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Introduction

Topological cyclic homology associates to a ring R a spectrum $\mathrm{TC}(R)$. The homotopy groups $\mathrm{TC}_i(R)$ are connected to Connes' cyclic homology groups of R, but they are stronger invariants than the cyclic homology groups. There is a map, called the cyclotomic trace, from Quillen's K-theory spectrum $K(R)$ into $\mathrm{TC}(R)$. This map is conjectured to be a p-homotopy equivalence for certain complete semi local rings, and in particular for rings of integers in local fields with positive residue characteristic p. We refer the reader to [25] for further discussion of the cyclotomic trace. In this paper we set up a general strategy for calculating topological cyclic homology, and we apply it to the key case where the ring in question is the ring of p-adic integers.

Let us very briefly describe the construction of topological Hochschild and cyclic homology. If we in the standard simplicial Hochschild complex of R, whose homotopy groups are the Hochschild homology groups, replace the ring with the Eilenberg-MacLane spectrum it determines, and the tensor product (over $\mathbb{Z}$) with smash product of spectra, then we obtain the topological Hochschild homology spectrum $\mathrm{THH}(R)$. There are severe technical difficulties in carrying out the indicated substitutions, but they were overcome in [8] by the introduction of functors with smash product. A ring R gives in particular rise to such a functor. The resulting spectrum $\mathrm{THH}(R)$ turns out to be an equivariant S^1-spectrum with deloops in the direction of every representation. Following [24] one would then expect that the topological cyclic homology to be closely related to the homotopy orbit spectra $EC_{n+} \wedge_{C_n} \mathrm{THH}(R)$. This is indeed the case, but instead of taking homotopy quotients with respect to the finite cyclic groups it is better to take fixed sets $\mathrm{THH}(R)^{C_n}$. The fixed sets contain many strata, one for each subgroup of C_n with the homotopy quotient above corresponding to the free strata. The spectrum $\mathrm{TC}(R)$ is a certain homotopy inverse limit of the fixed sets over a category which contains the inclusions of fixed sets and certain maps which mix the strata, cf. [11], [22] and sect.1 below.

The content of the paper is as follows. In the first section we introduce the concept of a p-cyclotomic spectrum. It is an equivariant S^1-spectrum with some extra structure,

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and THH (R) belongs to this class of spectra for any functor with smash product and any prime p. The extra structure consists of S^1-maps

$$\varphi_{C_p} : \mathrm{THH}\,(R)^{C_p} \to \mathrm{THH}\,(R)$$

different from the inclusion of fixed sets, where C_p is the cyclic group of order p. For a p-cyclotomic spectrum T there are the important cofibrations

$$EC_{p^n+} \wedge_{C_{p^n}} T \to T^{C_{p^n}} \xrightarrow{\Phi} T^{C_{p^{n-1}}} \tag{0.1}$$

with Φ equal to the $C_{p^{n-1}}$-fixed set of φ_{C_p}. The sphere spectrum is a cyclotomic spectrum, and the above cofibrations are split for $T = S^0$, giving the standard decomposition from [34] of the fixed point spectra.

The second section of the paper discusses the norm cofibration

$$EG_+ \wedge_G T \to \mathrm{Map}_G(EG_+, T) \to \hat{\mathbb{H}}(G; T) \tag{0.2}$$

for an equivariant G-spectrum T. Special cases of this fibration have been considered by various authors. We need for our purpose the Greenlees-May foundations from [19]. There are spectral sequences for each of the terms of (0.2). For example one has

$$\hat{H}^*(G; \pi_* T) \Rightarrow \pi_* \hat{\mathbb{H}}(G; T) \tag{0.3}$$

where $\hat{H}^*$ denotes the Tate cohomology. The main result in sect. 2 is Theorem 2.15, which relates the differential structure of (0.3) to the exact homotopy sequences of (0.2).

In section three we tabulate the spectral sequence

$$H^*(C_{p^n}; \pi_* J) \Rightarrow \pi_* \mathrm{Map}\,(EC_{p^n+}, J) \tag{0.4}$$

where J is the periodic image of J space, and where we use homotopy groups with $\mathbb{F}_p$ coefficients for an odd prime p.

The next two sections four and five discuss the structure of (0.3) when $T = \mathrm{THH}\,(\mathbb{Z}_p)$ and $G = C_{p^n}$. The natural map from $S^0 = \mathrm{THH}\,(*)$ to $\mathrm{THH}\,(\mathbb{Z}_p)$ is trivial on homotopy groups, but induces a highly non-trivial map

$$\mathrm{Map}_G\left(EC_{p^n+}, S^0\right) \to \mathrm{Map}_{C_{p^n}}(EC_{p^n+}, \mathrm{THH}\,(\mathbb{Z}_p)) \tag{0.5}$$

It just changes filtration. The E^2-term $H^*\left(C_{p^{n-1}}, \pi_* J\right)$ of (0.4) is a direct summand in theE^2-term of the domain of (0.5), and injects into theE^2-term of the range. Due to the filtration shift there are problems however in proving that the E^r-terms of (0.4) injects into the E^r-term of the range in (0.5). Assuming this to be true, however, the structure of (0.4) implies the structure of

$$H^*(C_{p^n}; \pi_* \mathrm{THH}(\mathbb{Z}_p)) \Rightarrow \pi_* \mathrm{Map}_{C_{p^n}}(EC_{p^n+}, \mathrm{THH}(\mathbb{Z}_p)) \tag{0.6}$$

for all n. This is our Conjecture 4.3. In section five it is shown that Conjecture 4.3 is indeed true if the unit map from S^0 to $K(\mathbb{Z}_p)$ factors over the J-spectrum. Such a factorization is known to exist on the level of the 0-th spaces of the spectra by [28]. There are other possible attacks on Conjecture 4.3, than to prove the factorization. The most promising is to use that the unit maps into the cyclic 1-skeleton of THH $(\mathbb{Z}_p)$ when composed with the trace map, but at the time of writing we have not been able to carry this to a definite conclusion.

