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ON THE DEFINITION OF THE GALOIS GROUPOID

by

Hiroshi Umemura

For José Manuel Aroca on the occasion of his 60th birthday

Abstract. — We sketch a proof of equivalence of two general differential Galois theories, Malgrange's theoy and ours, if the base field consists only of constants.

***Résumé* (Sur la définition du groupoïde de Galois).** — Nous esquissons la démonstration du fait que deux théories de Galois, la théorie de Malgrange et la nôtre, sont équivalentes dans le cas absolu, i.e. quand le corps de base consiste uniquement en des constantes.

1. Introduction

Today we have two general differential Galois theories [**4**] and [**3**]. While the first published in 1996 is a Galois theory of differential field extensions, the latter proposed in 2001 is a Galois theory of foliations on varieties. They look somehow different but specialists observed coincidence in examples. The aim of this note is to sketch in fact they are equivalent in the absolute case, by which we mean the case where the base field K of the differential field extension L/K consists of only constants. For the relative case or for a general differential field extension L/K, there may be a similar result but there are subtle questions. First of all we must have an adequate definition of the Galois groupoid for the extension L/K in terms of foliations in the spirit of [**3**].[(1)]

We show by analyzing a non-trivial interesting example, the equivalence. Given a differential field, it is an algebraic counter part of a dynamical system on a algebraic variety. If we observe this dynamical system closely by algebraic method, or if an algebraist observes the dynamical system, then we get as a natural object Galois groupoid of the dynamical system, or of the given differential field. This procedure of

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(1) Added on 25 August 2008, we can apply this method also to the reative case.

observation is done through the universal Taylor morphism and ties Malgrange's idea and ours.

2. Differential fields and dynamical systems

A differential field (L, δ) consists of a field L and a derivation $\delta : L \to L$. So we have $\delta(a+b)) = \delta(a)+\delta(b)$ and $\delta(ab) = \delta(a)b + a\delta(b)$ for every $a, b \in L$. Similarly we define a differential ring (R, δ). An element a of a differential field or a differential ring is called a constant if $\delta(a) = 0$. The set C_L or C_R of constants forms respectively a subfield or subring.

Now we consider a differential field that is finitely generated as an abstract field over the complex number field $\mathbb{C}$ in such a way that the complex number field $\mathbb{C}$ is a subfield of the field C_L of constants.

Remark 2.1. — *In the sequel, we work over the complex number field $\mathbb{C}$ so that the reader has a concrete image, we may replace, however, the complex number field $\mathbb{C}$ by any field of characteristic* 0.

We explain by examples that a differential field is an algebraic counter part of a differential dynamical system on an algebraic variety.

Example 2.1. Let us consider the differential field $(\mathbb{C}(x), d/dx)$, where x is a variable over $\mathbb{C}$ and hence $\mathbb{C}(x)$ is the rational function field of one variable. A geometric model of the differential field $(\mathbb{C}(x), d/dx)$ is a dynamical system $(\mathbb{A}^1, d/dx) =$ (Spec $\mathbb{C}[x]$, d/dx). In other words, the field of rational functions of the affine line $\mathbb{A}^1$ with derivation d/dx gives the differential field $(\mathbb{C}(x), d/dx)$.

Remark 2.2. — *Since for any non-empty Zariski open subset U of $\mathbb{A}^1$, $(U, d/dx)$ satisfies the condition required above, the general model of the differential field $(\mathbb{C}(x), d/dx)$ is ($\mathbb{A}^1-$ (a finite number of points), d/dx). The model is determined up to birational equivalence.*

Example 2.2. Let x, y be two independent variables over $\mathbb{C}$ so that $\mathbb{C}[x, y]$ is a polynomial ring over $\mathbb{C}$. Let us consider the differential field

$$(\mathbb{C}(x, y), \partial/\partial x + y\partial/\partial y).$$

A model of this differential field is the (x, y)-plane $\mathbb{A}^2$ or Spec $\mathbb{C}[x, y]$ with vector field $\partial/\partial x + y\partial/\partial y$. A general flow on the affine plane $\mathbb{A}^2$ is given by $(t, c\exp t)$, $t \in \mathbb{C}$ for a fixed $c \in \mathbb{C}$. In this Example we may replace the affine plane $\mathbb{A}^2$ by any non-empty Zariski open set of $\mathbb{A}^2$.

Generally we can prove the following proposition.

