Astérisque

FRANK LORAY
DAVID MARÍN PÉREZ

Projective structures and projective bundles over compact Riemann surfaces

Astérisque, tome 323 (2009), p. 223-252
<http://www.numdam.org/item?id=AST_2009__323__223_0>

Astérisque
323, 2009, p. 223–252

PROJECTIVE STRUCTURES AND PROJECTIVE BUNDLES OVER COMPACT RIEMANN SURFACES

by

Frank Loray & David Marín Pérez

To José Manuel Aroca for his 60th birthday

Abstract. — A projective structure on a compact Riemann surface C of genus g is given by an atlas with transition functions in $\mathrm{PGL}(2,\mathbb{C})$. Equivalently, a projective structure is given by a $\mathbb{P}^1$-bundle over C equipped with a section σ and a foliation $\mathcal{F}$ which is both transversal to the $\mathbb{P}^1$-fibers and the section σ. From this latter geometric bundle picture, we survey on classical problems and results on projective structures. By the way, we will recall some basic facts about $\mathbb{P}^1$-bundles. We will give a complete description of projective (actually affine) structures on the torus with an explicit versal family of foliated bundle picture.

***Résumé* (Structures projectives et fibrés projectifs sur les surfaces de Riemann compactes)**
Une structure projective sur une surface de Riemann C de genre g est donnée par un atlas dont les applications de transition sont à valeurs dans $\mathrm{PGL}(2,\mathbb{C})$. De manière équivalente, une structure projective est donnée par un fibré en $\mathbb{P}^1$ sur C équipé d'une section σ et d'un feuilletage $\mathcal{F}$ transverse à la fois aux fibres $\mathbb{P}^1$ et à la section σ. À partir de cette dernière description géométrique, nous survolons quelques problèmes et résultats classiques sur les structures projectives. Nous rappelons quelques propriétés de base sur les fibrés en $\mathbb{P}^1$. Nous donnons une description complète des structures projectives (qui sont en fait affines) sur le tore avec une famille verselle explicite de fibrés feuilletés.

2000 Mathematics Subject Classification. — 34Mxx, 37F75, 53C10.
Key words and phrases. — Riemann surfaces, ordinary differential equations, projective structures, foliations.

The first author was partially supported by ANR SYMPLEXE BLAN06-3-137237.
The second author was partially supported by FEDER / Ministerio de Educación y Ciencia of Spain, grant MTM2004-00566.

1. Projective structures

1.1. Definition and examples. — Denote by Σ_g the orientable compact real surface of genus g. A *projective structure* on Σ_g is given by an atlas $\{(U_i, f_i)\}$ of coordinate charts (local homeomorphisms) $f_i : U_i \to \mathbb{P}^1$ such that the transition functions $f_i = \varphi_{ij} \circ f_j$ are restrictions of Moebius transformations $\varphi_{ij} \in \mathrm{PGL}(2, \mathbb{C})$.

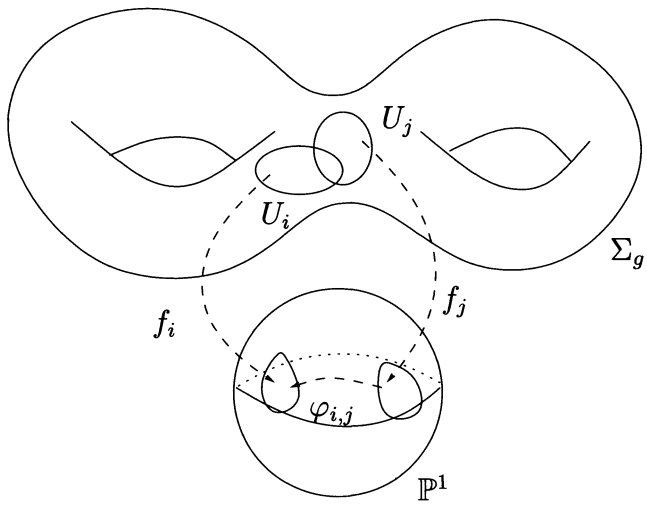

FIGURE 1. Projective atlas

There is a unique maximal atlas defining the projective structure above, obtained from the previous one by adding all charts $\{(U_i, \varphi \circ f_i)\}$ when φ runs over $\mathrm{PGL}(2, \mathbb{C})$.

A projective structure induces a *complex structure* on Σ_g, just by pulling-back that of $\mathbb{P}^1$ by the projective charts. We will denote by C the corresponding Riemann surface (complex curve).

***Example 1.1* (The Universal cover).** — Let C be a compact Riemann surface having genus g and consider its universal cover $\pi : U \to C$. By the Riemann Mapping Theorem, we can assume that U is either the Riemann sphere $\mathbb{P}^1$, or the complex plane $\mathbb{C}$ or the unit disk Δ depending wether $g = 0$, 1 or ≥ 2. We inherit a representation of the fundamental group $\rho : \pi_1(C) \to \mathrm{Aut}(U)$ whose image Λ is actually a subgroup of $\mathrm{PGL}(2, \mathbb{C})$. All along the paper, by abuse of notation, we will identify elements $\gamma \in \pi_1(C)$ with their image $\rho(\gamma) \in \mathrm{PGL}(2, \mathbb{C})$. The atlas defined on C by all local determinations of $\pi^{-1} : C \dashrightarrow \mathbb{P}^1$ defines a projective structure on C compatible with the complex one. Indeed, any two determinations of π^{-1} differ by left composition with an element of Λ.

We thus see that any complex structure on Σ_g is subjacent to a projective one. In fact, for $g \geq 1$, we will see that there are many projective structures compatible to a

given complex one (see Theorem 1.2). We will refer to the projective structure above as the *canonical projective structure* of the Riemann surface C: it does not depend on the choice of the uniformization of U. We now give other examples.

***Example 1.2* (Quotients by Kleinian groups).** — Let $\Lambda \subset \mathrm{PGL}(2,\mathbb{C})$ be a subgroup acting properly, freely and discontinuously on some connected open subset $U \subset \mathbb{P}^1$. Then, the quotient map $\pi : U \to C := U/\Lambda$ induces a projective structure on the quotient C, likely as in Example 1.1. There are many such examples where U is neither a disk, nor the plane. For instance, *quasi-Fuchsian groups* are obtained as image of small perturbations of the representation ρ of Example 1.1; following [**35**], such perturbations keep acting discontinuously on some *quasi-disk* (a topological disk whose boundary is a Jordan curve in $\mathbb{P}^1$).

***Example 1.3* (Schottky groups).** — Pick $2g$ disjoint discs $\Delta_1^-, \ldots, \Delta_g^-$ and $\Delta_1^+, \ldots, \Delta_g^+$ in $\mathbb{P}^1$, $g \geq 1$. For $i = 1, \ldots, n$, let $\varphi_i \in \mathrm{PGL}(2,\mathbb{C})$ be a loxodromic map sending the disc Δ_i^- onto the complement $\mathbb{P}^1 - \Delta_i^+$.

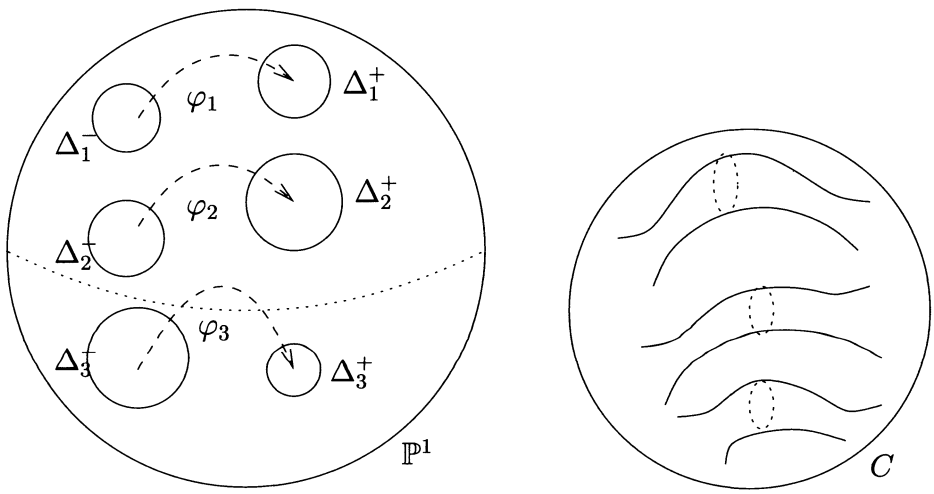

FIGURE 2. Schottky groups

The group $\Lambda \subset \mathrm{PGL}(2,\mathbb{C})$ generated by $\varphi_1, \ldots, \varphi_g$ acts properly, freely, and discontinuously on the complement U of some closed set contained inside the disks (a Cantor set whenever $g \geq 2$). The fundamental domain of this action on U is given by the complement of the disks and the quotient $C = U/\Lambda$ is obtained by gluing together the boundaries of Δ_i^+ and Δ_i^- by means of φ_i, $i = 1, \ldots, g$. Therefore, C is a compact Riemann surface of genus g. This picture is clearly stable under small deformation of the generators φ_i and we thus obtain a complex $3g-3$ dimensional family of projective structures on the genus g surface Σ_g (we have divided here by the action of $\mathrm{PGL}(2,\mathbb{C})$ by conjugacy).

1.2. Developping map and monodromy representation. — Given a projective atlas and starting from any initial coordinate chart (U_0, f_0), one can extend it analytically along any path γ starting from $p_0 \in U_0$.

Indeed, after covering γ by finitely many projective coordinate charts, say (U_0, f_0), (U_1, f_1), ... ,(U_n, f_n), one can modify them step by step in the following way. First of all, since $f_0 = \varphi_{01} \circ f_1$ on $U_0 \cap U_1$, one can replace the chart f_1 by $\tilde{f}_1 := \varphi_{01} \circ f_1$ which is well-defined on U_1, extending f_0. Next, we replace f_2 by $\tilde{f}_2 := \varphi_{01} \circ \varphi_{12} \circ f_2$ which, on $U_1 \cap U_2$, coincide with $\tilde{f}_1$. Step by step, we finally arrive at the chart $\tilde{f}_n := \varphi_{01} \circ \cdots \circ \varphi_{n-1\ n} \circ f_n$ which, by construction, is the analytic continuation of f_0 along γ.

Therefore, the local chart (U_0, f_0) extends (after lifting on the universal covering) as a global submersion on the universal cover

$$f : U \to \mathbb{P}^1$$

which is called the *developping map* of the projective structure. The developping map is moreover holomorphic with respect to the complex structure subjacent to the projective one. By construction, the monodromy of f along loops takes the form

$$f(\gamma.u) = \varphi_\gamma \circ f, \quad \varphi_\gamma \in \mathrm{PGL}(2,\mathbb{C}) \quad \forall \gamma \in \pi_1(\Sigma_g, p_0) \tag{1}$$

(u is the coordinate on U and $\gamma.u$, the canonical action of $\pi_1(\Sigma_g, p_0)$ on U). In fact, φ_γ is the composition of all transition maps $\varphi_{i,j}$ encoutered along γ for a given finite covering of projective charts: with notations above, setting $(U_n, f_n) = (U_0, f_0)$, we have $\varphi_\gamma = \varphi_{01} \circ \cdots \circ \varphi_{n-1\ n}$. It turns out that φ_γ only depends on the homotopy class of γ and we inherit a *monodromy representation*

$$\rho : \pi_1(\Sigma_g, p_0) \to \mathrm{PGL}(2,\mathbb{C}) \ ; \ \gamma \mapsto \varphi_\gamma. \tag{2}$$

The image Λ of ρ will be called monodromy group. The developping map f is defined by the projective structure up to the choice of the initial chart (U_0, f_0) above: it is unique up to left composition $\varphi \circ f$, $\varphi \in \mathrm{PGL}(2,\mathbb{C})$. Therefore, the monodromy representation is defined by the projective structure up to conjugacy: the monodromy of $\varphi \circ f$ is $\gamma \mapsto \varphi \circ \varphi_\gamma \circ \varphi^{-1}$.

Conversely, any global submersion $f : U \to \mathbb{P}^1$ on the universal covering $\pi : U \to \Sigma_g$ satisfying (1) is the developping map of a unique projective structure on Σ_g. We note that condition (1) forces the map $\gamma \to \varphi_\gamma$ to be a morphism.

Example 1.4. — The developping map of the canonical projective structure (see example 1.1) is the inclusion map $U \hookrightarrow \mathbb{P}^1$ of the universal cover of C. More generally, when the projective structure is induced by a quotient map $\pi : U \to C = U/\Lambda$ like in example 1.2, then the developping map f is the universal cover $\tilde{U} \to U$ of U and the monodromy group is Λ. In example 1.3, the open set U is not simply connected (the complement of a Cantor set) and the developping map is a non trivial covering. Thus the corresponding projective structure is not the canonical one. Similarly, the developping map of a quasi-Fuchsian group is the uniformization map of the corresponding

quasi-disk and is not trivial; the projective structure is neither the canonical one, nor of Schottky type.

