z\ ,$$

and so the theorem can be rephrased as saying that image of $[Z]$ under the map:

$$\mathrm{Ext}^r(K^*(\tau), \Omega^r_X) \to \mathrm{Ext}^r(\mathcal{O}_X, \Omega^r_X)$$

induced by the isomorphism $\mathcal{O}_X \simeq K^0(\tau)$, is the top Chern class of $E^\vee$. Our first result is to show that a choice of connection $\widetilde{\nabla}$ on E, determines, via Chern-Weil theory applied to superconnections, an *explicit* map of complexes from the Koszul complex $K^*(\tau)$ to the Dolbeault complex of Ω^r_X, which represents the Grothendieck fundamental class and the restriction of which to the degree zero component $\mathcal{O}_X$ of the Koszul complex is *precisely* multiplication by the r-th Chern form of $E^\vee$.

One motivation for the current paper was to obtain a better understanding of the proof by Toledo and Tong of the Hirzebruch-Riemann-Roch theorem in [**12**]. In that paper the authors used local Koszul resolutions of the structure sheaf of the diagonal $\Delta_X \subset X \times X$ to construct the Grothendieck fundamental class $[\Delta_X]$, and then to compute $\chi(X, \mathcal{O}_X)$ as the degree of the restriction of the appropriate Kunneth-component of $[\Delta_X]$ to the diagonal. For such a computation one needs only the existence of a "nice" representative of the Grothendieck fundamental class in some neighborhood of the diagonal. However the diagonal Δ_X is *not* in general the zero set of a holomorphic section of a vector bundle. Instead one can use the "holomorphic exponential map" (see the article [**10**] for an exposition) to construct, in a neighborhood of the diagonal, a *real analytic* section of $p^*(T_X)$, which vanishes exactly on the diagonal. (Here $p : X \times X \to X$ is the projection onto the first factor.) Thus we are led to consider what happens if we ask only that τ be real analytic rather than holomorphic. In our second main result, we use the theory of superconnections and twisted complexes in the style of Brown [**5**], and of Toledo and Tong (*op. cit.*) to construct a map from the Dolbeault resolution of $K^*(\tau)$ to that of Ω^r_X representing the Grothendieck fundamental class and which restricts to the r-th Chern form of $E^\vee$. An important tool in this construction is a non-commutative version of the supertrace for endomorphisms of Grassman algebras.

We should also remark that instead of working in the real analytic category, one can make a very similar argument in the algebraic category, using formal schemes.

Let us now give a more detailed outline of the paper. Recall that the section τ gives rise to a natural Koszul resolution $K(\tau)^* \to \mathcal{O}_Z$, in which $K(\tau)^{-j} = \bigwedge^j \mathcal{E}$. Here $\mathcal{E}$ is the sheaf of holomorphic sections of E. Choose a connection $\nabla : \mathcal{A}_X \otimes \mathcal{E} \to \mathcal{A}_X^{1,0} \otimes \mathcal{E}$ of type $(1,0)$ ($\mathcal{A}_X^{1,0}$ being the sheaf of real analytic $(1,0)$-forms on X) on $\mathcal{E}$, such that $\nabla^2 = 0$. Let $\widetilde{\nabla} = \nabla + \overline{\partial}$ be the associated connection. We view ∇ as acting not only on $\mathcal{E}$, but on all tensor constructions on $\mathcal{E}$. Then our first result is:

***Theorem* (A).** — *The connection ∇ and the section τ determine a map of complexes, from the Koszul resolution $K(\tau)^*$ of $\mathcal{O}_Z$, to the Dolbeault resolution $\mathcal{A}_X^{r,*}[r]$ of $\Omega_X^r[r]$*

$$\psi : K(\tau)^* \to \mathcal{A}_X^{r,*}[r]$$

the degree $-r$ component of which is $\frac{1}{r!}(\imath_{\nabla(\tau)})^r$, and the degree 0 component $K(\tau)^0 = \mathcal{O}_X \to \mathcal{A}_X^{r,*}[r]^0 = \mathcal{A}_X^{r,r}$ of which is represented by the r-th Chern form of $(E^\vee, \widetilde{\nabla})$. In general ψ is given by a linear algebra construction involving ∇ and the curvature $R = [\nabla, \overline{\partial}]_s$ of $\widetilde{\nabla}$, and we have:

- *The class in $\mathrm{Ext}^r_{\mathcal{O}_X}(\mathcal{O}_Z, \Omega_X^r)$ represented by ψ is the Grothendieck fundamental class $[Z]$.*
- *The image of $[Z]$ in $\mathrm{Ext}^r_{\mathcal{O}_X}(\mathcal{O}_X, \Omega_X^r) \simeq H^{r,r}(X, \mathbb{C})$, is represented by the degree zero component of ψ, which is equal to the r-th Chern form $c_r(E^\vee, \widetilde{\nabla})$*

It follows immediately that the image of $[Z]$ in $H^{r,r}(X, \mathbb{C})$ is equal to $c_r(E^\vee)$.

The proof of Theorem A is contained in Section 5. (cf. Theorem 5.5 and Corollary 5.6).

In the second half of the paper, we extend Theorem A to the case where Z is the zero locus of a real analytic section of $E^\vee$. It is no longer the case that τ determines a Koszul resolution of $\mathcal{O}_Z$, but instead we get a resolution of $\mathcal{A}_X^{0,*} \otimes \mathcal{O}_Z$. In order to get a complex that is quasi-isomorphic to $\mathcal{O}_Z$, we construct a resolution of the Dolbeault resolution $\mathcal{A}_X^{0,*} \otimes \mathcal{O}_Z$ of $\mathcal{O}_Z$, by constructing a *twisted differential*, δ, in the sense of Toledo and Tong [**13**], on $\mathcal{A}_X^{0,*} \otimes \bigwedge^* \mathcal{E}$.

A key tool in extending Theorem A to this situation is the notion of the "generalized supertrace" of an endomorphism of the exterior algebra of a finitely generated projective module. Suppose that V is a (locally) free module of finite rank r over a commutative ring k. Then the generalized supertrace is a map

$$\mathrm{Tr}_\Lambda : \mathrm{End}_k(\textstyle\bigwedge^* V) \to \bigwedge^* V^\vee$$

(c.f. Definition 6.1). If A is a graded-commutative algebra over k, we can extend this to a map

$$\mathrm{Tr}_\Lambda : \mathrm{End}_A(A \widehat{\otimes} \textstyle\bigwedge^* V) \to A \widehat{\otimes} \bigwedge^* V^\vee$$

Here $\widehat{\otimes}$ denotes the "super" or graded tensor product. If $\varphi \in \mathrm{End}_A(\bigwedge^* V)$, then the degree 0 component of $\mathrm{Tr}_\Lambda(\varphi)$ is the usual super-trace of φ. The key property of Tr_Λ (which is proved in Section 3.) is:

Proposition. — *Assume that* $\varphi \in \mathrm{End}_A(\bigwedge^* V)$*, and let* $\delta \in \mathrm{End}_A(\bigwedge^* V)$ *be an* A*-linear superderivation. Then*

$$\mathrm{Tr}_\Lambda[\delta, \varphi]_s = [\delta, \mathrm{Tr}_\Lambda(\varphi)]_s$$

***Theorem* (B)**. — *Let* Z *be a complex submanifold of* X *such that there exists a holomorphic vector bundle* $\pi : E \to X$ *and* $\tau \in \Gamma(X, \mathcal{a}_X \otimes \mathcal{E}^\vee)$ *such that* $\iota_\tau : \mathcal{a}_X \otimes \mathcal{E} \to \mathcal{a}_X \otimes \mathcal{J}_Z$ *is surjective. Then*

- *There is a superconnection* δ *of type (0,1), on the super-bundle* $\bigwedge^* E$*, such that:*
  1. $\delta^2 = 0$*, so* δ *defines a differential on* $\mathcal{a}_X^{0,*} \otimes \bigwedge^* \mathcal{E}$*,*
  2. *the component of* δ *of degree* -1 *with respect to the grading on* $\bigwedge^* E$ *is the Koszul differential* ι_τ*,*
  3. *If we write* δ *for the induced differential on* $\mathcal{a}_X^{0,*} \otimes \bigwedge^* \mathcal{E}$*, then the map* $\bigwedge^0 \mathcal{E} = \mathcal{O}_X \to \mathcal{O}_Z$ *induces a quasi-isomorphism of complexes:*

$$(\mathcal{a}_X^{0,*} \otimes \textstyle\bigwedge^* \mathcal{E}, \delta) \xrightarrow{\sim} (\mathcal{a}_X^{0,*} \otimes \mathcal{O}_Z, \bar{\partial}) \xleftarrow{\sim} \mathcal{O}_Z$$

- *Let* R_A *be the curvature of the superconnection* $A = \nabla + \delta$ *on* $\bigwedge^* E$*. Then the generalized supertrace of* $\frac{1}{r!} R_A^r$ *defines a map of complexes*

$$\mathcal{a}_X^{0,*} \otimes \textstyle\bigwedge^* \mathcal{E} \to \mathcal{a}_X^{r,*}[r],$$

  which, via the quasi-isomorphisms in part 1), represents the Grothendieck fundamental class $[Z]$*,*
- *The image of* $[Z]$ *in* $H^{r,r}(X, \mathbb{C})$ *is represented by the degree 0 component of the generalized supertrace of* $\frac{1}{r!} R_A^r$*, i.e., by the super-trace of* $\frac{1}{r!} R_A^r$*, which by Quillen* **[11]** *is an* (r,r)*-form representing the Chern character* $\mathrm{ch}_r(\bigwedge^* E)$*.*

The proof of Theorem B is contained in Proposition 8.4, Theorem 10.3, and Corollary 10.5. We would like to thank the referee for comments which let to a substantial improvement in the organization of the paper.

1. Superobjects

Throughout this paper we will use the language of *super-objects*. We include here basic definitions and properties for the convenience of the reader and to fix notation. We omit the details and proofs, which may be found in **[11]** and **[4]**.

Let k be a commutative ring with unity .

Definition 1.1. — *A* k*-module* V *with a* $\mathbb{Z}/2\mathbb{Z}$*-grading is called a* k*-supermodule.*

Remark 1.2. — In the same spirit, a $\mathbb{Z}/2\mathbb{Z}$-graded object in an additive category is called a *superobject*. As realizations of this general definition, we will be dealing with super algebras, super vector bundles on a smooth manifold, and sheaves of superalgebras on a topological space, etc.

We will write V^+ and V^- for the degree 0 (mod 2) and degree 1 (mod 2) parts of V and we will call them the even and the odd parts of V respectively. Let $\nu \in V$ be a homogeneous element. We say $|\nu| = 0$ if $\nu \in V^+$ and $|\nu| = 1$ if $\nu \in V^-$.