Proposition 2.1. — *Let (L, δ) be a differential field such that the field L is of finite type over the complex number field $\mathbb{C}$ and $\mathbb{C}$ is a subfield of the field C_L of constants of (L, δ). Then there exists a smooth algebraic variety V over $\mathbb{C}$, with regular algebraic vector field X such that (V, X) is a model of the differential field (L, δ). In other* *words , the rational function field $\mathbb{C}(V)$ of V is isomorphic to the field L and the vector field X is identified with the derivation δ through this isomorphism.*

See Lemma (1.5), [**5**].

3. Groupoids

We need a seemingly abstract definition of groupoid but it is as concrete as vector space.

Definition 3.1. — *A groupoid is a small category G in which all morphisms are isomorphisms. An object of G is called a vertex and a morphism in G is called an element of G.*

The groupoid was introduced by Brandt in 1926. In 1950's Ehresmann used groupoids in theory of foliations. In 1960's Grothendieck studied quotients by groupoids in algebraic geometry. Here are examples of groupoids to have an image of groupoids.

Example 3.1. A group G is a groupoid. We define a category $\mathcal{C}$ that is a groupoid. The object of the category $\mathcal{C}$ is one point P, i.e. $ob\,\mathcal{C} = \{P\}$. We set

$$\mathrm{Hom}\ (P,\ P) = G$$

and compose two morphisms of $\mathrm{Hom}\ (P,\ P) = G$ according as the group law of G.

Example 3.2. Equivalence relation $\sim$ on a set X. The set $ob\,G$ of the objects of the groupoid G is the set X. For $x,\ y \in ob\,G$, we define

$$\mathrm{Hom}\ (x,\ y) = \begin{cases} 1 \text{ morphism}, & \text{if } x \sim y, \\ \varnothing, & \text{otherwise.} \end{cases}$$

Since every element x is equivalent to itself, we have the identity Id_x. Since equivalence relation is reflexive, every morphism is an isomorphism. Since equivalence relation is transitive, we can compose two morphisms. So the above definition yields us a groupoid.

Example 3.3. Group operation (G, X) of a group G on a set X is a groupoid. The set $ob\,C$ of the groupoid C is the set X. For $x,\ y \in X = ob\,C$, we set $\mathrm{Hom}\ (x,\ y) = \{g \in G | gx = y\}$. If $g \in\ \mathrm{Hom}\ (x,\ y)$ and $h \in\ \mathrm{Hom}\ (y,\ z)$, then $gx = y$ and $hy = z$ by definition so that $z = hy = h(gx) = (hg)x$ and consequently $hg \in\ \mathrm{Hom}\ (x, z)$. So we can compose two morphisms. If $gx = y$, then $hy = x$, h being g^{-1} so that every morphism is an isomorphism.

Example 3.4. Poincaré groupoid. Let X be a topological space. Let $ob\,G$ of the category be the set X. A path from a point $x \in X$ to another point $y \in X$ is a continuous map $\varphi : [0, 1] \to X$ from the interval $[0, 1]$ to the topological space X such that $\varphi(0) = x$ and $\varphi(1) = y$. We set in the category G,

$$\text{Hom }(x, y) := \text{ the set of paths from } x \text{ to } y \text{ modulo homotopy equivalence.}$$

Then it is well-know that the category G is a groupoid, which is called a Poincar groupoid.

Now let G be a groupoid . We set

$$Y := \{\text{morphisms in the category } G\}$$

and

$$X := ob\, G.$$

Let $\varphi \in Y$ so that $\varphi \in$ Hom (A, B) for some $A, B \in ob\, G$. Let us denote the sourc A of φ by $s(\varphi)$ and the target B of φ by $t(\varphi)$. So we get two maps $s : Y \to X$ an $t : Y \to X$. Let $(Y, t) \times (Y, s)$ be the fiber product of $t : Y \to X$ and $s : Y \to X$ s that

$$(Y, t) \times (Y, s) = \{(\varphi, \psi) \in Y \times Y | s(\varphi) = t(\psi)\}.$$

The composition of morphisms defines a map

$$\Phi : (Y, t) \times (Y, s) \to Y, \qquad (\varphi, \psi) \mapsto \psi \circ \varphi.$$

The associativity of the composition is described by a commutative diagram that w do not make precise. See [**2**]. The existence of the identity map Id_A for every $A \in ob$ (as well as the property called symmetry that every morphism is an isomorphism i also characterized in terms of maps and commutative diagrams.

Here is a summary of the above observation. Groupoid is described by two sets Y and X, two maps $s : Y \to X$ and $t : Y \to X$ and the composition maps

$$\Phi : (Y, t) \times (Y, s) \to Y, \qquad (\varphi, \psi) \mapsto \psi \circ \varphi.$$

that satisfy certain commutative diagrams and so on.

