Astérisque

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Astérisque, tome 228 (1995), p. 19-40
<http://www.numdam.org/item?id=AST_1995__228__19_0>

Trinomial Equations in Function Fields

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I. INTRODUCTION. Siegel [Si] conjectured in 1929 that when a curve defined by the diophantine equation $f(x,y)=0$ with rational integral coefficients is irreducible and of positive genus, then the number of rational integral points in the curve can be bounded in terms of the number of monomials appearing in the polynomial $f(x,y)$. This conjecture is not true as it stands. In fact, one finds in Fermat the formulation and solution of the problem of finding two rational cubes expressible as the sum of two other rational cubes in infinitely many ways. Fermat used the method of the tangent, which goes back to Diophantus, on the cubic elliptic curve $x^3+y^3=h$. By clearing denominators one then obtains values of h for which the equation $x^3+y^3-h=0$ has an arbitrarily large number of integral solutions.

On the other hand, Siegel's conjecture has been verified in some cases. In particular, the binomial Thue equation $ax^r+by^r=1$ has for $r \geqq 5$ at most 2 solutions in rational integers (up to sign, if r is even) [Do], and the Thue

* The second author supported in part by NSF Grant DMS–900-2876

equation $F(x, y) = 1$, where F is a binary form of degree $r \geqq 3$, has at most cr solutions in rational integers, for a computable absolute constant c [B–S]. In fact, $c \leqq 430$ if r is sufficiently large.

A natural question to ask is whether Siegel's conjecture has some validity in function fields. The case of a binomial equation was studied by Mueller [M1], [M2]. Let K/k be a function field of one variable with constant field k, of characteristic 0 and genus g. Then Mueller proved that the equation $ax + by = 1$ where $a, b \in K^*$ and either $a \notin k^*$ or $b \notin k^*$ has at most two solutions $(x, y) \in (K^*)^r \times (K^*)^s$ up to equivalence, provided that either $r = s > 30 + 20g$ (see [M1]) or $r \neq s$ and $\min(r, s) > 120 + 40g$ (see [M2]).

The main result of this paper is the following theorem on trinomial Thue equations. Let $K = k(t)$ be the field of rational functions in one variable t, with constant field k algebraically closed of characteristic 0, let $\mathcal{O} = k[t]$ be the ring of polynomials in t with coefficients in k. By a trinomial Thue equation we mean the equation

$$ax^r + bx^s y^{r-s} + cy^r = h \tag{1.1}$$

with coefficients in $\mathcal{O}$ and $abch \neq 0$, with $0 < s < r$.

We say that a polynomial solution (x, y) of (1.1) is *primitive* if $xy \neq 0$ and the polynomials x, y have no common non-constant factors, and we say that two solutions (x, y) and (x', y') are *proportional* if both x/x' and y/y' are constants in k.

THEOREM 1. *The trinomial Thue equation* (1.1) *has at most* 3 *primitive, non-proportional polynomial solutions provided* $\min(s, r - s, |r - 2s|) > 6072$.

The constant 3 in our theorem is clearly optimal. In fact, we may prescribe in advance three solutions (x_i, y_i), $i = 1, 2, 3$ and solve the linear system $ax_i^r + bx_i^s y_i^{r-s} + cy_i^r - h = 0$, $i = 1, 2, 3$ in the unknowns a, b, , c, h. The same argument also shows that in case $r = 2s$, $a = c$ the number of inequivalent solutions may increase to 4 because of the automorphism $(x, y) \mapsto (y, x)$ of the associated trinomial. We conjecture that Theorem 1 continues to hold provided only that r is sufficiently large and $r \neq 2s$.

The proof of Theorem 1 relies on the well-known *abc*-inequality in function fields, which was used for similar purposes in the two binomial cases [M1], [M2]. It has also been employed by the present authors [B–M] and [M3] to obtain an explicit upper bound on the number of families of solutions of the so-called generalized Fermat equation in function fields, with a bound which depends only on the number of monomials of the equation provided the degree is large enough.

Our approach to treating trinomial equations may be applied to other situations. Consider for example the trinomial equation

$$ax^r + by^s + cz^t = 0 \tag{1.2}$$

with coefficients in $\mathcal{O}$ and $abc \neq 0$. A polynomial solution (x, y, z) is primitive if x, y, z are coprime in pairs and not 0, and two solutions (x, y, z) and (x', y', z') are proportional if x/x', y/y', z/z' are all constants in k.

THEOREM 2. *The trinomial equation* (1.2) *has at most* 2 *primitive, non-proportional solutions provided* $\min(r, s, t) > 60$.

We have stated our results only for the function field $K = k(t)$, rather than for a general function field of genus g, but there is no doubt that they admit such an extension.

In the present state, our method is too clumsy to use for treating much more general situations, since for each equation one is obliged to analyze separately various "degenerate" cases of application of the *abc*-inequality, and the number of such cases increases very rapidly with the number of monomials. Thus it would be of definite interest to find a more conceptual approach, perhaps involving combinatorial ideas and algebraic geometry, to the study of "fewnomial" diophantine equations in function fields and number fields.

II. APPLICATION OF THE *abc*-INEQUALITY. Suppose that the trinomial equation (1.1) has four solutions (x_i, y_i), $i = 1, 2, 3, 4$. We follow an idea of Chowla [C]. From the system of equations

$$ax_i^r + bx_i^s y_i^{r-s} + cy_i^r - h = 0\,, \quad i = 1, \ldots, 4$$

we obtain the relation

$$\det\begin{pmatrix} x_1^r & x_1^s y_1^{r-s} & y_1^r & 1 \\ x_2^r & x_2^s y_2^{r-s} & y_2^r & 1 \\ x_3^r & x_3^s y_3^{r-s} & y_3^r & 1 \\ x_4^r & x_4^s y_4^{r-s} & y_4^r & 1 \end{pmatrix} = 0\,. \tag{2.1}$$

In a similar way, suppose that equation (1.2) has three solutions (x_i, y_i), $i = 1, 2, 3$. Then we obtain

$$\det\begin{pmatrix} x_1^r & y_1^s & z_1^t \\ x_2^r & y_2^s & z_2^t \\ x_3^r & y_3^s & z_3^t \end{pmatrix} = 0\,. \tag{2.2}$$

We expand the determinants by Laplace's rule and write the corresponding equations in the form

$$\sum_{\sigma\in\mathcal{S}} \varepsilon_\sigma m_\sigma = 0 \tag{2.3}$$

where σ runs over the set $\mathcal{S}(4)$ or $\mathcal{S}(3)$, or briefly $\mathcal{S}$, of permutations of $\{1,2,3,4\}$ or $\{1,2,3\}$, where ε_σ is the parity of the permutation and where m_σ denotes the corresponding monomial in the Laplace expansion. By a *block* $\mathcal{B}$ we mean a non-empty subset of $\mathcal{S}$ for which

$$\sum_{\sigma\in\mathcal{B}} \varepsilon_\sigma m_\sigma = 0\,. \tag{2.4}$$

A block $\mathcal{B}$ is *irreducible* if it does not contain a proper subblock.