$\mathrm{End}_k(V)$ is also a k-supermodule with the grading

$$\begin{aligned} \mathrm{End}_k(V)^+ &= \mathrm{Hom}_k(V^+, V^+) \oplus \mathrm{Hom}_k(V^-, V^-) \\ \mathrm{End}_k(V)^- &= \mathrm{Hom}_k(V^+, V^-) \oplus \mathrm{Hom}_k(V^-, V^+) \end{aligned}$$

Moreover, the algebra of endomorphisms $\mathrm{End}_k(V)$ is a k-superalgebra with this grading. If no confusion is likely to arise, we will suppress the mention of the ring k from now on.

Definition 1.3. — *Let A be a superalgebra. The supercommutator of two elements of A is*

$$[a, b]_s = ab - (-1)^{|a||b|} ba$$

where a and b are homogeneous. The supercommutator is extended bilinearly to non-homogeneous a and b.

If the supercommutator $[\,,]_s : A \otimes A \to A$ is the zero map, then A is called a commutative superalgebra. The exterior algebra of a free module M with the $\mathbb{Z}/2\mathbb{Z}$-grading $\bigwedge^+ M = \bigoplus_{p\ even} \bigwedge^p M$ and $\bigwedge^- M = \bigoplus_{p\ odd} \bigwedge^p M$ is a commutative superalgebra.

Let V be finitely generated and projective. Assume that $\frac{1}{2} \in k$. Giving a $\mathbb{Z}/2\mathbb{Z}$-grading on V is equivalent to giving an involution $\epsilon \in \mathrm{End}_k(V)$, that is $\epsilon^2 = I$. The even and the odd parts are the eigenspaces corresponding to the eigenvalues $+1$ and -1 respectively. In the same fashion, the $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathrm{End}_k(V)$ can be given by the involution

$$\rho(\varphi) = \epsilon \circ \varphi \circ \epsilon$$

where $\varphi \in \mathrm{End}_k(V)$.

Definition 1.4. — *Let $\varphi \in \mathrm{End}_k(V)$. The supertrace of φ, denoted by $\mathrm{tr}_s(\varphi)$, is defined to be*

$$\mathrm{tr}_s(\varphi) = \mathrm{tr}(\epsilon \circ \varphi)$$

where 'tr' is the usual trace map.

Lemma 1.5. — *The supertrace vanishes on supercommutators.*

Proof. — Cf. [**11**]. □

Let A and B be superalgebras. We define the *super tensor product* of A and B, denoted by $A \widehat{\otimes} B$, to be the k-module $A \otimes B$ with the $\mathbb{Z}/2\mathbb{Z}$-grading

$$\begin{aligned} (A \widehat{\otimes} B)^+ &= (A^+ \otimes B^+) \oplus (A^- \otimes B^-) \\ (A \widehat{\otimes} B)^- &= (A^+ \otimes B^-) \oplus (A^- \otimes B^+) \end{aligned}$$

and the algebra structure

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|b_1||a_2|} a_1 a_2 \otimes b_1 b_2$$

for homogeneous elements $a_1, a_2 \in A$ and $b_1, b_2 \in B$. As usual the product is extended bilinearly.

Definition 1.6. — *Let A be a superalgebra and $\delta \in \mathrm{End}_k(A)$ be homogeneous. We will call δ a superderivation if it satisfies the super-Leibniz formula*

$$\delta(a_1a_2) = \delta(a_1)a_2 + (-1)^{|\delta||a_1|}a_1\delta(a_2)$$

for homogeneous $a_1, a_2 \in A$. We will call a non-homogeneous element of $\mathrm{End}_k(A)$ *a superderivation, if its even and odd components are superderivations.*

2. Sheaves on Real Analytic Manifolds

While we could use the C^∞ Dolbeault complex in the proof of the first main theorem, for consistency we will work with real-analytic forms throughout this paper. In this section, we shall recall the results that we need.

Theorem 2.1. — *Let M be a real analytic manifold which is countable at infinity and let $\mathcal{F}$ be a coherent analytic sheaf on M. Then*

$$H^p(M, \mathcal{F}) = 0 \qquad \textit{for} \quad p > 0$$

Proof. — Cf. Proposition 2.3 of [**3**]. □

We denote the sheaf of real analytic functions on M by $\mathcal{A}_M$, while if X is a complex manifold, we shall write $\mathcal{A}_X^{p,q}$ for the sheaf of (p,q)-forms with real analytic coefficients. It is a classical result (see [**6**]) that the real-analytic Dolbeault complex is a resolution of the sheaf Ω_X^p of holomorphic p-forms. It follows from the theorem, therefore, that if $\mathcal{E}$ is a locally free sheaf of $\mathcal{O}_X$-modules, then the cohomology groups $H^q(X, \mathcal{E})$ may be computed as the cohomology of the real analytic Dolbeault complex $\mathcal{A}_X^{0,*} \otimes_{\mathcal{O}_X} \mathcal{E}(X)$.

Corollary 2.2. — *Let $\mathcal{F}$ be a locally free sheaf of $\mathcal{A}_M$-modules of finite rank. Then $\mathcal{F}$ is a projective object in the category of coherent sheaves of $\mathcal{A}_M$-modules.*

Proof. — Cf. Lemma 2.7 of [**3**]. □

It follows immediately that any vector bundle on a complex manifold admits a real analytic connection, since to give such a connection is the same as splitting the Atiyah sequence.

Proposition 2.3. — *Let X_1 and X_2 be complex spaces. The canonical projection*

$$\pi_1 : X_1 \times X_2 \to X_1$$

is flat.

Proof. — Cf. [**7**] (Proposition 3.17 on page 155). □

Corollary 2.4. — *Let X be a complex manifold. The sheaf $\mathcal{A}_X$ is a flat sheaf of $\mathcal{O}_X$-algebras.*

Proof. — Let $\overline{X}$ denote the complex manifold with the opposite complex structure and $\triangle : X \to X \times \overline{X}$ be the diagonal embedding. Let $\pi_1 : X \times \overline{X} \to X$ be the projection onto the first component. Let $x \in X$ be any point. The stalks $\mathcal{A}_{X,x}$ and $\mathcal{O}_{X\times\overline{X},\triangle(x)}$ are canonically isomorphic. Hence the result follows from the proposition applied to the map $\pi_1 : X \times \overline{X} \to X$. □

It follows immediately that if $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules, then the cohomology groups $H^q(X, \mathcal{F})$ may be computed as the cohomology of the real analytic Dolbeault complex $\mathcal{A}_X^{0,*} \otimes_{\mathcal{O}_X} \mathcal{F}(X)$.

3. Superconnections and the Chern Character

Let us recall the definition and basic properties of superconnections from [**11**].

We assume that X is a real analytic manifold of dimension n. However, everything in this section applies verbatim to the smooth case. We denote the exterior algebra of the sheaf of real analytic differential forms on X, which is a sheaf of commutative superalgebras, by $\mathcal{A}_X^*$. Let $E = E^+ \oplus E^-$ be a real analytic super vector bundle on X. We will write $\mathcal{E}$ for the sheaf of real analytic sections of E, and $\mathcal{A}_X^*(\mathcal{E})$ for $\mathcal{A}_X^* \widehat{\otimes}_{\mathcal{A}_X} \mathcal{E}$.

Definition 3.1. — *A $\mathbb{C}$-linear endomorphism A of $\mathcal{A}_X^*(\mathcal{E})$ of odd degree is called a superconnection on E if it satisfies the super-Leibniz rule*

$$A(\omega \otimes s) = d\omega \otimes s + (-1)^{|\omega|}\omega \wedge A(s)$$

for local sections ω, s of $\mathcal{A}_X^*$ and $\mathcal{E}$ respectively.

If X is an almost complex manifold and if A satisfies the following version of the super-Leibniz rule

$$A(\omega \otimes s) = \overline{\partial}\omega \otimes s + (-1)^{|\omega|}\omega \wedge A(s)$$

then, it is called a superconnection of type $(0,1)$ (or simply a $(0,1)$-superconnection).

A^2 is called the curvature of the superconnection and is denoted by R_A. The curvature of A satisfies the identity

$$R_A(\omega \otimes s) = \omega \wedge R_A(s)$$

for local sections ω and s of $\mathcal{A}_X^*$ and $\mathcal{E}$ respectively. Thus R_A can be thought as a section of the sheaf of superalgebras $\mathcal{A}_X^* \widehat{\otimes}\, \mathcal{E}nd_{\mathcal{A}_X}(E)$ where $\mathcal{E}nd_{\mathcal{A}_X}(E)$ denotes the sheaf of endomorphisms of the bundle E.

We extend the supertrace to a map $\mathrm{tr}_s : \mathcal{A}_X^* \widehat{\otimes}\, \mathcal{E}nd_{\mathcal{A}_X}(E) \to \mathcal{A}_X^*$ by the formula

$$\mathrm{tr}_s(\omega \otimes \varphi) = \omega\, \mathrm{tr}_s(\varphi)$$

for local sections ω and φ of $\mathcal{A}_X^*$ and $\mathcal{E}nd_{\mathcal{A}_X}(E)$.

Proposition 3.2. — *Let n be a non-negative integer. The differential form $\mathrm{tr}_s(R_A^n)$ is closed and its cohomology class does not depend on the choice of the superconnection A.*

Proof. — Cf. [**11**]. □

Theorem 3.3. — *The differential form*

$$\operatorname{tr}_s(\exp(R_A)) \tag{3.1}$$

represents the class $\operatorname{ch}(E^+) - \operatorname{ch}(E^-)$ *in cohomology.*

Proof. — Cf. [**11**]. □

Remark 3.4. — The reader is warned that we omit the usual factor of $(\frac{\mathrm{i}}{2\pi})$ from (3.1), following the convention in algebraic geometry.

4. The Grothendieck Fundamental Class

General references for this section are [**9**] and [**1**].

Let X be a compact complex manifold of dimension n. We denote the sheaf of holomorphic functions and the sheaf of holomorphic k-forms on X by $\mathcal{O}_X$ and Ω^k_X respectively. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are sheaves of $\mathcal{O}_X$−modules. We write $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ for the sheaf of $\mathcal{O}_X$-morphisms from $\mathcal{F}$ to $\mathcal{G}$ and $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ for $\Gamma(X,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}))$. The derived functors of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ (resp. $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$) will be denoted by $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ (resp. $\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$). We simply write $\mathcal{F}^\vee$ for the dual of $\mathcal{F}$.