The rest of the paper is based upon Conjecture 4.3, at least in part. Section six and section seven compare (0.1) and (0.2) and show that (0.1) is homotopy equivalent to the 0-connected cover of (0.2) via the obvious maps which inject fixed sets into homotopy fixed sets. This uses Conjecture 4.3 for general n. For $n = 1$ and $n = 2$, however, we can prove the conjecture, and for these values of n, (0.1) and the (-1)-connected cover of (0.2) do agree. Combining the results of section two and section four one then obtains the homotopy groups with $\mathbb{F}_p$ coefficients of TC $(\mathbb{Z}_p)$, p odd. Section eight proves periodicity: multiplication with v_1 induces an isomorphism between $\pi_r(\mathrm{TC}(\mathbb{Z}_p); \mathbb{F}_p)$ and $\pi_{r+2(p-1)}(\mathrm{TC}(\mathbb{Z}_p); \mathbb{F}_p)$ for $r \geq 0$. In section nine we use the linearization map from TC $(*)$ to TC $(\mathbb{Z}_p)$, the known structure of TC $(*)$, from [11], and a theorem of J. Rognes [33], to show that

$$\mathrm{TC}(\mathbb{Z}_p)_p^\wedge \simeq (\mathrm{Im}\, J \times \mathbb{Z})_p^\wedge \times B(\mathrm{Im}\, J \times \mathbb{Z}_p)_p^\wedge \times (\Sigma bu)_p^\wedge \tag{0.7}$$

This result, however is dependent on Conjecture 4.3.

We note that (0.7) is the expected structure of $K(\mathbb{Z}_p)_p^\wedge$ according to the generalized Lichtenbaum-Quillen conjecture as formulated by Dwyer and Friedlander. It lends credit to the belief that the cyclotomic trace is a homotopy equivalence, after p-completion, for these kind of rings.

The final section ten has the character of an appendix. Its main result shows that relative K-theory is mapped monomorphically to the relative topological Hochschild homology in the first non-trivial dimensions. As a consequence we derive an unpublished result of the first named author, which is used in a critical way in section six.

The scheme set up here for evaluating TC $(-)$ has been applied to a number of other rings. For example, one knows by now TC $(R)_p^\wedge$ for $R = \mathbb{F}$, $\mathbb{F}[[t]]$, $\mathbb{F}[t, t^{-1}]$ and $\mathbb{F}_p[\varepsilon]/(\varepsilon^2)$, cf. [22] and [25]. For example, TC($\mathbb{F}$) $\simeq H\mathbb{Z}_p$, the Eilenberg-MacLane spectrum of $\mathbb{Z}_p$, when $\mathbb{F}$ is a finite field with p^a elements. Furthermore the only non-zero homotopy groups of $\mathrm{TC}\left(\mathbb{F}[\epsilon]/(\epsilon^2)\right)$ are:

$$p \text{ odd: } \mathrm{TC}_{2n-1}\left(\mathbb{F}[\epsilon]/\left(\epsilon^2\right)\right) = \widehat{W}_{2n-1}(\mathbb{F})^{\langle -1\rangle} \text{ and } \mathrm{TC}_0\left(\mathbb{F}[\epsilon]/\left(\epsilon^2\right)\right) = \mathbb{Z}_p$$

$$p = 2: \quad \mathrm{TC}_{2n-1}\left(\mathbb{F}[\epsilon]/\left(\epsilon^2\right)\right) = \mathbb{F}^{\oplus n} \text{ and } \mathrm{TC}_0\left(\mathbb{F}[\epsilon]/\left(\epsilon^2\right)\right) = \mathbb{Z}_2$$

Here $\widehat{W}_n(\mathbb{F})$ denotes the Witt-vectors of length n, i.e.

$$\widehat{W}_n(\mathbb{F}) = (1 + X\mathbb{F}[[X]])^*/\left(1 + X^{n+1}\mathbb{F}[[X]]\right)^*,$$

and the superscript $< -1 >$ indicates the -1 eigenspace for the involution on $\widehat{W}(\mathbb{F})$ which changes sign on X. This predicts then the values for the "tangent space of $K(\mathbb{F})$".

Further advancement in the understanding of TC (R) is dependent upon a more thorough understanding of THH (R) than is available at present. We refer the reader to the discussion given in [25]. Apart from the obvious unsolved problem of proving Conjecture 4.3, the present paper raises at least two other issues, namely to calculate TC $(\mathbb{Z}_2)$ and to calculate TC (A) for rings of integers in local fields of positive residue characteristic. Ideally, one might hope to describe TC (A) for a Galois extension $A/\mathbb{Z}_p$ as a functor TC $(\mathbb{Z}_p)$ and the extension.

The calculations performed in sections six to eight are somewhat unpleasant, and not well understood from an algebraic point of view. One feels that some algebraic notion could be developed to explain and streamline them. In particular one would like to have a good description of the structure of the homotopy groups of the homotopy S^1-orbit of THH $(\mathbb{Z}_p)$. This will be needed for example in the calculation of $\mathrm{TC}(\mathbb{Z}_p[C_{p^n}])$.

The present paper has been a long time under way, and several people have contributed with very helpful comments. We in particular want to acknowledge the help we have had from L. Hesselholt and J. Rognes. J. Rognes read a draft of the entire paper, corrected several mistakes, and gave many valuable suggestions for improvements in the exposition. Finally we owe to him the characterization of the spectrum $\Sigma bu_p^\wedge$ used in section nine.

Added in January 1994. It appears that Stavros Tsalidis in his 1994 Ph.D thesis from Brown University has proved Conjecture 4.3 below, so that (0.7) and the other conditional results of this paper are in fact theorems. See Remark 6.9 below for a little more details.

Combined with a second recent result due to R. McCarthy that the diagram

$$\begin{array}{ccc} K(R)_p^\wedge & \to & \mathrm{TC}(R)_p^\wedge \\ \downarrow & & \downarrow \\ K(R/I)_p^\wedge & \to & \mathrm{TC}(R/I)_p^\wedge \end{array}$$

is homotopy Cartesian when I is a nilpotent ideal in R, one gets from [22] and [25] that $K(R)_p^\wedge \simeq \mathrm{TC}(R)_p^\wedge$ for rings of integers in p-local number fields. In particular, $K(\mathbb{Z}_p)_p^\wedge \simeq \mathrm{TC}(\mathbb{Z}_p)_p^\wedge$ is given by (0.7), when p is odd.