We shall use the *abc*-inequality of Mason, Voloch, Brownawell and Masser in the precise form obtained in [B–M], Theorem B. Let k and K be as in section I. For every absolute value v of K we denote by $v(x)$ the order of x at v. Suppose $\mathcal{B}$ is an irreducible block, let m_{σ_0} be a monomial originating from this block and let S be the set of places where some rational function m_σ/m_{σ_0}, for some $\sigma \in \mathcal{B}$, is not an unit. We divide (2.4) by m_{σ_0} and deduce from the *abc*-inequality that

$$H(\mathcal{B}) \leqq \frac{1}{2}(|\mathcal{B}| - 1)(|\mathcal{B}| - 2)|S| \tag{2.5}$$

where H is the projective height

$$H(\mathcal{B}) = -\sum_v \min_{\sigma\in\mathcal{B}} v(m_\sigma),$$

where S is the set of places where some m_σ/m_{σ_0} is not an unit, and where $|E|$ denotes the cardinality of the finite set E.

By the sum formula in K, we can rewrite this as

$$H(\mathcal{B}) = \sum_v \bigl(v(m_\sigma) - \min_{\sigma\in\mathcal{B}} v(m_\sigma)\bigr) \tag{2.6}$$

for every $\sigma \in \mathcal{B}$.

For the purpose of transforming (2.5) into a more workable version we define for every v the closed interval

$$I_v(\mathcal{B}) = [\min_{\sigma\in\mathcal{B}} v(m_\sigma), \max_{\sigma\in\mathcal{B}} v(m_\sigma)]$$

and

$$|I_v(\mathcal{B})| = \max_{\sigma\in\mathcal{B}} v(m_\sigma) - \min_{\sigma\in\mathcal{B}} v(m_\sigma).$$

We certainly have

$$|I_v(\mathcal{B})| \leqq \sum_{\sigma\in\mathcal{B}} \bigl(v(m_\sigma) - \min_{\sigma\in\mathcal{B}} v(m_\sigma)\bigr),$$

therefore, summing over v and using (2.5) and (2.6), we obtain

$$\sum_v |I_v(\mathcal{B})| \leqq \frac{1}{2}|\mathcal{B}|(|\mathcal{B}|-1)(|\mathcal{B}|-2)|S(\mathcal{B})| \tag{2.7}$$

where

$$S(\mathcal{B}) = \bigcup\{v \in M_K \mid |I_v(\mathcal{B})| > 0\}.$$

Now we decompose $\mathcal{S}$ into disjoint irreducible blocks $\mathcal{B}_i$, hence

$$\mathcal{S} = \sum_i \mathcal{B}_i. \tag{2.8}$$

In the sequel, all blocks $\mathcal{B}$ will originate from this decomposition, thus $\mathcal{B} = \sum_{i\in I} \mathcal{B}_i$ with $I \subseteq \{1, \ldots, n\}$.

We apply (2.7) to each irreducible block $\mathcal{B}_i$, sum over the blocks noting that

$$\sum_i |\mathcal{B}_i|(|\mathcal{B}_i| - 1)(|\mathcal{B}_i| - 2) \leqq |\mathcal{S}|(|\mathcal{S}| - 1)(|\mathcal{S}| - 2),$$

and obtain a fortiori

$$\sum_v \sum_i |I_v(\mathcal{B}_i)| \leqq \frac{1}{2}|\mathcal{S}|(|\mathcal{S}| - 1)(|\mathcal{S}| - 2)N, \tag{2.9}$$

where N is the number of absolute values v such that $\sum_i |I_v(\mathcal{B}_i)| > 0$.

DEFINITION. *With respect to v, an irreducible block $\mathcal{B}$ is:*

good, if $|I_v(\mathcal{B})| > \frac{1}{2}|\mathcal{S}|(|\mathcal{S}| - 1)(|\mathcal{S}| - 2)$;

neutral, if $|I_v(\mathcal{B})| = 0$;

bad, otherwise.

An absolute value v is:

good, if there is at least one good irreducible block;

neutral, if every irreducible block is neutral;

bad, otherwise.

The corresponding sets of v's are denoted by V_g, V_b, V_n.

First of all, it is clear that in (2.9) we have $N = |V_g| + |V_b|$. We abbreviate $K = |\mathcal{S}|(|\mathcal{S}| - 1)(|\mathcal{S}| - 2)/2$ and deduce from (2.9) that

$$(K + 1)|V_g| \leqq K(|V_g| + |V_b|). \tag{2.10}$$

LEMMA 1. *If there are no bad absolute values then all monomials in an irreducible block $\mathcal{B}$ are proportional.*

PROOF. From (2.10) we see that if $V_b = \emptyset$ then $V_g = \emptyset$ too. Hence every v is neutral, thus $|I_v(\mathcal{B})| = 0$ for every irreducible block. This means that

$v(m_\sigma/m_{\sigma'}) = 0$ for every v and any two $\sigma, \sigma' \in \mathcal{B}$. Hence $m_\sigma/m_{\sigma'}$ is a constant in k, as asserted.

LEMMA 2. *In the case of equation* (1.2), *if all monomials in each irreducible block are proportional then at least two solutions* (x_i, y_i, z_i) *are also proportional.*

PROOF. Let us abbreviate $X_i = x_i^r$, $Y_i = y_i^s$, $Z_i = z_i^t$ and $[ijk] = X_iY_jZ_k$. Suppose first that there is an irreducible block with two elements corresponding to permutations of different parity, for example $[123] - [132] = 0$. Then we get $Y_2Z_3 = Y_3Z_2$ and since Y_i and Z_i are coprime we find $Y_3 = \lambda Y_2$, $Z_3 = \lambda Z_2$ with $\lambda \in k$, and now the equation $aX_i + bY_i + cZ_i = 0$ shows that $X_2 = \lambda X_3$, yielding proportionality. Thus Lemma 2 is true in this case, so we may assume that every block of two elements consists of monomials with the same parity.