This allows us to generalize the notion of groupoid in a category in which fibe product exists. This is exactly by the same way as we define an algebraic group (requiring that, first of all, G is an algebraic variety, the composition law $G \times G \to$ (is a morphism of algebraic varieties and so on.

Definition 3.2. — *Let C be a category in which fiber product exists. A groupoid i the category C consists of two objects $Y, X \in ob\, C$, two morphisms $s : Y \to X$ an $t : Y \to X$ and a morphism*

$$\Phi : (Y, t) \times (Y, s) \to Y$$

etc, satisfying the above conditions (cf. Grothendieck [**2**]*)*

Example 3.5. Let C be the category of algebraic varieties defined over a field and let (G, V) be an operation of an algebraic group on an algebraic variety V define over k. We have two morphisms p, h from $G \times V$ to V, namely the second projectio

p and the group operation $h(g, v) = gv$. Then $Y = G \times X$, $X = V$ $s = p$ and $t = h$ is a groupoid in the category C. Compare to Example 3.3.

We need a tool, an algebraic D-groupoid that generalizes Example 3.5.

4. Lie groupoids and D-groupoids

For a complex manifold V, we can attach its invertible jets $J^*(V \times V)$ that is a groupoid over $V \ \times V$ in the category of analytic spaces. We recall the definition for $V = \mathbb{C}$. The jet space $J(\mathbb{C} \times \mathbb{C})$ is an infinite dimensional analytic space $\mathbb{C} \times \mathbb{C}^{\mathbb{N}}$ with coordinate system $(x, , y_0, y_1, y_2, \dots)$, We have two morphisms $s : J(\mathbb{C} \times \mathbb{C}) \to \mathbb{C}$ and $t : J(\mathbb{C} \times \mathbb{C}) \to \mathbb{C}$ given by

$$s((x, , y_0, y_1, y_2, \dots)) = x \qquad \textit{and} \qquad t((x, , y_0, y_1, y_2, \dots)) = y_0.$$

So we have a morphism $(s, t) : J(\mathbb{C} \times \mathbb{C}) \to \mathbb{C} \times \mathbb{C}$ that makes $J(\mathbb{C} \times \mathbb{C})$ an infinite dimensional affine space over $\mathbb{C} \times \mathbb{C}$. The invertible jet space $J^*(\mathbb{C} \times \mathbb{C})$ is , by definition, the Zarisiki open set of $J(\mathbb{C} \times \mathbb{C})$. Namely,

$$J^*(\mathbb{C} \times \mathbb{C}) := \{(x, y_0, y_1, y_2, \dots) \in J(\mathbb{C} \times \mathbb{C}) | \, y_1 \neq 0\}.$$

We simply denote $J^*(\mathbb{C} \times \mathbb{C})$ by J^* and we write the restrictions of the morphisms s, t to the Zariski open set J^* by the same letters. Now we explain J^* with two morphisms $s : J^* \to \mathbb{C}$ and $t : J^* \to \mathbb{C}$ is a groupoid. To this end we must define the composite morphism $\Phi : (J^*, t) \times (J^*, s) \to J^*$. Let

$$\varphi = (x, y_0, y_1, \dots), \qquad \psi = (u, v_0, v_1, \dots),$$

be points of J^* such that $y_0 = t(\varphi) = s(\psi) = u$, i.e. (φ, ψ) is a point of $(J^*, t) \times (J^*, s)$. Then we set

$$\Phi(\psi, \varphi) := (x, v_0, y_1 v_1, y_2 v_1 + y_1^2 v_2, \dots). \tag{1}$$

The n-th component of $\Phi(\psi, \varphi)$ is given by the following rule. Imagine formally that φ were a function of x taking the value y_0 at x, or $\varphi(x) = y_0$, with $\varphi'(x) = y_1, \varphi''(x) = y_2 \dots$. Similarly consider as if ψ were a function of u with $\psi(u) = v_0$, $\psi'(u) = v_1$, $\psi''(u) = v_2, \dots$. Then $\Phi(\psi, \varphi)$ is the composite function $\psi \circ \varphi$, which is a function of x, so that its n-th component is the value of $d^n \psi \circ \varphi / dx^n$ at x. For example ,

$$d(\psi \circ \varphi)/dx = \psi_u \varphi_x = y_1 v_1, d^2(\psi \circ \varphi)/dx^2 = \varphi_{xx} \psi_u + \varphi_x^2 \psi_{uu} = y_2 v_1 + y_1^2 u_2, \dots$$

One can check this composition law is associative and the inverse of

$$\varphi = (x, y_0, y_1, \dots)$$

is given by the inverse function $x(y_0)$ and its derivatives $d^n x(y_0)/dy_0^n$ for $n \in \mathbb{N}$, namely by

$$(y_0, x, 1/y_1, -y_2/y_1^3, \dots).$$

We can very naturally extend this construction over a complex manifold of any dimension.