***Example 1.5* (The Sphere).** — Given a projective structure of the Riemann sphere $\mathbb{P}^1$, we see that the developping map $f : \mathbb{P}^1 \to \mathbb{P}^1$ is uniform (no monodromy since $\pi_1(\mathbb{P}^1)$ is trivial). Therefore, f is a global holomorphic submersion (once we have fixed the complex structure) and thus $f \in \mathrm{PGL}(2, \mathbb{C})$. Consequently, the projective structure is the canonical one and it is the unique projective structure on $\mathbb{P}^1$.

For similar reason, we remark that the monodromy group of a projective structure on a surface of genus $g \geq 1$ is never trivial.

***Example 1.6* (The Torus).** — Let $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ be a lattice in $\mathbb{C}$, $\tau \in \mathbb{H}$, and consider the elliptic curve $C := \mathbb{C}/\Lambda$. The monodromy of a projective structure on C is abelian; therefore, after conjugacy, it is in one of the following abelian groups:

- the linear group $\{\varphi(z) = az \ ; \ a \in \mathbb{C}^*\}$,
- the translation group $\{\varphi(z) = z + b \ ; \ b \in \mathbb{C}\}$,
- the finite abelian dihedral group generated by $-z$ and $1/z$.

The canonical projective structure on C has translation monodromy group Λ. On the other hand, for any $c \in \mathbb{C}^*$ the map

$$f_c : \mathbb{C} \to \mathbb{P}^1 \ ; \ u \mapsto \exp(c.u) \tag{3}$$

is the developping map of a projective structure on C whose monodromy is linear, given by

$$f_c(u+1) = e^c \cdot f(u) \quad \text{and} \quad f_c(u+\tau) = e^{c\tau} \cdot f(u). \tag{4}$$

We inherit a 1-parameter family of projective structures parametrized by $c \in \mathbb{C}^*$ (note that $f_0 \equiv 1$ is not a submersion). We will see latter that this list is exhaustive. In particular, all projective structures on the torus are actually affine (transition maps in the affine group).

The projective structures listed in example 1.6 are actually *affine structures*: the developping map takes values in $\mathbb{C}$ with affine monodromy.

***Theorem 1.1* (Gunning [12]).** — *All projective structures on the elliptic curve $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, are actually affine and listed in example 1.6 above. There is no projective structure having affine monodromy on surfaces Σ_g of genus $g \geq 2$.*

In particular, the dihedral group is not the holonomy group of a projective structure on the torus.

Partial proof. — Here, we only prove that the list of example 1.6 exhausts all affine structures on compact Riemann surfaces. In example 1.7, we will see that there are no other projective structure on tori than the affine ones.

Let $f : U \to \mathbb{P}^1$ be the developping map of a projective structure with affine monodromy on the compact Riemann surface $C \neq \mathbb{P}^1$: f is a holomorphic local homeomorphism satisfying

$$f(\gamma \cdot u) = a_\gamma \cdot f(u) + b_\gamma,\ a_\gamma \in \mathbb{C}^*,\ b_\gamma \in \mathbb{C}, \quad \forall \gamma \in \pi_1(C, p_0)$$

Choose a holomorphic 1-form ω_0 on C and write $\omega_0 = \phi \cdot df$. Here, we identify ω_0 with its lifting on the universal covering. Since f is a local diffeomorphism, df has no zeroes and ϕ is holomorphic on U, vanishing exactly on zeroes of ω_0 and poles of f. Moreover, the monodromy of ϕ is that of df, given by $\phi(\gamma \cdot u) = a_\gamma^{-1} \cdot \phi(u)$. Therefore, the meromorphic 1-form $\omega_1 = \frac{d\phi}{\phi}$ has no monodromy: it defines a meromorphic 1-form on C having only simple poles, located at the zeroes of ω_0 and poles of f, the residues of which are positive integers. Following Residue Theorem, ω_1 has actually no poles: f is holomorphic, ω_0 does not vanish and thus genus $g = 1$. This proves the second assertion of the statement.

Now, assume $g = 1$, $U = \mathbb{C}$ and $C = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. The 1-form ω_1 above is holomorphic and thus takes the form $\omega_1 = -c \cdot du$ for some constant $c \in \mathbb{C}$. In other words, we have $f''/f' = c$ and we obtain after integration

- $f(u) = a \cdot e^{cu} + b$ when $c \neq 0$,
- $f(u) = a \cdot u + b$ when $c = 0$

for constants $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$. After left composition by an affine map, which does not affect the affine structure, we can set $a = 1$ and $b = 0$ and f belongs to the list of example 1.6. □

Remark 1.1. — We see from the proof that the projective structures on $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ are naturally parametrized by $\mathbb{C}$, namely the constant map $\phi = f''/f' \equiv c$, which is not clear from the description of example 1.6 (we see $\mathbb{C}^*$ plus one point). One can recover this by choosing conveniently the integration constants a and b in the proof above. Indeed, consider the alternate family of developping maps given by

$$F : \mathbb{C}^2 \to \mathbb{C}\ ;\ (c, u) \mapsto f_c(u) := \begin{cases} \frac{e^{cu}-1}{c}, & c \neq 0 \\ u, & c = 0 \end{cases} \tag{5}$$

The map F is clearly holomorphic on $\mathbb{C}^2$ and makes the developping maps f_c into a holomorphic family parametrized by $c \in \mathbb{C}$. Moreover, the corresponding holonomy representations are given by

$$f_c(u + \gamma) = \begin{cases} e^{c\gamma} f_c(u) + \frac{e^{c\gamma}-1}{c}, & c \neq 0 \\ u + \gamma, & c = 0 \end{cases} \qquad \forall \gamma \in \mathbb{Z} + \tau\mathbb{Z}$$

and we see the affine motions with common fixed point $-1/c$ converging to translations while $c \to 0$.

Remark 1.2. — When we set $g = 1$ in example 1.3, we have $U = \mathbb{C}^*$ and Λ is generated by a single map $\varphi(z) = e^{2i\pi\lambda} z$. The quotient $C = U/\Lambda$ is the elliptic curve with lattice $\mathbb{Z} + \lambda\mathbb{Z}$. The complex structure varies with λ and very few projective structures on

the torus are obtained by this way. In fact, we see in example 1.6 that, for generic values of c, the monodromy group of the corresponding projective structure is not discrete (c is not $\mathbb{Z}$-commensurable with 1 and τ).

1.3. Quadratic differentials. — In order to generalize the arguments involved in the proof of Theorem 1.1 for genus $g \geq 2$ Riemann surfaces, we have to replace f''/f' by the *Schwartzian derivative* of f

$$\mathcal{S}(f) := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2. \tag{6}$$

Recall that, for any holomorphic functions f and g, we have

$$\mathcal{S}(f \circ g) = \mathcal{S}(f) \circ g \cdot (g')^2 + \mathcal{S}(g). \tag{7}$$

Given a projective structure on a Riemann surface C, consider the Schwartzian derivative of the corresponding developping map $\phi := \mathcal{S}(f)$. For any $\gamma \in \Lambda = \pi_1(C)$, we deduce from property (1) of f that

$$\phi \circ \gamma \cdot (\gamma')^2 = \mathcal{S}(f \circ \gamma) = \mathcal{S}(\varphi_\gamma \circ f) = \phi.$$

In other words, the *quadratic differential* $\omega = \phi(u) \cdot du^2$ is invariant under Λ and gives rise to a quadratic differential on the Riemann surface C. We note that ω is holomorphic. Indeed, outside the poles of f, $\phi_1 := f'$ is not vanishing, thus $\phi_2 := \phi_1'/\phi_1$ is holomorphic and $\phi = \phi_2' - (\phi_2)^2/2$ well. On the other hand, at a pole of f, one can replace f for instance by $1/f$, which is not relevant for the Schwartzian derivative, and go back to the previous argument. By this way, we canonically associate to any projective structure on C a holomorphic quadratic differential ω on C, i.e. a global section of $K_C^{\otimes 2}$, where K_C is the canonical line bundle over C.

Conversely, given any holomorphic quadratic differential $\omega = \phi(u) \cdot du^2$ on the Riemann surface C, one can solve locally the differential equation $\mathcal{S}(f) = \phi$ in f and recover the coordinate charts of a projective structure on C (compatible with the complex one): the fact is that any two (local) solutions of $\mathcal{S}(f) = \phi$ differ by left composition by a Moebius transformation.

Example 1.7. — In genus 1 case, any holomorphic quadratic differential takes the form $\omega = c \cdot du^2$ for a constant $c \in \mathbb{C}$ ($K_C^{\otimes 2} = K_C$ is still the trivial bundle). In fact, $\omega = \tilde{\omega}^2$, where $\tilde{\omega} = \sqrt{c}du$. On the other hand, any solution of $f''/f' = \tilde{c}$ gives rise to a solution of $\mathcal{S}(f) = -\tilde{c}^2/2 = c$; therefore, the projective structure defined by ω is actually subjacent to the affine structure defined by $\tilde{c}$. This concludes the proof of Theorem 1.1. We note that the space of affine structures forms a two fold covering of the space of projective structures (the choice of the square root $\tilde{c}$). Of course, this comes from the fact that the 2 affine structures given by f_c and $1/f_c$ (with notations of example 1.6) do not define distinct projective structures.

For genus $g \geq 2$ Riemann surfaces, the dimension of $H^0(C, K_C^{\otimes 2})$ can be computed by Riemann-Roch Formula, and we obtain

***Theorem 1.2* (Gunning [12]).** — *The set of projective structures on a complex Riemann surface C of genus $g \geq 2$ is parametrized by the $3g-3$-dimensional complex vector space $H^0(C, K_C^{\otimes 2})$.*

In this vector space, 0 stands for the canonical structure of example 1.1.

1.4. The monodromy mapping. — A natural question arising while studying projective structures is to understand, for a given surface Σ_g, the nature of the Monodromy (or Riemann-Hilbert) Mapping

$$\mathcal{P}_g \longrightarrow \mathcal{R}_g.$$

On the left-hand side, $\mathcal{P}_g$ denotes the set of all projective structure on Σ_g up to isomorphism; on the right-hand side, $\mathcal{R}_g$ is the set of representations of the fundamental group in $\mathrm{PGL}(2,\mathbb{C})$ up to conjugacy:

$$\mathcal{R}_g = \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PGL}(2,\mathbb{C}))/_{\mathrm{PGL}(2,\mathbb{C})}.$$

Let us first consider the genus $g = 1$ case. From Gunning's Theorem 1.1, the left-hand side can be viewed as a $\mathbb{C}$-bundle over the modular orbifold $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$ where $\mathbb{H}$ denotes the upper-half plane whose fiber at a given complex structure is the affine line of holomorphic differentials. Nevertheless, to avoid dealing with orbifold points, we prefer to deal with the parametrization of affine structures by $\mathbb{H} \times \mathbb{C}$ given by the map

$$(\tau, c) \mapsto (C, \omega) \text{ where } \begin{cases} C = & \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \\ \omega = & c \cdot du \end{cases}$$

Here, the base $\mathbb{H}$ is the space of marked complex structures on the torus, up to isomorphism, and the fiber over τ is the affine line of differentials $\mathbb{C} \cdot du$, u the variable of $\mathbb{C}$. Since all projective structures are actually affine, we can replace $\mathcal{R}_1$ by $\mathcal{A}_1 := \mathrm{Hom}(\pi_1(C), \mathrm{Aff}(\mathbb{C}))/_{\mathrm{Aff}(\mathbb{C})}$ where

$$\mathrm{Aff}(\mathbb{C}) := \{\varphi(z) = az + b,\ a \in \mathbb{C}^*,\ b \in \mathbb{C}\}$$

is the group of affine transformations. Once we have fixed generators 1 and τ for the fundamental group of $C = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, the set $\mathrm{Hom}(\pi_1(C), \mathrm{Aff}(\mathbb{C}))$ identifies with the complex 3-dimensional subvariety

$$\{(a_1 z + b_1, a_\tau z + b_\tau)\ ;\ (a_1 - 1)b_\tau = (a_\tau - 1)b_1\} \subset (\mathbb{C}^* \times \mathbb{C})^2$$

(here, we see the condition for the commutativity). The $\mathrm{Aff}(\mathbb{C})$)-action by conjugacy commutes with the projection on the linear part $\mathbb{C}^* \times \mathbb{C}^*$. For any $(a_1, a_\tau) \neq (1,1)$, the action is transitive on the fibre: it is the usual action of $\mathrm{Aff}(\mathbb{C})$) on the line $\{(b_1, b_\tau)\ ;\ (a_1 - 1)b_\tau = (a_\tau - 1)b_1\} \subset \mathbb{C}^2$. The fibre over $(1,1)$ is $\mathbb{C}^2$ and $\mathrm{Aff}(\mathbb{C})$) acts by homothecy: after deleting the point $(b_1, b_\tau) = (0,0)$ corresponding to the trivial representation, the quotient by the action is $\mathbb{P}^1$. Since the trivial representation does not occur as monodromy representation of an affine structure on the torus, we consider the quotient $\mathcal{B}_1 \subset \mathcal{A}_1$ of its complement in $\mathrm{Hom}(\pi_1(C), \mathrm{Aff}(\mathbb{C}))$. By the

previous remarks, $\mathcal{B}_1$ is the blowing-up of $\mathbb{C}^* \times \mathbb{C}^*$ at the point $(1,1)$ and can be identified with the 2-dimensional complex manifold

$$\mathcal{B}_1 = \{(a_1, a_\tau, [b_1 : b_\tau]) \ ; \ (a_1 - 1)b_\tau = (a_\tau - 1)b_1\} \subset \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{P}^1$$

where $[z : w]$ denotes homogeneous coordinates on $\mathbb{P}^1$.