By factoring out the greatest common divisor we may assume that the first coordinates are coprime, and so are the second and third coordinates, while by hypothesis (x_i, y_i, z_i) are coprime in pairs. By permuting the variables and the solutions, we remain with the following possible types for $v(x_i)$, $v(y_i)$, $v(z_i)$, numbered I, II, III, IV, V:

TABLE OF TYPES

$$\begin{pmatrix} e & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & g \end{pmatrix} \begin{pmatrix} e & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e & 0 & 0 \\ f & 0 & 0 \\ 0 & g & 0 \end{pmatrix} \begin{pmatrix} e & 0 & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with matrix rows $v(x_i)$, $v(y_i)$, $v(z_i)$, $i = 1, 2, 3$ and where $e > 0$, $f > 0$, $g > 0$.

For types I and II, examination of $v(m_\sigma) = \max$ gives the unique element $[123]$, a contradiction. For types III and IV, we see that the block with $v(m_\sigma) = \min$ is necessarily $[3**]$, which was excluded at the beginning of the proof. For type V, examination of the monomials with $v(m_\sigma) = \max$ gives the block $[1**]$, which again was excluded at the beginning of the proof. This completes the proof of Lemma 2.

LEMMA 3. *In the case of equation* (1.1), *if* $r \neq 2s$ *and all monomials in* *each irreducible block are proportional then at least two solutions* (x_i, y_i) *are also proportional.*

PROOF. We write $[ijkl] = x_i^r x_j^s y_j^{r-s} y_k^r$ and denote proportionality by $\sim$.

We exploit the hypothesis of Lemma 3 as follows. This hypothesis implies that all monomials in an irreducible block have the same order at every v. On the other hand, the order of the elements x_i, y_i, $i = 1, 2, 3, 4$ is restricted by the coprimality condition, so that either $v(x_i)$ or $v(y_i)$ is 0, for every i. Moreover, if we had $v(x_i) > 0$ for every i then we could factor out the greatest common divisor of the x_i's, obtaining a new trinomial equation with four solutions but with at least one of the $v(x_i)$ equal to 0. The same remark of course applies to the y_i's. This leaves us, up to permutations and exchange of x with y and s with $r - s$, with the following seven types of $v(x_i)$'s and $v(y_i)$'s, numbered I, II, III, IV, V, VI, VII:

TABLE OF TYPES

$$\begin{pmatrix} a & 0 \\ b & 0 \\ c & 0 \\ 0 & e \end{pmatrix} \begin{pmatrix} a & 0 \\ b & 0 \\ c & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & 0 \\ 0 & e \\ 0 & f \end{pmatrix} \begin{pmatrix} a & 0 \\ b & 0 \\ 0 & e \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & e \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

with matrix rows $v(x_i)$, $v(y_i)$, $i = 1, 2, 3, 4$ and where $a \geqq b \geqq c > 0$ and $e \geqq f > 0$.

Now for each type we create the *basic table* which groups together monomials with a same absolute value, and note that the hypothesis of Lemma 3 implies that monomials with a same absolute value, taken with the appropriate sign, form a block. The strategy is the following: starting with the largest and smallest absolute values, we form two blocks, which are then removed from the basic table. Usually these two blocks are small, and we can deduce some non-trivial information about the solutions. This information is fed back into the basic table, often producing new small blocks, which are then removed. Then the process starts again with the smaller new basic table, until at the end we reach either proportionality of solutions or a contradiction.

Let us consider type VII. The basic table is

BASIC TABLE, TYPE VII

$\mathcal{U}_1$	$[1 * **]$	ra
$\mathcal{U}_2$	$[*1 * *]$	sa
$\mathcal{U}_3$	$[* * 1*]\ [* * *1]$	0

where $\mathcal{U}_i$ are the corresponding groupings and where at the right we have $v(m_\sigma)$. In what follows, we say that $\mathcal{U}_i + \mathcal{U}_j + \dots$ is a block if the monomials from $\mathcal{U}_i \bigcup \mathcal{U}_j \bigcup \dots$, taken with the appropriate parity sign, form a block.

The block $\mathcal{U}_2$ yields

$$x_1^s y_1^{r-s} \det \begin{pmatrix} x_2^r & y_2^r & 1 \\ x_3^r & y_3^r & 1 \\ x_4^r & y_4^r & 1 \end{pmatrix} = 0 .$$

The condition det $= 0$ is precisely the condition which arises from a binomial Thue equation, which we know has at most two solutions except in degenerate cases, and the conclusion is that we must have either $x_2^r = x_3^r = x_4^r$ or $y_2^r = y_3^r = y_4^r$ or have two proportional solutions (x_i, y_i) and (x_j, y_j), which would be the conclusion of Lemma 3. If we apply the same argument to the block $\mathcal{U}_1$ we obtain the relation

$$x_1^r \det \begin{pmatrix} x_2^s y_2^{r-s} & y_2^r & 1 \\ x_3^s y_3^{r-s} & y_3^r & 1 \\ x_4^s y_4^{r-s} & y_4^r & 1 \end{pmatrix} = 0 ,$$

which combined with the first shows that we may suppose $y_2^r = y_3^r = y_4^r$. Since x_i^r, $x_i^s y_i^{r-s}$, 1, $i = 1, 2, 3, 4$ satisfy a linear relation with non-zero coefficients, we see that x_i^r, $x_i^s y_i^{r-s}$, 1, $i = 2, 3, 4$ lie in a hyperplane in K^3. Therefore we have

$$\det \begin{pmatrix} x_2^r & x_2^s y_2^{r-s} & 1 \\ x_3^r & x_3^s y_3^{r-s} & 1 \\ x_4^r & x_4^s y_4^{r-s} & 1 \end{pmatrix} = 0 .$$

If we set up the basic table for this determinant, we have only two cases to consider besides $x_2^r = x_3^r = x_4^r$, and they all lead to proportionality of solutions. This proves Lemma 3 if type VII occurs.

