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Abstract. — Let $X(t)$, $0 \leq t \leq 1$, be a bridge from 0 to 0 with exchangeable increments on $D[0,1]$. We obtain the n.a.s.c. for the sojourn below 0 to be uniformly distributed, or equivalently for X to have a uniform index of the (unique) supremum. This is applied to Lévy bridges.

It seems particularly fitting for the present author to be given an opportunity to contribute to a volume in honor of Meyer and Neveu. Professor Meyer alone, over the years, has rewritten, revised, and expanded not fewer than five of our research papers, mostly as part of his herculean efforts on behalf of the Seminaire de Probabilités. There are various anecdotes concerning these papers which, if space permitted, we would gladly include. However, it seems fair to say that Meyer always put business before amusement, and following his lead we must be content to do likewise. Suffice it to say that both the subject and the author are lastingly indebted for his contributions. The present paper, however, is already indebted to a referee, so we can hope that it, at least, will not merit his revision.

In his famous paper [8], P. Lévy obtained the arcsine law for the positive sojourn of Brownian motion, and also the uniform law for the positive sojourn of Brownian bridge. Very recently ([5]) R. K. Getoor and M. J. Sharpe have obtained the necessary and sufficient conditions for the same arcsine law to hold for a diffuse Lévy process X on R. One purpose of the present paper is to do the analogous thing (but without the "diffuse" assumption) for the uniform law, at least if we understand by "bridge" the process $X_t - tX_1$, $t \leq 1$.

Also in the paper [8], Lévy obtained the arcsine law for the distribution of the last exit time g from 0 before $t = 1$. Since Lévy knew that $M(t) - B(t) \stackrel{d}{=} |B(t)|$, where $M(t) = \max_{s \leq t} B(s)$ ($B(s)$ being a Brownian motion) it followed immediately (although he does not mention it) that the location (abscissa) of the maximum of B in $0 < t < 1$ again has the arcsine law. He also probably realized that the location of the maximum of the bridge B^0 is uniformly distributed.

Both of these facts extend to processes with exchangeable increments whenever the corresponding laws for the positive sojourn are valid, by virtue of the identity that the law of the positive sojourn is the same as that of the location of the first supremum in [0,1]. This identity has a combinatorial basis in the analogous discrete parameter case, due to E. Sparre-Andersen. It was extended by a limit procedure to Lévy processes by Pecherskii and Rogozin [14], and to Lévy bridges by J. Bertoin [12]. In the present paper it is extended to processes with exchangeable increments (Theorem 1.4*). We wish to thank a referee for sketching this proof, based on the discrete parameter case (see Theorem 2 of W. Feller [13, XII. 8] for this case). However since this is a rather hard result, and the others seem much more intuitive, we have indicated it and the results depending on it with an asterisk.

For diffuse Lévy processes, the necessary and sufficient condition for the arcsine law of positive sojourn is $P\{X_t > 0\} = \frac{1}{2}, t > 0$. By contrast, for diffuse Lévy **bridges** the uniform law of positive sojourn **always** holds. In both cases the surprising level of generality goes back to Sparre-Andersen's work in the discrete parameter setting [1,2]. Indeed, a formula of [2] is used in [5]. Our debt is less concrete, although our reasoning is already implicit in [1]. It seems that for bridges the set-up of a discrete parameter, as in [1], only obscures the relative simplicity of the continuous parameter case.

Both the uniform sojourn law and the uniform location of the maximum are first obtained, in Section 1, for processes with exchangeable increments, where we rely on a representation given in O. Kallenberg [6]. Here it seems natural to replace the notion of bridge by the process linearly centered to vanish at $t = 1$. For Lévy processes, however, usage favors using the term "bridge" for a process conditioned to vanish at $t = 1$. Accordingly, we treat the two concepts separately in Section 2, although, generally speaking, the same uniform laws hold for both. In fact the two concepts coincide only in the Gaussian case (Theorem 2.2), and the definition by conditioning of course requires some supplementary hypothesis. We have found Condition (C) of Kallenberg [7] to be most adaptable to our needs at this point (but see the Remarks after Lemma 2.6).

Section 1. The uniform law for linearly centered processes with exchangeable increments.

A certain part of the theorems we wish to prove can be formulated for an arbitrary measurable function $f(t)$, $0 \le t < 1$. We set $S(x,f) = \int_0^1 I_{(-\infty,x]}(f(t))dt$, $-\infty < x < \infty$. Noting that $\lim_{x\to-\infty} S(x,f) = 0$, $\lim_{x\to+\infty} S(x,f) = 1$, and $S(x,f)$ is non-decreasing and continuous to the right, we call $S(x,f)$ the "sojourn distribution function" of f. More generally, if $X_t(w)$ is a measurable stochastic process, $0 \le t < 1$, we call $S(x, X(w))$ the (random) sojourn distribution of X, and when X is understood from context we abbreviate to simply $S(x)$. In that case, it is clear that $S(x)$ is a stochastic process associated with X. We say that f (or X) has continuous sojourn distribution if $S(x,f)$ (or $S(x,X)$, P-a.s.) is continuous in x. Now a critical result for the sequel is

Lemma 1.1(a). *Let f have continuous sojourn distribution, and let U be a uniformly distributed random variable on (0,1). Let*

$$X(t,w) = f((t+U)\mathrm{mod}1) - f(U) = \begin{cases} f(t+U) - f(U); & t < 1-U \\ f(t+U-1) - f(U); & 1-U \le t < 1 \end{cases}$$

Then $P\{S(0,X) \le x\} = x$, $0 < x < 1$, that is $S(0,X)$ has the same law as U.

Proof. Since $S(x,f)$ is continuous, for $0 < p < 1$ there is a number x_p for which $S(x_p, f) = p$. Then if $f(t) < x_p$ we have

$$\begin{aligned} &\int_0^1 I_{(-\infty,0]}(f((t+s)\mathrm{mod}1) - f(t))ds \\ &\le \int_0^1 I_{(-\infty,0]}(f((t+s)\mathrm{mod}1) - x_p)ds \\ &= \int_0^1 I_{(-\infty,x_p]}f((t+s)\mathrm{mod}1)ds \\ &= p. \end{aligned}$$

Similarly, if $f(t) > x_p$, then

$$\begin{aligned} &\int_0^1 I_{(-\infty,0]}(f((t+s)\mathrm{mod}1) - f(t))ds \\ &\ge \int_0^1 I_{(-\infty,x_p]}f((t+s)\mathrm{mod}1)ds \\ &= p. \end{aligned}$$

Thus we have

$$\begin{aligned} &S(0,X) \le p \quad \text{if} \quad f(U) < x_p, \quad \text{and} \\ &S(0,X) \ge p \quad \text{if} \quad f(U) > x_p. \end{aligned}$$

Now $P\{f(U) \le x_p\} = p$, and since f has continuous sojourn distribution,

$$P\{f(U) = x_p\} = \int_0^1 I_{\{x_p\}}(f(s))ds = 0.$$

So it follows that $P\{S(0,X) \le p\} \ge p$ and $P\{S(0,X) \ge p\} \ge 1-p$. By addition we get $P\{S(0,X) = p\} = 0$, and finally $P\{S(0,X) \le p\} = p$, as asserted.