The abelian groups $\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ and sheaves $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ are related by the following spectral sequence

$$E_2^{i,j} = H^i(X,\mathcal{E}xt^j_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})) \Rightarrow \operatorname{Ext}^{i+j}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}))$$

Let Y be a complex submanifold of X of codimension p. We denote the sheaf of ideals defining Y by $\mathcal{I}$. In this situation, one has that

$$\begin{aligned}
\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{O}_Y,\Omega^p_X) &= 0 \qquad \text{for } i<p \qquad \text{and}\\
\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{O}_Y,\Omega^p_X) &= \textstyle\bigwedge^p(\mathcal{I}/\mathcal{I}^2)^\vee\otimes\mathcal{O}_Y\otimes\Omega^p_X\\
&= \mathcal{H}om_{\mathcal{O}_X}(\textstyle\bigwedge^p(\mathcal{I}/\mathcal{I}^2),\mathcal{O}_Y\otimes\Omega^p_X).
\end{aligned}$$

All tensor products are taken over $\mathcal{O}_X$ unless stated otherwise. It follows that the edge homomorphism

$$\operatorname{Ext}^p_{\mathcal{O}_X}(\mathcal{O}_Y,\Omega^p_X) \to H^0(\mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{O}_Y,\Omega^p_X))$$

is an isomophism, and so

$$\operatorname{Ext}^p_{\mathcal{O}_X}(\mathcal{O}_Y,\Omega^p_X) = \operatorname{Hom}_{\mathcal{O}_X}(\textstyle\bigwedge^p(\mathcal{I}/\mathcal{I}^2),\mathcal{O}_Y\otimes\Omega^p_X).$$

Therefore there is a class $[Y]$ in $\operatorname{Ext}^p_{\mathcal{O}_X}(\mathcal{O}_Y,\Omega^p_X)$ which corresponds to the homomorphism of sheaves

$$\begin{aligned}
\textstyle\bigwedge^p(\mathcal{I}/\mathcal{I}^2) &\to \mathcal{O}_Y\otimes\Omega^p_X\\
f_1\wedge\cdots\wedge f_p &\mapsto df_1\wedge\cdots\wedge df_p.
\end{aligned}$$

The class $[Y]$ is called the Grothendieck fundamental class of Y in X.

5. Koszul Factorizations

In this section, we prove Theorem A of the Introduction. The proof is contained in Proposition 5.4 and Theorem 5.5.

Let $\pi : E \to X$ be a holomorphic vector bundle of rank r and let ∇ be a flat real analytic connection of type $(1,0)$ on E. For instance, ∇ can be taken as the $(1,0)$ part of the canonical connection associated to a real analytic hermitian structure on E. We write $\widetilde{\nabla}$ for $\overline{\partial} + \nabla$. We will denote the induced connection, and the $(1,0)$-connection on the dual bundle $E^\vee$ using the same symbols. However R will be used exclusively to denote the curvature of the induced connection on $E^\vee$. Throughout this section we assume that $\tau \in \Gamma(X, \mathcal{E}^\vee)$. Let $\imath_\tau : \mathcal{A}_X \otimes \bigwedge^p \mathcal{E} \to \mathcal{A}_X \otimes \bigwedge^{p-1} \mathcal{E}$ be contraction by τ as usual. We extend $\imath_\tau$ to an odd superderivation of the sheaf of commutative superalgebras $\mathcal{A}^*_X \widehat{\otimes} \bigwedge \mathcal{E}$. Note that $\nabla(\tau) = \imath_\tau \circ \nabla + \partial \circ \imath_\tau : \mathcal{A}_X \otimes \mathcal{E} \to \mathcal{A}^{1,0}_X$ and therefore $\nabla(\tau)$ can be considered as an element of $\Gamma(X, \mathcal{A}^{1,0}_X \otimes \mathcal{E}^\vee)$. We write $\imath_{\nabla(\tau)} : \mathcal{A}_X \otimes \bigwedge^p \mathcal{E} \to \mathcal{A}^{1,0}_X \otimes \bigwedge^{p-1} \mathcal{E}$ and $\imath_{R(\tau)} : \mathcal{A}_X \otimes \bigwedge^p \mathcal{E} \to \mathcal{A}^{1,1}_X \otimes \bigwedge^{p-1} \mathcal{E}$ for the contractions with $\nabla(\tau) \in \Gamma(X, \mathcal{A}^{1,0}_X \otimes \mathcal{E}^\vee)$ and $R(\tau) \in \Gamma(X, \mathcal{A}^{1,1}_X \otimes \mathcal{E}^\vee)$ respectively. We extend $\imath_{\nabla(\tau)}$ (resp. $\imath_{R(\tau)}$) to an even (resp. odd) superderivation of $\mathcal{A}^*_X \widehat{\otimes} \bigwedge \mathcal{E}$.

We state two facts without proof

$$\begin{aligned} [\overline{\partial}, \imath_{\nabla(\tau)}]_s &= \imath_{R(\tau)} \\ [\imath_{\nabla(\tau)}, \imath_{R(\tau)}]_s &= 0. \end{aligned}$$

Lemma 5.1. — *For $1 \leqslant p \leqslant r$ the following diagram is commutative*

$$\begin{tikzcd} \mathcal{Z}_X^{r-p,r-p} \otimes \bigwedge^p \mathcal{E} \arrow[r, "\frac{1}{p!}(\imath_{\nabla(\tau)})^p"] \arrow[d, "\imath_{R(\tau)}"'] & \mathcal{A}_X^{r,r-p} \arrow[d, "\overline{\partial}"] \\ \mathcal{Z}_X^{r-p+1,r-p+1} \otimes \bigwedge^{p-1} \mathcal{E} \arrow[r, "\frac{1}{(p-1)!}(\imath_{\nabla(\tau)})^{p-1}"] & \mathcal{A}_X^{r,r-p+1} \end{tikzcd}$$

where $\mathcal{Z}^{p,p}_X$ denote the sheaf of $\overline{\partial}$-closed (not necessarily ∂-closed) forms of type (p,p).

Proof. — We have

$$\begin{aligned}\bar{\partial}\circ\frac{1}{p!}\,(\imath_{\nabla(\tau)})^p &= \frac{1}{p!}\,[\bar{\partial},(\imath_{\nabla(\tau)})^p]_s\\ &= \frac{1}{p!}\sum_{j=0}^{p-1}(\imath_{\nabla(\tau)})^j\circ[\bar{\partial},\imath_{\nabla(\tau)}]_s\circ(\imath_{\nabla(\tau)})^{p-j-1}\\ &= \frac{1}{p!}\sum_{j=0}^{p-1}(\imath_{\nabla(\tau)})^j\circ\imath_{R(\tau)}\circ(\imath_{\nabla(\tau)})^{p-j-1}\\ &= \frac{1}{p!}\sum_{j=0}^{p-1}(\imath_{\nabla(\tau)})^{p-1}\circ\imath_{R(\tau)}\\ &= \frac{1}{p!}\,p\,(\imath_{\nabla(\tau)})^{p-1}\circ\imath_{R(\tau)} = \frac{1}{(p-1)!}\,(\imath_{\nabla(\tau)})^{p-1}.\end{aligned}$$ □

Let $\phi_p : \bigwedge^p \mathscr{E} \to \mathscr{H}om_{\mathscr{O}_X}(\bigwedge^{r-p}\mathscr{E}, \bigwedge^r\mathscr{E})$ be the isomorphism given by

$$\phi_p : \alpha \mapsto (\beta \mapsto \beta\wedge\alpha) \qquad \text{for} \qquad \alpha\in\textstyle\bigwedge^p\mathscr{E},\quad \beta\in\bigwedge^{r-p}\mathscr{E},\ \text{and } 0\leqslant p\leqslant r$$

We will identify the sheaves $\mathscr{H}om_{\mathscr{O}_X}(\bigwedge^{r-p}\mathscr{E}, \bigwedge^r\mathscr{E})$ and $\bigwedge^{r-p}\mathscr{E}^\vee\otimes\bigwedge^r\mathscr{E}$ via the canonical isomorphism between them.

Lemma 5.2. — *The following diagram is commutative for* $1\leqslant p\leqslant r$

$$\begin{array}{ccc}\bigwedge^p\mathscr{E} & \xrightarrow{(\phi_p^{-1}\otimes 1)\circ(\bigwedge^{r-p}R\otimes 1)\circ\phi_p} & \bigwedge^p\mathscr{E}\otimes\mathscr{Z}_X^{r-p,r-p}\\ {\scriptstyle\imath_\tau}\big\downarrow & & \big\downarrow{\scriptstyle\imath_{R(\tau)}}\\ \bigwedge^{p-1}\mathscr{E} & \xrightarrow{(\phi_{p-1}^{-1}\otimes 1)\circ(\bigwedge^{r-p+1}R\otimes 1)\circ\phi_{p-1}} & \bigwedge^{p-1}\mathscr{E}\otimes\mathscr{Z}_X^{r-p+1,r-p+1}\end{array}$$

where $\bigwedge^p R : \bigwedge^p\mathscr{E}^\vee\to\bigwedge^p\mathscr{E}^\vee\otimes\mathscr{Z}_X^{p,p}$ *is defined by* $\bigwedge^p R\,(e^1\wedge\cdots\wedge e^p) = R(e^1)\wedge\cdots\wedge R(e^p)$ *for local sections* $e^1,\dots,e^p$ *of* $\mathscr{E}^\vee$.