Table of Contents

§ 1. Cyclotomic spectra at p

We introduce a special class of S^1-equivariant spectra and show that the topological Hochschild homology of any functor with smash product belongs to this class. This, we hope, will help to clarify the topological cyclic homology of such a functor.

Let T be an S^1-prespectrum indexed on all representations, or more precisely on a *complete universe* $\mathcal{U}$ in the language of [23], §I.2. It associates to each finite dimensional subspace $V \subset \mathcal{U}$ a based space $T(V)$, and for two such $V \subset W$ there is an S^1-map $\sigma : T(V) \wedge S^{W-V} \to T(W)$ so that the obvious associativity conditions are satisfied. Here $W - V$ denotes the orthogonal complement to V in W; T is called an S^1-spectrum if the adjoints $\hat{\sigma} : T(V) \to \Omega^{W-V} T(W)$ are homeomorphisms.

Let C be a finite subgroup of S^1 and $\rho_C : S^1 \to S^1/C$ the isomorphism $\rho_C(z) = \sqrt[c]{z}, c = |C|$. If X is an S^1-space then X^C is an S^1/C-space and we can form the induced S^1-space $\rho_C^*(X^C)$.

Let us recall from [2], (7.1) or from [23], p.111 that for any S^1-spectrum indexed on $\mathcal{U}$ and for any (finite) subgroup $C \subset S^1$, one can define an S^1/C-spectrum $\Phi^C T$

indexed on $\mathcal{U}^C$ by setting

$$\Phi^C T(Z) = \varinjlim_{V \subset \mathcal{U}} \Omega^{V^C - Z} T(V)^C.$$

The induced S^1-spectrum indexed on $\rho_C^* \mathcal{U}^C$ is denoted $\rho_C^\# \Phi^C T$:

$$\rho_C^\# \Phi^C T(\rho_C^* Z) = \rho_C^* \Phi^C T(Z).$$

We remember that $\mathcal{U}$ is an inner product space which contains each irreducible S^1-representation a countable number of times. Then the same is the case for the S^1-universe $\rho_C^* \mathcal{U}^C$ and we may pick an identification of $\mathcal{U}$ with $\rho_C^* \mathcal{U}^C$. This done, $\rho_C^\# \Phi^C T$ becomes an S^1-spectrum indexed on $\mathcal{U}$.
Let $\mathcal{F}$ be the family of all finite subgroups of S^1 and $\mathcal{F}_p$ the subfamily of all p-groups. A map of S^1-(pre)spectra $f : T_1 \to T_2$ is called an $\mathcal{F}$-equivalence resp. $\mathcal{F}_p$-equivalence if the induced map on fixed point spectra $f^C : T_1^C \to T_2^C$ is a homotopy equivalence for all $C \in \mathcal{F}$ resp. $C \in \mathcal{F}_p$. Here T_1^C is the usual fixed point spectrum indexed on $\mathcal{U}^C$, $T_1^C(Z) = T_1(Z)^C$ for $Z \subset \mathcal{U}^C$. All our spectra are CW-spectra, and $f : T_1 \to T_2$ is an $\mathcal{F}$-equivalence if and only if the induced map

$$f \wedge \text{ id } : T_1 \wedge E\mathcal{F}_+ \to T_2 \wedge E\mathcal{F}_+$$

is an S^1-homotopy equivalence.

Definition 1.1. Let p be any prime. A p-cyclotomic spectrum is an S^1-spectrum T together with an $\mathcal{F}_p$-equivalence

$$\varphi : \rho_{C_p}^\# \Phi^{C_p} T \to T$$

where C_p is the cyclic group of order p.

There is a similar concept of course of p-cyclotomic prespectra, and the spectrification of a p-cyclotomic prespectrum is a p-cyclotomic spectrum.
Let F be a functor with smash product (FSP) with $F(S^i)$ $(i-1)$-connected for all i. Recall that $\mathrm{THH}_\bullet(F)$ is the cyclic space

$$[k] \mapsto \operatorname{holim}_{\to} \operatorname{Map}(S^{i_0} \wedge \ldots \wedge S^{i_k}, F(S^{i_0}) \wedge \ldots \wedge F(S^{i_k}))$$

cf. [8], [11], [22] for details. Its realization $\mathrm{THH}(F)$ has an S^1-action. More generally we have for any based space Y the cyclic space $\mathrm{THH}_\bullet(F;Y)$ with k-simplices

$$[k] \mapsto \operatorname{holim}_{\to} \operatorname{Map}(S^{i_0} \wedge \ldots \wedge S^{i_k}, F(S^{i_0}) \wedge \ldots \wedge F(S^{i_k}) \wedge Y)$$

Its realization is denoted $\mathrm{THH}(F;Y)$. If Y is a based S^1-space then $\mathrm{THH}(F;Y)$ has an $S^1 \times S^1$-action where one factor acts via the cyclic structure and the other factor is the action induced from the action on Y, using that $\mathrm{THH}(F;Y)$ is a functor in Y. In the sequal we shall always consider $\mathrm{THH}(F;Y)$ with its induced diagonal S^1-action. If Z is a second based S^1-space then there is an obvious S^1-map $\mathrm{THH}(F;Y) \wedge Z \to \mathrm{THH}(F;Y \wedge Z)$ and we may define an S^1-prespectrum by setting

$$t(F)(V) = \mathrm{THH}(F;S^V)$$

Its associated S^1-spectrum is denoted $T(F)$. The spectrum $T(F)$ is different from $t(F)$, but not very much as we have:

Proposition 1.2. *The adjoints*

$$\sigma : t(F)(V) \to \Omega^W t(F)(V \oplus W)$$

are $\mathcal{F}$-equivalences, where $\mathcal{F}$ is the family of finite subgroups of S^1.

Proposition 1.3. *$T(F)$ is a p-cyclotomic spectrum.*

Before we can prove these results we need to prove the following lemma, well-known when G is the trivial group, cf. [27], where the terminology "proper" is explained.