Type VI is also easy. The basic table is

BASIC TABLE, TYPE VI

$\mathcal{U}_1$	$[1 * 2*]$	$ra + re$
$\mathcal{U}_2$	$[12 * *]$	$ra + (r - s)e$
$\mathcal{U}_3$	$[* 12 *]$	$sa + re$
$\mathcal{U}_4$	$[1 * *2]$	ra
$\mathcal{U}_5$	$[* * 21]$	re
$\mathcal{U}_6$	$[21 * *]$ $[*1 * 2]$	sa
$\mathcal{U}_7$	$[* 21 *]$ $[*2 * 1]$	$(r - s)e$
$\mathcal{U}_8$	$[2 * *1]$ $[* * 12]$ $[2 * 1*]$	0

By inspection, we see that $\mathcal{U}_1$ and $\mathcal{U}_8$ are blocks. Now the block $[1 * 2*]$ implies that $x_3^s y_3^{r-s} = x_4^s y_4^{r-s}$.

If $\mathcal{U}_2$ were a block we would derive $y_3^r = y_4^r$, which combined with the preceding relation gives $(x_3, y_3) \sim (x_4, y_4)$. Thus we suppose that $\mathcal{U}_2$ is not a block and we see that $\mathcal{U}_2 + \mathcal{U}_3$ must be a block and moreover $(r-s)a - se = 0$. Since e is determined it is now easy to compare the various values of $v(m_\sigma)$. We remove the blocks we have found and look at terms with $v(m_\sigma) = \min$. They can arise only from $\mathcal{U}_6$ and $\mathcal{U}_7$. We cannot have $sa = (r - s)e$ because we assume $r \neq 2s$, hence one of them is a block and by symmetry we may suppose that $\mathcal{U}_7$ is a block. This leaves us with $\mathcal{U}_4$, $\mathcal{U}_5$ and $\mathcal{U}_6$. We have $a \neq e$, again because $r \neq s$, and we conclude that $\mathcal{U}_4$ is a block. This implies that $x_3^s y_3^{r-s} y_4^r = x_4^s y_4^{r-s} y_3^r$ and $(x_3, y_3) \sim (x_4, y_4)$.

For type IV it suffices to look at the monomials with $v(m_\sigma) = \max$. We see that $[* * 34]$ must be a block, and $(x_1, y_1) \sim (x_2, y_2)$ follows easily from this.

Type V is a little more complicated. The basic table is

BASIC TABLE, TYPE V

$\mathcal{U}_1$	$[12 * *]$	$ra + sb$
$\mathcal{U}_2$	$[21 * *]$	$rb + sa$
$\mathcal{U}_3$	$[1 * 2*]$ $[1 * *2]$	ra
$\mathcal{U}_4$	$[2 * 1*]$ $[2 * *1]$	rb
$\mathcal{U}_5$	$[* 12 *]$ $[*1 * 2]$	sa
$\mathcal{U}_6$	$[* 21 *]$ $[*2 * 1]$	sb

$\mathcal{U}_7$ $[**12]\ [**21]$ 0

The terms with $v(m_\sigma) = \min$ show that $\mathcal{U}_7$ is a block, giving

$$(x_3^s y_3^{r-s} x_4^r - x_4^s y_4^{r-s} x_3^r)(y_1^r - y_2^r) = 0\,.$$

The vanishing of the first factor yields $(x_3, y_3) \sim (x_4, y_4)$, hence we may suppose $y_1^r = y_2^r$.

A similar argument with $v(m_\sigma) = \max$ shows that $y_3^r = y_4^r$ and that $\mathcal{U}_1$, $\mathcal{U}_2$ are blocks.

Next, we see that $\mathcal{U}_6$ or $\mathcal{U}_5 + \mathcal{U}_6$ are blocks, according as $a > b$ or $a = b$. If $\mathcal{U}_6$ is a block then using the relations $y_1^r = y_2^r$ and $y_3^r = y_4^r$ we find

$$(x_3^r - x_4^r) x_2^s y_2^{r-s} (y_4^r - y_2^r) = 0\,,$$

while if $\mathcal{U}_5 + \mathcal{U}_6$ is a block we find

$$(x_3^r - x_4^r)(x_2^s y_2^{r-s} - x_1^s y_1^{r-s})(y_4^r - y_2^r) = 0\,.$$

The alternatives $x_1^s y_1^{r-s} = x_2^s y_2^{r-s}$ and $x_3^r = x_4^r$ lead to proportionality, so we are left with $y_1^r = y_2^r = y_3^r = y_4^r$. It follows that x_i^r, $x_i^s y_i^{r-s}$, 1 are linearly dependent over K and we get

$$\det \begin{pmatrix} x_2^r & x_2^s y_2^{r-s} & 1 \\ x_3^r & x_3^s y_3^{r-s} & 1 \\ x_4^r & x_4^s y_4^{r-s} & 1 \end{pmatrix} = 0\,.$$

This case has been considered already in the analysis for type VII, thus disposing of type V.

For type II we distinguish the subcases $a > b > c$, $a = b > c$, $a > b = c$, $a = b = c$.

Subcase $a > b > c$. The blocks with $v(m_\sigma) = \max$ or min are $[12**]$ and $[43**]$, which imply $y_3^r = y_4^r$ and $y_1^r = y_2^r$. This in turn implies that $[21**]$ and $[34**]$ are also blocks. After removing these four blocks the terms with $v(m_\sigma) = \min$ show that $[42**]$ is a block. This yields $y_1^r = y_3^r$ and finally $y_1^r = y_2^r = y_3^r = y_4^r$. The analysis ends as for type V.

Subcase $a = b > c$. The basic table is

BASIC TABLE, TYPE II, $a = b > c$

$\mathcal{U}_1$	$[12**]$ $[21**]$	$(r+s)a$
$\mathcal{U}_2$	$[13**]$ $[23**]$	$ra+sc$
$\mathcal{U}_3$	$[31**]$ $[32**]$	$rc+sa$
$\mathcal{U}_4$	$[14**]$ $[24**]$	ra
$\mathcal{U}_5$	$[34**]$	rc
$\mathcal{U}_6$	$[41**]$ $[42**]$	sa
$\mathcal{U}_7$	$[43**]$	sc

Inspection of the basic table shows that $[43**]$ is a block, hence $y_1^r = y_2^r$ and $[34**]$ is also a block. After removing these two blocks we see that $\mathcal{U}_6$ is a block. The corresponding sum of monomials is

$$x_4^r(x_1^s y_1^{r-s} y_3^r - x_1^s y_1^{r-s} y_2^r + x_2^s y_2^{r-s} y_1^r - x_2^s y_2^{r-s} y_3^r) = 0$$

which is transformed into

$$x_4^r(x_1^s y_1^{r-s} - x_2^s y_2^{r-s})(y_3^r - y_2^r) = 0$$

because of the relation $y_1^r = y_2^r$. If $x_1^s y_1^{r-s} = x_2^s y_2^{r-s}$ we get $(x_1, y_1) \sim (x_2, y_2)$, again because of the relation $y_1^r = y_2^r$. Thus we remain with $y_2^r = y_3^r$. This implies that $[314**]$ and $[24**]$ are blocks and after removing them we see that $\mathcal{U}_3$ is the block with the remaining monomials with smallest $v(m_\sigma)$. The analysis of this block is identical to that for $\mathcal{U}_6$, the only difference being the exchange of 3 and 4. The conclusion is that $y_2^r = y_4^r$ and $y_1^r = y_2^r = y_3^r = y_4^r$. The analysis ends as for type V.