The second appearance of the uniform law which we intend to treat concerns the location, or argument, of the supremum. Here we shall assume that all functions or processes considered are right-continuous and left limited, so that their paths are in the space D [0, 1]. In the case of processes, we use the coordinate filtration, augmented by all P-null sets. This is equivalent to the augmented topological filtration of the complete separable metric space–see [3] for more details. This approach has the advantage that the supremum and the essential supremum coincide, so we need not treat them separately.

For $f \in D\,[0, 1]$, we adjust the definition at 1 by setting $f(1) = f(1-)$, and we define $f(0-) = f(1)$, so that f can be viewed as defined on a circle. Let $Mf = \sup_{0 \le t < 1} f(t)$. We say that f has unique location of supremum (or just unique supremum) if there is a unique t_0, $0 \le t_0 < 1$, with $Mf = f(t_0-) \vee f(t_0)$, and we write $t_0 = ArgMaxf = AMf$. If this holds $P - a.s.$ for a process X, we write $AM(X)$ for its location (set=0 where not unique). We note that for any f there exists at least one t_0 with $Mf = f(t_0-) \vee f(t_0)$, so there is no problem as to existence.

Lemma 1.1(b). *If f has unique supremum, and $X = X(t, w)$ is as in Lemma 1.1(a), then $AM(X)$ has the same law as U.*

Proof. If $t_0 = AMf$, then one sees that $AM(X) = (1 + t_0 - U)(\text{mod}1)$, so the result follows.

We will apply this to certain processes with exchangeable increments. From now on, all processes considered will be assumed to have paths $X(\cdot, w) \in D[0, 1]$, where $D[0, 1]$ is the measurable space of right-continuous, left-limited real-valued functions (see [3] for details). We recall ([6]) that X_t has exchangeable increments if, for each n, the joint law of $\left\{X\left(\frac{k}{n}\right) - X\left(\frac{k-1}{n}\right); 1 \le k \le n\right\}$ is that same as that of $\left\{X\left(\frac{\sigma(k)}{n}\right) - X\left(\frac{\sigma(k)-1}{n}\right); 1 \le k \le n\right\}$ for every permutation σ of $\{1, 2, \ldots, n\}$. We will need to use the

Representation Theorem. *(Kallenberg, [6]). The process X_t, $X_0 = 0$, has exchangeable increments if and only if it may be represented in the form*

$$(1.1) \qquad X_t = \alpha t + \sigma B_o(t) + \sum_{j=1}^{\infty} \beta_j (1(t - t_j) - t); \quad 1(s) \doteq \begin{cases} 0; & s < 0 \\ 1; & s \ge 0 \end{cases},$$

where

(a) $B_o(t)$ is a Brownian bridge, $0 \le t \le 1$,

(b) α, σ and $\beta_1, \beta_2, \ldots$ are real-valued random variables (on the probability space of X), independent of $B_o(\cdot)$, $0 \le \sigma$, and $\sum_{j=1}^{\infty} \beta_j^2 < \infty$, P-a.s.

(c) t_j, $1 \le j$, are uniformly distributed on (0, 1), independent of each other and of the random variables in (a) and (b).

Remark. Any or all of the variables in (a) and (b) may assume the value 0. The series, if infinite, converges a.s. uniformly in $t \le 1$.

Given such a process X_t, we set $Y_t \doteq X_t - tX_1$, $0 \le t < 1$. The following Lemma is the key to applying Lemma 1.1(a) to Y_t.

Lemma 1.2 *(a) Y_t has exchangeable increments. (b) X_t has continuous sojourn distribution if and only if, in the representation (1.1),*
$P\left(\cup_{i=1}^{3} S_i\right) = 1$, *where* $S_1 = \{\sigma \neq 0\}$,
$S_2 = \{$*infinitely many* $\beta_j \neq 0\}$, *and*
$S_3 = \{$*only finitely many* $\beta_j \neq 0$ *and* $\sum_{finite} \beta_j \neq \alpha\}$ *(here we define an empty sum to equal 0).*

Remark. In the representation (1.1) for Y_t, we have $\alpha \equiv 0$.

Proof. (a) More generally, for each n let $f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)$ be $B^n/B-$measurable functions. Then the joint law of $\{f_j\left(X\left(\frac{1}{n}\right), X\left(\frac{2}{n}\right) - X\left(\frac{1}{n}\right)\right), \ldots, X(1) - X\left(\frac{n-1}{n}\right), \quad 1 \leq j \leq m\}$ is invariant under permutation of the n increments. Since $Y\left(\frac{k}{n}\right) - Y\left(\frac{k-1}{n}\right) = -\sum_{i \neq k}\left(X\left(\frac{i}{n}\right) - X\left(\frac{i-1}{n}\right)\right)$, exchangeability for Y follows as a consequence.

Turning to (b), which is much less obvious, let us first examine the set $S = \{$only finitely many $\beta_j \neq 0$ and $\sigma = 0\}$. Since there are only countably many finite subsets of j, it is seen that S is measurable. Now on the subset $S \cap \left\{\sum_j \beta_j = \alpha\right\}$ we have $Y = \sum_j \beta_j 1(t - t_j)$, the sum being finite, which is a random step function. Obviously it does not have continuous sojourn distribution. On the other hand, over $S \cap \left\{\sum_j \beta_j \neq \alpha\right\}$ we have a step function plus the line $t\left(\alpha - \sum_j \beta_j\right)$. The sojourn time at x is that of $\sum_j \beta_j 1\left(t - t_j\right)$ on the line $-t\left(\alpha - \sum_j \beta_j\right) + x, \ \ 0 < t < 1$, which includes at most one point in each step of the former. Hence it is of Lebesque measure 0, and we have proved (b) in the case $\Omega = S$, or more generally on the set S.

For the general case, we need to distinguish between fixed and mobile discontinuities of $S(x, Y)$. By definition, x is a fixed discontinuity if $E(S(x, Y) - S(x-, Y)) > 0$, while x is a mobile discontinuity at $w \in \Omega$ if $S(x, Y(w)) - S(x-, Y(w)) > 0$ but $E(S(x, Y) - S(x-, Y)) = 0$. We will first show that, if $P(U_{i=1}^{3} S_i) = 1$, $S(x, Y)$ has no fixed discontinuities. By Fubini's Theorem, $E(S(x, Y) - S(x-, Y)) = E \int_0^1 I_{\{x\}}(Y_s) ds = \int_0^1 (F_Y(x, s) - F_Y(x-, s)) ds$, where $F_Y(x, s)$ is the marginal distribution function of $Y(s)$. Hence it suffices to show that the marginal distributions are continuous. Since $B_o(s)$ is independent of the other variables and continuously distributed, the conditional marginal distributions, given $\sigma \neq 0$ and the other variables, are continuous, hence their expectation over the other variables is also continuous. Thus the expectations are continuous over $\{\sigma \neq 0\}$, i.e. there are no fixed discontinuities over this set. Now on $\{\sigma = 0\}$ we have infinitely many $\beta_i \neq 0$, hence it now suffices to treat the case $P\{$infinitely many $\beta_i \neq 0\} = 1$. We use the observation that for any two distribution functions F and G, if $*$ denotes convolution and Δ_x denotes the jump at x (possibly 0), then $\sup_x \Delta_x(F * G) = \sup_x \int (\Delta_{x-y} G) dF_y \leq \sup_x \Delta_x G$.