Lemma 1.4. *Let $Y_\bullet$ be a G-simplicial space with Y_k^H $n(H)$-connected for all k. Suppose X is a based G-space with finitely many orbit types, and such that $dim X^H \leq n(H) + 1$. If $Y_\bullet^H$ is proper for the occuring orbit types then*

$$|\,\mathrm{Map}(X, Y_\bullet)\,| \to \mathrm{Map}(X, |\,Y_\bullet\,|)$$

is a G-homotopy equivalence

Proof. Consider first the special case of $X = G/H_+ \wedge S^n$ where

$$\begin{aligned}\mathrm{Map}_G(X, |\,Y_\bullet\,|) &= \mathrm{Map}(S^n, |\,Y_\bullet\,|^H) = \mathrm{Map}(S^n, |\,Y_\bullet^H\,|)\\ \mathrm{Map}_G(X, Y_k) &= \mathrm{Map}(S^n, Y_k^H)\end{aligned}$$

The result now is implied by [27], Theorem 12.3. In general we can induct over the G-cells. Suppose given a cofibration

$$X_0 \to X \to S^n \wedge G/H_+$$

and the theorem proved for X_0. The cofibration induces a simplicial Hurewicz fibration

$$\mathrm{Map}(S^n, Y_\bullet^H) \to \mathrm{Map}_G(X, Y_\bullet) \to \mathrm{Map}_G(X_0, Y_\bullet)$$

whose realization is a Hurewicz fibration by [27], Theorem 12.7. Consider the diagram

$$\begin{array}{ccccc} |\mathrm{Map}(S^n, Y_\bullet^H)| & \to & |\mathrm{Map}_G(X, Y_\bullet)| & \to & |\mathrm{Map}_G(X_0, Y_\bullet)| \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Map}(S^n, |Y_\bullet^H|) & \to & \mathrm{Map}_G(X, |Y_\bullet|) & \to & \mathrm{Map}_G(X_0, |Y_\bullet|) \end{array}$$

We can inductively assume the outer vertical arrows are homotopy equivalences, and get the same conclusion in the middle.
The same argument works for $\mathrm{Map}_K(X, Y_\bullet)$ for every subgroup $K \subseteq G$, and the equivariant result follows from the equivariant Whitehead theorem, cf. [2]. □

Recall from [11], sect.1 the concept of (edgewise) subdivision. For a cyclic space $Z_\bullet$ and finite cyclic subgroup $C \subset S^1$, it associates a simplicial space $sd_C Z_\bullet$ with a simplicial C-action. Moreover, there is a homeomorphism

$$D : | sd_C Z_\bullet | \to | Z_\bullet |,$$

a natural $\mathbb{R}/c\mathbb{Z}$-action on $| sd_C Z_\bullet |$ which extends the given simplicial C-action, and D is S^1-equivariant when $\mathbb{R}/c\mathbb{Z}$ is identified with S^1 in the standard fashion. Here $c = | C |$. In the special case of $\mathrm{THH}_\bullet(F; Y)$ the k-simplices of $sd_C \mathrm{THH}_\bullet(F; Y)$ are given by

$$[k] \mapsto \operatorname{holim}_{\rightarrow} \mathrm{Map}(S^{i_0 R} \wedge \ldots \wedge S^{i_k R}, F(S^{i_o})^{(C)} \wedge \ldots \wedge F(S^{i_k})^{(C)} \wedge Y)$$

with $iR = \mathbb{R}C \oplus \ldots \oplus \mathbb{R}C$ and $F(S^i)^{(C)} = F(S^i) \wedge \ldots \wedge F(S^i)$, c factors.
The C-action on the mapping space is by cyclic permution of the factors in the target and by the induced action on $S^{i_v R}$ in the source ([11], (3.6)).

Proof of Proposition 1.2. By lemma 1.4 and the above discussion of subdivision it suffices to prove that the obvious map

$$\begin{array}{c} \operatorname{holim}_{\rightarrow} \mathrm{Map}(S^{i_o R} \wedge \ldots \wedge S^{i_k R}, F(S^{i_o})^{(C)} \wedge \ldots \wedge F(S^{i_k})^{(C)} \wedge S^V) \\ \downarrow \hat{\sigma} \\ \operatorname{holim}_{\rightarrow} \mathrm{Map}(S^{i_o R} \wedge \ldots \wedge S^{i_k R} \wedge S^W, F(S^{i_o})^{(C)} \wedge \ldots \wedge F(S^{i_k})^{(C)} \wedge S^W \wedge S^V) \end{array}$$

is a C-homotopy equivalence. Indeed, the simplicial spaces involved are "good" or "strictly proper", so [34], Appendix or [27], Theorem 11.13 applies.
It suffices to take $W = lR$. In this case we can compose $\hat{\sigma}$ with the map

$$\begin{array}{c} \operatorname{holim}_{\to} \operatorname{Map}(S^{i_oR} \wedge \ldots \wedge S^{i_kR} \wedge S^{lR}, F(S^{i_o})^{(C)} \wedge \ldots \wedge F(S^{i_k})^{(C)} \wedge S^{lR} \wedge S^V) \\ \downarrow \tau \\ \operatorname{holim}_{\to} \operatorname{Map}(S^{i_oR} \wedge \ldots \wedge S^{(i_k+l)R}, F(S^{i_o})^{(C)} \wedge \ldots \wedge F(S^{i_k+l})^{(C)} \wedge S^V) \end{array}$$

upon using the identification $S^{lR} = S^l \wedge \ldots \wedge S^l$ and the stabilization

$$F(S^{i_k}) \wedge S^l \to F(S^{i_k+l})$$

The map τ is a C-equivariant homotopy equivalence by [11], Lemma 3.11 and Lemma 3.12. Moreover, the composite $\tau \circ \hat{\sigma}$ is a C-equivalence so $\hat{\sigma}$ must also be a homotopty equivalence. □

Corollary 1.5. *The map $t(F)(V) \to T(F)(V)$ is an $\mathcal{F}$-equivalence.* □