Subcase $a > b = c$. The basic table is

BASIC TABLE, TYPE II, $a > b = c$

$\mathcal{U}_1$	$[12**]$ $[13**]$	$ra+sb$
$\mathcal{U}_2$	$[21**]$ $[31**]$	$rb+sa$
$\mathcal{U}_3$	$[23**]$ $[32**]$	$(r+s)b$
$\mathcal{U}_4$	$[14**]$	ra
$\mathcal{U}_5$	$[24**]$ $[34**]$	rb

$\mathcal{U}_6$	$[41 * *]$	sa
$\mathcal{U}_7$	$[42 * *]$ $[43 * *]$	sb

Inspection of the basic table shows that $\mathcal{U}_1$ and $\mathcal{U}_2$ are blocks, yielding

$$\begin{aligned} x_2^s y_2^{r-s}(y_3^r - y_4^r) - x_3^s y_3^{r-s}(y_2^r - y_4^r) &= 0 \\ x_2^r(y_3^r - y_4^r) - \qquad x_3^r(y_2^r - y_4^r) &= 0. \end{aligned}$$

This is a homogeneous linear system for $y_3^r - y_4^r$ and $y_2^r - y_4^r$, with determinant $x_3^r x_2^s y_2^{r-s} - x_2^r x_3^s y_3^{r-s}$. If the determinant vanishes we get $(x_2, y_2) \sim (x_3, y_3)$. Thus we remain with $y_2^r = y_3^r = y_4^r$, and the rest of the analysis proceeds very much in the same way as in the subcase $a = b > c$.

Subcase $a = b = c$. The monomials with $v(m_\sigma) = \min$ form a block $[4{*}{*}{*}]$ of six elements. After removing this block we see that the remaining monomials with $v(m_\sigma) = \min$ form a block $[*4 * *]$ with six elements. This means that

$$\det\begin{pmatrix} x_1^s y_1^{r-s} & y_1^r & 1 \\ x_2^s y_2^{r-s} & y_2^r & 1 \\ x_3^s y_3^{r-s} & y_3^r & 1 \end{pmatrix} = 0$$

and

$$\det\begin{pmatrix} x_1^r & y_1^r & 1 \\ x_2^r & y_2^r & 1 \\ x_3^r & y_3^r & 1 \end{pmatrix} = 0\,,$$

and we fall back into the analysis of type VII.

Type I is easy to study. If $a > b > c$ then $[1243]$ is the only monomial with $v(x_\sigma) = \max$, which is impossible.

If instead $a = b > c$ the terms with $v(m_\sigma) = \max$ yield the block $[* * 43]$, hence $x_1^r x_2^s y_2^{r-s} = x_2^r x_1^s y_1^{r-s}$, which implies proportionality $(x_1, y_1) \sim (x_2, y_2)$.

If $a > b = c$ the same argument shows that $[1{*}4{*}]$ is a block, therefore $[4{*}1{*}]$ is another block. After removing this block and looking at the terms with $v(m_\sigma) = \min$ we see that $[4{*}{*}1]$ is a block. This implies $x_2^s y_2^{r-s} y_3^r = x_3^s y_3^{r-s} y_2^r$ and $(x_2, y_2) \sim (x_3, y_3)$.

Thus we remain with the subcase $a = b = c$. The terms with $v(m_\sigma) = \max$ form the block $[* * 4*]$, and those with $v(m_\sigma) = \min$ form the block $[4 * **]$.

This yields

$$\det\begin{pmatrix} x_1^s y_1^{r-s} & y_1^r & 1 \\ x_2^s y_2^{r-s} & y_2^r & 1 \\ x_3^s y_3^{r-s} & y_3^r & 1 \end{pmatrix} = 0$$

and

$$\det\begin{pmatrix} x_1^r & x_1^s y_1^{r-s} & 1 \\ x_2^r & x_2^s y_2^{r-s} & 1 \\ x_3^r & x_3^s y_3^{r-s} & 1 \end{pmatrix} = 0$$

which is analyzed exactly as for type VII.

Type III is the hardest to analyze, and it is here that the condition $r \neq 2s$ appears again. First of all, the symmetry of the associated matrix shows that we may interchange x, y, s with $y, x, r-s$, thus we may assume $2s < r$ because $r \neq 2s$ by hypothesis. As for type II, we distinguish subcases.

Subcase $a > b$, $e > f$, $2s < r$. The monomials with $v(m_\sigma) = \max$ must form the block $[1*3*]$. It follows that $x_2^s y_2^{r-s} = x_4^s y_4^{r-s}$ and also $[3*1*]$ is a block. Note that we have also obtained the equation

$$sb - (r-s)f = 0\,. \tag{2.11}$$

The block with $v(m_\sigma) = \min$ therefore is $[3*1*]$ [3421] [4213]. After removing this block we verify using (2.11) that the two largest values of $v(m_\sigma)$ occur for [1342] and [2134], all other values being smaller. Hence [1342] and [2134] form a block, and we get

$$(r-s)a - rb - se + rf = 0\,. \tag{2.12}$$

If we write down the corresponding relation and use the equation $x_2^s y_2^{r-s} = x_4^s y_4^{r-s}$, we find the further relation $x_4^r y_1^r = x_3^r y_2^r$.

