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A PROBLEM ON THE II_1-FACTORS OF FUCHSIAN GROUPS

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Unexplained notations are as in the previous paper [Har].

Let $G = PSL_2(\mathbb{R})$ be viewed as the group of orientation preserving isometries of $\mathcal{P}$, the Poincaré half-plane (= the connected simply connected complete Riemannian manifold of dimension 2 and constant curvature -1). Consider in G a discrete subgroup Γ (= a Fuchsian group) which is finitely generated and not elementary (namely neither a finite group nor a finite extension of $\mathbb{Z}$). If Γ_1, Γ_2 are two such groups which are isomorphic as abstract groups, then it is known that their covolumes (in the sense defined below) are equal. Very briefly, the question we ask is : does one have $W^*_\lambda(\Gamma) \approx L(F_r)$ for the appropriate r ? Let us recall some classical facts, precise the meaning of "the appropriate", and list cases where the answer to the previous question is known to be positive.

The group Γ is described by the following data : integers $g, q, s, t \geq 0$ and integers $\nu_1, ..., \nu_q \geq 2$, submitted to the unique condition

$$\frac{1}{2\pi} Cov(\Gamma) \doteq 2g - 2 + s + t + \sum_{j=1}^{q} \left(1 - \frac{1}{\nu_j}\right) \ > \ 0 \ .$$

The data are summarized in the *signature* $(g : \nu_1, ..., \nu_q; s, t)$ of the group, written also $(g; s, t)$ if $q = 0$. In case Γ is a lattice in G (a lattice is automatically finitely generated and non elementary), then $t = 0$ and Γ is *completely* described by its signature, up to conjugation by an orientation preserving quasiconformal homeomorphism of $\mathcal{P}$.

Algebraically, the group Γ has a presentation with $2g + q + s + t$ generators

$$a_1, b_1, ..., a_g, b_g, e_1, ..., e_q, p_1, ..., p_s, h_1, ..., h_t$$

and $1 + q$ relations

$$\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} \prod_{j=1}^{q} e_j \prod_{k=1}^{s} p_k \prod_{l=1}^{t} h_l = 1 \ ,$$
$$e_j^{\nu_j} = 1 \quad (1 \leq j \leq q).$$

There are precisely q classes of maximal finite cyclic subgroups of Γ, each of these classes containing a group generated by one of the e_j 's. (In particular, Γ is torsionfree if and only if $q = 0$.) Each parabolic element of Γ is conjugated to exactly one power of exactly one of the p_j 's.

It is easy to check that the conjugacy classes of Γ distinct from $\{1\}$ are all infinite, so that the von Neumann algebra $W^*_\lambda(\Gamma)$ is a II_1-factor. Moreover, it is a full factor (see [Har]).

Geometrically, the group Γ has a limit set $\mathcal{L}_\Gamma$ in the boundary $\partial\mathcal{P}$ of $\mathcal{P}$. (The space $\mathcal{P} \cup \partial\mathcal{P}$ is naturally homeomorphic to a closed 2-disc, and $\partial\mathcal{P} = \mathbb{R} \cup \{\infty\} \approx \mathbb{S}^1$ if $\mathcal{P} = \{z \in \mathbb{C} | Im(z) > 0\}$.) The *Nielsen region* $\overline{N}$ is the hyperbolic convex hull of $\mathcal{L}_\Gamma$ in $\mathcal{P} \cup \partial\mathcal{P}$. The inside part $N = \overline{N} \cap \mathcal{P}$ is a Γ-invariant closed subset of $\mathcal{P}$. The quotient N/Γ is a Riemann surface obtained from a compact surface of genus g with t connected components in its boundary by removing s inner points. Moreover there are distinct points $x_1, ..., x_q$ in the interior of N/Γ such that the covering $N \to N/\Gamma$ has ramification of order ν_j over x_j (and no other ramification point). The number $Cov(\Gamma)$ defined above is then the hyperbolic area of N/Γ, and one has $Cov(\Gamma) = -2\pi\chi(N/\Gamma)$, where χ denotes the appropriate Euler-Poincaré characteristics, in the sense of orbifolds. (This is a consequence of Gauss-Bonet formula, and this explains also why $Cov(\Gamma_1) = Cov(\Gamma_2)$ if Γ_1, Γ_2 are isomorphic as abstract groups.) There is also a purely algebraic way to define the Euler-Poincaré characteristics of the virtually torsion-free group Γ, described in [Ser]. (This is straightforward if G/Γ is not compact, because then Γ contains a free subgroup of finite index; if G/Γ is compact, see n^o 3.2 in [Ser].)