Thus the maximum jump size is reduced through convolution, and by conditioning on the sequence β_j it suffices to show that the marginal distributions of $\sum_{j=1}^{\infty}\beta_j(1(t-t_j)-t)$ are continuous when β_j are constants with $\sum_j \beta_j^2 < \infty$. To this effect, since the t_j are independent, it is enough to find a subsequence $j_n \to \infty$ such that $\lim_{m\to\infty} \max_x P\left\{\sum_{n=1}^{m}\beta_{j_n}(1(t-t_{j_n})-t)=x\right\}=0$. A single term $\beta_j 1(t-t_j)$ has a law with jumps (= point masses) of size t at $\beta_j(1-t)$, and of size $1-t$ at $-\beta_j t$. The sum of n such terms has jumps at the 2^n possible sums of these points, which may not be distinct. However, they become distinct if we choose only j_n such that $(t\vee(1-t))\sum_{k=n+1}^{\infty}|\beta_{j_k}| < (t\wedge(1-t))|\beta_{j_n}|$. Then a sum of jump positions of index $j_k, k\geq n+1$ cannot equal the separation in the jump position of two sums of size n which differ only at the nth term. Such a sequence j_n is easily constructed by induction on n. Beginning with a fixed subsequence, also denoted $\beta_n(\neq 0)$ such that $\sum_n \beta_n < \infty$ we set $\beta_{j_1}=\beta_1$ and, having chosen the further subsequence $(\beta_{j_1},\ldots,\beta_{j_n})$ we let $j_{n+1} > j_n$ be any index for which

$$(t\vee(1-t))\sum_{k=j_n+1}^{\infty}|\beta_k| < (t\wedge(1-t))|\beta_{j_n}|.$$

Then clearly a sum $\sum_{k=1}^{n}\beta_{j_k}(1(t-t_{j_k})-t)$ has all of its 2^n jump points distinct, and hence its maximum jump (point mass) is $(t\vee(1-t))^n$. Since this tends to 0, the proof of absence of fixed sojourn discontinuities is complete.

It remains to consider the mobile discontinuities. For $0<a<b<1$, let $I=(a,b)$; $S_I(x)=\int_a^b I_{(-\infty,x]}(X_s)ds$ is the sojourn distribution of X in the interval I. We assume, in accordance with the case at hand, that $P\{\sigma\neq 0$ or infinitely many $\beta_j\neq 0\}=1$. We now consider the conditional law of $X_{a+t}-X_a$, $0\leq t\leq b-a$, given the processes $(X_s, s\leq a)$ and $(X_s, b\leq s\leq 1)$. Slightly redundantly, we also treat as given $B_o(b)-B_o(a)$ and the sequence $(t_j I_j, 1\leq j)$ where $I_j \doteq \begin{cases} 0; & t_j\in I \\ 1; & t_j\notin I \end{cases}$. It is not hard to see that this amounts to being given a countable number of random variables, so the conditional joint distributions of $X_{a+t}-X_a$ are well-defined (P-a.s.) on the space D[a, b], and extend to a conditional probability on the coordinate σ-field. We claim that this conditional process again has exchangeable increments, P-a.s. (i.e. $X_{a+(b-a)t}-X_a$ does so, as a process on D[0, 1]). Indeed, even given α, σ, and $(\beta_j,\ 1\leq j)$ as well as the other assumed data, the t_j for which $I_j=0$ are conditionally independent and uniformly distributed on (a, b). Actually, it is not hard to see that the β_j with $I_j=1$ are already given, along with X and the corresponding t_j, i.e. the jumps of X outside of I are all given, and consequently so is α and the quadratic variation coefficient σ, and consequently $B_o(s)$ is also determined outside of I. Now inside I we can write

$$\begin{aligned} X_{a+t}-X_a &= (\alpha+\sigma(B_o(b)-B_o(a)))t \\ &+\sigma(B_o(a+t)-B_o(a)-(B_o(b)-B_o(a)t) \\ &+\sum_{t_j\in I}\beta_j(1(a+t-t_j)-t), \quad 0<t<b-a \end{aligned} \tag{1.2}$$

where the second term on the right is σ times a conditional Brownian bridge and the t_j are conditionally uniform on (a, b) and independent of the former. Then, when we give to the β_j such that $t_j \in I$ the conditional distribution given all of $(t_jI_j, 1 \le j)$, the process $(X_s,\ s \notin I)$, and $B_o(b) - B_o(a)$, we see that (1.2) becomes a representation (1.1) for a process with exchangeable increments on $(0,\ b-a)$, conditional on the given quantities, P-a.s. Furthermore our hypothesis, that either $\sigma \neq 0$ or infinitely many $\beta_j \neq 0$, also holds for the conditional process, P-a.s. Indeed, σ is the same for both, and it is clear that infinitely many $\beta_j \neq 0$ implies, P-a.s., infinitely many $\beta_j \neq 0$ with corresponding $t_j \in I$. Consequently, by what was already shown, the conditional process with probability 1 has no fixed sojourn discontinuities.

Now fix n and let $I(k) = (\frac{k-1}{n}, \frac{k}{n})$. Setting $I = I(k)$, the sojourn processes $S_{I(j)}(x)$, $1 \le j \neq k \le n$, are all given along with the process $(X_t, t \notin I_k)$. Hence their discontinuities are also given, and at most denumerable in number. Hence it follows that, P-a.s. for the conditional distribution of $S_{I(k)}(x)$, there are no discontinuities at any values of x where $S_{I(j)}(x)$ is discontinuous for some $j \neq k$. It is easy to see that the event that any two $S_{I(j)}(x)$ and $S_{I(k)}(x)$ have a discontinuity at the same x is measurable; indeed, it is $\bigcup_N\bigcap_m$ {some $S_{I(j)}$ and $S_{I(k)}$ both have a discontinuity of size at least N^{-1} in some interval $((l-1)m^{-1},\ lm^{-1}), -\infty < l < \infty\}$. Consequently, since the conditional probability of any jump shared by $S_{I(k)}$ is 0, this is a null event. On the other hand, $S(x) = \sum_{j=1}^{n} S_{I(j)}(x)$, and each $S_{I(j)}(x)$ can have jumps of size at most n^{-1}. Since they have no jumps in common, $S(x)$ also has jump size limited by n^{-1}. Letting $n \to \infty$, the absence of mobile discontinuities is proved.