Let

$$\varphi : sd_C\mathrm{THH}_k(F;Y)^C \to \mathrm{THH}_k\ (F;Y^C)$$

be the map which restricts a k-simplex

$$f : S^{i_oR} \wedge \ldots \wedge S^{i_kR} \to F(S^{i_o})^{(C)} \wedge \ldots F(S^{i_k})^{(C)} \wedge Y$$

to its C-fixed points, $\varphi(f) = f^C$. The fixed set $sd_C\mathrm{THH}_\bullet(F;Y)^C$ has an obvious cyclic structure and φ is a cyclic map. The resulting S^1-action on $\mid sd_C\mathrm{THH}_\bullet\,(F;Y)\mid^C$ is the quotient $\mathbb{R}/\mathbb{Z}$-action associated to the natural $\mathbb{R}/c\mathbb{Z}$-action on$\mid sd_C\mathrm{THH}_\bullet(F;Y)\mid$. Now [11], Lemma 1.11 shows that the composite

$$(1.6)\quad \varphi_C : \rho_C^* \mid \mathrm{THH}_\bullet(F;Y)\mid^C \xrightarrow{D^{-1}} \mid sd_C\mathrm{THH}_\bullet(F;Y)\mid^C \xrightarrow{\varphi} \mid \mathrm{THH}_\bullet(F;\rho_C^*(Y^C))\mid$$

is an S^1-map; here $\rho_C^*(-)$ was defined in the beginning of this section.

Proof of Proposition 1.3. We first consider the S^1-prespectrum $t(F)$. Using (1.6) we get an S^1-map

$$\varphi : \rho_C^* t(F)(V)^C \to t(F)(\rho_C^*(V^C)).$$

We show first that it induces an $\mathcal{F}$-equivalence

$$\varinjlim \Omega^{V^C}\varphi : \varinjlim \Omega^{V^C} t(F)(V+Z)^C \to \varinjlim \Omega^{V^C} t(F)(\rho_C^*(V^C+Z))$$

for any $Z \subset \mathcal{U}^C$. Let $C \subseteq G \subset S^1$. We must argue that $\varinjlim \Omega^{V^C}\varphi$ is a G-homotopy equivalence. To simplify we consider only the case $Z = 0$ - the general case is only notationally more complicated.
Suppose $\Gamma = \rho_C^{-1}(G/C)$ so that $|G|=|\Gamma| \cdot |C|$. Then $sd_G = sd_\Gamma sd_C$ and there is a commutative diagram

$$\begin{array}{ccc} \rho_C^* \mid sd_G \mathrm{THH}_\bullet(F; S^V) \mid^C & \xrightarrow{\hat{\varphi}} & \mid sd_\Gamma \mathrm{THH}_\bullet(F; \rho_C^*(S^{V^C})) \mid \\ \cong\downarrow D_\Gamma & & \cong\downarrow D_\Gamma \\ \rho_C^* \mid sd_C \mathrm{THH}_\bullet(F; S^V) \mid^C & \xrightarrow{\varphi} & \mid \mathrm{THH}_\bullet(F; \rho_C^*(S^{V^C})) \mid \end{array}$$

with $\hat{\varphi}$ defined on simplices as above, $\hat{\varphi}(f) = f^C$. It suffices to see that $\varinjlim \Omega^{V^C}\hat{\varphi}$ is a Γ-homotopy equivalence. As in the proof of Proposition 1.2, it is enough to prove the corresponding statement for $\varinjlim \Omega^{V^C}\hat{\varphi}_k$. The fiber of $(\Omega^{V^C}\varphi_k)^G$ is

(*) $$\operatorname{holim}_{\to} \mathrm{Map}_G(S^{V^C} \wedge S^{iR}/S^{iR^C}, F(S^{i_o})^{(G)} \wedge \ldots \wedge F(S^{i_k})^{(G)} \wedge S^V)$$

where $R = \mathbb{R}G, R^C = (\mathbb{R}G)^C \cong \mathbb{R}\Gamma$ and $i = \sum i_\nu$. We must prove that the connectivity of this mapping space tends to ∞ as V runs through $\mathcal{U}$.
Given a space A, we write $\mathrm{conn}(A)$ for the maximal i with $\pi_{i-1}(A) = 0$. When $A = \mathrm{Map}_G(X, Y)$ it is easily seen that

$$\mathrm{conn}(A) \geq \min\Big\{\mathrm{conn}(Y^K) - \mathrm{dim}X^K | K \subseteq G\Big\}$$

e.g. by equivariant obstruction theory. We want to apply this to the space in (*). If $K \supseteq C$ then $\dim(S^{V^C} \wedge S^{iR}/S^{iR^C})^K = \dim V^K$ whereas

$$[\mathrm{conn}(S^V \wedge L(S^{i_o})^{(G)} \wedge \ldots \wedge L(S^{i_k})^{(G)})^K = \dim V^K + i \mid G : K \mid$$

and the difference tends to ∞ with $(i_o, \ldots, i_k)$. If $K \not\supseteq C$ then

$$\dim(S^{V^C} \wedge S^{iR}/S^{iR^C})^K = \dim V^{CK} + i\dim R^K = \dim V^{CK} + i \mid G : K \mid .$$

In this case, CK is strictly larger than K so that the difference $\dim V^K - \dim V^{CK}$ tends to infinity when V runs through $\mathcal{U}$. This proves that $\varinjlim \Omega^{V^C}\hat{\varphi}$ is a G-homotopy equivalence. The spectrification of $(t(F), \varphi)$ is the pair $(T(F), \varphi)$. It follows from Proposition 1.2 that this is a p-cyclotomic spectrum. □.

Given a G-spectrum T indexed on $\mathcal{U}$, the inclusion $i : \mathcal{U}^G \to \mathcal{U}$ induces a G-spectrum i^*T indexed on $\mathcal{U}^G$, i.e. on G-trivial representations. Conversely, a G-spectrum E indexed on $\mathcal{U}^G$ induces a G-spectrum i_*E indexed on $\mathcal{U}$ by setting

$$(i_*E)(V) = \varinjlim \Omega^W S^{V+W-V^G-W^G} E(V^G + W^G)$$

cf. [23], ch II.1. These constructions are adjoint. In particular if we begin with a G-spectrum T- indexed on $\mathcal{U}$ we have the G-equivariant counit

$$\varepsilon : i_*i^*T \to T$$

This need not be a G-homotopy equivalence. However we always have

Lemma 1.7. *There is a natural G-homotopy equivalences of spectra indexed on $\mathcal{U}$*

$$\varepsilon : i_*(i^*T \wedge EG_+) \to T \wedge EG_+$$

provided i^*T is bounded from below.