We remove these three blocks, use the fact that $2s < r$, (2.11) and (2.12) imply

$$sa < (r-s)e, \qquad sb + rf < rb + (r-s)f,$$

and see that the terms with $v(m_\sigma) = \min$ form a subblock of $[*12*]$ [3241]. If $[*12*]$ were a block then $x_3^r = x_4^r$, therefore from $x_4^r y_1^r = x_3^r y_2^r$ we would get $y_1^r = y_2^r$. Hence $[43**]$ would be a block too. After removing these blocks and looking at the terms with $v(m_\sigma) = \min$ the only possibility is the block [3241] [2413], which implies $sb + rf = rb + (r-s)f$ and $(r-s)b = sf$. On the other hand, we have $sb = (r-s)f$ which together with the preceding equation gives $r = 2s$, which was excluded. Thus we conclude that [∗12∗] [3241] is an irreducible block, and in particular

$$sa - sb - rf = 0. \tag{2.13}$$

At this stage, using equations (2.11), (2.12) and (2.13), the basic table simplifies to

SIMPLIFIED BASIC TABLE, TYPE III, $a > b$, $e > f$

$\mathcal{U}_1$	[1324] [2431]	$A(t^3 + t - 1)$
$\mathcal{U}_2$	[4132]	$A(t^3 - t^2 + 3t - 1)$
$\mathcal{U}_3$	[2341] [4231]	$A(t^3 - t^2 + 2t - 1)$
$\mathcal{U}_4$	[2314]	$A(t^3 - t^2 + t - 1)$
$\mathcal{U}_5$	[43 ∗ ∗]	$A(t^3 - 2t^2 + 2t - 1)$
$\mathcal{U}_6$	[1243]	$A(2t^2 + t - 1)$
$\mathcal{U}_7$	[1423]	$A(2t^2 - 1)$
$\mathcal{U}_8$	[2143]	$A(t^2 + 2t - 1)$
$\mathcal{U}_9$	[2413]	$A(t^2 - 1)$
$\mathcal{U}_{10}$	[3142]	$A(3t - 1)$

where we have abbreviated $t = r/s$ and $A = sa/(2t-1)$. Note that our hypotheses imply $t > 2$. Finally we see that if $t > 2$ the term [4132] cannot form a block with any other term, a contradiction.

Subcase $a = b$, $e > f$, $2s \neq r$. The basic table is

BASIC TABLE, TYPE III, $a = b$, $e > f$

$\mathcal{U}_1$	[∗ ∗ 34]	$(r+s)a + re$
$\mathcal{U}_2$	[∗ 43 ∗]	$ra + (r-s)f + re$
$\mathcal{U}_3$	[∗ 34 ∗]	$ra + (r-s)e + rf$
$\mathcal{U}_4$	[4 ∗ 3∗]	$sa + re$
$\mathcal{U}_5$	[∗3 ∗ 4]	$ra + (r-s)e$
$\mathcal{U}_6$	[43 ∗ ∗]	$(r-s)e$

$\mathcal{U}_7$	$[**43]$	$(r+s)a+rf$
$\mathcal{U}_8$	$[*4*3]$	$ra+(r-s)f$
$\mathcal{U}_9$	$[3*4*]$	$sa+rf$
$\mathcal{U}_{10}$	$[*12*]$ $[*21*]$	sa
$\mathcal{U}_{11}$	$[34**]$	$(r-s)f$

Inspection of this table shows that the terms with $v(m_\sigma)=\max$ come only from $\mathcal{U}_1$ and $\mathcal{U}_2$. If $\mathcal{U}_1$ or $\mathcal{U}_7$ were a block we would get $x_1^r x_2^s y_2^{r-s} = x_2^r x_1^s y_1^{r-s}$ and $(x_1,y_1)\sim(x_2,y_2)$. Thus we may assume that $\mathcal{U}_1$ is not a block. If now $\mathcal{U}_2$ were a block then $\mathcal{U}_3$ would also be a block, and after removing them we verify that $\mathcal{U}_1$ must be the block of largest order, a contradiction. It follows that $\mathcal{U}_1+\mathcal{U}_2$ is a block and also

$$sa-(r-s)f=0\,. \tag{2.14}$$

Moreover, $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{U}_3$ and $\mathcal{U}_7$ are not blocks.

Next, we claim that $\mathcal{U}_{10}+\mathcal{U}_{11}$ is a block. For if $\mathcal{U}_{11}$ is a block then so is $\mathcal{U}_6$, which forces $\mathcal{U}_{10}$ to be a block. Otherwise $\mathcal{U}_{11}$ must form a block with something else, and this can only be $\mathcal{U}_{10}$.

Since $\mathcal{U}_1+\mathcal{U}_2$ is a block, we find

$$x_1^r(x_2^s y_2^{r-s} - x_4^s y_4^{r-s}) = x_2^r(x_1^s y_1^{r-s} - x_4^s y_4^{r-s})$$

Now $\mathcal{U}_3$ can form a block only with elements of $\mathcal{U}_4$ and $\mathcal{U}_7$. Also $\mathcal{U}_4$ and $\mathcal{U}_7$ cannot have elements in a same irreducible block, because otherwise $sa=(r-s)f$ and $ra+(r-s)e+rf=sa+re=(r+s)a+rf$ lead to a contradiction. Thus either $\mathcal{U}_3+\mathcal{U}_4$ or $\mathcal{U}_3+\mathcal{U}_7$ is a block. If $\mathcal{U}_3+\mathcal{U}_7$ were a block we would find

$$x_1^r(x_2^s y_2^{r-s} - x_3^s y_3^{r-s}) = x_2^r(x_1^s y_1^{r-s} - x_3^s y_3^{r-s})$$

which combined with the previous displayed equation gives

$$(x_1^r - x_2^r)(x_3^s y_3^{r-s} - x_4^s y_4^{r-s}) = 0\,.$$

The alternative $x_1^r = x_2^r$ implies that $\mathcal{U}_3$ is a block, which is not the case, and the alternative $x_3^s y_3^{r-s} = x_4^s y_4^{r-s}$ gives $e=f$, which is not the case. Hence

$\mathcal{U}_3 + \mathcal{U}_4$ is a block and $(r-s)a - se + rf = 0$. Next, we verify using $r \neq 2s$ that we cannot have an irreducible block with elements from $\mathcal{U}_8$ and either $\mathcal{U}_5$, or $\mathcal{U}_7$, or $\mathcal{U}_9$. Also, if $\mathcal{U}_8$ were a block we would get $x_1^r y_2^r = x_2^r y_1^r$, implying proportionality. Thus there is an irreducible block with elements from $\mathcal{U}_8$ and $\mathcal{U}_6$, whence $ra - (r-s)e + (r-s)f = 0$. The three equations $sa - (r-s)f = 0$, $(r-s)a - se + rf = 0$, $ra - (r-s)e + (r-s)f = 0$ imply $r = 2s$, completing the analysis of this subcase.