The projection $\mathcal{B}_1 \to \mathbb{C}^* \times \mathbb{C}^*$ is just the blow-up of the point $(1,1)$ and the exceptional divisor consists in euclidean representation. Finally, the monodromy map is described by

$$\mathbb{H} \times \mathbb{C} \to \mathcal{B}_1 \ ; \ (\tau, c) \mapsto \begin{cases} (e^c, e^{c\tau}, [\frac{e^c - 1}{c} : \frac{e^{c\tau} - 1}{c}]), & c \neq 0 \\ (1, 1, [1 : \tau]), & c = 0 \end{cases}$$

Looking at the differential of the Monodromy Map above, we see that it has always rank 2 and *the Monodromy Map is a holomorphic local diffeomorphism*; it is moreover *injective and proper in restriction to each fiber* $\tau \times \mathbb{C}$. Its image is the complement of the preimage on $\mathcal{B}_1$ of the real torus $\mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{C}^* \times \mathbb{C}^*$ plus the complement of $\mathbb{P}^1(\mathbb{R})$ inside the exceptional divisor $\mathbb{P}^1(\mathbb{C})$ over the point $(1,1) \in \mathbb{C}^* \times \mathbb{C}^*$.

But the Monodromy Map *is neither injective, nor a covering map onto its image*: for instance, for any $\tau, \tau' \in \mathbb{H}$, $\tau' \neq \tau$, and for any $(m,n) \in \mathbb{Z}^2 - \{(0,0)\}$, the two affine structures

$$(\tau, 2i\pi \frac{m\tau' + n}{\tau' - \tau}) \quad \text{and} \quad (\tau', 2i\pi \frac{m\tau + n}{\tau' - \tau})$$

have the same monodromy representation. In particular, the injectivity is violated for arbitrarily close complex structures. On the other hand, the monodromy of the canonical structure $(\tau, 0)$ occurs only for this structure.

Consider now the genus $g \geq 2$ case. It follows from Gunning's Theorem 1.2 above that the set $\mathcal{P}_g$ of projective structures on the genus $g \geq 2$ surface Σ_g can be viewed as a complex $6g - 6$-dimensional space. Indeed, if we denote by $\mathcal{T}_g$ the Teichmüller space of complex marked structures on Σ_g viewed as an open subset of $\mathbb{C}^{3g-3}$, then $\mathcal{P}_g$ is parametrized by the rank $3g - 3$-vector bundle $\tilde{\mathcal{P}}_g$ over $\mathcal{T}_g$ whose fiber over a given complex structure C is the space of quadratic differentials $H^0(C, K_C^{\otimes 2})$.

By Theorem 1.1, the monodromy representation cannot be affine in the case $g \geq 2$. The image of $\mathcal{P}_g$ by the Monodromy Map is thus included in the subset of *irreducible* representations

$$\mathcal{R}_g^{\text{irr}} := \mathcal{R}_g - \mathcal{A}_g$$

where $\mathcal{A}_g = \text{Hom}(\pi_1(\Sigma_g), \text{Aff}(\mathbb{C}))/_{\text{PGL}(2,\mathbb{C})}$ is the set of affine representations *up to* $\text{PGL}(2,\mathbb{C})$*-conjugacy*. One can check (see [**13**]) that $\mathcal{R}_g^{\text{irr}}$ forms a non-singular complex manifold of dimension $6g - 6$. Thus, the Monodromy Map can locally be described as a holomorphic map between open subsets of $\mathbb{C}^{6g-6}$ and the following result makes sense (see proof in section 1.5).

Theorem 1.3 **(Hejhal [16, 7, 18])**. — *The Monodromy Map is a local diffeomorphism.*

In [**20**], it is moreover proved that the Monodromy Map is symplectic with respect to symplectic structures that can be respectively canonically defined on both spaces (see [**10**]).

The restriction of the Monodromy Map to each fiber $H^0(C, K_C^{\otimes 2})$ of $\tilde{\mathcal{P}}_g$ over $C \in \mathcal{T}_g$ is injective. In other words, we have the following result whose proof will be given in section 2.2.

***Theorem 1.4* (Poincaré [28]).** — *Given a compact Riemann surface C, any two projective structures are the same if, and only if, they have the same monodromy representation (up to* $\mathrm{PGL}(2,\mathbb{C})$*).*

It is clear that the Monodromy Map is not surjective. First of all, by Theorem 1.1, its image is contained in $\mathcal{R}_g^{\mathrm{irr}} \subset \mathcal{R}_g$. On the other hand, the space of representations $\mathrm{Hom}(\pi_1(C), \mathrm{PGL}(2,\mathbb{C}))$ falls into 2 connected components, namely the component of those that can be lifted as $\mathrm{Hom}(\pi_1(C), \mathrm{SL}(2,\mathbb{C}))$ and the other ones. Since the Monodromy Map is continuous (actually holomorphic) and since the monodromy of canonical projective structures can be lifted to $\mathrm{SL}(2,\mathbb{R})$, it becomes clear that the image of the Monodromy Map will be in the former component. Finally, notice that the monodromy representation cannot be in $\mathrm{PSU}(2,\mathbb{C})$, i.e. conjugated to a group of rotations of the sphere, otherwise we could pull-back the invariant spherical metric of $\mathbb{P}^1$ by the developping map, giving rise to a curvature $+1$ metric on the surface, impossible except in the trivial case $g = 0$ (see Example 1.5). The main result in the field, which has been conjectural for decenies, is the following.

***Theorem 1.5* (Gallo-Kapovich-Marden [9]).** — *Consider the genus g surface Σ_g, $g \geq 2$. A homomorphism $\rho \in \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PGL}(2,\mathbb{C}))$ is the monodromy representation of a projective structure on the Σ_g if, and only if, ρ can be lifted as $\tilde{\rho} \in \mathrm{Hom}(\pi_1(C), \mathrm{SL}(2,\mathbb{C}))$ and the image of ρ is, up to* $\mathrm{PGL}(2,\mathbb{C})$*-conjugacy, neither in the affine group* $\mathrm{Aff}(\mathbb{C})$*, nor in the rotation group* $\mathrm{PSU}(2,\mathbb{C})$*.*

1.5. The fibre bundle picture. — Let $f : U \to \mathbb{P}^1$ be the developping map of a projective structure on C (here we fix the underlying complex structure) and consider its graph $\{(u, f(u)) \ ; \ u \in U\} \subset U \times \mathbb{P}^1$. The fundamental group $\pi_1(C)$ acts on the product $U \times \mathbb{P}^1$ as follows: for any $\gamma \in \pi_1(C)$, set

$$\gamma : (u, y) \mapsto (\gamma \cdot u, \varphi_\gamma(y))$$

where $u \mapsto \gamma \cdot u$ is the canonical action of $\pi_1(C)$ on the universal cover and φ_γ is the monodromy of the projective structure along γ. This action of $\pi_1(C)$ is proper, free and discontinuous since its projection on U is so. By consequence, we can consider the quotient:

$$P := U \times \mathbb{P}^1 /_{\pi_1(C)}.$$

The projection $U \times \mathbb{P}^1 \to C$ defined by $(u, y) \mapsto \pi(u)$, where $\pi : U \to C$ is the universal cover, is preserved by the action and induces a global submersion

$$\pi : P \to C$$

making P *into a* $\mathbb{P}^1$*-bundle over* C. *The graph of* f *also is invariant under the action (consequence of (1)) thus defining a section*

$$\sigma : C \to P.$$

Finally, the horizontal foliation defined by $\{y = \text{constant}\}$ is also preserved and defines a foliation $\mathcal{F}$ transversal to all $\mathbb{P}^1$-fibres on P. Since the developping map f is regular, its graph is transversal to the horizontal foliation and σ is transversal to $\mathcal{F}$. In this situation, we say that the $\mathbb{P}^1$-bundle P is flat. The triple $(\pi : P \to C, \mathcal{F}, \sigma)$ is well-defined by the projective structure up to analytic isomorphism of $\mathbb{P}^1$-bundles.

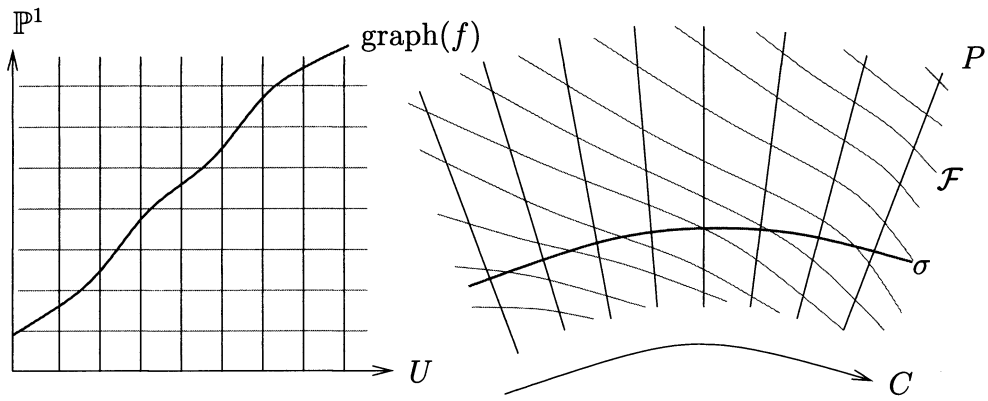

FIGURE 3. From projective structure to bundle picture

Conversely, given a $\mathbb{P}^1$-bundle $\pi : P \to C$, a foliation $\mathcal{F}$ on P transversal to π and a section $\sigma : C \to P$ transversal to $\mathcal{F}$, then the (unique) projective structure on $\mathbb{P}^1$-fibres can be transported, transversely to the foliation $\mathcal{F}$, inducing a projective structure on the section $\sigma(C)$, and thus on its π-projection C.

In the recent terminologoly of [**3**], such triple $(\pi : P \to C, \mathcal{F}, \sigma)$ are called $\mathfrak{sl}(2, \mathbb{C})$-opers.

Remark 1.3. — Given a homomorphism $\rho \in \mathrm{Hom}(\pi_1(C), \mathrm{PGL}(2, \mathbb{C}))$, one can at least construct the pair $(\pi : P \to C, \mathcal{F})$ as above. This foliated surface is called the *suspension* of the representation ρ, also known as the *flat* $\mathbb{P}^1$*-bundle* associated to ρ. Conversely, consider a flat $\mathbb{P}^1$-bundle, i.e. a pair $(\pi : P \to C, \mathcal{F})$ where $\pi : P \to C$ is a $\mathbb{P}^1$-bundle and $\mathcal{F}$ is a foliation transversal to π. Then one can associate to it a representation ρ in the following way.

Over any sufficiently small open subset $U_i \subset C$, one can construct a trivializing coordinate $F_i : \pi^{-1}(U_i) \to \mathbb{P}^1$ for the flat bundle, that is to say inducing an isomorphism in restriction to each fibre and such that the level curves $F_i^{-1}(y_0)$ are local leaves of the foliation $\mathcal{F}$. In fact, F_i is uniquely determined after choosing the local

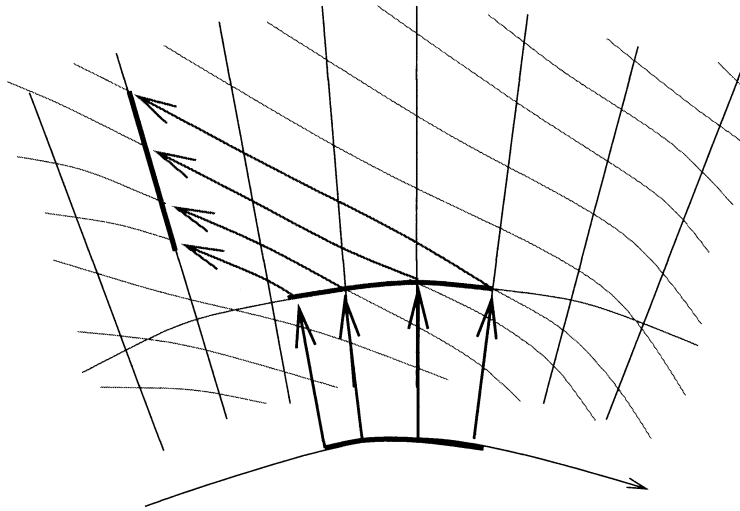

FIGURE 4. From bundle picture to projective structure

$\mathcal{F}$-invariant sections $\sigma_0, \sigma_1, \sigma_\infty : U_i \to P$ along which F_i takes values 0, 1 and ∞ respectively. Such flat coordinate is well defined up to left-composition by a Moebius transformation; likely as in section 1.2, after fixing a flat local coordinate F over some neighborhood of the base point $x_0 \in C$, we inherit a monodromy representation $\rho : \pi_1(C, x_0) \to \mathrm{PGL}(2, \mathbb{C})$ where the analytic continuation of F along any loop γ satisfies $F(\gamma \cdot u) = \rho(\gamma) \circ F(u)$.