Remark 4.1. — *We considered above the n-th coordinate to be $d^n y/dx^n$. But it is more natural to use $(1/n!)d^n y/dx^n$. In this way we can work over $\mathbb{Z}$.*

The above construction of Lie groupoids, in the category of analytic spaces, also works in the category of algebraic varieties, or to be more correct in the category of schemes over a field C. The most important ingredient in the algebraic constriction is the universal extension of derivations [**9**]. We do not go into the detail because it is technical and will be published elsewhere. So for a non-singular algebraic variety V defined over the field C of characteristic 0, we can define its invertible jet space $J^*(V \times V)$ that is an algebraic variety of infinite dimension, i.e. an affine scheme over $V \times V$.

Definition 4.1. — *An algebraic D-groupoid is a sbugroupoid of J^* defined by a differential ideal.*

We are going to show what the definition means by concrete Examples.

Let $V = \mathbb{A}^1$ and we consider the Lie groupoid $J^*(\mathbb{C} \times \mathbb{C})$. Recall that the construction above in this case is purely algebraic. The coordinate ring or the ring of (algebraic) regular function on $J^*(\mathbb{C} \times \mathbb{C})$ is $\mathbb{C}[x, y_0, y_1, \dots, 1/y_0]$. The derivation $\delta = \partial/\partial x + \sum_{i=0}^{\infty} y_{i+1}\partial/\partial y_i)$ operates on the coordinate ring $\mathbb{C}[x, y_0, y_1, \dots, 1/y_1]$ of $J^*(\mathbb{C} \times \mathbb{C})$. So $(\mathbb{C}[x, y_0, y_1, \dots, 1/y_1], \delta)$ is a differential algebra. Consider the differential ideal I of the coordinate ring generated by $y_1 - 1$ so that $y_{n+1} = \delta^n(y_1) = \delta^n(y_1 - 1) \in I$ for $n = 1, 2, 3, \dots$. Hence the algebraic subvariety of $J^*(\mathbb{C} \times \mathbb{C})$ defined by the ideal I is

$$Y = \{(a, b, 1, 0, \dots) \in J^*(\mathbb{C} \times \mathbb{C})\}.$$

Let

$$((a, b, 1, 0, \dots), (c, d, 1, 0, \dots)) \in (J^*, t) \times (J^*, s)$$

so that $b = c$, then

$$\Phi((a, b, 1, 0, \dots), (c, d, 1, 0, \dots)) = (a, d, 1, 0, \dots)$$

by (1). This shows that the subvariety $Y \subset J^*$ is closed by the composition. Since the inverse of $(a, b, 1, 0, \dots)$ is $(b, a, 1, 0, \dots)$, Y is an algebraic groupoid. This groupoids is nothing but the strongest equivalence relation on $\mathbb{C}$ according which arbitrary two points of $\mathbb{C}$ are equivalent. See Example 3.2. Another interpretation is the operation of the additive group $\mathbb{C}$, which is an algebraic group, on it self. See Examples 3.3 and 3.5.

Example 4.2. Let $V = \mathbb{A}^1 - \{0\}$ and n an integer . Then by construction $J^*(V \times V) = \{(x, y_0, y_1, \dots) \in \mathbb{C} \times \mathbb{C}^{\mathbb{N}} | \, x, y_0, y_1 \neq 0\}$. Let I_n be the differential ideal of

$$\mathbb{C}[x, y_0, y_1, \dots, 1/x, 1/y_0, 1/y_1]$$

generated by $y_1 - (x/y_0)^n$. Let us assume that $\varphi = (x, y_0, y_1, \dots) \in J^*(V \times V)$ satisfies the differential equation

$$\varphi_x = (x/\varphi)^n \tag{2}$$

and $\psi = (u, v_0, v_1, \dots) \in J^*(V \times V)$ satisfies the differential equation

$$\psi_u = (u/\psi)^n \tag{3}$$

with $y_0 = u$, then by (2) and (3)

$$\frac{d(\psi \circ \varphi)}{dx} = \frac{d\psi}{du}\frac{d\varphi}{dx} = \left(\frac{u}{\psi}\right)^n \left(\frac{x}{\varphi}\right)^n = \left(\frac{x}{\psi}\right)^n.$$

So the ideal I_n defines a groupoid G_n. By considering the automorphism $V \to V$, $u \mapsto u^{-1}$, the groupoid G_n is isomorphic to G_{-n+2}. In fact this follows from

$$\frac{d(\varphi(x)^{-1})}{dz} = \left(\frac{z}{\varphi(x)^{-1}}\right)^{2-n}$$

if φ satisfies (2), where $z = x^{-1}$.