Observe that N/Γ is compact if and only if $s = 0$. Observe also that, for Fuchsian groups which are finitely generated (as discussed here), the four following are equivalent : $N = \mathcal{P}$, the group Γ is a lattice in G (namely the space $\mathcal{P}/\Gamma$ is of finite aerea), the Fuchsian group Γ is "of the first kind" (namely $\mathcal{L}_\Gamma = \partial\mathcal{P}$), $t = 0$ in the signature of Γ.

Here are a few examples of signatures and related groups:

(1) $(g; 0, 0) \Longrightarrow \Gamma \approx \Pi_1(\Sigma_g)$, where Σ_g denotes a closed surface of genus g,

(2) $(g; s, t)$ with $s + t > 0 \Longrightarrow \Gamma$ is isomorphic to the free group $F_{2g+s+t-1}$,

(3) $(0 : 2, \nu; 1, 0)$ with $\nu \geq 3 \Longrightarrow \Gamma \approx (\mathbb{Z}/\nu\mathbb{Z}) \star (\mathbb{Z}/2\mathbb{Z})$ is one of the *Hecke groups*,

(4) $(g : \nu_1, ..., \nu_q; s, t)$ with $s + t > 0 \Longrightarrow \Gamma$ is a free product of cyclic groups $F_n \star (\text{F}\ _{1 \leq j \leq q}(\mathbb{Z}/\nu_j\mathbb{Z}))$, with $n = 2g + s + t - 1$.

Of course, (4) generalizes both (2) and (3). For all this, see e.g. [Bea] and [Gre].

The following is a wild formulation of the problem to understand how far the factors W^*_λ(Fuchsian groups) are from the W^*_λ(free groups). The factors $L(F_r)$ are defined by K. Dykema [Dy1] and F. Radulescu [Rad] for all real numbers r such that $r > 1$ and for $r = \infty$.

Problem. *Let $\Gamma \subset PSL_2(\mathbb{R})$ be a finitely generated Fuchsian group as above, and set $r = 1 + \frac{1}{2\pi}Cov(\Gamma)$. Does one have $W^*_\lambda(\Gamma) \approx L(F_r)$?*

In the torsion-free case (the group is then either free or the fundamental group of a closed surface), the problem can be rephrased as

$$W^*_\lambda(\Pi_1(\Sigma_g)) \overset{?}{\approx} W^*_\lambda(F_{2g-1}) .$$

For the groups of example (4) above with $q = 1$, the conjecture holds by [Voi, Theorem 3.3]. For the Hecke groups of (3), it holds by [Dy2, Corollary 5.3]; more precisely, if $\Gamma \approx (\mathbb{Z}/\nu\mathbb{Z}) \star (\mathbb{Z}/2\mathbb{Z})$, then $W^*_\lambda(\Gamma) \approx W^*_\lambda\left(F_{3/2-1/\nu}\right)$. The general case of (4) holds by [Dy2, Proposition 2.4].

REMARKS. (i) A factor M of type II_1 with trace τ is said to have the *Haagerup Approximation Property* if the identity on the Hilbert space $L^2(M,\tau)$ can be approximated by compact unital trace-preserving completely positive maps. Let Γ be an infinite conjugacy class group, and assume that there exists a function $\psi : \Gamma \to \mathbb{R}$ which is conditionally of negative type and which tends to infinity at infinity. For each integer $n \geq 1$, the function $\phi_n = exp\left(-\frac{1}{n}\psi\right)$ is of positive type by Schoenberg's Theorem, and thus defines a multiplier on $W^*_\lambda(\Gamma)$ which is completely positive (see e.g. [CaH, Proposition 4.2]). It follows that the factor $W^*_\lambda(\Gamma)$ has the Haagerup Approximation Property. Examples of such groups Γ include non abelian free groups [Haa], various groups acting on trees or real trees, infinite Coxeter groups, and discrete subgroups of Lie groups in the families $SO(1,n)$ and $SU(1,n)$; see e.g. [HaV, chapitres 5, 6].

The problem phrased above is a way of testing how much the class of full II_1-factors which have the Haagerup Approximation Propery is larger than the class of free group factors. A similar problem appears in [Pop].

(ii) As covolumes of lattices in G are related to Murray-von Neumann coupling constants [GHJ, Section 3.3.d], there may be an approach to the problem above using the index of appropriate subfactors.

(iii) It is known that a Fuchsian group Γ which is not finitely generated is a free product of an infinite sequence of cyclic groups. For such a group, one has $W^*_\lambda(\Gamma) \approx W^*_\lambda(F_\infty)$ by [Dy2, Corollary 5.4].

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