Proof. The map ε is induced from the structure maps

$$S^{l(R-\mathbb{R})} \wedge T(\mathbb{R}^l) \wedge EG_+ \to T(lR) \wedge EG_+$$

where $R = \mathbb{R}G$. As both sides are free G-spaces it suffices to show that, non-equivariantly, the listed map is $l\cdot|\, G|\, + k(l)$ connected with $k(l) \to \infty$ as $l \to \infty$. Now, $EG_+ \simeq S^0$ and we have the diagram

$$\begin{tikzcd}
\Omega^{l(R-\mathbb{R})}(S^{l(R-\mathbb{R})} \wedge T(\mathbb{R}^l)) \arrow[rr] & & \Omega^{l(R-\mathbb{R})}T(lR) \\
& T(\mathbb{R}^l) \arrow[ul, "\mathrm{susp}"'] \arrow[ur, "\simeq G"'] &
\end{tikzcd}$$

Since the suspension is approximately $2l$-connected the map in (*) becomes roughly $l\cdot|G|\, +1$-connected. $\square$

The functors $i_*(-)$ and $(-) \wedge EG_+$ commute since, quite trivially, this is the case for their adjoint functors i^* and $F(EG_+, -)$. The equivariant transfer induces a homotopy equivalence

$$i^*T \wedge_G EG_+ \to [i_*(i^*T) \wedge EG_+]^G$$

by [23], p.97. Thus we obtain

Corollary 1.8. *The equivariant transfer gives a homotopy equivalence between* $i^*T \wedge_G EG_+$ *and* $(T \wedge EG_+)^G$. □

For a p-cyclotomic spectrum (T, φ) we have for $Z \subset \mathcal{U}^{C_{p^n}}$ and $V \subset \mathcal{U}$ the composition

$$\begin{aligned}
\rho^*_{C_p} T(Z)^{C_p} &= \rho^*_{C_p} \varinjlim_{V \subset \mathcal{U}} \left[\Omega^{V-Z} T(V)\right]^{C_p} \\
&\xrightarrow{\text{fix}} \varinjlim_{V \subset \mathcal{U}} \Omega^{\rho^*_{C_p}(V^{C_p}-Z)} T(V)^{C_p} \\
&\xrightarrow{\varphi} \varinjlim_{V \subset \mathcal{U}} \Omega^{\rho^*_{C_p}(V^{C_p}-Z)} T\left(\rho^*_{C_p} V^{C_p}\right) \\
&= T\left(\rho^*_{C_p} Z\right).
\end{aligned}$$

It induces in particular a map of non-equivalent spectra

$$(1.9) \qquad \Phi : T^{C_{p^n}} \longrightarrow T^{C_{p^{n-1}}}$$

Let $T_{hC_{p^n}}$ denote the homotopy orbit spectrum

$$T_{hC_{p^n}} = i^*T \wedge_{C_{p^n}} EC_{p^n+}$$

The next result will play an important role throughout the rest of this paper.

Theorem 1.10. *For each bounded below p-cyclotomic spectrum there is a cofibration of non-equivariant spectra*

$$T_{hC_{p^n}} \longrightarrow T^{C_{p^n}} \xrightarrow{\Phi} T^{C_{p^{n-1}}}$$

Proof. The cofibration

$$EC_{p^n+} \longrightarrow S^0 \longrightarrow \Sigma EC_{p^n}$$

induces a cofibration of C_{p^n}-equivariant spectra

$$T \wedge EC_{p^n+} \longrightarrow T \longrightarrow T \wedge \Sigma EC_{p^n}$$

so induces a cofibration upon taking C_{p^n} fixed sets. According to Corollary 1.8,

$$(T \wedge EC_{p^n+})^{C_{p^n}} \simeq T_{hC_{p^n}}$$

so it remains to calculate $(T \wedge \Sigma EC_{p^n})^{C_{p^n}}$. We shall use the general fact that for arbitrary C_{p^n}-spaces X and Y,

$$\mathrm{Map}_{C_{p^n}}(X, Y \wedge \Sigma EC_{p^n}) \simeq \mathrm{Map}_{C_{p^n}/C_p}(X^{C_p}, Y^{C_p}) \tag{1.1}$$

by the map which restricts a map f to its C_p-fixed set. This follows from equivariant obstruction theory upon using that X/X^{C_p} is C_{p^n}-free (in the based sense) and that ΣEC_{p^n} is non-equivariantly contractible. Now we have:

$$\begin{aligned}(T \wedge \Sigma EC_{p^n+})^{C_{p^n}}(\mathbb{R}^l) &= \varinjlim_V \mathrm{Map}_{C_{p^n}}(S^V, T(V \oplus \mathbb{R}^l) \wedge \Sigma EC_{p^n}) \\ &\simeq \varinjlim_V \mathrm{Map}_{C_{p^{n-1}}}(S^{V^{C_p}}, T(V \oplus \mathbb{R}^l)^{C_p}) \\ &\simeq \varinjlim_V \mathrm{Map}_{C_{p^{n-1}}}(S^{V^{C_p}}, T(V^{C_p} \oplus \mathbb{R}^l))\end{aligned}$$

This conclude the proof. □.

For a p-cyclotomic spectrum we have the two commuting maps

$$D, \Phi : T^{C_{p^n}} \to T^{C_{p^{n-1}}}$$

with D being the inclusion of fixed sets, and Φ the map from (1.9).

Definition 1.12. The topological cyclic homology associated to a p-cyclotomic spectrum T is the spectrum

$$\mathrm{TC}\,(T, p) = (\operatorname*{holim}_{\overleftarrow{D}} \ T^{C_{p^n}})^{h\Phi}$$

In the above definition the superscript $h\Phi$ indicates homotopy fixed set, i.e. the homotopy fiber of

$$\Phi - \ \mathrm{id} \ : \operatorname{holim} T^{C_{p^n}} \to \operatorname{holim} \ T^{C_{p^n}}$$

It is easy to see that the role of D and Φ can be interchanged, i.e. that

$$(\operatorname*{holim}_{\overleftarrow{D}} T^{C_{p^n}})^{h\Phi} \simeq (\operatorname*{holim}_{\overleftarrow{\Phi}} T^{C_{p^n}})^{hD}$$

For a functor F with smash product we have the cyclotomic spectrum $T(F)$ of Proposition 1.3 and we define

$$\mathrm{TC}(F, p) = \mathrm{TC}(T(F), p)[0, \infty) \tag{1.13}$$

the $(-1)-$connected cover. It is called the *topological cyclic homology at* p of F.