Example 4.3. The Schwarzian defines a D-groupoid over $\mathbb{C}$. To be more precise let us recall the Schwarzian derivative

$$\{y;\, x\} := \left(\frac{d^3y}{dx^3}\right) / \left(\frac{dy}{dx}\right) - \frac{3}{2}\left[\frac{d^2y}{dx^2} / \frac{dy}{dx}\right]^2,$$

where y is a function of x. We know that when y is a function of x and z is a function of y and consequently z is a function of x, we have a formula

$$\{z;\, x\} = \left(\frac{dy}{dx}\right)^2 \{z;\, y\} + \{y;\, x\} \tag{4}$$

The formula (4) shows that the differential ideal I of $\mathbb{C}[x, y_0, y_1, \dots, 1/y_1]$ generated by $y_2/y_1 - (3/2)y_2/y_1)^2$ defines a D-groupoid that is a subgroupoid of $J^*(\mathbb{C} \times \mathbb{C})$.

5. Examples of Galois groupoids

Example 5.1. Let t, x, y, be independent variables over $\mathbb{C}$. Let $\delta : \mathbb{C}(t, x, y) \to \mathbb{C}(t, x, y)$ be the $\mathbb{C}$-derivations of the rational function field $\mathbb{C}(t, x, y)$ such that

$$\delta(t) = 0, \qquad \delta(x) = 1, \qquad \delta(y) = \frac{t}{x}y, \tag{5}$$

In other words

$$\delta = \frac{\partial}{\partial x} + \frac{ty}{x}\frac{\partial}{\partial y}.$$

This is the differential field studied by Cassidy and Singer [**1**]. We analyzed this example in [**8**] . We present here a new point of view that ties our previous definition of Galois group *Inf-gal* and Galois groupoid of Malgrange.([**4**], [**3**]). The first is, as we briefly review in §6, an automorphism group of a certain differential field and the latter is a D-groupoid over an algebraic variety. See also §6.

We explain from the view point of dynamical system what is the Galois groupoid

$$\mathrm{Gal}\,(\mathbb{C}(t,\,x,\,y)/\mathbb{C})$$

of the differential field

$$(\mathbb{C}(t,\,x,\,y),\,\delta) \tag{6}$$

over $\mathbb{C}$. First of all, $R := \mathbb{C}[t,\,x,\,y,\,1/x]$ is δ-invariant so that

$$(\mathrm{Spec}\;R,\,\delta)$$

is a model of the differential field (6) (cf. §2). The Galois groupoid $\mathrm{Gal}((\mathbb{C}(t,\,x,\,y),\,\delta)/\mathbb{C})$ is a D-groupoid over the algebraic variety $V = \mathrm{Spec}\;R = \{(t,\,x,\,y) \in \mathbb{C}^3 \,|\, x \neq 0\}$. Let us recall the universal Taylor morphism

$$\iota : R \to R^\natural[[X]]$$

that sends an element $a \in R$ to its formal Taylor expansion

$$\sum_{n=0}^{\infty} \frac{1}{n!}\delta^n(a)X^n, \tag{7}$$

where X is a variable. Recall according to our convention that $R^\natural$ denotes the ring R without derivation δ. So logically we must write Spec $R^\natural$. It follows from (7)

$$\iota(t) = t, \qquad \iota(x) = x + X, \qquad \iota(y) = y\left(1 + \frac{X}{x}\right)^t,$$

where

$$\left(1 + \frac{X}{x}\right)^t = \sum_{n=0}^{\infty} \frac{t(t-1)\cdots(t-n+1)}{n!} X^n.$$

See [**4**]. We set

$$\tilde{T}(t,\,x,\,y;\,X) := \iota(t), \qquad \tilde{X}(t,\,x,\,y;\,X) := \iota(x), \qquad \tilde{Y}(t,\,x,\,y;\,X) := \iota(y).$$