Proof. — The lemma is an immediate consequence of the following commutative diagrams. (Note that $\wedge\tau$ and $\wedge R(\tau)$ denote right multiplication by τ and $R(\tau)$ respectively).

$$\begin{array}{ccccc}\bigwedge^p\mathscr{E} & \xrightarrow{\phi_p} & \bigwedge^{r-p}\mathscr{E}^\vee\otimes\bigwedge^r\mathscr{E} & \xrightarrow{\bigwedge^{r-p}R\otimes 1} & \bigwedge^{r-p}\mathscr{E}^\vee\otimes\bigwedge^r\mathscr{E}\otimes\mathscr{Z}_X^{r-p,r-p}\\ {\scriptstyle\imath_\tau}\big\downarrow & & {\scriptstyle\wedge\tau}\big\downarrow & & \big\downarrow{\scriptstyle\wedge R(\tau)}\\ \bigwedge^{p-1}\mathscr{E} & \xrightarrow{\phi_{p-1}} & \bigwedge^{r-p+1}\mathscr{E}^\vee\otimes\bigwedge^r\mathscr{E} & \xrightarrow{\bigwedge^{r-p+1}R\otimes 1} & \bigwedge^{r-p+1}\mathscr{E}^\vee\otimes\bigwedge^r\mathscr{E}\otimes\mathscr{Z}_X^{r-p+1,r-p+1}\end{array}$$

and

$$\begin{array}{ccc}
\bigwedge^{r-p}\mathcal{E}^\vee\otimes\bigwedge^r\mathcal{E}\otimes\mathcal{Z}_X^{r-p,r-p} & \xrightarrow{\phi_p^{-1}\otimes 1} & \bigwedge^p\mathcal{E}\otimes\mathcal{Z}_X^{r-p,r-p} \\
\wedge R(\tau)\Big\downarrow & & \imath_{R(\tau)}\Big\downarrow \\
\bigwedge^{r-p+1}\mathcal{E}^\vee\otimes\bigwedge^r\mathcal{E}\otimes\mathcal{Z}_X^{r-p+1,r-p+1} & \xrightarrow{\phi_{p-1}^{-1}\otimes 1} & \bigwedge^{p-1}\mathcal{E}\otimes\mathcal{Z}_X^{r-p+1,r-p+1}.
\end{array}$$

It is worth mentioning that R can be written as a matrix of $\overline{\partial}$-closed forms of type $(1,1)$ with respect to any given local holomorphic framing. Consequently the image of the mapping $\bigwedge^p R:\bigwedge^p\mathcal{E}^\vee\to\bigwedge^p\mathcal{E}^\vee\otimes\mathcal{A}_X^{p,p}$ lies in $\bigwedge^p\mathcal{E}^\vee\otimes\mathcal{Z}_X^{p,p}$. Since τ is a holomorphic section, a similar remark applies to the mappings $\imath_{R(\tau)}$ and $\wedge R(\tau)$. □

Proposition 5.3. — *The following diagram is commutative*

$$\begin{array}{ccccccccc}
\bigwedge^r\mathcal{E} & \xrightarrow{\imath_\tau} & \bigwedge^{r-1}\mathcal{E} & \xrightarrow{\imath_\tau} & \cdots & \xrightarrow{\imath_\tau} & \mathcal{E} & \xrightarrow{\imath_\tau} & \mathcal{O}_X \\
\psi_r=\frac{1}{r!}(\imath_{\nabla(\tau)})^r\Big\downarrow & & \psi_{r-1}\Big\downarrow & & & & \psi_1\Big\downarrow & & \psi_0=\det(R)\Big\downarrow \\
\mathcal{A}_X^{r,0} & \xrightarrow{\overline{\partial}} & \mathcal{A}_X^{r,1} & \xrightarrow{\overline{\partial}} & \cdots & \xrightarrow{\overline{\partial}} & \mathcal{A}_X^{r,r-1} & \xrightarrow{\overline{\partial}} & \mathcal{A}_X^{r,r}.
\end{array}$$

where $\psi_p=\frac{1}{p!}(\imath_{\nabla(\tau)})^p\circ(\phi_p^{-1}\otimes 1)\circ(\bigwedge^{r-p}R\otimes 1)\circ\phi_p$.

Proof. — This is an immediate result of the previous two lemmas. □

The symbol $f:C^*\xrightarrow{\sim}D^*$ is used to denote that f is a quasi-isomorphism.

Proposition 5.4. — *The morphism* $\psi\in\mathcal{H}om_{\mathcal{O}_X}(\bigwedge^*\mathcal{E},\mathcal{A}_X^{r,*}[r])$ *represents the Grothendieck fundamental class of* Z *in* $\mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{O}_Z,\Omega_X^r)$.

Proof. — The morphism (which is a map of complexes) $\psi\in\mathcal{H}om_{\mathcal{O}_X}(K(\tau)^*,\mathcal{A}_X^{r,*}[r])$ gives us an r-cocycle, denoted by $[\psi]$, in the double complex $\mathcal{H}om^*_{\mathcal{O}_X}(K(\tau)^\cdot,\mathcal{A}_X^{r,\cdot})$. We have the quasi-isomorphisms

$$\mathcal{H}om^*_{\mathcal{O}_X}(K(\tau)^\cdot,\mathcal{A}_X^{r,\cdot})\xrightarrow{\sim}K(\tau)^*\otimes\det(\mathcal{E}^\vee)\otimes\mathcal{A}_X^{r,*}\xrightarrow{\sim}\mathcal{O}_Z\otimes\det(\mathcal{E}^\vee)\otimes\mathcal{A}_X^{r,*}.$$

Under these maps, $[\psi]$ is mapped to $\psi_r:\bigwedge^r\mathcal{E}\to\mathcal{A}_X^{r,0}$ (mod $\mathcal{I}$).

We have $\tau=\sum_i\alpha_i e^i$ with respect to some local holomorphic framing $\{e^1,\dots,e^r\}$ of $E^\vee$. Then

$$\begin{aligned}
\psi_r(e_1\wedge\cdots\wedge e_r) &= \frac{1}{r!}(\imath_{\nabla(\tau)})^r(e_1\wedge\cdots\wedge e_r)\\
&= \imath_{\nabla(\tau)}(e_1)\wedge\cdots\wedge\imath_{\nabla(\tau)}(e_r)\\
&= \partial\alpha_1\wedge\cdots\wedge\partial\alpha_r \pmod{\mathcal{I}}\\
&= d\alpha_1\wedge\cdots\wedge d\alpha_r \pmod{\mathcal{I}}.
\end{aligned}$$

Then the result follows from the fact that the morphism defining Grothendieck fundamental class of Z in $\mathrm{Hom}_{\mathcal{O}_X}(\bigwedge^r(\mathcal{I}/\mathcal{I}^2),\mathcal{O}_Z\otimes\Omega_X^r)$ is mapped to $\psi_r:\bigwedge^r\mathcal{E}\to\mathcal{A}_X^{r,0}$ (mod $\mathcal{I}$) under the following sequence of quasi-isomorphisms

$$\bigwedge^r(\mathcal{I}/\mathcal{I}^2)^\vee\otimes\Omega_X^r\xrightarrow{\sim}\mathcal{O}_Z\otimes\det(\mathcal{E}^\vee)\otimes\Omega_X^r\xrightarrow{\sim}\mathcal{O}_Z\otimes\det(\mathcal{E}^\vee)\otimes\mathcal{A}_X^{r,*}.$$ □

Theorem 5.5. — *The map of complexes* $\psi \in \mathrm{Hom}_{\mathcal{O}_X}(\bigwedge^* \mathcal{E}, \mathcal{A}_X^{r,*}[r])$ *represents the Grothendieck fundamental class of* Z *in* $\mathrm{Ext}^r_{\mathcal{O}_X}(\mathcal{O}_Z, \Omega^r_X)$. *Moreover the image of* ψ *in* $H^r(X, \Omega^r_X)$ *is the* r*-th Chern form* $c_r(E^\vee, \widetilde{\nabla})$.

Proof. — Since ψ represents the Grothendieck fundamental class locally, it does so globally. The second result follows from the fact that $\psi_0 = \det(R)$ by Proposition 5.3 and that $\det(R)$ is the r-th Chern form of the pair $(E^\vee, \widetilde{\nabla})$. □

We obtain immediately

Corollary 5.6. — *Let* $\pi : E \to X$ *be a holomorphic vector bundle of rank* r *and* $\tau : X \to E^\vee$ *be a holomorphic section which is transverse to the zero section. If* Z *is the complex submanifold where* τ *vanishes, then the fundamental class of* Z *in Dolbeault cohomology is represented by the* r*-th Chern form* $c_r(E^\vee, \widetilde{\nabla})$.

Notice that the standard proofs of this result (for example in [**8**]) implicitly use the axioms defining Chern classes.

6. Generalized Supertraces

The heart of this section is Proposition 6.4 which will be used in Section 9 to construct a map of complexes that represents the Grothendieck fundamental class.

Let V be a finitely generated projective module over a commutative ring with unity k, and let $V^\vee$ be its dual. Let $\langle\, ,\rangle : V^\vee \otimes_k V \to k$ be the pairing defined by $\langle s, t\rangle = s(t)$ for $s \in V^\vee$ and $t \in V$. We extend $\langle\, ,\rangle$ to a pairing between $\bigwedge^m V^\vee$ and $\bigwedge^m V$ by

$$\langle u, v\rangle = \det\langle u^i, v_j\rangle$$

where $u = u^1 \wedge \cdots \wedge u^m \in \bigwedge^m V^\vee$ and $v = v_1 \wedge \cdots \wedge v_m \in \bigwedge^m V$. It is easy to check that

$$\langle u, v\rangle = (\imath_{u^m} \circ \cdots \circ \imath_{u^1})(v_1 \wedge \cdots \wedge v_m)$$

where $\imath_{u^j}$ denotes contraction by u^j. We denote the exterior algebra of V by $\bigwedge V$ with the usual grading for which $\bigwedge^n V$ has degree n. Then

$$\mathrm{Hom}_k(\textstyle\bigwedge V, \bigwedge V)$$

is naturally graded with $\mathrm{Hom}_k(\bigwedge^m V, \bigwedge^n V)$ having degree $(n - m)$.

Definition 6.1. — *Let* $\varphi \in \mathrm{Hom}_k(\bigwedge V, \bigwedge V)$. *If* φ *has degree* $(-n)$ *with* $n \geqslant 0$, *we define the generalized supertrace of* φ, *denoted by* $\mathrm{Tr}_\Lambda(\varphi) \in \bigwedge^n V^\vee$ *(as opposed to the supertrace* tr_s*), as follows:*

$$(\mathrm{Tr}_\Lambda(\varphi))\,(\eta) = (-1)^{|\eta|}\mathrm{tr}_s(l_\eta \circ \varphi)$$

where $l_\eta \in \mathrm{End}_k(\bigwedge V)$ *is left multiplication by* η *for some* $\eta \in \bigwedge^n V$. *If* φ *has positive degree, then* $\mathrm{Tr}_\Lambda(\varphi)$ *is defined to be* 0.

Clearly when $n = 0$, we have $\mathrm{Tr}_\Lambda = \mathrm{tr}_s$.

Let $i : \bigwedge V^\vee \to \mathrm{End}_k(\bigwedge V)$ be the inclusion defined by $i(\alpha)(\beta) = \langle \alpha, \beta \rangle$ for $\alpha \in \bigwedge^m V^\vee$ and $\beta \in \bigwedge^m V$, and $i(\alpha)(\beta) = 0$ if $\beta \notin \bigwedge^m V$. We will often identify $\bigwedge V^\vee$ with its image under i, and think of $\mathrm{Tr}_\Lambda(\varphi)$ as belonging to $\mathrm{End}_k(\bigwedge V)$.