Subcase $a = b$, $e = f$, $2s < r$. The basic table is

BASIC TABLE, TYPE III, $a = b, \quad e = f$

$\mathcal{U}_1$	$[**34]\ [**43]$	$(r+s)a + re$
$\mathcal{U}_2$	$[*43*]\ [*34*]$	$ra + (2r-s)e$
$\mathcal{U}_3$	$[4*3*]\ [3*4*]$	$sa + re$
$\mathcal{U}_4$	$[*3*4]\ [*4*3]$	$ra + (r-s)e$
$\mathcal{U}_5$	$[3**4]\ [4**3]$	sa
$\mathcal{U}_6$	$[34**]\ [43**]$	$(r-s)e$

Inspection of the table shows at once that $\mathcal{U}_1 + \mathcal{U}_2$, $\mathcal{U}_3 + \mathcal{U}_4$, $\mathcal{U}_5 + \mathcal{U}_6$ are blocks.

Suppose first that $sa + re = ra + (r-s)e$. Then using the hypothesis $r \neq 2s$ we verify that $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_5, \mathcal{U}_6$ are distinct blocks. The blocks $\mathcal{U}_1$ and $\mathcal{U}_6$ give

$$(x_4^s y_4^{r-s} y_3^r - x_3^s y_3^{r-s} y_4^r)(x_1^r - x_2^r) = 0\,,$$

$$(x_4^s y_4^{r-s} y_3^r - x_3^s y_3^{r-s} y_4^r)(y_1^r - y_2^r) = 0\,.$$

The vanishing of the first factor implies $(x_3, y_3) \sim (x_4, y_4)$, while the vanishing of the second factors implies $(x_1, y_1) \sim (x_2, y_2)$. Thus we may suppose that $sa + re \neq ra + (r-s)e$, whence $\mathcal{U}_3$ and $\mathcal{U}_4$ are distinct blocks, giving

$$(x_2^s y_2^{r-s} - x_1^s y_1^{r-s})(x_4^r y_3^r - x_3^r y_4^r) = 0\,,$$

$$(x_3^s y_3^{r-s} - x_4^s y_4^{r-s})(x_1^r y_2^r - x_2^r y_1^r) = 0\,.$$

The vanishing of either of the second factors yield once again proportionality of solutions, therefore we remain with $x_1^s y_1^{r-s} = x_2^s y_2^{r-s}$, $x_3^s y_3^{r-s} = x_4^s y_4^{r-s}$.

Now, using this last relation, the block $\mathcal{U}_1 + \mathcal{U}_2$ gives

$$(x_1^r - x_2^r)(y_3^r - y_4^r)(x_2^s y_2^{r-s} - x_3^s y_3^{r-s}) = 0\,.$$

The vanishing of one of the first two factors implies again proportionality, and we remain with $x_1^s y_1^{r-s} = x_2^s y_2^{r-s} = x_3^s y_3^{r-s} = x_4^s y_4^{r-s}$. Since x_i^r, $x_i^s y_i^{r-s}$, y_i^r, 1, $i = 1,2,3,4$ satisfy a linear relation with non-zero coefficients, we see that x_i^r, y_i^r, 1, $i = 1,2,3$ lie in a hyperplane in K^3. It follows that

$$\det \begin{pmatrix} x_1^r & y_1^r & 1 \\ x_2^r & y_2^r & 1 \\ x_3^r & y_3^r & 1 \end{pmatrix} = 0\,,$$

and the analysis ends as for case VII. This completes the proof of Lemma 3.

III. CONCLUSION OF PROOFS. It is immediate that Theorem 1 and Theorem 2 follow from Lemma 1, Lemma 2 and Lemma 3 provided we show that the set V_b of bad absolute values v is empty. In the proofs of Lemma 2 and Lemma 3 we have already paved the way for verifying this fact, since if v is bad then the oscillation of the order of monomials in every irreducible block is bounded by $|\mathcal{S}|(|\mathcal{S}|-1)(|\mathcal{S}|-2)/2$, and Lemma 2 and Lemma 3 dealt with the case in which this oscillation was 0.

LEMMA 4. *In the case of equation* (1.2) *if* $\min(r,s,t) > K$ *we have* $V_b = \emptyset$.

PROOF. We assume the reduction to types as in Lemma 2. Note also that the statement about blocks of two elements is still valid.

Suppose that not every block is neutral and that $\min(r,s,t) > K$. Then the proof parallels that of Lemma 2, checking that for the corresponding types an irreducible block containing the corresponding elements [123], $[3**]$, $[1**]$ is necessarily a good block.

LEMMA 5. *In the case of equation* (1.1) *if* $\min(s, r-s, |r-2s|) > K$ *we have* $V_b = \emptyset$.

PROOF. We assume the reduction to types as in Lemma 3. We also refer to Lemma 3 for the basic tables associated to each type. The strategy of proof consists in assuming that v is bad, and checking that the arguments given in the proof of Lemma 3 carry on with a few modifications.

For type VII, a block with $|I_v(\mathcal{B})| > 0$ must have $|I_v(\mathcal{B})| > K$, and we are done.

For type VI with $a = e$ we have the same argument as for type VII. Thus, by symmetry, we may assume $a > e$. As in Lemma 3, we have again that $\mathcal{U}_1$, $\mathcal{U}_8$ and $\mathcal{U}_2 + \mathcal{U}_3$ are blocks, but now we have only $|(r-s)a - se| \leqq K$ instead of $(r-s)a - se = 0$. The rest of the argument proceed in exactly the same way, replacing the condition $r \neq 2s$ with $|r - 2s| > K$ in order to dispose of the possibility that $|sa - (r-s)e| \leqq K$.

The arguments for types IV, V, II and I are the same as in Lemma 3, always using the hypothesis $\min(s, r-s, |r-2s|) > K$ in order to deduce, from the hypothesis that v is bad, the same irreducible blocks as in the case v neutral.

For type III, we follow again the same argument except that now we have to be careful in checking that the lower bound $\min(s, r-s, |r-2s|) > K$ suffices for the proof.