Since ι is an algebra morphism compatible with δ and d/dX , we have by (5)

$$\frac{d\tilde{T}}{dX} = 0, \qquad \frac{d\tilde{X}}{dX} = 1, \qquad \frac{d\tilde{X}}{dX} = \frac{\tilde{T}}{\tilde{X}}\tilde{Y}. \tag{8}$$

In other words $\tilde{T}$, $\tilde{X}$, $\tilde{Y}$ is a solution to (5) with initial conditions

$$\tilde{T}(t,\,x,\,y;0) = t, \qquad \tilde{X}(t,\,x,\,y;0) = x, \qquad \tilde{Y}(t,\,x,\,y;0) = y$$

at $X = 0$. We are interested in the dynamical system

$$(t,\,x,\,y) \mapsto (\tilde{T},\,\tilde{X},\,\tilde{Y}) \tag{9}$$

on the algebraic variety $V = \mathrm{Spec}\;R = \{(u,\,v,\,w) \in \mathbb{C}^3 \,|\, v \neq 0\}$.

Suppose now that an algebraist, who knows nothing about transcendental functions, lives on the variety Spec R and he observes the dynamical system (9). As he can not recognize what is transcendental or analytic, he tries to understand the dynamical system(9) through algebraic differential equations. Above all, he will take the first derivatives to get

$$
\begin{aligned}
&\tilde{T} = t, \\
&\partial_t \tilde{X} = 0, \qquad \partial_x \tilde{X} = 1, \qquad \partial_y \tilde{X} = 0, \\
&(\partial_t \tilde{Y})/\tilde{Y} = \log(1 + \tfrac{X}{x}), \quad (\partial_x \tilde{Y})/\tilde{Y} = t(\tfrac{1}{\tilde{X}} - \tfrac{1}{x}), \quad y\partial_y \tilde{Y} = \tilde{Y}.
\end{aligned} \tag{10}
$$

The first equation of the last line of (10) contains the transcendental function log so that the poor algebraist can not understand it. So he will take the second derivative of the expression to conclude

$$\partial_t \left((\partial_t \tilde{Y})/\tilde{Y}\right) = 0, \qquad \partial_x \left((\partial_t \tilde{Y})/\tilde{Y}\right) = \frac{1}{\tilde{X}} - \frac{1}{x}, \qquad \partial_y \left((\partial_t \tilde{Y})/\tilde{Y}\right) = 0.$$

These are algebraic relations so that the algebraist can understand but these three equations except for the first are consequences of the other equations of (10) and hence they are superfluous. So all the necessary algebraic differential relations are

$$
\begin{aligned}
&\tilde{T} = t, \\
&\partial_t \tilde{X} = 0, \qquad \partial_x \tilde{X} = 1, \qquad \partial_y \tilde{X} = 0, \\
&\partial_t \left((\partial_t \tilde{Y})/\tilde{Y}\right) = 0, \quad (\partial_x \tilde{Y})/\tilde{Y} = t(\tfrac{1}{\tilde{X}} - \tfrac{1}{x}), \quad y\partial_y \tilde{Y} = \tilde{Y}.
\end{aligned} \tag{11}
$$

Let us summarize the analysis above. The algebraist observed the dynamical system (9) and arrived at the system (11) of algebraic partial differential equations. What is this system? The answer is this is the defining equation of the Galois groupoid $\mathrm{Gal}\left((\mathbb{C}(t, x, y), \delta)/\mathbb{C}\right)$ that is a D-groupoid on $V = \mathrm{Spec}\ R = \{(t, x, y) \in \mathbb{C}^3 \mid x \neq 0\}$.

To be more precise, we introduce variables $\mathcal{T}$, $\mathcal{X}$, $\mathcal{Y}$, which you might imagine functions of t, x, y, and their formal derivatives

$$\partial_t^l \partial_x^m \partial_y^n \mathcal{T}, \qquad \partial_t^l \partial_x^m \partial_y^n \mathcal{X}, \qquad \partial_t^l \partial_x^m \partial_y^n \mathcal{Y}, \qquad \text{for } (l, m, n) \in \mathbb{N}^3 \tag{12}$$

that are also variables. We identify

$$\partial_t^0 \partial_x^0 \partial_y^0 \mathcal{T} = \mathcal{T}, \qquad \partial_t^0 \partial_x^0 \partial_y^0 \mathcal{X} = \mathcal{X}, \qquad \partial_t^0 \partial_x^0 \partial_y^0 \mathcal{Y} = \mathcal{Y}.$$