In the above definitions we have singled out a fixed prime p to start with, but there is also a topological cyclic homology functor TC(F), defined for so called cyclotomic spectra rather than p-cyclotomic spectra, which makes use of all the finite subgroups of the circle. This is explained in [22], sect. 2 where it its also proved that one has

$$\mathrm{TC}(F)_p^{\wedge} \simeq \mathrm{TC}(F,p)_p^{\wedge}. \tag{1.14}$$

We shall often use the notation $\mathrm{TC}(F)_p^{\wedge}$ instead of $\mathrm{TC}(F,p)_p^{\wedge}$.

In [11] we defined for each functor with smash product a map

$$\mathrm{Trc} : K(F) \to \mathrm{TC}(F,p)$$

and showed it was a map of spectra. The spectrum structure on $K(F)$ is constructed by giving $BGL(F)^+$ a Γ-structure (in the sense of Segal) and then using the infinite loop space machine of [27] or [34]. There is a compatible Γ-space structure on $\mathrm{TC}(F,p)$ so that Trc becomes a map of Γ-spaces, and hence a map of spectra. The Γ-structure on $\mathrm{TC}(F,p)$ is induced from a set of C_{p^n}-equivariant Γ-structures (or $\Gamma_{C_{p^n}}$-structures) on $\mathrm{THH}\,(F)$, one for each n. Thus one must show that these equivariant Γ-space structures lead to C_{p^n}-equivariant spectra, homotopy equivalent to the restrictions of the S^1-spectrum $T(F)$. This is explained in [22], sect. 1.6.

§ 2 Skeleton spectral sequences

This section presents the spectral sequences on which the rest of the paper is based, namely the spectral sequences associated to the terms in the norm fibration of an equivariant spectrum. We refer the reader to [23] for questions about the stable G-category. Our G-spectrum will be indexed on a complete universe, and G will be finite unless otherwise specified.

For a G-spectrum X and a pointed G-space E (with $* \in E^G$), we can form the G-spectra $X \wedge E$ and $\mathrm{Map}\,(E,X)$:

$$(X \wedge E)_W = \varinjlim \Omega^V (X_{V+W} \wedge E), \quad \mathrm{Map}(E,X)_W = \mathrm{Map}(E, X_W)$$

One also has the *fixed point* - and *orbit* spectra of X defined by

$$X^G = |X|^G\ , \quad X/G = |X|/G \tag{2.1}$$

where $|X|$ denotes the spectrum indexed by G-trivial vector spaces, also sometimes denoted i^*X. Following Greenlees and May, [19], one defines

$$\begin{aligned} \mathbb{H}^\bullet(G,X) &= X^{hG} = \mathrm{Map}(EG_+,X)^G && \text{(homotopy fixed points)} \\ \mathbb{H}_\bullet(G,X) &= X_{hG} = |X| \wedge EG_+/G && \text{(homotopy orbit)} \\ \hat{\mathbb{H}}(G,X) &= (\Sigma(EG) \wedge \mathrm{Map}(EG_+,X))^G && \text{(Tate spectrum)} \end{aligned} \tag{2.2}$$

where EG is the free, contractible G-space as usual. The standard cofibration

$$EG_+ \to S^0 \to \Sigma(EG)$$

then defines a cofibration of spectra (cf. [19])

$$X_{hG} \xrightarrow{N^h} X^{hG} \xrightarrow{\Psi} \hat{\mathbb{H}}(G, X) \tag{2.3}$$

Indeed, one has only to identify $[EG_+ \wedge \operatorname{Map}(EG_+\,,\, X)]^G$ with X_{hG}. This uses the transfer homotopy equivalence

$$\tau : |X| \wedge_G EG_+ \xrightarrow{\simeq} (i_*|X| \wedge EG_+)^G$$

from [23, p.97] or [1], the G-homotopy equivalence

$$\varepsilon \wedge 1 : i_*|X| \wedge EG_+ \xrightarrow{\simeq} X \wedge EG_+ \tag{2.4}$$

and the G-homotopy equivalence

$$EG_+ \wedge X \xrightarrow{\simeq} EG_+ \wedge \operatorname{Map}(EG_+, X),$$

induced from the projection $EG_+ \to S^0$. Notice that (2.4) spells out to the assertion that

$$\varinjlim_V \Omega^V \left(S^{V-V^G} X_{V^G+\mathbb{R}^n} \wedge EG_+\right) \to \varinjlim_V \Omega^V (X_{V+\mathbb{R}^n} \wedge EG_+).$$

is a G-equivalence in a range $n + k(n)$ where $k(n) \to \infty$. This is the case for $\mathbb{R}G$-modules W, as the structure maps

$$S^W \wedge X_{\mathbb{R}^n} \to X_{\mathbb{R}^n+\mathbf{W}}$$

are $n + |W| + k(n + |W|)$ connected. This connectivity condition is satisfied when $|X|$ is bounded below (e.g. connected) which we shall always assume.
It is important to notice for our applications later in the paper that

$$N^h : X_{hG} \to X^{hG}$$

factors over $X^G \to X^{hG}$; this follows directly from the definitions.
For the equivariant sphere spectrum $X = S^0_G$, the terms in (2.3) can be evaluated from the affirmed Segal conjecture [12] to be

$$\begin{aligned} X_{hG} &= \Sigma^\infty BG_+ \\ X^{hG} &= \bigvee_{(H)} \Sigma^\infty BWH_+\ ,\ WH = N_G H/H \\ \hat{\mathbb{H}}(G; X) &= \bigvee_{(H), H\neq 1} \Sigma^\infty BWH_+ \end{aligned} \tag{2.5}$$

where $\Sigma^\infty BG_+$ etc. denote the suspension spectra, and where the wedge sum runs over conjugacy classes of subgroups.