It turns out that any flat $\mathbb{P}^1$-bundle is isomorphic to the suspension of its monodromy representation just defined. In fact, *any two flat $\mathbb{P}^1$-bundles are isomorphic if, and only if, they have the same monodromy representation up to* $\mathrm{PGL}(2, \mathbb{C})$ *conjugacy.* Indeed, let $(\pi : P \to C, \mathcal{F})$ and $(\pi' : P' \to C, \mathcal{F}')$ be flat $\mathbb{P}^1$-bundles having flat coordinates F and F' over $U_0 \subset C$ giving rise to the same monodromy representation; then the local isomorphism $\Phi : \pi^{-1}(U_0) \to \pi'^{-1}(U_0)$ sending any point p to the unique point p' satisfying $(\pi(p), F(p)) = (\pi'(p'), F'(p'))$ extends uniformly as a global isomorphism of flat $\mathbb{P}^1$-bundles $\Phi : P \to P'$, i.e. conjugating $\mathcal{F}$ to $\mathcal{F}'$ and satisfying $\pi' \circ \Phi = \pi$.

Proof of Hejal's Theorem 1.3. — In fact, since the Monodromy Map is clearly holomorphic, it is enough to prove that it is locally bijective.

Let $(\pi : P \to C, \mathcal{F}, \sigma)$ be the triple associated to a projective structure having monodromy representation $\rho \in \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PGL}(2, \mathbb{C}))$. For any perturbation $\rho' \in \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PGL}(2, \mathbb{C}))$ of ρ, the corresponding suspension $(\pi' : P' \to C, \mathcal{F}')$ is close to the foliated bundle $(\pi : P \to C, \mathcal{F})$; if the perturbation is small enough, one can find a real C^∞ section $\sigma' : C \to P'$ close to $\sigma : C \to P$ and still transversal to $\mathcal{F}'$ (all of this makes sense and can be checked on the neighborhood of a fundamental domain

of the universal cover $U \times \mathbb{P}^1$). The foliation $\mathcal{F}'$ still induces a projective structure on the real surface $\sigma'(C)$ that, by construction, has the required monodromy. This proves the surjectivity.

Let $(\pi : P \to C, \mathcal{F}, \sigma)$ be the triple associated to a projective structure $\mathcal{P}$ and consider another projective structure $\mathcal{P}'$ close to this $\mathcal{P}$ having the same monodromy representation. The fibre bundle construction can be done in the real C^∞ setting so that one can associate to $\mathcal{P}'$ a triple $(\pi : P \to C, \mathcal{F}, \sigma')$ where C is still the complex curve attached to $\mathcal{P}$ and $\sigma' : C \to P$ is now a real C^∞ section transversal to $\mathcal{F}$; we note that the pair $(\pi : P \to C, \mathcal{F})$ is the same for $\mathcal{P}$ and $\mathcal{P}'$ since they have the same monodromy representation. If $\mathcal{P}'$ is close enough to $\mathcal{P}$, say in the C^∞ category, then σ' is close to σ; one can therefore unambiguously define a C^∞ diffeomorphism $\phi : \sigma'(C) \to \sigma(C)$ by following the leaves of the foliation from one section to the other one. By construction, the projective structures induced by $\mathcal{F}$ on both sections are conjugated by ϕ. The diffeomorphism $\pi_*\phi := \pi \circ \phi \circ \sigma'$ actually integrates the quasi-conformal structure induced by $\mathcal{P}'$ on C; it is close to the identity. □

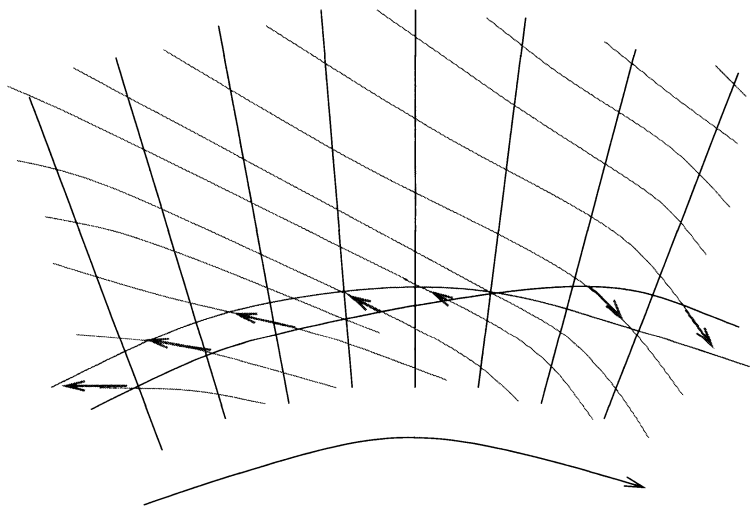

FIGURE 5. Local injectivity of the Monodromy Map

Remark 1.4. — Since the Monodromy Map is not globally injective, the injectivity argument of the previous proof cannot be carried out for sections σ and σ' that are not closed enough: the set of C^∞ sections transversal to $\mathcal{F}$ may have infinitely many connected components as it so happens in the case of affine structures on the torus. Similarly, the surjectivity argument of the proof cannot be globalized: when the monodromy representation ρ' eventually becomes reducible for instance, there does not exist C^∞ section transversal to $\mathcal{F}$ anymore. Following Theorem 1.5, the existence

of a C^∞ section transversal to $\mathcal{F}$ is possible if, and only if, $\mathcal{F}$ is the suspension of a non elementary representation ρ (lifting to $\mathrm{SL}(2,\mathbb{C})$) ! From this point of view, Theorem 1.5 looks like a very subtle transversality result.

2. $\mathbb{P}^1$-bundles and Riccati foliations

Motivated by the fibre bundle picture of section 1.5, we developp here the study of Riccati foliations on $\mathbb{P}^1$-bundles over compact Riemann surfaces.

2.1. Classification of $\mathbb{P}^1$-bundles. — Let $\pi : P \to C$ be a $\mathbb{P}^1$-bundle over a compact Riemann surface C: P is a smooth surface and the fibers of π are rational, isomorphic to $\mathbb{P}^1$. We also say that P is a *ruled surface*. Another $\mathbb{P}^1$-bundle $\pi' : P' \to C$ is *analytically equivalent* to the previous one if there is a holomorphic diffeomorphism $\phi : P \to P'$ such that $\pi' \circ \phi = \pi$. We recall some basic facts (see [**15, 24**]).

On open charts $u_i : U_i \to \mathbb{C}$ on C, the bundle becomes analytically trivial (see [**8**]): we have holomorphic diffeomorphisms (trivializing coordinates)

$$\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{P}^1 \ ; \ p \mapsto (\pi(p), \varphi_i(p)).$$

On overlapping charts $U_i \cap U_j$, the transition maps take the form $\phi_i = \phi_{i,j} \circ \phi_j$ where $\phi_{i,j}(u,y) = (u, \varphi_{i,j}(u,y))$ and

$$\varphi_{i,j} \in \mathrm{PGL}(2, \mathcal{O}(U_{i,j})).$$

The $\mathbb{P}^1$-bundle is equivalently defined by the collection

$$(\varphi_{i,j})_{i,j} \in H^1(C, \mathrm{PGL}(2,\mathcal{O})).$$

By lifting conveniently the transition maps into $H^1(C, \mathrm{GL}(2,\mathcal{O}))$, we may view a $\mathbb{P}^1$-bundle as the projectivization $P = \mathbb{P}V$ of a *rank* 2 *vector bundle* V over C. Moreover, another vector bundle V' will give rise to the same $\mathbb{P}^1$-bundle if, and only if, $V' = L \otimes V$ for a line bundle L over C. The classification of $\mathbb{P}^1$-bundles is thus equivalent to the classification of rank 2 vector bundles up to tensor product by a line bundle.

From the *topological* point of view, due to the fact that $\pi_1(\mathrm{PGL}(2,\mathbb{C})) = \mathbb{Z}/2\mathbb{Z}$, there are exactly 2 distinct $\mathbb{S}^2$-bundles over a compact real surface.

From the *birational* point of view, any $\mathbb{P}^1$-bundle is equivalent to the trivial bundle: there are infinitely many holomorphic sections $\sigma : C \to P$; after choosing 3 distinct ones σ_0, σ_1 and σ_∞, one defines a birational transformation $\phi : P \dashrightarrow C \times \mathbb{P}^1$ commuting with π by sending those sections respectively to $\{y = 0\}$, $\{y = 1\}$ and $\{y = \infty\}$. When the 3 sections are *disjoint*, the transformation ϕ is actually biregular and P is the trivial bundle $C \times \mathbb{P}^1$.

The *analytic classification* is a much more subtle problem. If P admits 2 disjoint sections, say $\sigma_0, \sigma_\infty : C \to P$, we then say that the bundle is *decomposable*: one can choose trivialization charts sending those two sections respectively onto $\{y = 0\}$ and $\{y = \infty\}$, so that P may be viewed as the compactification $\overline{L}$ of a line bundle L. Recall that line bundles are analytically classified by the Picard group $\mathrm{Pic}(C)$. Any

two elements $L, L' \in \mathrm{Pic}(C)$ have the same compactification P if, and only if, $L' = L$ or $L^{\otimes(-1)}$: we just exchange the role of σ_0 and σ_∞ (see proof of Proposition 3.1).

For instance, on $C = \mathbb{P}^1$, $\mathrm{Pic}(\mathbb{P}^1) \simeq \mathbb{Z}$ and the compactification of $\mathcal{O}(e)$ (or $\mathcal{O}(-e)$), $e \in \mathbb{N}$, gives rise to the Hirzebruck surface $\mathbb{F}_e$. It follows from Birkhoff's Theorem [**5**] that all $\mathbb{P}^1$-bundle is decomposable on $\mathbb{P}^1$ and is thus one of the $\mathbb{F}_e$ above.

An important analytic invariant of a $\mathbb{P}^1$-bundle over a curve C is the minimal self-intersection number of a section

$$e(P) := -\min\{\sigma.\sigma \ ; \ \sigma : C \to P\} \in \mathbb{Z}.$$

For a decomposable bundle $P = \overline{L}$, $L \in \mathrm{Pic}(C)$, we have $e(\overline{L}) = |\deg(L)| \geq 0$. For an undecomposable bundle, Nagata proved in [**25**] that $-g \leq e \leq 2g-2$ and all those possibilities occur.

From the *homological* point of view, $H^2(P, \mathbb{Z})$ is generated by the homology class of σ_0 and f where σ_0 is any holomorphic section and f any fibre. Let us choose σ_0 with minimal self-intersection:

$$\sigma_0 \cdot \sigma_0 = -e, \quad f \cdot f = 0 \quad \text{and} \quad \sigma_0 \cdot f = 1.$$

The homology class of any other holomorphic section is $\sigma = \sigma_0 + n \cdot f$ with $n \in \mathbb{N}$: it has self-intersection

$$\sigma \cdot \sigma = \sigma_0 \cdot \sigma_0 + 2n \cdot \sigma_0 \cdot f + n \cdot f \cdot f = -e + 2n \geq -e.$$

In particular, the intersection number of holomorphic sections are either all even, either all odd: e mod 2 is the *topological invariant* of the bundle.

On the other hand, if σ_0 and σ are not homologous then the intersection number $\sigma_0 \cdot \sigma = n - e$ must be non negative and we deduce that $\sigma \cdot \sigma \geq e$: when $e > 0$, this implies that σ_0 *is the unique holomorphic section having negative self-intersection*; there is a gap between $-e$ and e.

Theorem 2.1 **(Atiyah [1]).** — *Beside compactifications of line bundles, there are exactly* 2 *undecomposable* $\mathbb{P}^1$*-bundles over an elliptic curve,* P_0 *and* P_1*, with invariant* $e = 0$ *and* -1 *respectively.*

A $\mathbb{P}^1$-bundle P is *flat* (in the sense of Steenrod [**29**]) when a trivializing atlas can be choosen with constant transition maps $\varphi_{i,j} \in \mathrm{PGL}(2, \mathbb{C})$ (not depending on u). This means that this atlas defines by the same time a foliation $\mathcal{F}$ transversal to the fibres on P, namely the horizontal foliation defined by $\{y = \text{constant}\}$ in trivializing coordinates, see Remark 1.3.

Theorem 2.2 **(Weil [34]).** — *The flat* $\mathbb{P}^1$*-bundles over* C *are all the undecomposable bundles and all those arising as compactification of elements of* $Pic_0(C)$*.*

The pairs $(\pi : P \to C, \mathcal{F})$ are classified by $H^1(C, \mathrm{PGL}(2, \mathbb{C}))$. All triples $(\pi : P \to C, \mathcal{F}, \sigma)$ associated to a projective structure on $\mathcal{P}$ are actually supported on the same flat $\mathbb{P}^1$-bundle P, namely

– the Hirzebruch surface $\mathbb{F}_1$ when $g = 0$,

- the trivial bundle $C \times \mathbb{P}^1$ when $g = 1$,
- the unique maximally unstable ($e = 2g - 2$) and undecomposable bundle when $g > 1$ (see [**14**]).

A birational description of the bundle will be given in Remark 2.2.