The invertible jet space $J^*(V \times V)$ over $V = \mathrm{Spec}\ R$ is by definition

$$J(V \times V) := \mathrm{Spec}\ \mathbb{S},$$

where

$$\mathbb{S} := \mathbb{C}[t, x, y, \mathcal{T}, \mathcal{X}, \mathcal{Y}, \partial_t^l \partial_x^m \partial_y^n \mathcal{T}, \partial_t^l \partial_x^m \partial_y^n \mathcal{X}, \partial_t^l \partial_x^m \partial_y^n \mathcal{Y}, 1/x, 1/\mathcal{X}, 1/\mathrm{Jac}]_{(l, m, n) \in \mathbb{N}^3},$$

and where Jac is the Jacobian

$$\mathrm{Jac} = \begin{vmatrix} \partial_t \mathcal{T} & \partial_x \mathcal{T} & \partial_y \mathcal{T} \\ \partial_t \mathcal{X} & \partial_x \mathcal{X} & \partial_y \mathcal{X} \\ \partial_t \mathcal{Y} & \partial_x \mathcal{Y} & \partial_y \mathcal{Y} \end{vmatrix}.$$

The algebraic differential relations (11) gives us a differential ideal I of the ring $\mathbb{S}$ generated by

$$
(13) \qquad \begin{array}{lll} \mathcal{T} - t, & & \\ \partial_t \mathcal{X}, & \partial_x \mathcal{X} - 1, & \partial_y \mathcal{X}, \\ \mathcal{Y}^2 \partial_t \left((\partial_t \mathcal{Y})/\mathcal{Y} \right), & \mathcal{Y}[(\partial_x \mathcal{Y})/\mathcal{Y} - t(\frac{1}{\mathcal{X}} - \frac{1}{x})], & y\partial_y \mathcal{Y} - \mathcal{Y}. \end{array}
$$

Here the ring $\mathbb{S}$ is a partial differential algebra with respect to the three derivations ∂_t, ∂_x, ∂_y that operate on the variables (12) just formally as in the one variable case treated in §3. Then the ideal I defines a D-groupoid on V that is the Galois groupoid $\mathrm{Gal}\,(\mathbb{C}(t, x, y)/\mathbb{C})$.

An easy calculation leads us to the following

Proposition 5.1. — *For a general point $(t, x, y) \in V$, the solution $(\mathcal{T}, \mathcal{X}, \mathcal{Y})$ to the differential ideal I or equivalently to the partial differential system (11) is*

$$
(14) \qquad \mathcal{T} = t, \qquad \mathcal{X} = x + c_1, \qquad \mathcal{Y} = y \left(\frac{x + c_1}{x} \right)^t \exp(c_2 t + c_3)
$$

where c_1, c_2, c_3 are constants.

Differentiating the solution (14) with respect to the parameters c_1, c_2, c_3 at $(c_1, c_2, c_3) = (0, 0, 0)$, we get the vector fields

$$
(15) \qquad D_1 = \frac{\partial}{\partial x} + \frac{ty}{x} \frac{\partial}{\partial y}, \qquad D_2 = ty \frac{\partial}{\partial y}, \qquad D_3 = y \frac{\partial}{\partial y}
$$

on the variety $V = \mathrm{Spec}\, R = \{(t, x, y) \in \mathbb{C}^3 \mid x \neq 0\}$. The vector space spanned by these vector fields is closed under the bracket and forms a 3-dimensional commutative Lie algebra. This is the Lie algebra of the Galois groupoid $\mathrm{Gal}\,(\mathbb{C}(t, x, y)/\mathbb{C})$. We have thus proved

Corollary 5.1. — *The Lie algebra of the Galois groupoid* $\mathrm{Gal}\,(\mathbb{C}(t, x, y)/\mathbb{C})$ *is a 3 dimensional Abelian Lie algebra spanned by D_1, D_2, D_3.*

Example 5.2. Let $n \geq$ be an integer and we consider a field extension $L := \mathbb{C}(x, y)/\mathbb{C}(x)$ of the rational function field $\mathbb{C}(x)$ such that y is algebraic over $\mathbb{C}(x)$ with minimal polynomial

$$
(16) \qquad y^n - x = 0.
$$

With derivation $\delta = d/dx$, $L/\mathbb{C}(x)$ is a differential field extension. We have

$$
(\mathbb{C}(x, y), \delta) = \left(\mathbb{C}(x), \frac{1}{n} y^{1-n} \frac{d}{dy} \right).
$$

Then the Galois groupoid $\mathrm{Gal}\,((L, \delta)/\mathbb{C})$ is the groupoid G_{n-1} of Example 4.2.