We can now state and prove

Theorem 1.3(a). *Let X_t have exchangeable increments on $D[0,1]$, $X_0 = 0$, and let $Y_t = X_t - tX_1$. Then $S(0,Y)$ is uniformly distributed on (0, 1) if and only if, in a representation (1.1) for $X, P\{\sigma = 0$ and there are only finitely many $\beta_j \neq 0$ and $0 = \sum_{finite} \beta_j\} = 0$, where $\sum_{null} \beta_j \doteq 0$.*

Proof. To obtain a representation (1.1) of Y_t, we simply set $\alpha = 0$ in that of X_t. Hence by Lemma 1.2.(b) the condition is necessary and sufficient for Y_t to have continuous sojourn distribution. Now assuming this condition, let U be adjoined to Ω as a uniformly distributed random variable entirely independent of X (by product space construction, for example). We claim that $Y((t+U)\mathrm{mod}1) - Y(U)$, $0 \le t < 1$, has the same law as $Y(t)$ on $D[0,\ 1]$. Indeed, this is true even if U is given, say $U = t_0$. To prove this it suffices to take $t_0 = k/n$, since by right-continuity of path we can obtain the general case from this by letting $\frac{k}{n} \to t_0+$. Similarly, by writing $\frac{k}{n} = \frac{Nk}{Nm}$ for large N, we see that it is enough to prove that the joint law of $Y(\frac{j}{n})$, $1 \le j \le n$, is the same as that of $Y(\frac{(k+j)}{n} \text{ mod} 1) - Y(\frac{k}{n})$, $1 \le j \le n$. Now the map $\sigma : Y(\frac{j}{n}) \to Y(\frac{(k+j)}{n} \text{ mod} 1)$, $1 \le j \le n$, induces a map of increments $Y(\frac{j}{n}) - Y(\frac{(j-1)}{n})$, $1 \le j \le n$, which is in fact a cyclic permutation. Since Y_t has exchangeable increments, this map preserves the joint distribution of the increments. On the other hand, each $Y(\frac{j}{n})$ is a sum from 1 to j of these increments, and, thanks to the fact that $Y(0) = Y(1) = 0$, $Y(\frac{(k+j)}{n}\text{mod} 1) - Y(\frac{k}{n})$ is simply the corresponding sum of the images of these increments by σ. It follows just as in the proof of Lemma 1.2 (a) that the joint law of $Y(\frac{j}{n})$ is also preserved, proving the sufficiency of the condition.

Conversely, suppose that $P\{\sigma = 0 \text{ and } 0 = \sum_{finite} \beta_j\} > 0$, so that there is positive probability that Y_t is a random step function $\sum_{j=1}^{n} \beta_j 1(t - t_j)$, or identically 0. Let us order the β_j so that $\beta_1 \le \beta_2 \le \cdots \le \beta_n$. Clearly the new t_j remain independent and uniform on (0, 1). But then, on the further subset $\{t_1 < t_2 < \cdots < t_n\}$, which has non-zero probability, we have $Y_t \le 0$ for all t. Indeed, either $\beta_1 = \beta_2 = \cdots = \beta_n = 0$, or $0 > \beta_1$. In the second case, let $\hat{k} = min\{k \le n : \sum_1^k \beta_j > 0\}$, or 0 if the set is empty. Then if $\hat{k} \neq 0$ we have $\beta_{\hat{k}} > 0$, and hence $Y(1) > 0$ since the subsequent β_j's are all at least $\beta_{\hat{k}}$. This contradiction implies that the law of $S(0, Y)$ has an atom at 1, hence it is not uniform, as asserted.

Let us note, finally, the

Theorem 1.3(b). *Suppose that Y in Theorem 1.3(a) has unique location of supremum. Then AM(Y) is uniformly distributed on (0, 1).*

Proof. This follows from Lemma 1.1(b) in the same way as Theorem 1.3(a) followed from Lemma 1.1(a).

We turn now to the extension of Sparre-Andersen's result, mentioned in the introduction.

Theorem 1.4*. *Let X_t have exchangeable increments on $D[0,1]$, $X_0 = 0$, and let $LAMX \doteq \inf\{t : X_{t-} \vee X_t = \max_t\{X_{t-} \vee X_t\}\}$, i.e. the first location of the maximum. Then LAMX has the same law as $1 - S(0, X)$.*

Proof. Let $X_n(t) \doteq X(k2^{-n})$, $(k-1)2^{-n} \le t < k2^{-n}$, $1 \le k \le 2^n$, $1 \le n$. Then $X_n(k2^{-n}) = \sum_{j=0}^{k}(X_n(j2^{-n}) - X_n((j-1)2^{-n}))$, for each n is an exchangeable 2^n-tuple of random variables. Hence by Sparre-Andersen's theorem (see Feller, loc. sit., Theorem 2) we have

$$LAMX_n \stackrel{d}{=} 1 - S(0, X_n). \tag{1.3}$$

Let us assume, for the moment, that $S(0-, X) = S(0, X)$ P-a.s., i.e. $E\int_0^1 I_{\{0\}}(X_s)ds = 0$, and that X has an a.s. unique maximum at AMX. Then since $\lim_{n\to\infty} X_n = X$

for all (t, w), and hence $\lim_{n\to\infty} I_{(0,\infty)}(X_n(t)) = I_{(0,\infty)}(X(t))$ on $\{X(t) \neq 0\}$, by dominated convergence we have $\lim_{n\to\infty} S(0, X_n) = S(0, X)$, P-a.s. On the other hand, it is clear that $\lim_{n\to\infty}(X_n(LAMX_n-) \vee X_n(LAMX_n)) = MX$ ($\doteq \max_t(X_{t-} \vee X_t)$), and since AMX is unique it follows that $\lim_{n\to\infty} LAMX_n = AMX$. Hence by (1.3) we obtain the result in this case.

Turning to the general case, we will replace $X(t)$ by $X(\epsilon, \delta, t) \doteq X(t) + \epsilon B(t) - \delta t$, where $B(t)$ is an independent Brownian motion adjoined to the probability space (product construction if necessary) and ϵ, δ are positive constants. This process has exchangeable increments, and it will be seen to satisfy our two extra assumptions. Indeed, by Lemma 1.2 it has continuous sojourn distribution. As to the uniqueness of $AM(X(\epsilon, \delta, t))$, if not there would be rationals $0 < r_1 < r_2 < 1$ with $P\{\sup_{t\le r_1} X(\epsilon, \delta, t) = \sup_{t\ge r_2} X(\epsilon, \delta, t)\} > 0$. Denoting these suprema by M_1 and M_2, respectively, let us suppose given $X(t)$, $t \le 1$, along with $B(t)$, $t \le r_1$, and also $B(t) - B(r_2)$, $r_2 < t \le 1$. Then M_1 is given, and $M_2 = X(\epsilon, \delta, r_1) + X(r_2) - X(r_1) - \delta(r_2 - r_1) + \sup_{t\ge r_2} X(\epsilon, \delta, t) - X(\epsilon, \delta, r_2) + \epsilon(B(r_2) - B(r_1))$, where all terms on the right are given except the last, which is independent. Clearly the conditional probability is 0 that $M_1 = M_2$, hence the same holds unconditionally. Thus, by the previous argument, we have

$$AMX(\epsilon, \delta, \cdot) \stackrel{d}{=} 1 - S(0, X(\epsilon, \delta, \cdot)). \tag{1.4}$$