2.2. Riccati foliations on $\mathbb{P}^1$-bundles. — A Riccati foliation on the bundle $\pi : P \to C$ is a singular foliation (see definition in [**6**]) $\mathcal{F}$ on P which is transversal to a generic fibre. In trivialization charts (u_i, y), it is defined by a Riccati differential equation $\frac{dy}{du_i} = a(u_i)y^2 + b(u_i)y + c(u_i)$, a, b, c meromorphic in u, whence the name. The poles of the coefficients correspond to vertical invariant fibres for the foliation. Outside of those poles, the leaves of the foliation are graphs of solutions for the Riccati equation. The foliation $\mathcal{F}$ arising in the fibre bundle picture of section 1.5 is a *regular* Riccati foliation. Nevertheless, we will need to deal with singular foliations later.

One can define the monodromy representation of a Riccati foliation as

$$\rho : \pi_1(C - \{\text{projection of invariant fibres}\}) \to \mathrm{PGL}(2, \mathbb{C}).$$

A classical Theorem due to Poincaré asserts that, in the regular case, the monodromy representation characterizes the Riccati foliation as well as the $\mathbb{P}^1$-bundle supporting it up to analytic equivalence.

Remark 2.1. — One can view a Riccati foliation $\mathcal{F}$ on the $\mathbb{P}^1$-bundle $P = \mathbb{P}V$ as the projectivization of a meromorphic linear connection ∇ on the vector bundle V. In fact, given a (meromorphic linear) connection ζ on the determinant bundle $\det V = \bigwedge^2 V \to C$, there is a unique connection ∇ on V lifting $\mathcal{F}$ and such that $\mathrm{trace}(\nabla) = \zeta$. Indeed, over a local coordinate $u_i : U_i \to \mathbb{C}$, the bundle V is trivial and a connection ∇ is just a meromorphic system

$$\nabla : \quad \frac{d}{du_i}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \alpha(u_i) & \beta(u_i) \\ \gamma(u_i) & \delta(u_i) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

and the trace of ∇ is the rank 1 connection defined by

$$\zeta := \mathrm{trace}(\nabla) : \quad \frac{d\lambda}{du_i} = (\alpha(u_i) + \delta(u_i))\,\lambda.$$

The projection of ∇ on $\mathbb{P}V$ is therefore the Riccati equation defined in affine coordinate $(y : 1) = (y_1 : y_2)$ by

$$\mathcal{F} := \mathbb{P}\nabla : \quad \frac{dy}{du_i} = -\gamma(u_i)y^2 + (\alpha(u_i) - \delta(u_i))y + \beta(u_i).$$

Clearly, ∇ is uniquely defined by $\mathcal{F}$ and ζ. Notice finally that the line bundle $\det V$ admits a linear connection ζ without poles if and only if it belongs to $Pic_0(C)$.

We start recalling some usefull homological formulae from [**6**]. First of all, let us introduce $T_{\mathcal{F}}$, the *tangent bundle* of $\mathcal{F}$, which is a line bundle on the total space P defined as follows. In trivializing charts (u_i, y), the Riccati foliation is also defined by the meromorphic vector field $V_i := \partial_{u_i} + (a(u_i)y^2 + b(u_i)y + c(u_i))\partial_y$. The leaves of

$\mathcal{F}$ are just complex trajectories of the vector field V_i. After choosing a global (meromorphic) vector field v on C, one can write $v = f_i \cdot \partial_{u_i}$ for a meromorphic function f_i on the chart U_i so that the new meromorphic vector fields $f_i \cdot V_i$ glue together into a global meromorphic vector field V on P still defining $\mathcal{F}$ at a generic point. One can think of V as the lifting of v by the (meromorphic) projective connection defined $\mathcal{F}$ on the bundle. Then, $T_{\mathcal{F}}$ is the line bundle defined by the divisor of V, i.e. $T_{\mathcal{F}} = \mathcal{O}((V)_0 - (V)_\infty)$. If d denotes the number of invariant fibre (counted with the multiplicity of the corresponding pole for V_i), then the homology class of $T_{\mathcal{F}}$ is given by $T_{\mathcal{F}} = (2 - 2g - d) \cdot f$.

Given a curve σ on P, each component of which is not invariant by $\mathcal{F}$, then the number of *tangencies* $\mathrm{Tang}(\mathcal{F}, \sigma)$ counted with multiplicities is given by (see [**6**], p. 23)

$$\mathrm{Tang}(\mathcal{F}, \sigma) = \sigma \cdot \sigma - T_{\mathcal{F}} \cdot \sigma. \tag{8}$$

For instance, if $\sigma = \sigma_0 + n \cdot f$ is a section, we immediately deduce that $\mathrm{Tang}(\mathcal{F}, \sigma) = 2n - e - 2 + 2g + d$.

Proof of Poincaré Theorem 1.4. — Consider two projective structures on C (compatible with the complex structure of C) having the same monodromy representation: by the construction given in section 1.5, they correspond to triples $(\pi : P \to C, \mathcal{F}, \sigma)$ and $(\pi : P \to C, \mathcal{F}, \sigma')$ with common $\mathbb{P}^1$-bundle and Riccati foliation. Since $\mathcal{F}$ is regular and the section σ defining the first projective structure is transversal to $\mathcal{F}$, we have $d = 0$, $\mathrm{Tang}(\mathcal{F}, \sigma) = 0$, and we deduce that $e = 2n + 2g - 2$. On the other hand, $\mathrm{Tang}(\mathcal{F}, \sigma_0) = 2g - 2 - e$ should be non negative and we obtain $e = 2g - 2$ and $n = 0$: in the genus $g \geq 2$ case, $\sigma = \sigma_0$ is the unique section having negative self-intersection in P, and by the way $\sigma' = \sigma$. In genus 0 case, there is nothing to show; in genus 1 case, the result follows directly from formula (4) and Theorem 1.1. □

Another important formula is the Camacho-Sad Index Theorem (see [**6**]). Given a curve σ on P invariant by $\mathcal{F}$, the self-intersection number of σ equals the sum of Camacho-Sad index of $\mathcal{F}$ along this curve. When $\mathcal{F}$ is regular, all invariant curves are smooth and all Camacho-Sad index vanish: when $\mathcal{F}$ is regular, any invariant curve σ has zero self-intersection.

For instance, if $\mathcal{F}$ has affine monodromy, then the fixed point gives rise to an invariant section $\sigma_\infty : C \to P$. We deduce that $e = 0$ and σ_∞ realizes this minimal self-intersection number. In particular, we recover the fact that a projective structure on a genus $g \geq 2$ curve cannot have affine monodromy since the corresponding bundle has invariant $e = 2g - 2 > 0$.

More generally, if the monodromy of $\mathcal{F}$ has a finite orbit (e.g. a finite group of the infinite dihedral group), then $\mathcal{F}$ has an invariant curve $\sigma = m \cdot \sigma_0 + n \cdot f$ and formulae $\sigma \cdot \sigma = m(2n - em) = 0$ together with $\sigma \cdot f = m \geq 0$ and $\sigma \cdot \sigma_0 = n - em \geq 0$ show that $e \leq 0$. Again, this is not the monodromy of a projective structure whenever $g \geq 2$.

In the particular case where the monodromy of $\mathcal{F}$ is linear, we have 2 invariant disjoint sections σ_0 and σ_∞ showing that the bundle P is actually a compactification of a line bundle $L \in \mathrm{Pic}_0(C)$.

We should emphasize that any two line bundles L and L' have the same compactification if, and only if, $L' = L$ or $L^{\otimes(-1)}$. Indeed, we first note that, for a bundle P satisfying $e(P) = 0$, any two sections σ and σ' are disjoint if, and only if, they have 0 self-intersection (and are distinct). The compactification $\overline{L}$ of a line bundle L always has the two canonical disjoint sections σ_0 and σ_∞. Now, a diffeomorphism $\phi : \overline{L} \to \overline{L}'$ between the compactifications of 2 non trivial line bundles has to preserve or permute the two canonical sections; in the former case, ϕ is actually an equivalence of line bundles; in the latter case, ϕ restricts to the fibres as $1/z$ and invert the monodromy. Of course, $\overline{L}$ is trivial if, and only if, L is trivial as a line bundle. It follows that when C has genus 1 the corresponding set of equivalence classes of $\mathbb{P}^1$-bundles with $e = 0$ may be thought as $C/\{\pm 1\} \simeq \mathbb{P}^1$.

A bundle P obtained by suspension of a representation $\rho : \pi_1(C) \to \mathrm{PGL}(2,\mathbb{C})$ is topologically trivial (e even) if, and only if, ρ can be lifted as a representation $\tilde{\rho} : \pi_1(C) \to \mathrm{SL}(2,\mathbb{C})$.

There is an algebraic and somewhat technical notion of (semi-) stability of vector bundles of arbitrary rank on Riemann surfaces due to Mumford, see for instance [**27**]. We can define the (semi-) stability of a $\mathbb{P}^1$-bundle $\mathbb{P}V$ by the same requirement to the rank 2 vector bundle V. It turns out that a $\mathbb{P}^1$-bundle is *stable* (resp. *semi-stable*) when $e < 0$ (resp. $e \leq 0$). It is known that if such a bundle occur along an algebraic (resp. analytic) family, it occurs for a Zariski open subset of the family. There is a theorem of Narashimhan and Seshadri characterizing stable bundles on a compact Riemann surface C by means of a precise, but some technical, construction in terms of unitary representations of the fundamental group of C. We present here a more comprehensible consequence, see Corollaries 1 and 2 of [**27**]:

Theorem 2.3 **(Narasimhan-Seshadri [27]).** — *Let C be a compact Riemann surface of genus $g \geq 2$. Then a holomorphic vector bundle of degree zero is stable if and only if it arises from an irreducible unitary representation of the fundamental group $\pi_1(C)$ of C. A holomorphic vector bundle on C arises from a unitary representation of the fundamental group if and only if each of its undecomposable components is of degree zero and stable.*

Applying this general result to our situation we obtain that the map $\rho \mapsto (\pi : P \to C, \mathcal{F}) \mapsto (\pi : P \to C)$ which to a representation $\rho \in \mathrm{Hom}(\pi_1(C), \mathrm{PGL}(2,\mathbb{C}))$ associate the $\mathbb{P}^1$-bundle obtained by suspension (forgetting the flat structure) induces a bijection from the set of irreducible representations $\rho : \pi_1(C) \to \mathrm{PSU}(2,\mathbb{C})$ up to $\mathrm{PSU}(2,\mathbb{C})$ conjugacy onto the set of isomorphism class of $\mathbb{P}^1$-bundles with invariant $e < 0$ and even (not fixed).

The complete analytic classification of $\mathbb{P}^1$-bundles (including unstable ones) over curves of genus 2 has been achieved by the works of Atiyah [**1**] and Maruyama [**24**]. The analytic classification of rank 2 stable vector bundles over curves of arbitrary

genus from the algebraic point of view (in contrast with Narasimhan-Seshadri's approach) has been done by Tyurin in [**31**] (see also [**32**] for a survey in arbirtrary rank).

2.3. Birational geometry of $\mathbb{P}^1$-bundles. — Given a point p on (the total space of) a $\mathbb{P}^1$-bundle $\pi : P \to C$, we will denote by $\mathrm{elm}_p P$ the new $\mathbb{P}^1$-bundle obtained after elementary transformation centered at p: after blowing-up the point p, $\mathrm{elm}_p P$ is obtained by contracting the strict transform of the fiber passing through p. The strict transform of a section σ passing through p (resp. not passing through p) is a section of the new bundle having self-intersection $\sigma \cdot \sigma - 1$ (resp. $\sigma \cdot \sigma + 1$). All birational transformations between $\mathbb{P}^1$-bundles over curves are obtained by composing finitely many elementary transformations. On the other hand, any $\mathbb{P}^1$-bundle over a curve is birational to the trivial bundle.

Example 2.1. — For instance, let D be a divisor on C and let p_0 be the point on the zero section of the (total space of the) line bunde $\mathcal{O}(D)$ over $x \in C$. Denote by $\overline{\mathcal{O}(D)}$ the $\mathbb{P}^1$-bundle obtained after compactification (adding a section at infinity). Then

$$\mathrm{elm}_{p_0} \overline{\mathcal{O}(D)} = \overline{\mathcal{O}(D - [x])}.$$

Similarly, if p_∞ lies on the infinity section of $\overline{\mathcal{O}(D)}$ over x, then

$$\mathrm{elm}_{p_\infty} \overline{\mathcal{O}(D)} = \overline{\mathcal{O}(D + [x])}.$$

Now, recall that, as a consequence of Abel Theorem, the map

$$C^g \to \mathrm{Pic}_0(C) \ ; \ (x_1, \ldots, x_g) \mapsto \mathcal{O}(g[x_0] - [x_1] - \cdots - [x_g])$$

is surjective for any $x_0 \in C$: it follows that (compactification of) line bundles of degree 0 can be obtained after applying at most $2g$ elementary transformations to the trivial bundle.

In [**26**], Maruyama and Nagata proved that an undecomposable bundle can be obtained from the trivial one after at most $2g+1$ elementary transformations. On the other hand, we note that the minimal number of elementary transformations needed to trivialize all decomposable bundle is unbounded: for a line bundle of large degree $d >> 0$, one need at least d elementary transformations.