Remark 6.2. — We have the identity

$$\mathrm{Tr}_\Lambda \circ i = \mathrm{id}_{\bigwedge V^\vee}.$$

Remark 6.3. — Let $\tau \in V^\vee$ and let $\imath_\tau$ denote contraction by τ, which is a superderivation of $\bigwedge V$ of degree -1. It is straightforward to check that

$$\mathrm{Tr}_\Lambda(\imath_\tau) = \begin{cases} 0 & \text{if} \quad \mathrm{rank}\, V \geqslant 2 \\ \tau & \text{if} \quad \mathrm{rank}\, V = 1. \end{cases}$$

An important feature of the supertrace is that it vanishes on supercommutators. However, this is *not* true of the generalized trace Tr_Λ. Instead we have:

Proposition 6.4. — *Assume that $\varphi \in \mathrm{End}_k(\bigwedge V)$, and let $\delta \in \mathrm{End}_k(\bigwedge V)$ be a k-linear superderivation. Then*

$$\mathrm{Tr}_\Lambda[\delta, \varphi]_s = [\delta, \mathrm{Tr}_\Lambda(\varphi)]_s.$$

In order to prove the proposition we need a lemma.

Lemma 6.5. — *Assume that $\varphi \in \mathrm{Hom}_k(\bigwedge V, \bigwedge V)$ is of degree $-n \leqslant 0$, and let δ be a k-linear superderivation of degree j with $-n + j \leqslant 0$. Then*

$$\mathrm{Tr}_\Lambda[\delta, \varphi]_s = [\delta, \mathrm{Tr}_\Lambda(\varphi)]_s.$$

Proof. — Let $\eta \in \bigwedge^{n-j} V$ be any element. Then

$$\begin{aligned}
\mathrm{Tr}_\Lambda[\delta, \varphi]_s(\eta) &= \mathrm{Tr}_\Lambda(\delta \circ \varphi - (-1)^{-nj} \varphi \circ \delta)(\eta) \\
&= (-1)^{|\eta|} \mathrm{tr}_s(l_\eta \circ \delta \circ \varphi) - (-1)^{|\eta| - nj} \mathrm{tr}_s(l_\eta \circ \varphi \circ \delta) \\
&\quad \text{by definition of } \mathrm{Tr}_\Lambda \\
&= (-1)^{|\eta|} \mathrm{tr}_s(l_\eta \circ \delta \circ \varphi) - (-1)^{n - nj} \mathrm{tr}_s(\delta \circ l_\eta \circ \varphi) \\
&\quad \text{since } \mathrm{tr}_s([\delta, l_\eta \circ \varphi]_s) = 0 \\
&= (-1)^{n - nj + 1} \mathrm{tr}_s\left([\delta, l_\eta]_s \circ \varphi\right) \\
&= (-1)^{n - nj + 1} \mathrm{tr}_s(l_{\delta(\eta)} \circ \varphi) \\
&= -(-1)^{-nj} \mathrm{Tr}_\Lambda(\varphi)(\delta(\eta)) \quad \text{by definition of } \mathrm{Tr}_\Lambda \\
&= [\delta, \mathrm{Tr}_\Lambda(\varphi)]_s(\eta) \quad \text{since } \delta \circ \mathrm{Tr}_\Lambda(\varphi) = 0.
\end{aligned}$$

□

Proof of Proposition 6.4. — We can write $\varphi = \sum_n \varphi_n$ and $\delta = \sum_j \delta_j$ where φ_n is the degree n component of φ, and δ_j is the degree j component of δ. Note that a superderivation of $\bigwedge V$ is necessarily of degree greater than or equal to -1. Moreover, $\mathrm{Tr}_\Lambda[\delta_j, \varphi_n]_s = 0$ unless $n + j \leqslant 0$. Then

$$\begin{aligned}
\mathrm{Tr}_\Lambda[\delta, \varphi]_s &= \sum_{j\geqslant -1} \mathrm{Tr}_\Lambda[\delta_j, \varphi]_s \\
&= \sum_{j\geqslant -1} \sum_{n\leqslant -j} \mathrm{Tr}_\Lambda[\delta_j, \varphi_n]_s \\
&= \sum_{j\geqslant -1} \sum_{n\leqslant -j} [\delta_j, \mathrm{Tr}_\Lambda(\varphi_n)]_s \quad \text{by Lemma 6.5} \\
&= [\delta, \mathrm{Tr}_\Lambda(\varphi)]_s.
\end{aligned}$$

□

7. Twisted Complexes

In this section, we give a brief exposition of twisted complexes, which were introduced by E. Brown in [**5**]. These shall be used in Section 8 to construct "global resolutions" of $\mathcal{O}_Z$ by locally free $\mathcal{O}_X$-modules. (cf. Proposition 8.4). The reader is referred to [**14**] for an extensive study of the use of twisted complexes in the duality theory of complex manifolds.

Let **A** be an abelian category.

Definition 7.1. — *Let $M = \{M^{p,q}\}_{p,q\in\mathbb{Z}}$ be a bigraded object in* **A**, *and let δ be a differential of total degree +1 on the associated graded object $T(M)^i = \bigoplus_{p+q=i} M^{p,q}$. The pair (M, δ) is called a twisted complex if δ preserves the filtration with respect to the grading by the first degree on M. In this case, δ is called the twisting differential of the pair (M, δ).*

We can write $\delta = \sum_{k\geqslant 0} a_k$ where $a_k \in \prod_{p,q\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{O}}(M^{p,q}, M^{p+k,q-k+1})$. The fact that $\delta^2 = 0$ entails the following, which is called the *twisting cocycle condition*,

$$\sum_{i=0}^{n} a_i a_{n-i} = 0 \qquad \text{for } n \geqslant 0. \tag{7.1}$$

Consider the cases where n=0, 1, 2

$$a_0^2 = 0 \tag{7.2}$$

$$a_0 a_1 + a_1 a_0 = 0 \tag{7.3}$$

$$a_0 a_2 + a_1^2 + a_2 a_0 = 0 \tag{7.4}$$

$(M^{p,*}, a_0)$ is a cochain complex for each $p \in \mathbb{Z}$ by (7.2). The totality of maps $(-1)^q a_1^{p,q} : M^{p,q} \to M^{p+1,q}$ gives us a map of complexes from $(M^{p,*}, a_0)$ to $(M^{p+1,*}, a_0)$ by (7.3). Equation (7.4) entails that $-a_1^2 : (M^{p,*}, a_0) \to (M^{p+2,*}, a_0)$ is chain homotopic to the zero map, and the chain homotopy is given by $a_2 \in \prod_{p,q\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{O}}(M^{p,q}, M^{p+2,q-1})$.

By definition $(T(M)^*, \delta)$ is a filtered differential object with the filtration given by $F^k = \bigoplus_{i \geqslant k} M^{i,j}$. Hence there exists a spectral sequence with

$$E_1^{p,q} = H^q(M^{p,*}, a_0) \Rightarrow H^{p+q}(T(M)^*, \delta).$$

We note that $(H^*(M^{p,*}, a_0), a_1)$ is a cochain complex because of (7.4). As a result the E_2 terms of the spectral sequence are

$$E_2^{p,q} = H^p(H^q(M^{**}, a_0), a_1).$$

Remark 7.2. — The reader may observe that there is a formal similarity between twisting cocyles and flat superconnections.

Example 7.3. — Every double complex (with anticommuting differentials) is a twisted complex with $a_k = 0$ for $k \geqslant 2$.

Example 7.4. — Let (C^*, d) be a bounded cochain complex in $\mathbf{A}$, and for simplicity assume that $C^k \neq 0$ only for $0 \leqslant k \leqslant n$ for some $n \in \mathbb{N}$. Suppose we are given projective resolutions $(P^{p,*}, \alpha_p)$ of C^p for each p with augmentation maps $\epsilon_p : (P^{p,*}, \alpha_p) \to C^p$. Then one has cochain maps $\beta_p : (P^{p,*}, \alpha_p) \to (P^{p+1,*}, \alpha_{p+1})$ lifting $d : C^p \to C^{p+1}$.

$$\begin{tikzcd}
 & \vdots \arrow[d, "\alpha_0"'] & \vdots \arrow[d, "\alpha_1"'] & & \vdots \arrow[d, "\alpha_n"'] & \\
0 \arrow[r] & P^{0,-1} \arrow[r, "\beta_{-1}"] \arrow[d, "\alpha_0"'] & P^{1,-1} \arrow[r, "\beta_{-1}"] \arrow[d, "\alpha_1"'] & \cdots \arrow[r, "\beta_{-1}"] & P^{n,-1} \arrow[r] \arrow[d, "\alpha_n"'] & 0 \\
0 \arrow[r] & P^{0,0} \arrow[r, "\beta_0"] \arrow[d, "\epsilon_0"'] & P^{1,0} \arrow[r, "\beta_0"] \arrow[d, "\epsilon_1"'] & \cdots \arrow[r, "\beta_0"] & P^{n,0} \arrow[r] \arrow[d, "\epsilon_n"'] & 0 \\
0 \arrow[r] & C^0 \arrow[r, "d"] \arrow[d] & C^1 \arrow[r, "d"] \arrow[d] & \cdots \arrow[r, "d"] & C^n \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & & 0 &
\end{tikzcd}$$

Now we will construct maps a_k for $k \geqslant 0$ in order to make $(P^{**}, \delta = \sum_{k \geqslant 0} a_k)$ a twisted complex. (Note that $a_k = 0$ for $k \geqslant n+1$).

We set $a_0 = \{(-1)^p \alpha_p\}_{p \in \mathbb{Z}}$ and $a_1 = \{\beta_q\}_{q \in \mathbb{Z}}$. Then the twisting cocycle condition is satisfied for $n = 0$ and $n = 1$. Now assume that a_k is constructed. It is easy to check that $-\sum_{i=1}^k a_i a_{k-i+1} : (P^{p,*}, a_0) \to (P^{p+k+1,*}[-k+1], a_0)$ is a cochain map. If we consider $-\sum_{i=1}^k a_i a_{k-i+1}$ as a map from the complex $(P^{p,*}, a_0)$ to the augmented complex $(P^{p+k+1,*}[-k+1] \to C^{p+k+1}[-k+1] \to 0)$, then it is homotopic to the zero map being a chain map from a complex of projectives to an acyclic complex. Let a_{k+1}

be a chain homotopy between $-\sum_{i=1}^{k} a_1 a_{k-i+1}$ and the zero map, then one has

$$a_{k+1}a_0 + a_0 a_{k+1} + \sum_{i=1}^{k} a_i a_{k-i+1} = 0$$

which is equivalent to the twisting cocycle condition for $n = k+1$. This completes the induction.