We now pick a sequence $\epsilon_k \to 0$, $\delta_k \to 0$, such that $\epsilon_k \delta_k^{-1} \to 0$. Then denoting $X_k \doteq X(\epsilon_k, \delta_k, \cdot)$ we have, since $X_k \to X$ uniformly on $[0,1]$,

$$\begin{aligned} &\lim_{k\to\infty} |X(AMX_k) - X(LAMX)| \le \\ &\le \lim_{k\to\infty} |X_k(AMX_k) - X(LAMX)| + \lim_{k\to\infty} |X(AMX_k) - X(AMX_k)| \\ &= 0, \quad \text{P-a.s.} \end{aligned} \tag{1.5}$$

On the other hand, we note that

$$\begin{aligned} &X_k(t) - X(t) - (\epsilon_k B(LAMX) - \delta_k(LAMX)) \\ &= \epsilon_k(B(t) - B(LAMX)) - \delta_k(t - LAMX) \\ &< -2\epsilon_k(\max_t |B|) \end{aligned}$$

provided that $t > LAMX + 4\epsilon_k\delta_k^{-1}\max_t |B|$. For such t, then, we have

$$\begin{aligned} X_k(t) &< X(t) - \epsilon_k \max_t |B| - \delta_k(LAMX) \\ &< X(LAMX) - \epsilon_k \max_t |B| - \delta_k(LAMX). \end{aligned}$$

But for $t = AMX_k$, we have

$$\begin{aligned} X_k(AMX_k) &\ge X_k(LAMX) \\ &\ge X(LAMX) - \epsilon_k \max_t |B| - \delta_k(LAMX), \end{aligned}$$

whence it follows that

$$AMX_k \le LAMX + 4\epsilon_k\delta_k^{-1} \max_t |B|. \tag{1.6}$$

Since $\epsilon_k\delta_k^{-1} \to 0$, it follows from (1.5) and (1.6) that $\lim_{k\to\infty} AMX_k = LAMX$, P-a.s.

Finally, since we have $X_k < X$ uniformly in $\frac{1}{n} \le t \le 1$ for large $k > K(n, w)$, while $X_k \to X$ uniformly in t, we see that $\lim_{k\to\infty} S(0, X_k) = S(0, X)$, and by (1.4) the proof of Theorem 1.4* is complete.

We have next the following

Corollary 1.4*. *With X as before, $S(x, X)$ is continuous at $x = 0$, P-a.s., if and only if X has unique location of the supremum.*

Proof. Let us set $RAM(X) = \max\{s : X(s-) \vee X(s) = \sup_t X\}$. We introduce the process $X_{(1-t)-} - X_1$, which has exchangeable increments and starts at 0. Moreover, this process is identical in law to $-X_t$, $0 \le t \le 1$. To see this, we can apply the mapping $t \leftrightarrow 1-t$ on $[0,1]$, and note that this permutes the increments of X_t into those $X_{(1-t)} - X_1$ but reverses order of the endpoints. Multiplying by -1 gives the result. Now we have

$$\begin{aligned} RAM(X) &= 1 - LAM(X_{(1-t)-} - X_1) \\ &\overset{d}{=} S(0, X_{(1-t)-} - X_1) \\ &\overset{d}{=} S(0, -X) \\ &= 1 - S(0-, X). \end{aligned}$$

Thus we have, by Theorem 1.4 again,

$$\begin{aligned} E(S(0,X) - S(0-,X)) &= E(1 - S(0-,X) - (1 - S(0,X))) \\ &= E(RAMX - LAMX), \end{aligned}$$

which completes the proof.

In the special case that X is a bridge (i.e. $\alpha \equiv 0$ in (1.1)), we can complete Theorem 1.3 as follows.

Theorem 1.5*. *For an exchangeable bridge the following are equivalent (we omit P-a.s. in (b)-(d)).*

(a) *$S(0, X)$ is uniformly distributed.*
(b) *$S(x, X)$ is continuous.*
(c) *$S(x, X)$ is continuous at* 0.
(d) *X has a unique supremum (at AMX).*
(e) *$LAMX$ is uniformly distributed.*

Proof. The proof of the converse in Theorem 1.3(a) shows that if (a) fails then so does (c), while Lemma 1.2(b) in conjunction with Theorem 1.3(a) show that (a) and (b) are equivalent. Since (b) trivially implies (c), (a)-(c) are equivalent. Now (c) and (d) are equivalent by Corollary 1.4*, and (d) implies (e) by Theorem 1.3(b), since under (d) we have LAM $X = AMX$. Finally, (e) is equivalent to (a)-(c) by Theorem 1.4*, so (e) imples (d) and the argument is complete.

Section 2. The case of Lévy processes and Lévy bridges.

We first specialize to the case that X_t is a measurable process with homogeneous, independent increments, also called a Lévy process. Then its log characteristic function may be written in P. Lévy's form as

$$\begin{aligned}\log E\ exp(iuX_t) &\doteq t\psi(u) \\ &= t[iu\gamma - \sigma^2\frac{u^2}{2} + \int(e^{iux} - 1 - \frac{iux}{1+x^2})G(dx)],\end{aligned} \tag{2.1}$$

where the Lévy measure $G(dx)$ satisfies $G\{0\} = 0$ and $\int(1 \wedge x^2)G(dx) < \infty$. Conversely, any real γ, $\sigma^2 \geq 0$, and such G, determine a unique Lévy process as coordinate process on $D[0, 1]$, and this process determines them uniquely. (see M. Loeve, [9; Sec. 22 C and Sec 23, Ex. 9]).

Such a process X_t has exchangeable increments as in Section 1. We have indeed

Theorem 2.1(a). *The necessary and sufficient condition, with X_t as in (2.1), to have $S(0,Y)$ uniform on (0, 1) is that either $\sigma \neq 0$ or $G(R) = \infty$.*

Proof. It is clear that σ in (2.1) equals the σ of (1.1), which is therefore constant in the present situation. Thus by Theorem 1.3, $\sigma \neq 0$ suffices for $S(0,Y)$ to be uniform. Similarly, by writing X_t as the sum of a compound Poisson process with intensity (Lévy) measure $I_{(|X|>\epsilon)}(x)G(dx)$, and an independent process, for $\epsilon > 0$, we see that $\int_{|X|>\epsilon} G(dx)$ is a lower bound for the Poisson intensity of the jumps. Thus if $G(R) = \infty$ there are infinitely many jumps, so P-a.s. infinitely many $\beta_j \neq 0$ occur, and $S(0,Y)$ is uniform by Theorem 1.3(a).

Conversely, if $\sigma = 0$ and $G(R) < \infty$, then the process X_t is simply a compound Poisson process (possibly of rate 0) plus a uniform translation at rate $(\gamma - \int \frac{x}{1+x^2} dG)$. By Theorem 1.3(a), $S(0,Y)$ is not uniform unless $P\left\{0 = \sum_{finite} \beta_j\right\} = 0$. But, with probability $\exp(-G(R))$, this is an empty sum. Hence, $S(0,Y)$ has an atom at 1 and cannot be uniform, completing the proof.

Turning to the law of $AM(Y)$, we have

Lemma 2.1. *(a) If $G(R) < \infty$ and $\sigma = 0$, then Y does not have a unique supremum.*
(b) If $G(R) = \infty$ or $\sigma \neq 0$, then Y has a unique supremum.