Next section, we will give an explicit birational trivialization of the bundle P supporting all triples $(P, \mathcal{F}, \sigma)$ associated to projective structures. After birational trivialization, the Riccati foliation becomes singular, and the section σ no more transversal.

2.4. Riccati equation, schwartzian derivative and the 2^{nd} order linear differential equation. — First, we would like to make explicit the correspondance between the point of view of quadratic differentials, and that one of bundle triples.

Consider the triple $(\pi : P \to C, \mathcal{F}, \sigma)$ associated to a projective structure on the curve C. One can reduce P to the trivial bundle and σ to the infinity section $\{y = \infty\}$ either locally, by a fibre bundle isomorphism, or globally on C, by birational transformation. Here below, we adopt the later point of view; everything can be carried

out *mutatis mutandis* in the local regular setting. After a birational trivialization like above, $\mathcal{F}$ becomes possibly singular, but is now defined by a global Riccati equation

$$dy + \alpha \cdot y^2 + \beta \cdot y + \gamma = 0 \tag{9}$$

where α, β, γ are meromorphic 1-forms on C. This trivialization is unique up to birational transformation of the form $y = a\tilde{y} + b$ where a and b are meromorphic function on C, $a \not\equiv 0$. Let us see how such change of coordinate acts on the equation. A change of coordinate of the form $y = a\tilde{y}$ transforms the Riccati equation into

$$d\tilde{y} + a\alpha\tilde{y}^2 + (\beta + \frac{da}{a})\tilde{y} + \frac{\gamma}{a} = 0 \tag{10}$$

although a change of coordinate $y = \tilde{y} + b$ yields

$$d\tilde{y} + \alpha\tilde{y}^2 + (\beta + 2b\alpha)\tilde{y} + (db + b^2\alpha + b\beta + \gamma) = 0; \tag{11}$$

after a combination of those two transformations, we can choose α and β arbitrary (with $\alpha \not\equiv 0$) and then γ is uniquely determined by the projective structure. Let us show how to compute it from the developping map f of the projective structure.

Let us go back to the universal cover where the Riccati foliation is given by $dy_0 = 0$ and σ is the graph of f (see section 1.5). By a preliminary change of coordinate $y_0 = y_1 + f(u)$, we have now $\sigma = \{y_1 = 0\}$ and the equation becomes $\mathcal{F} : dy_1 + df = 0$. A second change of coordinate $y_1 = f'(u) \cdot y_2$ yields $\mathcal{F} : dy_2 + (1 + \frac{f''}{f'} y_2)du = 0$, and σ is still the zero section $y_2 = 0$. In the case of an affine structure on a torus, the later Riccati equation is well-defined: the corresponding triple $(\pi : P \to C, \mathcal{F}, \sigma)$ is then given by:

$$P = C \times \mathbb{P}^1 \ni (u, y), \quad \mathcal{F} : dy + (1 + cy)du = 0 \quad \text{and} \quad \sigma(u) \equiv 0$$

for some $c \in \mathbb{C}$.

In the general projective case, it is more convenient to send the section σ to the infinity: in the coordinate $\tilde{y}_2 = -1/y_2$, $\mathcal{F}$ is defined by $d\tilde{y}_2 + (\tilde{y}_2^2 - \frac{f''}{f'}\tilde{y}_2)du = 0$ and σ, by $\tilde{y}_2 = \infty$. We finally apply the change of coordinate $\tilde{y}_2 = y + \frac{1}{2}\frac{f''}{f'}$ and obtain

$$\mathcal{F} : dy + (y^2 + \frac{1}{2}\mathcal{S}_u(f))du = 0 \tag{12}$$

where $\mathcal{S}_u(f)$ is the schwartzian derivative of f with respect to the variable u. Unfortunately, du is not a global 1-form. Moreover, u is a transcendental variable that we do not want to deal with when we are considering a triple $(\pi : P \to C, \mathcal{F}, \sigma)$. In general, by birational trivialization of the bundle, one can assume σ at infinity and, after choosing a global holomorphic 1-form α on C, reduce the Riccati foliation to the special form

$$\mathcal{F} : d\tilde{y} + \alpha\tilde{y}^2 + \gamma = 0 \tag{13}$$

with γ meromorphic on C. Here, α plays the role of du, that is u is replaced by a variable v such that $\alpha = dv$; this makes sense at least at a generic point of C where everything is regular. Setting $u = \psi(v)$, the change of coordinate

$y = \frac{1}{\psi'}\left(\tilde{y} + \frac{1}{2}\frac{\psi''}{\psi'}\right)$ transforms equation (12) into (13); after computation we find $\gamma = \frac{1}{2}\left(\mathcal{S}_u(f)\circ\psi\cdot(\psi')^2 + \mathcal{S}_v(\psi)\right)dv$. Using (7), one finally obtains $\gamma = \mathcal{S}_v(f\circ\psi)dv$ where $\mathcal{S}_v$ is the schwarzian derivative with respect to v and deduce

Proposition 2.1. — *Let $(\pi : P \to C, \mathcal{F}, \sigma)$ be a triple defining a projective structure on C. Let $(v, y) \in U \times \mathbb{P}^1$ be bundle coordinates over $\pi^{-1}(U)$, $U \subset C$, such that*

$$\sigma : y = \infty \quad \text{and} \quad \mathcal{F} : dy + (y^2 + \frac{\phi(v)}{2})dv = 0.$$

Then the projective coordinates f on U are the solutions of $\mathcal{S}_v f = \phi$.

Remark 2.2. — Following [**26**], the maximally unstable undecomposable $\mathbb{P}^1$-bundle P corresponding to projective triples $(\mathcal{P}, \mathcal{F}, \sigma)$ can be trivialized after $2g$ elementary transformations (here $e = 2g - 2$ is even). The birational tranformation constructed above to put $\mathcal{F}$ into the normal form (13) however needs much more elementary transformations.

Indeed, at a point where $\alpha = dv \sim u^\nu du$ has a zero of order ν, i.e. $v \sim u^{\nu+1}$, the expression

$$\alpha \otimes \gamma = \frac{1}{2}\mathcal{S}_v(f)dv^{\otimes 2} \sim \left(\frac{dv}{v}\right)^{\otimes 2} \sim \left(\frac{du}{u}\right)^{\otimes 2}$$

has a pole of order 2 and thus $\gamma \sim \frac{du}{u^{\nu+2}}$ has a pole of order $\nu + 2$. In fact, $\psi' \sim \frac{1}{u^\nu}$ and the birational change of coordinate takes the form

$$y \sim u^\nu\left(\tilde{y} - \frac{\nu}{2}\frac{1}{u^{\nu+1}}\right) = \frac{1}{u^{\nu+1}}\left(u^{2\nu+1}\tilde{y} - \frac{\nu}{2}\right);$$

$3\nu + 2$ elementary transformations are needed at this point.

Now, we look for a sharp birational trivialization of P, that is to say with exactly $2g$ elementary transformations. For any choice of global meromorphic 1-forms α and β, there is a unique birational transformation of the form $y = a(\tilde{y} + b)$ putting the initial Riccati equation (12) into the form (9) and we have

$$\begin{cases} \alpha &= a du \\ \beta &= \frac{da}{a} + 2ab \\ \gamma &= db + ab^2 du + \frac{da}{a}b + \frac{\mathcal{S}_u(f)}{2a}du \end{cases}$$

Each zero or pole of α (or a) gives rise to an elementary transformation: if we choose α holomorphic, we already get $2g-2$ elementary transformations with the first change of coordinate. We would like now $ab = \frac{1}{2}(\beta - \frac{da}{a})$ be holomorphic (as much as possible). The sum of residues of $\frac{da}{a}$ is $2g-2$: we can construct a meromorphic 1-form β having the same principal part as $\frac{da}{a}$, plus one extra simple pole (at, say, p) with residue $2 - 2g$. The final change of coordinate $y = a\tilde{y} + ab$ is therefore a combination of $2g$ elementary transformations: the change of coordinate $y = a\tilde{y}$ goes from the trivial bundle to $\overline{K}$ with $2g - 2$ elementary transformations; the ultimate transformation $y = \tilde{y} + ab$ has one simple pole corresponding to a succession of 2 generic elementary transformations of the same fibre (compare [**24**]).

Setting $y = z'/z$, $z' = \frac{dz}{dv}$, the differential equation $dy + (y^2 + \frac{\phi(v)}{2})dv = 0$ is transformed into

$$z'' + \frac{\phi(v)}{2} z = 0. \tag{14}$$

Then the following goes back to Schwarz:

Proposition 2.2. — *Any solution f to the differential equation $\mathcal{S}_v(f) = \phi(v)$ takes the form $f = z_1/z_2$ where z_1 and z_2 are independant solutions of (14).*

Proof. — A straightforward computation shows that $\mathcal{S}(z_1/z_2) = -2\frac{z_2''}{z_2}$ provided that z_1 and z_2 are solutions of (14). Any other solution $\mathcal{S}_v(f) = \phi(v)$ takes the form $f = \frac{az_1+bz_2}{cz_1+dz_2}$, a quotient of two other solutions. In fact, one can take $f = z_1/z_2$ with $z_1 = \frac{f}{\sqrt{f'}}$ and $z_2 = \frac{1}{\sqrt{f'}}$. □

Remark 2.3. — One can easily generalize the notion of projective structure to the branching case by considering triples $(\pi : P \to C, \mathcal{F}, \sigma)$ with σ generically transversal to $\mathcal{F}$: branching points of the structure are those points $x \in C$ over which σ has a contact with $\mathcal{F}$. The local projective chart then takes the form $f \sim u^{\nu+1}$ where $\nu \in \mathbb{N}$ is the order of contact. More generally, one can consider a singular Riccati foliation $\mathcal{F}$ generically transversal to σ, or equivalently linear equation $dy + (y^2 + \frac{\phi(v)}{2})dv = 0$ with ϕ meromorphic. By the way, projective structures on the 3-punctured sphere (3 simple poles) correspond to the Gauss Hypergeometric equation, on the 4-punctured sphere, to the Heun equation and on the punctured torus, to the Lamé equation.

Remark 2.4. — Let $D = \sum_{i=1}^{k} \nu_i p_i$ be an effective divisor on Σ_g. Consider the set $\mathcal{P}_g(D)$ consisting in all the projective structures on Σ_g branched over the points p_i with ramification order $\nu_i \geq 0$ (see [**21, 22**]). Notice that the case $D = 0$ corresponds to genuine projective structures on Σ_g. As before we can describe the elements of $\mathcal{P}_g(D)$ as triples $(P, \mathcal{F}, \sigma)$, where $P \to \Sigma_g$ is a $\mathbb{P}^1$-bundle with structural group $\mathrm{PSL}(2, \mathbb{C})$, $\mathcal{F}$ is a transversely projective foliation transverse to the fibres, σ is a section such that $\sigma(p_i)$ is a tangency point with $\mathcal{F}$ of order ν_i for each $i = 1, \ldots, k$ and outside these points σ is transverse to $\mathcal{F}$. Projecting to Σ_g the branched projective structure induced by $\mathcal{F}$ on σ we obtain an orbifold complex structure C over (Σ_g, D). We can make a finite number of elementary transformations centered at the tangency points $\sigma(p_i)$ in order to obtain a birationally equivalent triple $(P', \mathcal{F}', \sigma')$, where $\mathcal{F}'$ is a singular Riccati foliation and $\sigma' : C \to P'$ is a holomorphic section everywhere transverse to $\mathcal{F}'$. Applying the same transversality arguments of the proof of Hejal's theorem to $(P', \mathcal{F}', \sigma')$ one shows that the monodromy mapping $\mathcal{M} : \mathcal{P}_g(D) \to \mathcal{R}_g$ is also a local diffeomorphism.

3. The genus 1 case

3.1. Monodromy and bundles

Proposition 3.1. — *Let $C = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ be an elliptic curve, $\rho : \pi_1(C) \to PGL(2,\mathbb{C})$ be any representation and $(\pi : P \to C, \mathcal{F})$ be the associated suspension. Then we are, up to conjugacy, in one of the following cases:*

- *$\rho : \pi_1(C) \to \mathbb{C}^*$ is linear and $P \in Pic_0(C)$ is the compactification of a line bundle; P is trivial if, and only if, $\rho(1,\tau) = (e^c, e^{\tau c})$.*
- *$\rho : \pi_1(C) \to \mathbb{C}$ is euclidean and either $P = P_0$ is the semi-stable undecomposable bundle, or P is the trivial bundle; we are in the latter case if, and only if, $\rho(1,\tau) = (c, \tau c)$.*
- *$\rho(1,\tau) = (-z, \frac{1}{z})$ and $P = P_{-1}$ is the stable undecomposable bundle.*

Proof. — It is easy to verify that all representation $\rho : \mathbb{Z}^2 \to \mathrm{PGL}(2,\mathbb{C})$ appear in the statement. We have already noticed that a linear representation gives rise to the compactification of a line bundle (this is almost the definition). In fact, for linear representations, we have the exact sequence of sheaves

$$0 \to \mathbb{C}^* \to \mathcal{O}^* \to \Omega \to 0$$

where $\mathcal{O}^*$ is the sheaf of invertible holomorphic functions and the morphism $\mathcal{O}^* \to \Omega$ is given by $f \mapsto \frac{df}{f}$. From the corresponding exact sequence of cohomology groups, we deduce the following one

$$0 \to H^0(C,\Omega) \to \mathrm{Hom}(\pi_1(C),\mathbb{C}^*) \to \mathrm{Pic}_0(C) \to 0.$$

The first non trivial morphism associates to a holomorphic 1-form ω the homomorphism $\gamma \to \exp(\int_\gamma \omega)$ while the second one is the suspension. In our particular case where C is an elliptic curve, we finally deduce

$$0 \to \mathbb{C}du \to \mathrm{Hom}(\pi_1(C),\mathbb{C}^*) \to C \to 0$$

and the first alternative of the statement follows.