Moreover, if we endow C^* with the *filtration bête*, i.e. the filtration defined by

$$\sigma_{\geqslant p}(C)^i = \begin{cases} 0 & \text{if} \quad i < p \\ C^i & \text{if} \quad i \geqslant p \end{cases}$$

then, the augmentation map $\epsilon : T(P^{**}) \to C^*$ respects the filtrations and is a quasi-isomorphism of the associated graded objects. Therefore, the cochain complex $(T(P^{**}), \delta)$ is quasi-isomorphic to (C^*, d).

Definition 7.5. — *Let C be an object in* $\mathbf{A}$. *A twisted complex (M, δ) is called a twisted resolution of C if*

$$H^i(T(M)^*, \delta) = \begin{cases} C & \textit{if} \quad i = 0 \\ 0 & \textit{if} \quad i \neq 0. \end{cases}$$

8. Koszul-Dolbeault twisted resolutions

The main result of this section is Proposition 8.4 wherein we use twisted complexes to construct a global resolution of $\mathcal{O}_Z$ by locally free $\mathcal{A}_X$-modules.

Let X be a compact complex manifold of dimension n. We write $\mathcal{A}_X$ and $\mathcal{A}_X^{p,q}$ for the sheaf of real analytic complex valued functions and for the sheaf of real analytic differential forms of type (p, q) respectively. Let $\pi : E \to X$ be a holomorphic vector bundle of rank r. We will denote the sheaf of holomorphic sections of E by $\mathcal{E}$. The dual bundle and its sheaf of sections will be denoted by $E^\vee$ and $\mathcal{E}^\vee$ respectively.

Definition 8.1. — *A connection of type (0,1) (or simply a (0,1)-connection) on E is a* $\mathbb{C}$*-linear map*

$$\overline{D} : \mathcal{A}_X \otimes \mathcal{E} \to \mathcal{A}_X^{0,1} \otimes \mathcal{E}$$

satisfying the Leibniz rule

$$\overline{D}(fs) = \overline{\partial} f \otimes s + f\overline{D}(s)$$

for local sections f of $\mathcal{A}_X$ and s of $\mathcal{E}$.

Let $\overline{D}$ be a $(0,1)$-connection on E, and let $\langle\, , \rangle$ denote the pairing between E and $E^\vee$. One defines a (0,1)-connection on $E^\vee$ (which will also be denoted by $\overline{D}$) by the formula

$$\overline{\partial}\langle s, t\rangle = \langle \overline{D}s, t\rangle + \langle s, \overline{D}t\rangle$$

for local sections s and t of $\mathcal{A}_X \otimes \mathcal{E}$ and $\mathcal{A}_X \otimes \mathcal{E}^\vee$ respectively. Finally we extend $\overline{D}$ to a $\mathbb{C}$-linear superderivation of odd degree of the sheaf of superalgebras $\mathcal{A}_X^* \widehat{\otimes} \bigwedge \mathcal{E}$. As a result, $\overline{D}$ is a $(0,1)$-superconnection on the bundle $\bigwedge E$. Let τ be a global section of $\mathcal{A}_X \otimes \mathcal{E}^\vee$. We extend $\imath_\tau$, the contraction by τ, to an odd degree superderivation of $\mathcal{A}_X^* \widehat{\otimes} \bigwedge \mathcal{E}$ which acts trivially on $\mathcal{A}(X)$.

Lemma 8.2. — *Let $\tau \in \Gamma(X, \mathcal{A}_X \otimes \mathcal{E}^\vee)$ be any section and let $\overline{D}$ be a $(0,1)$-connection on E such that $\overline{D}(\tau) = 0$. The following diagram is anti-commutative for $p, q \geqslant 0$*

$$\begin{array}{ccc} \mathcal{A}_X^{0,q} \otimes \bigwedge^p \mathcal{E} & \xrightarrow{\overline{D}} & \mathcal{A}_X^{0,q+1} \otimes \bigwedge^p \mathcal{E} \\ {\scriptstyle \imath_\tau}\downarrow & & {\scriptstyle \imath_\tau}\downarrow \\ \mathcal{A}_X^{0,q} \otimes \bigwedge^{p-1} \mathcal{E} & \xrightarrow{\overline{D}} & \mathcal{A}_X^{0,q+1} \otimes \bigwedge^{p-1} \mathcal{E}. \end{array}$$

Proof. — We have $\overline{D} \circ \imath_\tau + \imath_\tau \circ \overline{D} = [\overline{D}, \imath_\tau]_s = \imath_{\overline{D}(\tau)}$. □

Lemma 8.3. — *Let Z be a complex submanifold of X such that there exists a holomorphic vector bundle $\pi : E \to X$ and a section$\tau \in \Gamma(X, \mathcal{A}_X \otimes \mathcal{E}^\vee)$, vanishing along Z, such that the induced map $\imath_\tau : \mathcal{A}_X \otimes \mathcal{E} \to \mathcal{A}_X \otimes \mathcal{I}_Z$ is surjective. Then there exists a $(0,1)$-connection $\overline{D}$ on the bundle E such that $\overline{D}(\tau) = 0$.*

Proof. — We have the following diagram

$$\begin{array}{ccccccc} & & \mathcal{A}_X \otimes \mathcal{E} & & & & \\ & & {\scriptstyle \imath_{\overline{\partial}(\tau)}}\downarrow & & & & \\ \mathcal{A}_X^{0,1} \otimes \mathcal{E} & \xrightarrow{\imath_\tau} & \mathcal{A}_X^{0,1} & \xrightarrow{p} & \mathcal{A}_X^{0,1} \otimes \mathcal{O}_Z & \longrightarrow & 0. \end{array}$$

There exists an $\mathcal{A}_X$-linear map $\theta : \mathcal{A}_X \otimes \mathcal{E} \to \mathcal{A}_X^{0,1} \otimes \mathcal{E}$ such that $\imath_\tau \circ \theta = \imath_{\overline{\partial}(\tau)}$ since $p \circ \imath_{\overline{\partial}(\tau)} = 0$ and $\mathcal{A}_X \otimes \mathcal{E}$ is projective by Corollary 2.2. Then $\overline{D} = \overline{\partial} - \theta$ is the desired $(0,1)$-connection since

$$[\overline{D}, \imath_\tau]_s = [\overline{\partial}, \imath_\tau]_s - [\theta, \imath_\tau]_s = 0. \qquad \square$$

Proposition 8.4. — *Let Z be a complex submanifold of X such that there exists a holomorphic vector bundle $\pi : E \to X$ and $\tau \in \Gamma(X, \mathcal{A}_X \otimes \mathcal{E}^\vee)$ such that $\imath_\tau : \mathcal{A}_X \otimes \mathcal{E} \to \mathcal{A}_X \otimes \mathcal{I}_Z$ is surjective. There is a twisted resolution (cf. Definition 7.5) $(M^{l,m}, \delta = \sum_{k \geqslant 0} a_k)$ of $\mathcal{O}_Z$ with $M^{l,m} = \mathcal{A}_X^{0,l} \otimes \bigwedge^m \mathcal{E}$, $a_0 = \imath_\tau$, $a_1 = \overline{D}$, and where the a_k are $\mathcal{A}_X^*$-linear superderivations for $k \geqslant 2$.*

The quasi-isomorphisms are the augmentation map $\epsilon : (T(M)^*, \delta) \to (\mathcal{A}_X^{0,*} \otimes \mathcal{O}_Z, \overline{\partial})$ and the inclusion $\mathcal{O}_Z \to (\mathcal{A}_X^{0,*} \otimes \mathcal{O}_Z, \overline{\partial})$.

Proof. — The fact that $(\imath_\tau)^2 = 0$ and that $\overline{D}$ and $\imath_\tau$ anticommute implies that the twisting cocycle condition (defined in Section 7) is satisfied for $n = 0$ and $n = 1$. We will construct superderivations a_k, $k \geqslant 2$ satisfying the twisting cocycle condition by induction.

$$\begin{tikzcd}
\mathcal{A}_X \arrow[r, "\bar\partial"] & \mathcal{A}_X^{0,1} \arrow[r, "\bar\partial"] & \cdots \arrow[r, "\bar\partial"] & \mathcal{A}_X^{0,n} \\
\mathcal{A}_X \otimes \mathcal{E} \arrow[u, "\imath_\tau"] \arrow[r, "\overline{D}"] & \mathcal{A}_X^{0,1} \otimes \mathcal{E} \arrow[u, "\imath_\tau"] \arrow[r, "\overline{D}"] & \cdots \arrow[r, "\overline{D}"] & \mathcal{A}_X^{0,n} \otimes \mathcal{E} \arrow[u, "\imath_\tau"] \\
\vdots \arrow[u, "\imath_\tau"] & \vdots \arrow[u, "\imath_\tau"] & & \vdots \arrow[u, "\imath_\tau"] \\
\mathcal{A}_X \otimes \bigwedge^{r-1} \mathcal{E} \arrow[u, "\imath_\tau"] \arrow[r, "\overline{D}"] & \mathcal{A}_X^{0,1} \otimes \bigwedge^{r-1} \mathcal{E} \arrow[u, "\imath_\tau"] \arrow[r, "\overline{D}"] & \cdots \arrow[r, "\overline{D}"] & \mathcal{A}_X^{0,n} \otimes \bigwedge^{r-1} \mathcal{E} \arrow[u, "\imath_\tau"] \\
\mathcal{A}_X \otimes \bigwedge^{r} \mathcal{E} \arrow[u, "\imath_\tau"] \arrow[r, "\overline{D}"] & \mathcal{A}_X^{0,1} \otimes \bigwedge^{r} \mathcal{E} \arrow[u, "\imath_\tau"] \arrow[r, "\overline{D}"] & \cdots \arrow[r, "\overline{D}"] & \mathcal{A}_X^{0,n} \otimes \bigwedge^{r} \mathcal{E}. \arrow[u, "\imath_\tau"]
\end{tikzcd}$$

We have the diagram

$$\begin{tikzcd}
 & \mathcal{A}_X \otimes \mathcal{E} \arrow[r, "\imath_\tau"] \arrow[d, "-a_1^2"'] & \mathcal{A}_X \arrow[d, "0"] \\
\mathcal{A}_X^{0,2} \otimes \bigwedge^2 \mathcal{E} \arrow[r, "\imath_\tau"] & \mathcal{A}_X^{0,2} \otimes \mathcal{E} \arrow[r, "\imath_\tau"] & \mathcal{A}_X^{0,2}.
\end{tikzcd}$$

There exists an $\mathcal{A}_X$-linear map $a_2 : \mathcal{A}_X \otimes \mathcal{E} \to \mathcal{A}_X^{0,2} \otimes \bigwedge^2 \mathcal{E}$ such that $-a_1^2 = \imath_\tau \circ a_2$ since $\mathcal{A}_X^{0,q} \otimes \mathcal{E}$ is projective by Lemma 2.2. We extend a_2 to an odd superderivation of $\mathcal{A}_X^* \widehat{\otimes} \bigwedge \mathcal{E}$ which acts trivially on $\mathcal{A}_X^*$. The twisting cocycle condition for $n = 2$ is satisfied since both $-a_1^2$ and $a_2 \circ \imath_\tau + \imath_\tau \circ a_2$ are superderivations that act trivially on $\mathcal{A}_X^*$ and agree on $\mathcal{A}_X \otimes \mathcal{E}$.