*Remark**. This also follows from Corollary 1.4*.

Proof. (a) This is clear since X is a compound Poisson process, and Y vanishes identically with positive probability.

(b) (We are indebted to Bruce Hajek for the following argument). If $AM(Y(w))$ is not unique, then there are rational intervals (r_1, r_2) and (r_3, r_4), $r_2 < r_3$, such that $S_1 \doteq \sup_{r_1<s<r_2} Y_s = \sup_{r_3<s<r_4} Y_s \doteq S_2$, so it is enough to prove this has probability 0. We consider the two sides as a function of $X(r_3) - X(r_2)$, with X_t, $t \leq r_2$, and $X_t - X_{r_3}$, $t > r_3$, as given. Then $S_1 = \sup_{r_1<s<r_2} [X_s - s(X(r_2)+X(1)-X(r_3)) - s(X(r_3)-X(r_2))]$, where the first two terms on the right are given while the last is independent and has a continuous distribution. Analogously, $S_2 = \sup_{r_3<s<r_4} [(X_s - X_{r_3} + X_{r_2}) - s(X(r_2) + X(1)-X(r_3)) + (1-s)(X(r_3)-X(r_2))]$. We see that S_1 is decreasing in $X(r_3)-X(r_2)$, while S_2 is increasing. Consequently there is conditional probability 0 that they are equal, so the result is proved.

We now have immediately, by Lemma 1.1.(b),

Theorem 2.1.(b). *$AM(Y)$ is uniformly distributed if and only if $G(R) = \infty$ or $\sigma \neq 0$.*

In his original work, P. Lévy showed that if B_t is Brownian motion then the process $Y_t = B_t - tB_1$ is independent of B_1, hence we are justified in considering the law of Y_t as that of B_t conditioned by $\{B_1 = 0\}$ (or, indeed, that of $B_t - t\alpha$ conditioned by $\{B_1 = \alpha\}$, for any α). When we seek to generalize the idea of this conditional process to a general Lévy process, the first obstacle is that X_1 is not in general independent of Y_t. Indeed, we have

Theorem 2.2. *The random variable $X_{\frac{1}{2}} - \frac{1}{2}$ is independent of X_1 if and only if $G \equiv 0$, i.e. the process is Gaussian.*

Proof. If $G \equiv 0$, the process has the form $X_t = \gamma t + \sigma B_t$ for a Brownian motion B_t (or else $\sigma = 0$). Then $X_t - tX_1 = \sigma(B_t - tB_1)$, and the asserted independence follows by Lévy's result.

Conversely, suppose for <u>fixed</u> $t < 1$ that $X_t - tX_1$ and X_1 are independent (we will later take $t = \frac{1}{2}$). Then from (2.1) we have

$$\begin{aligned} &\log E \exp i(\alpha(X_t - tX_1) + \beta X_1) \\ &= \log E \exp i\alpha((1-t)X_t + t(X_t - X_1)) + \psi(\beta) \\ &= t\psi(\alpha(1-t)) + (1-t)\psi(-\alpha t) + \psi(\beta). \end{aligned} \tag{2.2}$$

On the other hand, (2.2) can also be expressed as

$$\begin{aligned} &\log E \exp i[(\beta + \alpha(1-t))X_t + (\beta - \alpha t)(X_1 - X_t)] \\ &= t\psi(\beta + \alpha(1-t)) + (1-t)\psi(\beta - \alpha t) \end{aligned} \tag{2.3}$$

Now specializing to $t = \frac{1}{2}$, setting (2.2) = (2.3) gives us

$$\frac{1}{2}\psi(\frac{\alpha}{2}) + \frac{1}{2}\psi(-\frac{\alpha}{2}) + \psi(\beta) = \frac{1}{2}\psi(\beta + \frac{\alpha}{2}) + \frac{1}{2}\psi(\beta - \frac{\alpha}{2}). \tag{2.4}$$

We will specialize further in two ways. First, taking $\alpha = \beta$ gives $\psi(\frac{-\alpha}{2}) + 2\psi(\alpha) = \psi(\frac{3}{2}\alpha)$, which implies that

$$|\exp\psi(\frac{-\alpha}{2})\exp 2\psi(\alpha)|^2 = |\exp(\frac{3}{2}\alpha)|^2, \tag{2.5}$$

where $|\exp\psi(u)|^2$ is the (symmetrized) characteristic function of $X_1 - X_1'$, with X_1' and X_1 i.i.d. Letting Y_i, $i = 1,2,3$, denote 3 independent such variables, this implies the identity in law

$$Y_1 + 2(Y_2 + Y_3) \stackrel{d}{=} 3Y_1. \tag{2.6}$$

Since we already know that this holds when X_1 is Gaussian, we assume $\sigma = \gamma = 0$, and writing $G^*(dx) = G(dx) + G(-dx)$ for the Lévy measure of Y_1 we obtain from (2.6) by the uniqueness of the Lévy representation

$$G^*(a,b] + 2G^*(\frac{a}{2},\frac{b}{2}] = G^*(\frac{a}{3},\frac{b}{3}]; \quad 0 < a < b. \tag{2.7}$$

This equation does have solutions–for example $dG^*_{(x)} = c|x|^{-(1+ln2/ln3)}\,dx$, yielding a symmetric stable process. However, we return to (2.4) and now set $\frac{\alpha}{2} = \beta$. Then, in the same way as (2.5)–(2.7) we obtain

$$3\psi(\frac{\alpha}{2}) + \psi(\frac{-\alpha}{2}) = \psi(3\frac{\alpha}{2}), \tag{2.8}$$

and with Y_i, $1 \le i \le 4$, as before this yields $\sum_{i=1}^{4} Y_i \stackrel{d}{=} 3Y_1$, and hence

$$G^*(a,b] = 4^{-1}G^*(\frac{a}{3},\frac{b}{3}]; \quad 0 < a < b. \tag{2.9}$$

Combining (2.7) and (2.9) gives

$$G^*(a,b] = \frac{2}{3}G^*(\frac{a}{2},\frac{b}{2}]. \tag{2.10}$$

By iteration, (2.9) and (2.10) give respectively

$$\begin{aligned} G^*(a,\infty) &= 4^{-n}G^*(3^{-n}a,\infty) \\ &= (\frac{2}{3})^m G^*(2^{-m}a,\infty), \quad \text{for all integers} \quad n,m > 0. \end{aligned} \tag{2.11}$$

Since $G^*(a,\infty)$ is monotone, this requires that $3^{-n} < 2^{-m}$ hold whenever $4^{-n} < (\frac{2}{3})^m$, unless $G^* \equiv 0$. For $n = 2$, $m = 4$ this breaks down, so we have $G^* \equiv 0$, and hence $G \equiv 0$ as was to be shown.

In view of Theorem 2.2, one cannot make any obvious sense out of $X_.$ conditioned by $X_1 = 0$ without supplementary hypothesis, even if $P\{-\epsilon < X_1 < \epsilon\} > 0$ for every ϵ. On the other hand, if $P\{X_1 = 0\} > 0$, there is no difficulty. We therefore treat this case first.