The beginning of the analysis of the subcase $a > b$, $e > f$ is identical to that in Lemma 3, except for the fact that instead of $2s < r$ we assume $2s < r - K$. We still obtain the three blocks [1 ∗ 3∗], [3 ∗ 1∗] [3421] [4213], [1342] [2134], the relation $x_2^s y_2^{r-s} = x_4^s y_4^{r-s}$ and the equations (2.11) and (2.12). The inequalities $sa < (r-s)e$ and $sb + rf < rb + (r-s)f$ are replaced by $sa < (r-s)e - 2K$ and $sb + rf < rb + (r-s)f - K$. Again we conclude that [∗12∗] [3241] is an irreducible block, but now (2.13) becomes

$$|sa - sb - rf| \leqq K\,. \tag{3.1}$$

The simplified basic table has the same blocks, with the difference that the order of monomials in the blocks is only approximately equal to the order appearing in the right-hand side.

The precise order of the various terms in the new simplified basic table is

New Simplified Basic Table, Type III, $a > b, \quad e > f$

$\mathcal{U}_1$	[1324]	$A(t^3 + t - 1) + X(t^3 - 3t^2 + 2t)$

$\mathcal{U}_1'$	[2431]	$A(t^3+t-1)+X(t^3-3t+1)$
$\mathcal{U}_2$	[4132]	$A(t^3-t^2+3t-1)+X(t^3-2t^2)$
$\mathcal{U}_3$	[2341]	$A(t^3-t^2+2t-1)+X(t^3-4t^2+2t)$
$\mathcal{U}_3'$	[4231]	$A(t^3-t^2+2t-1)+X(t^3-2t^2-t+1)$
$\mathcal{U}_4$	[2314]	$A(t^3-t^2+t-1)+X(t^3-4t^2+3t)$
$\mathcal{U}_5$	[43 * *]	$A(t^3-2t^2+2t-1)+X(t^3-3t^2+2t)$
$\mathcal{U}_6$	[1243]	$A(2t^2+t-1)-X(t-1)$
$\mathcal{U}_7$	[1423]	$A(2t^2-1)-X(2t-1)$
$\mathcal{U}_8$	[2143]	$A(t^2+2t-1)-Xt^2$
$\mathcal{U}_9$	[2413]	$A(t^2-1)-X(t^2-1)$
$\mathcal{U}_{10}$	[3142]	$A(3t-1)-Xt$

where $A=sa/(2t-1)$ and $X=(sa-sb-rf)/(2t-1)$. In the proof of Lemma 3 it was sufficient to check that the order of m_{4132} was different from that of any other element, but here we would like to check that $|v(m_{4132})-v(m_\sigma)|>K$ if $\sigma\neq 4132$. This is not true for every X, so some additional considerations are needed.

We begin with the remark that $r>2s+K$, $s>K$, (2.11), (2.12) and (3.1) imply $a\geqq 5$ and $s>\max(1,1/(t-2))K$. Also $(2t-1)|X|\leqq K$ and $t>2$.

This gives for example

$$v([1324])-v([4123])=(A-X)(t^2-2t)$$
$$>K(5\max(1,\frac{1}{t-2})-1)\frac{t^2-2t}{2t-1}>2K$$

and proceeding with rather similar considerations shows that $\mathcal{U}_1+\mathcal{U}_1'$ is a block. This implies $[1324]=[2431]$ and $X(t^3-3t^2+2t)=X(t^3-3t+1)$, hence $X=0$. Since $X=0$ the analysis of Lemma 3 holds.

The analysis in the subcase $a=b$, $e>f$ requires little modification. One obtains first that $\mathcal{U}_1+\mathcal{U}_2$ is a block, while (2.14) is modified into

$$|sa-(r-s)f|\leqq K\,. \tag{3.2}$$

Again we have

$$x_1^r(x_2^s y_2^{r-s}-x_4^s y_4^{r-s})=x_2^r(x_1^s y_1^{r-s}-x_4^s y_4^{r-s})$$

and $\mathcal{U}_3$ can form a block only with elements of $\mathcal{U}_4$ and $\mathcal{U}_7$. Next we check that either $\mathcal{U}_3 + \mathcal{U}_4$ or $\mathcal{U}_3 + \mathcal{U}_7$ is a block. If both $\mathcal{U}_3$ and $\mathcal{U}_7$ have an element in a same irreducible block then $|sa - (r-s)e| \leqq K$, which combined with (3.2) gives $e = f + 1$. Now $\mathcal{U}_7$ and $\mathcal{U}_4$ cannot have elements in a same irreducible block, otherwise $|r(a - e + f)| \leqq K$ and $a - e + f = 0$, and a contradiction is reached by looking at $\mathcal{U}_6$. Thus all of $\mathcal{U}_7$ forms a block with some elements of $\mathcal{U}_3$. If an element of $\mathcal{U}_3$ were also in another irreducible block with some element of $\mathcal{U}_4$, we would get $|r(a-e+f)| \leqq 2K$ and the same contradiction as before, so that we conclude that $\mathcal{U}_3 + \mathcal{U}_7$ is a block. A rather similar argument shows that the only other possibility is that $\mathcal{U}_3 + \mathcal{U}_4$ is a block.

If $\mathcal{U}_3 + \mathcal{U}_7$ is a block we argue as in the proof of Lemma 3. It follows that $\mathcal{U}_3 + \mathcal{U}_4$ is a block and moreover

$$|(r-s)a - se + rf| \leqq K. \tag{3.3}$$

Next, we verify using $\min(s, r-s, |r-2s|) > K$ that an irreducible block cannot contain an element of $\mathcal{U}_8$ and another element from one of $\mathcal{U}_5$, $\mathcal{U}_7$ and $\mathcal{U}_9$. Also, if $\mathcal{U}_8$ were a block we would get $x_1^r y_2^r = x_2^r y_1^r$, implying proportionality. Thus there is an irreducible block containing elements from $\mathcal{U}_8$ and $\mathcal{U}_6$, therefore

$$|ra - (r-s)e + (r-s)f| \leqq K. \tag{3.4}$$

If we eliminate e and f from the system of inequalities (3.2), (3.3) and (3.4) we find $|r-2s|a \leqq (2 - s/r)K$, hence $a = 1$ because $|r-2s| > K$ by hypothesis. Now (3.2) gives $|s - (r-s)f| \leqq K$, hence $(r-s)f \leqq 2K$ and $f = 1$ because $r - s > K$. Finally (3.2) becomes $|2s - r| \leqq K$, against our assumption, thus completing the analysis of this subcase.

The analysis of the last subcase, $a = b$ and $e = f$, is essentially identical to that of Lemma 3 except for substituting the inequality $|(sa + re) - (ra + (r-s)e)| \leqq K$ in place of the equation $sa + re = ra + (r-s)e$, and using the hypothesis $\min(s, r-s, |r-2s|) > K$ in place of $r \neq 2s$.

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