Now assume that a_k is constructed. Thus we have the diagram

$$\begin{tikzcd}
 & \mathcal{A}_X \otimes \mathcal{E} \arrow[r, "\imath_\tau"] \arrow[d, "\mu"'] & \mathcal{A}_X \arrow[d, "0"] \\
\mathcal{A}_X^{0,k+1} \otimes \bigwedge^{k+1} \mathcal{E} \arrow[r, "\imath_\tau"] & \mathcal{A}_X^{0,k+1} \otimes \bigwedge^{k} \mathcal{E} \arrow[r, "\imath_\tau"] & \mathcal{A}_X^{0,k+1} \otimes \bigwedge^{k-1} \mathcal{E}
\end{tikzcd}$$

where $\mu = -\sum_{i=1}^k a_i a_{k-i+1}$. It is straightforward to check that μ is $\mathcal{A}_X$-linear. Hence there exists a map $a_{k+1} : \mathcal{A}_X \otimes \mathcal{E} \to \mathcal{A}_X^{0,k+1} \otimes \bigwedge^{k+1} \mathcal{E}$ such that $\mu = \imath_\tau \circ a_{k+1}$. The extension of a_{k+1} to an odd superderivation of $\mathcal{A}_X^* \widehat{\otimes} \bigwedge \mathcal{E}$ which acts trivially on $\mathcal{A}_X^*$ satisfies the twisting cocycle condition by applying the argument used in the previous paragraph. Note that $a_k = 0$ for $k \geqslant n+1$, so the induction ends after a finite number of steps.

Let ϵ denote the augmentation map from the twisted complex $(T^* = T(M)^*, \delta)$ to the complex $(\mathcal{A}_X^{0,*} \otimes \mathcal{O}_Z, \bar\partial)$. If we endow the latter complex with *filtration bête*, then ϵ becomes a map of filtered complexes which is a quasi-isomorphism of the associated graded objects. Hence, (T^*, δ) is quasi-isomorphic to $(\mathcal{A}_X^{0,*} \otimes \mathcal{O}_Z, \overline{\partial})$, which in turn is quasi-isomorphic to $\mathcal{O}_Z$. □

Note that δ is a flat superconnection of type $(0,1)$ on the superbundle $\bigwedge E$.

9. Koszul Factorizations II

In this section, we will construct a map from the twisted complex $T^* = T(M)^*$ of Proposition 8.4 to the Dolbeault complex $\mathcal{A}_X^{r,*}[r]$ by using the generalized supertraces of Section 6. The precise argument is contained in Corollary 9.3. In the next section, we will prove that this map represents the Grothendieck fundamental class of Z in X.

We extend the generalized trace (cf. Definition 6.1) to a map

$$\mathrm{Tr}_\Lambda : \mathcal{A}_X^* \widehat{\otimes}\ \mathcal{E}nd_{\mathcal{O}_X}(\bigwedge E) \to \mathcal{A}_X^* \widehat{\otimes} \bigwedge \mathcal{E}^\vee$$

by the formula

$$\mathrm{Tr}_\Lambda(\omega \otimes \varphi) = \omega\, \mathrm{Tr}_\Lambda(\varphi)$$

for local sections ω and φ of $\mathcal{A}_X^*$ and $\mathcal{E}nd_{\mathcal{O}_X}(\bigwedge E)$ respectively.

Proposition 9.1. — *Let φ be a section of the sheaf of superalgebras $\mathcal{A}_X^* \widehat{\otimes}\ \mathcal{E}nd_{\mathcal{O}_X}(\bigwedge E)$ and δ be the twisting differential of Proposition 8.4. Then*

$$\mathrm{Tr}_\Lambda[\delta, \varphi]_s = [\delta, \mathrm{Tr}_\Lambda(\varphi)]_s$$

Proof. — We observe that, for local sections $\omega_1 \otimes \varphi_1, \omega_2 \otimes \varphi_2$ of $\mathcal{A}_X^* \widehat{\otimes}\ \mathcal{E}nd_{\mathcal{O}_X}(\bigwedge E)$, one has that

$$[\omega_1 \otimes \varphi_1, \omega_2 \otimes \varphi_2]_s = (-1)^{|\varphi_1||\omega_1|} \omega_1 \wedge \omega_2 \otimes [\varphi_1, \varphi_2]_s.$$

This follows from a straightforward computation and the fact that $\mathcal{A}_X^*$ is supercommutative. Therefore

$$\mathrm{Tr}_\Lambda[\omega_1 \otimes \varphi_1, \omega_2 \otimes \varphi_2]_s = (-1)^{|\varphi_1||\omega_1|} \omega_1 \wedge \omega_2 \otimes \mathrm{Tr}_\Lambda[\varphi_1, \varphi_2]_s.$$

Assume that $\tilde{\delta}$ is a section of $\mathcal{A}_X^* \widehat{\otimes}\ \mathcal{E}nd_{\mathcal{O}_X}(\bigwedge E)$ which is a superderivation. Since the supercommutator and Tr_Λ are additive, we may assume (without any loss of generality) that $\tilde{\delta} = \omega_1 \otimes \varphi_1$ and $\varphi = \omega_2 \otimes \varphi_2$. Then

$$\begin{aligned}
\mathrm{Tr}_\Lambda[\tilde{\delta}, \varphi]_s &= (-1)^{|\varphi_1||\omega_1|} \omega_1 \wedge \omega_2 \otimes \mathrm{Tr}_\Lambda[\varphi_1, \varphi_2]_s \\
&= (-1)^{|\varphi_1||\omega_1|} \omega_1 \wedge \omega_2 \otimes [\varphi_1, \mathrm{Tr}_\Lambda(\varphi_2)]_s \quad \text{by Proposition 6.4} \\
&= [\omega_1 \otimes \varphi_1, \omega_2 \otimes \mathrm{Tr}_\Lambda(\varphi_2)]_s \\
&= [\tilde{\delta}, \mathrm{Tr}_\Lambda(\varphi)]_s.
\end{aligned}$$

We further observe that $\delta - \overline{\partial}$ is a section of $\mathcal{A}_X^* \widehat{\otimes}\ \mathcal{E}nd_{\mathcal{O}_X}(\bigwedge E)$ which is a sum of superderivations. As a result we have

$$\mathrm{Tr}_\Lambda[\delta - \overline{\partial}, \varphi]_s = [\delta - \overline{\partial}, \mathrm{Tr}_\Lambda(\varphi)]_s.$$

We finally observe that

$$[\overline{\partial}, \varphi]_s = [\overline{\partial}, \omega_2 \otimes \varphi_2]_s = \overline{\partial}\omega_2 \otimes \varphi_2$$

and

$$[\overline{\partial}, \mathrm{Tr}_\Lambda(\varphi)]_s = [\overline{\partial}, \mathrm{Tr}_\Lambda(\omega_2 \otimes \varphi_2)]_s = [\overline{\partial}, \omega_2 \otimes \mathrm{Tr}_\Lambda(\varphi_2)]_s = \overline{\partial}\omega_2 \otimes \mathrm{Tr}_\Lambda(\varphi_2).$$

Then the assertion follows from the equation

$$\mathrm{Tr}_\Lambda[\overline{\partial}, \varphi]_s = [\overline{\partial}, \mathrm{Tr}_\Lambda(\varphi)]_s. \qquad \square$$

Recall that the differential δ is a flat superconnection of type $(0,1)$ on the super vector bundle $\bigwedge E$. If we let $A = \nabla + \delta$ (recall that ∇ is a flat $(1,0)$-connection on E), then A is a superconnection on $\bigwedge E$. The curvature of A, denoted by R_A, is given by the formula

$$R_A = A^2 = (\nabla + \delta)^2 = \nabla^2 + \nabla \circ \delta + \delta \circ \nabla + \delta^2 = [\nabla, \delta]_s$$

since $\nabla^2 = \delta^2 = 0$.

Corollary 9.2. — *Let $\psi = \frac{1}{r!}R_A^r$. Then*

$$[\delta, \mathrm{Tr}_\Lambda(\psi)]_s = 0.$$

Proof. — This follows from the fact that $[\delta, R_A]_s = 0$ and the proposition. $\square$

Corollary 9.3. — *Let $\psi = \frac{1}{r!}R_A^r$. Then*

$$\overline{\partial} \circ \mathrm{Tr}_\Lambda(\psi) = \mathrm{Tr}_\Lambda(\psi) \circ \delta.$$

In other words, $\mathrm{Tr}_\Lambda(\psi)$ is a cochain map from the twisted complex $(T^* = T(M)^*, \delta)$ of Proposition 8.4 (which is quasi-isomorphic to $\mathcal{O}_Z$) to the Dolbeault complex $(\mathcal{C}_X^{r,*}[r], \overline{\partial})$ (which is quasi-isomorphic to $\Omega_X^r[r]$).

Proof. — We have

$$\begin{aligned}
[\delta, \mathrm{Tr}_\Lambda(\psi)]_s &= \sum_{j\geqslant 0}\ \sum_{m-n\geqslant j-1} [a_j, \mathrm{Tr}_\Lambda(\psi_{m,n})]_s \\
&= \sum_{j\geqslant 0}\ \sum_{m-n\geqslant j-1} (a_j \circ \mathrm{Tr}_\Lambda(\psi_{m,n}) - \mathrm{Tr}_\Lambda(\psi_{m,n}) \circ a_j) \\
&\qquad \text{since } \mathrm{Tr}_\Lambda(\psi_{m,n}) \text{ are of even degree} \\
&= \overline{\partial} \circ \mathrm{Tr}_\Lambda(\psi) - \sum_{j\geqslant 0}\ \sum_{m-n\geqslant j-1} \mathrm{Tr}_\Lambda(\psi_{m,n}) \circ a_j \\
&= \overline{\partial} \circ \mathrm{Tr}_\Lambda(\psi) - \mathrm{Tr}_\Lambda(\psi) \circ \delta.
\end{aligned}$$

Then the assertion follows from Corollary 9.2. $\square$

Corollary 9.3 (combined with the Lemmas 10.1 and 10.4) can be seen as a generalization of Proposition 5.3 to the real-analytic case.