Theorem 2.3(a). *If X_t is a Lévy process and $P(X_1 = 0) > 0$, we define the "Lévy bridge" X_t^o to be (any) process with paths in $D[0,1]$ having the conditional law of $(X_t | X_1 = 0)$. Then X^o has exchangeable increments, and its positive sojourn $S(0, X^o)$ is uniformly distributed if and only if, in the representation (2.1), $\gamma - \int \frac{x}{1+x^2} dG \neq 0$ (we note that $P(X_1 = 0) > 0$ already implies $\sigma = 0$ and $G(R) < \infty$, so we need a compound Poisson process with non-zero drift; conversely, a compound Poisson process with zero drift satisfies $P(X_1 = 0) > 0$ but $S(0, X^o)$ is not uniformly distributed)*

Proof. It is clear that X^o has exchangeable increments and $X_1^o = 0$. Hence, in the notation of Theorem 1.3, $X_t^o = X_t^0 - tX_t^o = Y_t^o$, and the condition that $S(0, X^o)$ be uniform is $P\{0 = \sum \beta_j\} = 0$. However, by the definition of X^o we have $\gamma + (\sum \beta_j) - \int \frac{x}{1+x^2} G(dx) = 0$, i.e. $\sum \beta_j = \gamma - \int \frac{x}{1+x^2} G(dx)$ as asserted.

The corresponding result about $AM(X^o)$ is

Theorem 2.3(b). *If X_t is a Lévy process and $P\{X_1 = 0\} > 0$, then $AM(X^o)$ is unique and uniformly distributed on (0, 1) if and only if, in (2.1), $\gamma - \int \frac{x}{1+x^2} G(dx) \neq 0$.*

*Remark**. This also follows, of course, by Corollary 1.4*.

Proof. We need only show that this condition is necessary and sufficient for X^o to have unique location of supremum. Since $\sigma = 0$ and $G(R) < \infty$, we have a compound Poisson process with drift. If the drift is 0, the process will vanish identically with positive probability. Hence the condition is clearly necessary. Conversely, if the drift is non- zero, the paths of the process have the form of a step function plus a fixed line lt, $l \neq 0$. Suppose that the number of steps, say n, is given along with the ordered sizes of the jumps say $s_1 \leq s_2 \leq \cdots \leq s_n$. Then the corresponding times $t_1, \ldots, t_n$ of the jumps are independent and uniform on (0, 1). Clearly any point t at which $X_{t-}^o \vee X_t^o = \sup_s X_s^o$ must be in the set $\{0, t_1, \ldots, t_n, 1\}$. Indeed, it is not hard to see that 0 and 1 are excluded, i.e. $\sup_s X_s^0 > 0$ (note that there must be at least one jump). The right and left limit values at t_k are $\left(\sum_{t_j \leq t_k} s_j\right) - lt_k$ and $\left(\sum_{t_j < t_k} s_j\right) - lt_k$, respectively. These sums are all in a fixed finite set, while the lt_k are uniform and independent. Clearly no two coincide.

We turn to defining X^o when $P(X_1 = 0) = 0$. It is not hard to recognize that if $G(R) < \infty$ and $\sigma = 0$ the definition may be quite problematical; indeed, even for G concentrated at 2 points and $P\{-\epsilon < X_1 < \epsilon\} > 0$ for every $\epsilon > 0$, conditioning by $\{-\epsilon < X_1 < \epsilon\}$ as $\epsilon \to 0$ may lead to a process with infinitely many jumps of size bounded away from 0, in such a way that a limit process does not exist. Accordingly, we consider only the case that either $\sigma > 0$ or $G(R) = \infty$. Everything works smoothly if, when $\sigma = 0$, we strengthen $G(R) = \infty$ to Hypothesis (C) of Kallenberg [7], namely

Hypothesis (C). *For $t > 0$, $\int_{-\infty}^{\infty} \exp(t\psi(u))du$ exists, i.e. the Fourier transform of the law of X_t is in L_1.*

This implies L_2, hence the law of X_t has a density and $P\{X_1 = 0\} = 0$. It always holds if $\sigma > 0$, and, as argued in [7, §5], it is only "slightly" stronger than $G(R) = \infty$. Setting $\nu(u) \doteq \sigma^2 + \int_{-u}^{u} x^2 dG(x)$, indeed, Hypothesis (C) holds whenever $\lim_{u\to 0} u^{-2}|\log u|^{-1}\nu(u) = \infty$, whereas $\sigma > 0$ or $G(R) = \infty$ holds if $\limsup_{u\to 0} u^{-2}\nu(u) = \infty$, and $G(R) < \infty$ if $\lim_{u\to 0} u^{-2}|\log u|^r \nu(u) = 0$ for some $r > 1$ (see [7, §5]). Let us verify an assertion of [7], as

Lemma 2.4. *Under Hypothesis (C), the law of X_t has a density $p(t,x)$ continuous in (t,x) for $t \geq \epsilon > 0$.*

Proof. Let $f_t(u) \doteq \exp(t\psi(u))$ denote the Fourier transform. It is well-known (see [9, 12.1, Corollary]) that for each $t > 0$, there exists the continuous inverse transform $p(t,x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-uix} f_t(u)du$, which is a density for the law of X_t. Now we have $|f_t(u)| = \exp -t(\frac{\sigma^2 u^2}{2} + \int(1-\cos ux)dG(x))$, which is integrable in u by hypothesis, and monotone, continuous in t. Since we have

$$|p(t,x_2) - p(t,x_1)| \leq (2\pi)^{-1} \int |1 - \exp -u(x_2 - x_1)||f_t(u)|du,$$

with integrand dominated by $2|f_\epsilon(u)|$ for $t \geq \epsilon$, we see by dominated convergence that $p(t,x)$ is uniformly continuous in x, uniformly in $t \geq \epsilon$. On the other hand, for $\epsilon \leq t_1 < t_2$ we have

$$|p(t_2,x) - p(t_1,x)| \leq (2\pi)^{-1} \int |f_{t_2}(u) - f_{t_1}(u)|du,$$

which tends to 0 as $t_2 \to t_1$ uniformly in x. Hence Lemma 2.4 follows.

We now introduce the law of X_t given $X_1 = x$. Let $\mathcal{F}_s \doteq \sigma(X_u, u \leq s)$, augmented by all P–nullsets.

Definition 2.5. For x such that $p(1,x) > 0$, and $A \in \mathcal{F}_s$, $s < 1$, we set $P(A|X_1 = x) = E(p(1-s, x-X_s); A)p^{-1}(1,x)$.

Since, by Lemma 2.4, we have $\lim_{\epsilon\to 0+} (2\epsilon)^{-1} \int_{x-\epsilon}^{x+\epsilon} p(1-s, y-X_s)dy = p(1-s,\ x-X_s)$ uniformly on the probability space, it is easy to see that, for every $A \in \mathcal{F}_s$, we have

$$P(A|X_1 = x) = \lim_{\epsilon\to 0+} P(A|x-\epsilon < X_1 < x+\epsilon). \tag{2.12}$$

Indeed, by a Theorem of Vitali, Hahn, and Saks the convergence is uniform in A, and the limit is (clearly) a probability on $\mathcal{F}_s$, consistent in s for $s < 1$ by Chapman-Kolmogorov equation.