In fact let $\iota : \mathbb{C}[y, 1/y] \to \mathbb{C}[y, 1/y]^{\natural}[[X]]$ is the universal Taylor morphism. Applying the universal Taylor morphism ι to (16), we have

$$
\tilde{Y}^n - (x + X) = 0,
$$

where $\tilde{Y} := \iota(y)$. Then apply the derivation d/dy to this equation, we get

$$\frac{\tilde{Y}}{dy} = \left(\frac{y}{\tilde{Y}}\right)^{n-1}.$$

6. Relation with our previous definition

Using Example 5.1, we briefly explain the equivalence of the definition above of Galois groupoid through the dynamical system and our previous definition depending on the differential field automorphism group of a certain partial differential field extension.

We keeping the notation of Example 5.1, we apply the method of our previous paper([**4**]. We start from the ordinary differential field extension $(\mathbb{C}(t, x, y), \delta)/\mathbb{C}$ and construct a partial differential field extension $\mathcal{L}/\mathcal{K}$ and our infinitesimal Galois group $\mathit{Inf\text{-}gal}\,(\mathcal{L}/\mathcal{K})$ is the infinitesimal automorphism group $\mathit{Inf\text{-}aut}\,(\mathcal{L}/\mathcal{K})$.

In $R^{\natural}[[X]] of \S 5$, we have $R^{\natural}$ and $\iota(R)$. The derivations ∂_t, ∂_x, ∂_y of $R^{\natural}$ operate on the power series ring $R^{\natural}[[X]]$ through coefficients. Now $\mathcal{R}$ is the differential algebra generated by $R^{\natural}$ and $\iota(R)$ in $R^{\natural}[[X]]$. It follows from the definition of $\mathcal{R}$, $\mathbb{S}$ and the ideal I of $\mathbb{S}$

Proposition 6.1. — *The ring $\mathcal{R}$ is isomorphic to $\mathbb{S}/I$ as $\{\partial_t, \partial_x, \partial_y\}$-differential algebra.*

It follows from (11) that $\mathcal{R} = \mathbb{C}[t, x, y, 1/y][\tilde{X}, \tilde{Y}, \tilde{Y}_t]$ and $\tilde{X}, \tilde{Y}, \tilde{Y}_t]$ are transcendental over $\mathbb{C}[t, x, y, 1/y]$ The base ring is by definition $\mathbb{C}[t, x, y, 1/y]$. Our Galois group $\mathit{Inf\text{-}gal}\,((\mathbb{C}(t, x, y), \delta)/\mathbb{C})$ is the infinitesimal automorphism group of $\mathcal{R}/\mathbb{C}[t, x, y, 1/y]$. Propositions (5.1) and (6.1) give us 3 infinitesimal $\{\partial_t, \partial_x, \partial_y\}$-differential automorphisms σ_i, of $\mathcal{R}[\epsilon]/\mathbb{C}[t, x, y, 1/y]$ ($1 \le i \le 3$) with $\epsilon^2 = 0$ such that

$$\begin{array}{llll}
\sigma_1(\tilde{T}) = \tilde{T}, & \sigma_1(\tilde{X}) = \tilde{X} + \epsilon, & \sigma_1(\tilde{Y}) = \tilde{Y} + \frac{t\tilde{Y}}{\tilde{X}}\epsilon, & \sigma_1(\tilde{Y}_t) = \tilde{Y}_t + \frac{\tilde{Y} - t\tilde{Y}_t}{\tilde{X}}\epsilon \\
\sigma_2(\tilde{T}) = \tilde{T}, & \sigma_2(\tilde{X}) = \tilde{X}, & \sigma_2(\tilde{Y}) = \tilde{Y} + t\tilde{Y}\epsilon, & \sigma_2(\tilde{Y}_t) = \tilde{Y}_t + (\tilde{Y} - t\tilde{Y}_t)\epsilon \\
\sigma_3(\tilde{T}) = \tilde{T}, & \sigma_3(\tilde{X}) = \tilde{X}, & \sigma_3(\tilde{Y}) = \tilde{Y} + \tilde{Y}\epsilon, & \sigma_3(\tilde{Y}_t) = \tilde{Y}_t + \tilde{Y}_t\epsilon.
\end{array}$$

These infinitesimal automorphisms define infinitesimal automorphisms of $\mathcal{L}[\epsilon]/\mathcal{K}$, where $\mathcal{L}$ and $\mathcal{K}$ are respectively the quotient field of $\mathcal{R}$ and $\mathbb{C}[t, x, y, 1/y]$.

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