The suspension of an euclidean representation gives rise to a bundle with a section σ_∞ having 0 self-intersection. If there is another section σ_0 disjoint from σ_∞, then it should be either transversal, or invariant by $\mathcal{F}$ from (8): in the first case, σ_0 provides a projective structure on C and the monodromy satisfies $\rho(1,\tau) = (c,\tau c)$ by Guning Theorem 1.1; in the second case, the monodromy has two fixed points and is trivial, so is the bundle. In the remaining case where there is no disjoint section from σ_∞, the bundle is undecomposable with invariant $e = 0$ and we conclude with Atiyah Theorem 2.1 that $P = P_0$.

Finally, if $\rho(1,\tau) = (-z, \frac{1}{z})$ is the irreducible representation, we note that ρ cannot be lifted to $\mathrm{SL}(2,\mathbb{C})$ and thus e is odd. On the other hand, Weil Theorem 2.2 tells us that P must be undecomposable (being flat with $e \neq 0$). From Atiyah Theorem 2.1, the only possibility is $P = P_{-1}$. □

3.2. Algebraic families of bundles and Riccati foliations. — It follows from [24] that all degree 0 line bundles as well as P_0 can be obtained after 2 elementary transformations of the trivial bundle. In order to obtain P_{-1}, a third one is needed. We use this approach to provide an algebraic family of flat bundles and Riccati foliations.

Let $p \in \overline{\mathcal{O}}$, $q \in \mathrm{elm}_p(C \times \mathbb{P}^1)$ and consider $P = \mathrm{elm}_q \mathrm{elm}_p \overline{\mathcal{O}}$. Fix trivializing coordinates $(u, z) \in C \times \mathbb{P}^1$ and, for simplicity, set $p = (0, \infty)$. This is irrelevant since all flat $\mathbb{P}^1$-bundles over C admit a one parameter group of automorphism lifting the action of ∂_u (see description of Proposition 3.1). After elementary transformation at the point p, we obtain the bundle $\overline{\mathcal{O}(-[0])}$ having one section σ_∞ with -1 self-intersection and a special point $\tilde{p}$, on the fiber over $u = 0$ but not on σ_∞, through which all sections having $+1$ self-intersection intersect. Indeed, $+1$-sections come from horizontal sections of the trivial bundle. Here, we use the fact that there is no holomorphic section of homology type $\sigma_0 + f$ on $\overline{\mathcal{O}}$, otherwise it would be the graph of a regular covering $C \to \mathbb{P}^1$.

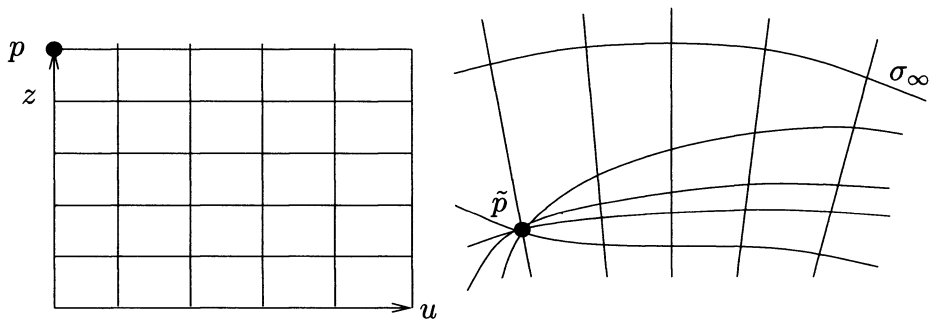

FIGURE 6. The bundle $\overline{\mathcal{O}(-[0])}$

Case 0: $q = \tilde{p}$. The elementary transformation centered at $\tilde{p}$ goes back to the trivial bundle: $P = \overline{\mathcal{O}}$.

Case 1: $q = (u_0, z_0)$ with $u_0 \neq 0$ and $z_0 \neq \infty$. After vertical automorphism, one may assume $z = 0$. The sections $\{z = 0\}$ and $\{z = \infty\}$ respectively give rise to disjoint sections σ_0 and σ_∞ on P having 0 self-intersection. We are in $\mathrm{Pic}_0(C)$ case: $P = \overline{L}$.

The generic horizontal section $\{z = c\}$ gives rise to a section σ on P intersecting σ_0 at $u = 0$ and σ_∞ at $u = u_0$; in other words, σ is a meromorphic section of L with divisor $\mathrm{Div}(\sigma) = [0] - [u_0]$ on C: $L = \mathcal{O}([u_0] - [0])$ corresponds to $u_0 \in \mathrm{Pic}_0(C) \simeq C$.

Case 2: q is on the fibre over $u = 0$ but is neither $\tilde{p}$, nor on σ_∞. Then, $P = P_0$ is the indecomposable bundle. Indeed, assume that there exists a section σ on P disjoint from σ_∞. It then comes from a section of $\overline{\mathcal{O}(-[0])}$ disjoint from σ_∞ and

passing through q, itself coming from a section of $\overline{\mathcal{O}}$ intersecting σ_∞ only at $u = 0$, without multiplicity. We have already seen that this cannot happen.

Case 3: q is on σ_∞, over u_0. We obtain the bundle $\overline{\mathcal{O}(-[0] - [u_0])}$.

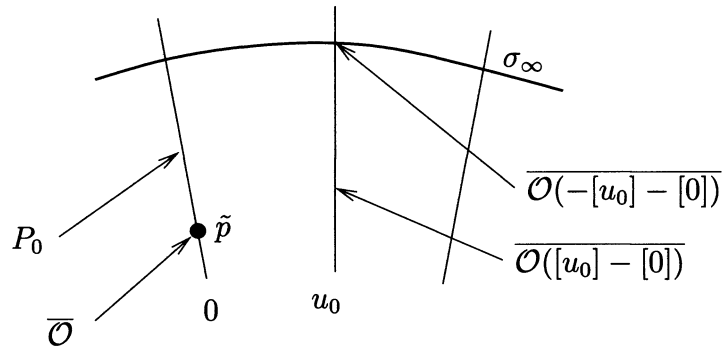

FIGURE 7. An algebraic family of topologically trivial bundles

Here, we have parametrized all topologically trivial flat bundles by the line bundle $\mathcal{O}(-[0])$, see [**30**]. We now want to parametrize all regular Riccati foliations on topologically trivial bundles. The natural way to do this is to provide an explicit family of Riccati equations on the trivial bundle $\overline{\mathcal{O}}$ having appearant singular fibres whose desingularization span all regular Riccati foliations. For instance, consider a linear Riccati foliation defined on the bundle $\overline{\mathcal{O}([u_0] - [0])}$, $u_0 \neq 0$. Apart from the invariant sections σ_0 and σ_∞, the leaves are multivalued sections without zero or pole; after trivialisation of the bundle, those multivalued sections $z(u)$ have now a simple pole over $u = 0$ and a simple zero over $u = u_0$ (and still have linear monodromy): their logarithmic derivative $\frac{dz(u)}{z(u)}$ is a meromorphic 1-form on C having exactly 2 simple poles, one at 0 with residue -1 and one at u_0 with residue $+1$. In other words, the Riccati equation defining the singular foliation after trivialization of the bundle is

$$\frac{dz}{du} = \left(\frac{\wp'(u) + \wp'(u_0)}{2(\wp(u) - \wp(u_0))} + c \right) \cdot z.$$

Indeed, the 1-form $\left(\frac{\wp'(u)+\wp'(u_0)}{2(\wp(u)-\wp(u_0))} + c \right) du$ has a simple pole at $u = 0$ with residue -1 since its principal part is given by $\frac{1}{2} \frac{\wp'(u)}{\wp(u)} du$ and $\wp$ has a double pole at $u = 0$; the other poles may come from the two zeroes of $\wp(u) - \wp(u_0)$, namely $u = \pm u_0$, but $u = -u_0$ is actually regular since the numerator $\wp'(u) + \wp'(u_0)$ also vanishes at this point: by Residue Theorem, $u = u_0$ is a simple pole with residue $+1$. Of course, any other 1-form having the same principal part must differ by a holomorphic 1-form, namely $c \cdot du$, $c \in \mathbb{C}$. We have omitted from our discussion the case $u_0 = -u_0$ is an order 2 point which can be treated like $u = 0$.

After two elementary transformations of $\overline{\mathcal{O}}$ centered at the points $(u,z)=(0,\infty)$ and $(u_0,0)$, we obtain by this way all (linear) foliations on the bundle $\mathcal{O}([u_0]-[0])$ while c runs over $\mathbb{C}$. This does not provide yet a universal family for linear connections on C since the limit of the Riccati foliation while $u_0\to 0$ is the vertical fibration: for (u,z) in a compact set not intersecting $\{u=0\}$, $\{z=0\}$ and $\{z=\infty\}$, we have

$$\frac{\wp'(u)+\wp'(u_0)}{2(\wp(u)-\wp(u_0))}+c\sim -\frac{1}{2}\frac{\wp'(u_0)}{\wp(u_0)}\sim\frac{1}{u_0}\quad\text{while}\quad u_0\sim 0.$$

In other words, the 1-form $u_0dz-u_0\left(\frac{\wp'(u)+\wp'(u_0)}{2(\wp(u)-\wp(u_0))}+c\right)\cdot zdu$ tends uniformly to $-zdu$ on the compact set, so does the foliation. We would like to complete this $\mathbb{C}$-bundle over $u_0\neq 0$ with the family $\frac{dz}{du}=c_0\cdot z$, $c_0\in\mathbb{C}$, of linear connections on the trivial bundle $\mathcal{O}$ ($u_0=0$). A way to obtain it from our large family is obviously to set $c=c(u_0)=c_0-\frac{1}{u_0}$ and take the limit while $u_0\to 0$ with $c_0\in\mathbb{C}$ fixed. In other words, in the parameter space (u_0,c) we consider only the limit at $(0,\infty)$ while $u_0\to 0$ with a special direction. The good global parameter space is obtained after separating the germs of curves $c+\frac{1}{u_0}=$ constant. This is done after 2 elementary transformations on $\overline{\mathcal{O}}$: first we blow-up $(u_0,c)=(0,\infty)$ by setting $c=t/u_0$, and then we blow-up $(u_0,t)=(0,-1)$ by setting $t+1=su_0$, so that $s=\frac{1}{u_0}+c$ coincides with the expected parameter c_0. The resulting parameter space is the *affine bundle* $A_0:=P_0-\sigma_\infty$ where σ_∞ is the unique 0-section of the undecomposable bundle P_0.

Now, we construct a fine moduli space as follows. Consider the product $\overline{\mathcal{O}}\times\overline{\mathcal{O}}$ with global coordinates $((u_0,c),(u,z))$, and equipp the bundle over (u_0,c) with the Riccati foliation

$$\frac{dz}{du}=\left(\frac{\wp'(u)+\wp'(u_0)}{2(\wp(u)-\wp(u_0))}+c\right)\cdot z.$$

This can be seen as an algebraic foliation on the total space. Now, apply the elementary transformations with center along the surfaces $\{u=0,z=\infty\}$ and $\{u=u_0,z=0\}$. Then, we modify the base $(u_0,c)\in\overline{\mathcal{O}}$ by two elementary transformations so that we obtain P_0 as a base and the foliation extends as a linear connections all along $u_0=0$.

The euclidean connections on P_0 are given by:

$$\frac{dz}{du}=\wp(u)+\gamma.$$

Indeed, one can check that the reduction of the singularity over $u=0$ yields P_0; on the other hand, it is clear that monodromy is given by translations. This can be obtained also as a limit of our previous family of connections, or better from

$$\frac{dz}{du}=\left(\frac{\wp'(u)+\wp'(u_0)}{2(\wp(u)-\wp(u_0))}+c\right)\cdot(z-c)$$

which is equivalent to the previous one by the change of coordinate $z\mapsto z+c$. Now, instead of taking limit along curves $c=c(u_0)=c_0-\frac{1}{u_0}$ as $u_0\to 0$ with c_0 constant, we take limit along $c=c(u_0)=\gamma u_0-\frac{1}{u_0}$, $\gamma\in\mathbb{C}$ constant, i.e. $c_0=\gamma u_0$. We have on

convenient compact sets:

$$\frac{\wp'(u)+\wp'(u_0)}{2(\wp(u)-\wp(u_0))}-\frac{1}{u_0}\sim u_0\wp(u) \quad \text{while} \quad u_0\sim 0$$

so that

$$\frac{dz}{du}=\left(\frac{\wp'(u)+\wp'(u_0)}{2(\wp(u)-\wp(u_0))}+c\right)\cdot(z-c)\sim(\wp(u)+\gamma)\cdot(u_0z+1-\gamma u_0^2)\sim\wp(u)+\gamma.$$

3.3. The Riemann-Hilbert Mapping. — For a given elliptic curve $C=\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, the Riemann-Hilbert Mapping provides an analytic isomorphism

$$\mathcal{M}:A_0\to\mathbb{C}^*\times\mathbb{C}^*$$

between two spaces of algebraic nature.