10. Comparison with the Grothendieck Class

As a result of Corollary 9.3, $\mathrm{Tr}_\Lambda(\psi)$ gives us an element in $\mathrm{Hom}_{\mathcal{O}_X}(T^*, \mathcal{A}_X^{r,*}[r])$, and therefore a class in $\mathrm{Ext}^r_{\mathcal{O}_X}(T^*, \mathcal{A}_X^{r,*})$. We can identify $\mathrm{Ext}^r_{\mathcal{O}_X}(T^*, \mathcal{A}_X^{r,*})$ with the group $\mathrm{Ext}^r_{\mathcal{O}_X}(\mathcal{O}_Z, \Omega^r_X)$ since T^* and $\mathcal{A}_X^{r,*}$ are quasi-isomorphic to $\mathcal{O}_Z$ and Ω^r_X respectively. Now we shall prove that the class of $\mathrm{Tr}_\Lambda(\psi)$ in $\mathrm{Ext}^r_{\mathcal{O}_X}(\mathcal{O}_Z, \Omega^r_X)$, denoted by $[\mathrm{Tr}_\Lambda(\psi)]$, is the Grothendieck fundamental class. But we need two preliminary lemmas first.

Lemma 10.1. — *Let* $\psi = \frac{1}{r!}R^r_A$. *We write* $\psi = \sum \psi_{m,n}$ *where* $\psi_{m,n} \in \mathcal{A}(X) \otimes \mathrm{Hom}_{\mathcal{A}_X}(\bigwedge^m \mathcal{E}, \bigwedge^n \mathcal{E})$. *Then*

$$\psi_{r,0} = \frac{1}{r!}(\imath_{\nabla(\tau)})^r.$$

Proof. — We have $R_A = [\nabla, \delta]_s = \imath_{\nabla(\tau)} + \sum_{k\geqslant 1} \nabla(a_k)$. In this sum, the only summand that lowers the Koszul degree is the term $\imath_{\nabla(\tau)}$. Therefore, the only term that lowers the Koszul degree by r in $\frac{1}{r!}R^r_A$ is $\frac{1}{r!}(\imath_{\nabla(\tau)})^r$. □

Therefore, we have the following equality for the restriction of $\mathrm{Tr}_\Lambda(\psi)$ to $\mathcal{A}_X \otimes \bigwedge^r \mathcal{E}$

$$\mathrm{Tr}_\Lambda(\psi)|_{\mathcal{A}_X \otimes \bigwedge^r \mathcal{E}} = \frac{1}{r!}(\imath_{\nabla(\tau)})^r.$$

Lemma 10.2. — *Let* A^*, B^*, *and* C^* *be cochain complexes; and* $f : A^* \to B^*$ *and* $g : B^* \to C^*$ *be maps of complexes such that the composition* $g \circ f$ *is homotopic to the zero map. There exists a map* $l(f) : A^* \to \mathrm{Cone}(g)^*[-1]$ *such that the following triangle is commutative*

$$\begin{tikzcd} A^* \arrow[r, "l(f)"] \arrow[dr, "f"'] & \mathrm{Cone}(g)^*[-1] \arrow[d, "\mathrm{Pr}"] \\ & B^* \end{tikzcd}$$

where $\mathrm{Pr} : \mathrm{Cone}(g)^*[-1] \to B^*$ *is the projection map.*

Proof. — Exercise. □

Theorem 10.3. — $[\mathrm{Tr}_\Lambda(\psi)] \in \mathrm{Ext}^r_{\mathcal{O}_X}(\mathcal{O}_Z, \Omega^r_X)$ *is the Grothendieck fundamental class.*

Proof. — Since $\mathrm{Ext}^r_{\mathcal{O}_X}(\mathcal{O}_Z, \Omega^r_X) \cong H^0(X, \mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{O}_Z, \Omega^r_X))$, we need only prove that the classes agree locally.

Let $x \in X$ be a point and U be a neighborhood of x such that the restriction of E to U is trivial, with $\{f_1, \dots, f_r\}$ a local holomorphic framing of E over U and $\{f^1, \dots, f^r\}$ the dual framing for $E^\vee$; we identify E and $E^\vee$ with the trivial bundle via these framings. Assume that Z has holomorphic equations $\{z_1, \dots, z_r\}$ in U, and is hence the zero set of the $\nu = z_1 f^1 + \cdots + z_r f^r$ of $E^\vee$. Then the Koszul complex $K(\nu)^*$ over U is quasi-isomorphic to $\mathcal{O}_Z|_U$ where $K(\nu)^{-i} = \bigwedge^i \mathcal{O}^r_U$ and the differentials are contractions by ν .

We will construct a quasi-isomorphism $\tilde{u} : K(\nu)^* \to T^*|_U$ such that the class of the composition $\mathrm{Tr}_\Lambda(\psi)|_U \circ \tilde{u}$ in

$$\mathcal{E}xt^r_{\mathcal{O}_X}(K(\nu)^*, \mathcal{A}_U^{r,*}[r]) \cong \mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{O}_Z|_U, \Omega_U^r)$$

is the restriction of the Grothendieck fundamental class of Z.

Step 1: We first define a map u from $K(\nu)^*$ to $K(\tau)^*|_U$. By the assumptions on Z and τ, there exist $u_{ji} \in \Gamma(U, \mathcal{A}_X)$ such that $z_i = \sum_j u_{ji}\alpha_j$ where $\tau = \alpha_1 e^1 + \cdots + \alpha_r e^r$. We let $u_0 : \mathcal{O}_U \to \mathcal{A}_U$ be the inclusion and $u_{-1} : \mathcal{O}_U^r \to \mathcal{A}_U \otimes \mathcal{E}|_U$ be the map which sends f_i to $\sum_j u_{ji} e_j$. Then we extend u_{-1} to a map of Koszul complexes by setting $u_{-k} = \bigwedge^k u_{-1}$. Therefore u_{-r} is given by multiplication by $\det(u_{ij})$.

Step 2: Next, we shall extend $u : K(\nu)^* \to K(\tau)^*|_U$ to a map $\tilde{u} : K(\nu)^* \to T^*|_U$. The twisted complex T^* is a filtered complex with respect to Dolbeault degree. Let us denote this (decreasing) filtration by F^i, i.e. $F^i = F^i(T^*) = \bigoplus_{j \geqslant i} \mathcal{A}_X^{0,j} \otimes \bigwedge \mathcal{E}$. Then one has $Gr_F^i = \mathcal{A}_X^{0,i} \otimes K(\tau)^*$.

The map $(-\delta + \imath_\tau) : K(\tau)^* \to F^1[1]$ is a map of complexes, since $(-\delta + \imath_\tau) \circ \imath_\tau = -\delta \circ (-\delta + \imath_\tau)$. Hence

$$(-\delta + \imath_\tau) \circ u : K(\nu)^* \to F^1[1]|_U$$

is a cochain map. Moreover, $(-\delta + \imath_\tau) \circ u$ is homotopic to the zero map since $K(\nu)^*$ is a complex of free $\mathcal{O}_X$-modules; $F^1[1]$ is acyclic in negative degrees; and $((-\delta + \imath_\tau) \circ u) : \mathcal{O}_U \to F^1[1]^0|_U$ is the zero map. (Note that u_0 is the inclusion of $\mathcal{O}_U$ into $\mathcal{A}_U$, and $(-\delta + \imath_\tau)_0 = \bar{\partial}$). Consequently, there exists an extension

$$\tilde{u} = l(u) : K(\nu)^* \to \mathrm{Cone}(-\delta + \imath_\tau)^*[-1]$$

by Lemma 10.2. This is the desired extension since $T^* = \mathrm{Cone}(-\delta + \imath_\tau)^*[-1]$.

Step 3: We now prove that $\tilde{u} : K(\nu)^* \to T^*|_U$ is a quasi-isomorphism. Let $i : \mathcal{O}_Z|_U \to \mathcal{A}_U^{0,*} \otimes \mathcal{O}_Z|_U$ be the inclusion and $p : K(\nu)^* \to \mathcal{O}_Z|_U$ be the augmentation map. Then we have a commutative diagram

$$\begin{array}{ccc} K(\nu)^* & \xrightarrow{\tilde{u}} & T^*|_U \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \epsilon} \\ \mathcal{O}_Z|_U & \xrightarrow{i} & \mathcal{A}_U^{0,*} \otimes \mathcal{O}_Z|_U \end{array}$$

in which the vertical arrows and the bottom horizontal arrow are quasi-isomorphisms. As a result, $\tilde{u}$ is a quasi-isomorphism.

Step 4: Let $\eta = \mathrm{Tr}_\Lambda(\psi) \circ \tilde{u}$. Thus the degree $(-r)$ component of η is given by the composition $\frac{1}{r!}(\imath_{\nabla(\tau)})^r \circ \det(u_{ij})$. Hence

$$\begin{aligned}
\eta_{-r}(f_1 \wedge \cdots \wedge f_r) &= \frac{1}{r!}(\imath_{\nabla(\tau)})^r(\det(u_{ij})e_1 \wedge \cdots \wedge e_r) \\
&= \det(u_{ij})\frac{1}{r!}(\imath_{\nabla(\tau)})^r(e_1 \wedge \cdots \wedge e_r) \\
&= \det(u_{ij})\partial\alpha_1 \wedge \cdots \wedge \partial\alpha_r \pmod{\mathscr{I}} \\
&= \partial z_1 \wedge \cdots \wedge \partial z_r \pmod{\mathscr{I}} \\
&= dz_1 \wedge \cdots \wedge dz_r \pmod{\mathscr{I}}.
\end{aligned}$$

By Proposition 5.4 η represents the Grothendieck fundamental class of $Z \cap U$ in U. Since $\tilde{u}$ is a quasi-isomorphism, so is $\mathrm{Tr}_\Lambda(\psi)|_U$. Since $\mathrm{Tr}_\Lambda(\psi)$ represents the Grothendieck fundamental class locally, it does so globally. □

Lemma 10.4. — *Let $\psi = \frac{1}{r!}R_A^r$. We have*

$$\mathrm{Tr}_\Lambda(\psi)|_{\mathscr{O}_X} = \frac{1}{r!}\mathrm{tr}_s(\psi).$$

Proof. — Omitted. □

Corollary 10.5. — *The image of the Grothendieck fundamental class in $H^r(X, \Omega_X^r)$ is represented by the (r, r) degree part of the Chern character form of the superbundle $\bigwedge E$ equipped with the superconnection $A = \nabla + \delta$.*

Proof. — This follows from the theorem, Lemma 10.4, and Theorem 3.3. □

This completes the proof of Theorem B. □

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