We next specialize to $x = 0$ in

Lemma 2.6. *If $p(1,0) > 0$, let $P^o(A) = P(A|X_1 = 0)$, $A \in \mathcal{F}_s$, $s < 1$. Then $P^o(A)$ extends from $\vee_{s<1}\mathcal{F}_s$ to $\mathcal{F}_{1-}$ uniquely, and $P^o\{X_{1-} = 0\} = 1$. Under $P^o(A)$, the* *process (set = 0 at t = 1) has paths in* $D[0,1]$, P^o*–a.s., and it is an inhomogeneous Markov process with transition density*

$$p^o(t_1, x; t_2, y) = p(t_2 - t_1, y - x)p(1 - t_2, -y)p^{-1}(1 - t_1, -x), \; 0 \le t_1 < t_2 < 1,$$

where we interpret $\frac{0}{0} = 0$

Remark. This Lemma is also proved, by somewhat different methods, in [4]. The hypotheses of [4] are considerably more general than ours, and the reader may refer to [4] for more details. However, we still need Hypothesis (C) below for Theorems 2.7 and 2.8, because of its consequence (2.12), which does not hold under the general hypothesis of [4].

Another Remark. After completion of this paper, we received the manuscript [15] of Fitzsimmons and Getoor, which proves Theorem 2.7 below in the setting of [4] restricted to Lévy processes. We then observed that our method also may be extended to that case. We have only to replace our uses of (2.12), which is employed to show that for $A \in \mathcal{F}_{1-\delta}$ with $P(A) = 1$ one has also $P^o(A) = 1$, by Definition 2.5, which gives the same implication.

Proof. The marginal density of X_t for P^o is $p(t,x)p(1-t,-x)p^{-1}(1,0)$ for $t < 1$. Indeed, this follows from the definition of P^o by taking $A = \{X_t \le x\}$ and differentiating in x (with $s = t$). From this, a routine Markov property of X_t, verifies the last assertion for $t \le 1 - \epsilon$, $\epsilon > 0$. It remains to examine the behavior at $t = 1$. Noting that the marginal densities are invariant under the transformation $t \leftrightarrow 1 - t$ and $x \leftrightarrow -x$, it follows by routine but rather tedious computation using the time-reversed transition density, that under P^o the processes X_t and $-X_{(1-t)-}$ have the same joint law for $\epsilon \le t \le 1-\epsilon$, $\epsilon > 0$. Since the law of X_t under P_o is also well-defined for $0 \le t \le \epsilon$, with $X_{0+} = 0$, we can consistently define its law for $1-\epsilon \le t \le 1$ as that of $-X_{(1-t)-}$, $1-\epsilon \le t \le 1$. Indeed, this gives a consistent family of joint distribution functions on $0 \le t \le 1$, with $X_1 = 0$. The fact that $X_{0+} = 0$, and the equivalence in law $X_t \leftrightarrow -X_{(1-t)-}$ assures us that $\lim_{t\to 1} X_t = 0$, P–a.s., so the Kolmogorov extension of the joint law is carried by D [0, 1], with $X_{1-} = X_1 = 0$. This completes the definition of P^o, and finishes the Lemma.

Theorem 2.7. *Assuming Hypothesis (C) and* $p(1,0) > 0$, *let* $X^o(t)$ *denote any realization of the process* X_t *for* P^o *of Lemma 2.6, having paths in D [0, 1]. We call* $X^o(t)$ *a "Lévy bridge" (from 0 to 0) corresponding to the Lévy process* X_t. *The positive sojourn* $S(0, X^o)$ *is uniformly distributed on (0, 1).*

Proof. For $\epsilon > 0$, the conditional law of X_t given $\{-\epsilon < X_1 < \epsilon\}$ is that of a process with exchangeable increments, since X_1 is invariant under permutation of increments. Letting $\epsilon \to 0+$, and using the fact that $P^o(A) = \lim_{\epsilon\to 0+} P(A|-\epsilon < X_1 < \epsilon)$ (as in (2.12)), we see that X^o has exchangeable increments. Let S = {elements in $D[0,1]$ having infinitely many jumps}. Then $\{X_{(\cdot)} \in S\} \in \mathcal{F}_1$: indeed it suffices that $\{X_{(\cdot)} \notin S\} \in \mathcal{F}_1$, and this can be written $\bigcup_N \limsup_{n\to\infty}\{X_{(\cdot)}$ is continuous in $((k-1)n^{-1}, (k+1)n^{-1})$ for all but at most N values of $k \le n\}$, so it suffices that $\{X_{(.)}$ is continuous in $((k-1)n^{-1},(k+1)n^{-1})\} \in \mathcal{F}_1$, which is clear. Now either $P\{X_{(.)} \in S\} = 0$ or $P\{X_{(.)} \in S\} = 1$, and either probability is unchanged for $P\{X_{(.)} \in S_\delta | -\epsilon < X_t < \epsilon\}$ where $S_\delta \doteq \{$infinitely many jumps in $0 \le t < 1-\delta\}$, $\delta < 1$. Letting $A_\delta = \{X_{(.)} \in S_\delta\}$, and letting $\epsilon \to 0$, it follows that this probability is also the same for P^o. Thus, if $P\{X_{(.)} \in S\} = 1$, then, noting that $X^o(t) = X^o(t) - tX^o(1) = Y^o(t)$, $S(0, X^o)$ is uniform by Theorem 1.3. On the other hand, if $P\{X_{(.)} \in S\} = 0$, then $\sigma > 0$ by Hypothesis (C) (since $G(R) < \infty$). Let $\hat{S} = \{$elements in D [0, 1] having infinite variation$\} \cap S^c$. Then it is not hard to see that, again, $\{X_{(.)} \in \hat{S}\} \in \mathcal{F}_1$. Since $P\{X_{(.)} \in \hat{S}\} = 1$, it follows as before that $P^o\{X^o_{(.)} \in \hat{S}\} = 1$. Therefore, in a representation (1.1) of X^o_t, since there are only finitely many $\beta_j \neq 0$, we must have $\sigma > 0$, P–a.s. Again by Theorem 1.3, $S(0, X^o)$ is uniformly distributed on (0, 1), as asserted.

We conclude with the result about $AM(X^o)$.

Theorem 2.8. *Assuming Hypothesis (C) and* $p(1,0) > 0$, $AM(X^o)$ *is uniformly distributed on* $(0,1)$

Proof. We need only show the uniqueness of supremum (or apply Corollary 1.4*). Let $S_\delta = \{$paths in $D[0, 1-\delta]$ having a unique supremum$\}$. Then S_δ is measurable, and it follows as in Lemma 2.1(b) (somewhat simplified) that $P(X_{(.)} \in S_\delta) = 1$. Thus $P(X^o_{(.)} \in S_\delta) = \lim_{\epsilon \to 0} P(X_{(.)} \in S_\delta | -\epsilon < X(1) < \epsilon) = 1$, and the proof is concluded by letting $\delta \to 0$.

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