The space of linear connections on C is an affine $\mathbb{C}$-bundle over $\mathrm{Pic}_0(C)\simeq C$ that we have identified with A_0: it is defined by gluing the chart $(u_0,c)\in(C-\{0\})\times\mathbb{C}$ with the chart $(u_0,c_0)\in(C,0)\times\mathbb{C}$ by the transition map

$$(u_0,c)\mapsto(u_0,c_0):=(u_0,c+\frac{1}{u_0}).$$

The space of linear representations of $\pi_1(C)$ is $\mathbb{C}^*\times\mathbb{C}^*$. In the main chart (u_0,c), the analytic connection is given by

$$\frac{dz}{z}=\left(\frac{\wp'(u)+\wp'(u_0)}{2(\wp(u)-\wp(u_0))}+c\right)\cdot du.$$

Introducing Weierstrass Zeta Function $\zeta(u)=-\int_0^u\wp(\xi)d\xi$, one can write $\frac{\wp'(u)+\wp'(u_0)}{2(\wp(u)-\wp(u_0))}=\zeta(u-u_0)-\zeta(u)+\zeta(u_0)$ and integrate the differential equation above by means of the Weierstrass Sigma Function: the general solution [1] is therefore given by $z(u)=a\frac{\sigma(u-u_0)}{\sigma(u)}e^{\zeta(u_0)\cdot u}$, $a\in\mathbb{C}^*$, and the monodromy is given by the homomorphism

$$\Lambda=\mathbb{Z}+\tau\mathbb{Z}\to\mathbb{C}^*\ ;\ \gamma\mapsto\exp(-u_0\zeta(\gamma)+\zeta(u_0)\gamma+c\gamma).$$

Finally we obtain the full monodromy mapping

$$\mathcal{M}:A_0\to\mathbb{C}^*\times\mathbb{C}^*\ ;\ \begin{cases}(u_0,c)&\mapsto&(e^{-u_0\zeta(1)+\zeta(u_0)+c},e^{-u_0\zeta(\tau)+\zeta(u_0)\tau+c\tau})\\(0,c_0)&\mapsto&(e^{c_0},e^{c_0\tau})\end{cases}$$

The image by the monodromy map of the algebraic fibration defined on A_0 is the holomorphic foliation defined on $\mathbb{C}^*\times\mathbb{C}^*$ by the linear vector field

$$x\partial_x+\tau y\partial_y.$$

As a particular case of Narasimhan-Seshadri Theorem, the unitary representations $\mathbb{S}^1\times\mathbb{S}^1$ form a smooth real 2-dimensional torus transversal to the foliation and cutting each leaf once. It is the space of the leaves. It inherits, from the transversal complex foliation, a complex structure, namely the structure of C. The euclidean foliations

[1] This computation was communicated to the first author by Frits Beukers; a similar computation but with a slightly different presentation was done in [**17**].

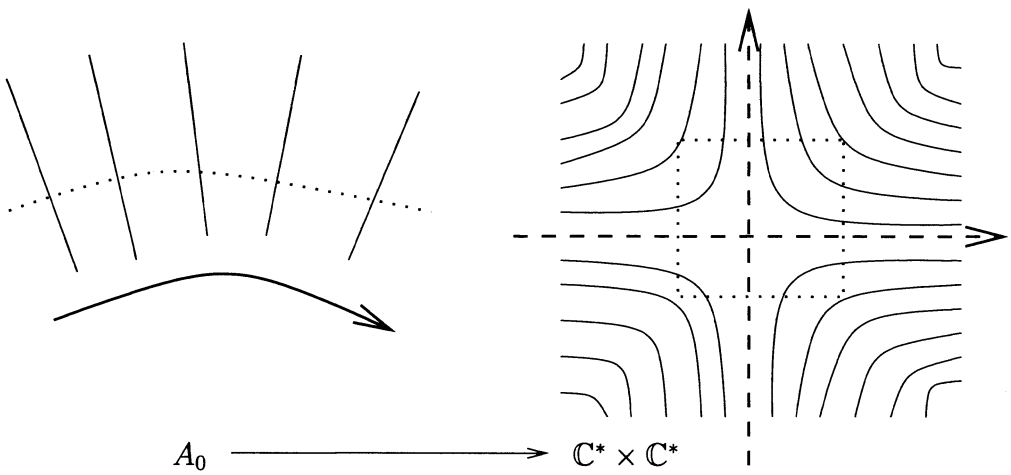

FIGURE 8. The Riemann-Hilbert Mapping

defined on A_0 by $\frac{dc}{du_0} = \wp(u_0) + \gamma$, $\gamma \in \mathbb{C}$, are sent to the linear foliations $x\partial x + \lambda y \partial y$, $\lambda \in \mathbb{P}^1 \setminus \{\tau\}$. The space of linear connections is equipped with a group law given by tensor product; it is just the pull-back of the natural group law on $\mathbb{C}^* \times \mathbb{C}^*$. We thus get an analytic isomorphism between two algebraic groups that are not algebraically equivalent.

One can compute the group law on A_0 as follows. Given two connections

$$\frac{dz}{z} = \left(\frac{\wp'(u) + \wp'(u_1)}{2(\wp(u) - \wp(u_1))} + c_1\right) \cdot du \quad \text{and} \quad \frac{dz}{z} = \left(\frac{\wp'(u) + \wp'(u_2)}{2(\wp(u) - \wp(u_2))} + c_2\right) \cdot du,$$

the tensor product is a connection of the form

$$\frac{dz}{z} = \left(\frac{\wp'(u) + \wp'(u_3)}{2(\wp(u) - \wp(u_3))} + c_3\right) \cdot du$$

with $u_3 = u_1 + u_2$ (group law on $\mathrm{Pic}_0(C)$). Then, c_3 is determined by the fact that

$$\frac{\wp'(u) + \wp'(u_1)}{2(\wp(u) - \wp(u_1))} + \frac{\wp'(u) + \wp'(u_2)}{2(\wp(u) - wp(u_2))} - \frac{\wp'(u) + \wp'(u_3)}{2(\wp(u) - \wp(u_3))} + c_1 + c_2 - c_3 = \frac{df}{f}$$

for a meromorphic function f on C. Looking at the principal part of the left hand side, one see that f must have divisor $\mathrm{Div}(f) = [u_1] + [u_2] - [u_3] - [0]$ so that, up to a scalar, we have

$$f = \frac{\wp'(u) - \wp'(u_1) - \frac{\wp'(u_2) - \wp'(u_1)}{\wp(u_2) - \wp(u-1)}(\wp(u) - \wp(u_1))}{\wp(u) - \wp(u_3)};$$

after computations, one finds that

$$c_3 = c_1 + c_2 - \frac{\wp'(u_2) - \wp'(u_1)}{2(\wp(u_2) - \wp(u_1))}.$$

References

[1] M. F. ATIYAH – "Complex fibre bundles and ruled surfaces", *Proc. London Math. Soc. (3)* **5** (1955), p. 407–434.

[2] W. BARTH, C. PETERS & A. VAN DE VEN – *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 4, Springer, 1984.

[3] A. BEILINSON & V. DRINFELD – "Opers", preprint arXiv:math/0501398, 2005.

[4] F. BEUKERS – "Solutions of a Lamé system", private notes, 2004.

[5] G. D. BIRKHOFF – "A theorem on matrices of analytic functions", *Math. Ann.* **74** (1913), p. 122–133.

[6] M. BRUNELLA – *Birational geometry of foliations*, Monografías de Matemática, Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2000, `http://www.impa.br/Publicacoes/Monografias/Abstracts/brunella.ps`.

[7] C. J. EARLE – "On variation of projective structures", in *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, Ann. of Math. Stud., vol. 97, Princeton Univ. Press, 1981, p. 87–99.

[8] O. FORSTER – *Lectures on Riemann surfaces*, Graduate Texts in Mathematics, vol. 81, Springer, 1981.

[9] D. GALLO, M. KAPOVICH & A. MARDEN – "The monodromy groups of Schwarzian equations on closed Riemann surfaces", *Ann. of Math. (2)* **151** (2000), p. 625–704.

[10] W. M. GOLDMAN – "The symplectic nature of fundamental groups of surfaces", *Adv. in Math.* **54** (1984), p. 200–225.

[11] X. GÓMEZ-MONT – "Holomorphic foliations in ruled surfaces", *Trans. Amer. Math. Soc.* **312** (1989), p. 179–201.

[12] R. C. GUNNING – "Special coordinate coverings of Riemann surfaces", *Math. Ann.* **170** (1967), p. 67–86.

[13] ———, "Analytic structures on the space of flat vector bundles over a compact Riemann surface", in *Several complex variables, II (Proc. Internat. Conf., Univ. Maryland, College Park, Md., 1970)*, Springer, 1971, p. 47–62. Lecture Notes in Math., Vol. 185.

[14] ———, "Affine and projective structures on Riemann surfaces", in *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, Ann. of Math. Stud., vol. 97, Princeton Univ. Press, 1981, p. 225–244.

[15] R. HARTSHORNE – *Algebraic geometry*, Springer, 1977, Graduate Texts in Mathematics, No. 52.

[16] D. A. HEJHAL – "Monodromy groups and linearly polymorphic functions", *Acta Math.* **135** (1975), p. 1–55.

[17] N. J. HITCHIN – "Twistor spaces, Einstein metrics and isomonodromic deformations", *J. Differential Geom.* **42** (1995), p. 30–112.

[18] J. H. HUBBARD – "The monodromy of projective structures", in *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, Ann. of Math. Stud., vol. 97, Princeton Univ. Press, 1981, p. 257–275.

[19] N. M. KATZ – "An overview of Deligne's work on Hilbert's twenty-first problem", in *Mathematical developments arising from Hilbert problems (Proc. Sympos. Pure Math., Vol. XXVIII, Northern Illinois Univ., De Kalb, Ill., 1974)*, Amer. Math. Soc., 1976, p. 537–557.

[20] S. KAWAI – "The symplectic nature of the space of projective connections on Riemann surfaces", *Math. Ann.* **305** (1996), p. 161–182.

[21] R. MANDELBAUM – "Branched structures on Riemann surfaces", *Trans. Amer. Math. Soc.* **163** (1972), p. 261–275.

[22] ———, "Branched structures and affine and projective bundles on Riemann surfaces", *Trans. Amer. Math. Soc.* **183** (1973), p. 37–58.

[23] ———, "Unstable bundles and branched structures on Riemann surfaces", *Math. Ann.* **214** (1975), p. 49–59.

[24] M. MARUYAMA – *On classification of ruled surfaces*, Lectures in Mathematics, Department of Mathematics, Kyoto University, vol. 3, Kinokuniya Book-Store Co. Ltd., 1970.

[25] M. NAGATA – "On self-intersection number of a section on a ruled surface", *Nagoya Math. J.* **37** (1970), p. 191–196.

[26] M. NAGATA & M. MARUYAMA – "Note on the structure of a ruled surface", *J. reine angew. Math.* **239/240** (1969), p. 68–73.

[27] M. S. NARASIMHAN & C. S. SESHADRI – "Stable and unitary vector bundles on a compact Riemann surface", *Ann. of Math. (2)* **82** (1965), p. 540–567.

[28] H. POINCARÉ – "Sur les groupes des équations linéaires", *Acta Math.* **4** (1884), p. 201–312.

[29] N. STEENROD – *The Topology of Fibre Bundles*, Princeton Mathematical Series, vol. 14, Princeton University Press, 1951.

[30] T. SUWA – "On ruled surfaces of genus 1", *J. Math. Soc. Japan* **21** (1969), p. 291–311.

[31] A. N. TJURIN – "On the classification of two-dimensional fibre bundles over an algebraic curve of arbitrary genus", *Izv. Akad. Nauk SSSR Ser. Mat.* **28** (1964), p. 21–52.

[32] ———, "Geometry of moduli of vector bundles", *Uspehi Mat. Nauk* **29** (1974), p. 59–88.

[33] ———, "Periods of quadratic differentials", *Uspekhi Mat. Nauk* **33** (1978), p. 149–195, 272.

[34] A. WEIL – "Généralisation des fonctions abéliennes", *J. Math. Pures Appl.* **17** (1938), p. 47–87.

[35] ———, "On discrete subgroups of Lie groups", *Ann. of Math. (2)* **72** (1960), p. 369–384.

F. LORAY, IRMAR, UFR de Mathématiques, Université de Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex (France), France • *E-mail :* `frank.loray@univ-rennes1.fr`
Url : `http://perso.univ-rennes1.fr/frank.loray`

D. MARÍN PÉREZ, Departament de Matemàtiques, Universitat Autònoma de Barcelona, Edifici Cc. Campus de Bellaterra, 08193 Cerdanyola del Vallès, Spain • *E-mail :* `davidmp@mat.uab.es`
Url : `http://mat.uab.es/~davidmp`