The homotopy groups of the terms in (2.2) can be approximated by spectral sequences. For X_{hG} and X^{hG} we can either filter EG by its skeletons or take a Postnikov decomposition of $|X|$. The resulting spectral sequences have

$$\begin{aligned} E^2_{s,t}(X_{hG}) &= H_s(G;\pi_t X) \Longrightarrow \pi_{s+t}(X_{hG}) \\ E^2_{s,t}(X^{hG}) &= H^{-s}(G;\pi_t X) \Longrightarrow \pi_{s+t}(X^{hG}) \end{aligned} \tag{2.6}$$

with E^2-terms being the homology (resp. cohomology) groups of the trivial G-modules $\pi_t X = \pi_t(|X|)$. The convergence of the second spectral sequence is dependent upon the vanishing of the usual $\varprojlim^{(1)}$-term,

$$\varprojlim{}^{(1)}[S^n \wedge E_k G_+\,,\, X]^G = 0 \tag{2.7}$$

Our main calculations below are of homotopy groups with finite coefficients where S^n in (2.7) gets replaced with the Moore space S^n/p. In this case (2.7) is satisfied when X has finite type because $\varprojlim^{(1)}$ vanishes on inverse systems of finite groups.

The homotopy groups of the third term in (2.3) is also approximated by a spectral sequence. This follows from Greenlees' "filtration" of the spectrum $\tilde{E}G = \Sigma EG$, [18]: There are maps of (suspension) spectra

$$\begin{tikzcd}[column sep=small] \dots \arrow[r] & F_{-2} \arrow[r] & F_{-1} \arrow[rr] \arrow[dr] & & F_0 \arrow[r] & F_1 \arrow[r] & F_2 \arrow[r] & \dots \\ & & & S^0 \arrow[ur] & & & & \end{tikzcd} \tag{2.8}$$

with $\varinjlim F_k = \tilde{E}G$. Here F_k is the cofiber of $E_{k-1}G_+ \to S^0$ for $k \geq 0$ and F_k is the Spanier-Whitehead dual of $\tilde{E}_{-k}G$ when $k \leq 0$; $E_kG \subset EG$ denotes the k-skeleton. The successive cofibers of (2.8) are

$$F_k/F_{k-1} = G_+ \wedge \bigvee S^k$$

and the chain complex of homology of the quotient spectra,

$$\dots \to H_k(F_k, F_{k-1}) \to H_{k-1}(F_{k-1}, F_{k-2}) \to \dots$$

is a complete resolution of $\mathbb{Z}$ by free $\mathbb{Z}$ G-modules in the sense of [13].

If M is any $\mathbb{Z}G$-module and $P_k = H_k(F_k, F_{k-1})$ then the homology of the complex $P_* \otimes_{\mathbb{Z}G} M$ is by definition the Tate homology groups

$$H_k(P_* \otimes_{\mathbb{Z}G} M) = \hat{H}_{k-1}(G;M)$$

These are equal to the usual group homology when $k > 1$ and are equal to the group cohomology $H^{-k}(G;M)$ when $k < 0$. We note that $\hat{H}_{k-1}(G;M) = \hat{H}^{-k}(G;M)$ for all k.
We use (2.8) to filter the G-spectrum $\tilde{E}G \wedge \mathrm{Map}(EG_+, X)$ and hence $\hat{\mathbb{H}}(G;X)$, and get a spectral sequence

$$\hat{H}^{-s}(G;\pi_t X) \Rightarrow \pi_{s+t}\Big(\hat{\mathbb{H}}(G;X)\Big) \tag{2.9}$$

It is a whole plane spectral sequence and is not always convergent, even when we use homotopy groups with finite coefficients and X has finite type. A general criteria for convergence can be found in an unpublished paper by Boardman [3], and in [21] where the spectral sequence is studied when G is the circle group and $X = J_G$ is the equivariant periodic image of J spectrum. For our use of (2.9), $G = C_{p^n}$ and $X = \mathrm{THH}(\mathbb{Z}_p)$, the spectral sequence does in fact converge as we shall see later on. To see that the E^2-term is as claimed we use that the functor $G_+ \wedge (-)$ is right adjoint to the functor which forgets the G-action, cf. [23, p.89]. Thus the E^1–term is

$$\pi_{s+t}((F_s/F_{s-1} \wedge \mathrm{Map}(EG_+, X))^G) = \hat{C}_{s-1}(G;\pi_t X)$$

and hence the E^2-term is the Tate homology groups

$$\hat{H}_{s-1}(G;\pi_t X) = \hat{H}^{-s}(G;\pi_t X)$$

as claimed in (2.9). Since in (2.8), $F_0 = S^0$ the filtration used to define the skeleton spectral sequence (2.9) restricts to a filtration

$$\ldots \to F_{-k} \to F_{-k+1} \to \ldots \to S^0 \tag{2.10}$$

which in turn induces a filtration on $S^0 \wedge \mathrm{Map}(EG_+, X)$ and hence a spectral sequence with abutment $\pi_*\mathrm{Map}(EG_+, X)$ and E^2-term equal to $\hat{H}^{-s}(G;\pi_t X)$, $s \leq 0$. Moreover, the map from $\mathrm{Map}_G(EG_+, X)$ to $\hat{\mathbb{H}}(G;X)$ induces a map from (2.9) to this new spectral sequence which on the E^2-level is the standard homomorphism

$$H^{-s}(G;\pi_t X) \to \hat{H}^{-s}(G;\pi_t X), \ s \leq 0 \tag{2.11}$$

This homomorphism, we remember, is an isomorphism when $s < 0$,

Lemma 2.12. *The spectral sequence induced from the filtration* (2.10) *of* $[S^0 \wedge \mathrm{Map}(EG_+, X)]^G$ *is isomorphic to the spectral sequence* (2.6) *induced from the skeleton filtration* $E_k G_+$ *of* EG_+

Proof. The spectral sequence (2.6) comes from the exact couple

$$\begin{tikzcd} \pi_*\big(\mathrm{Map}_G(E_kG_+,X),X^{hG}\big) \arrow[rr] & & \pi_*\big(\mathrm{Map}_G(E_{k-1}G_+,X),X^{hG}\big) \arrow[dl] \\ & \pi_*\mathrm{Map}_G(E_kG/E_{k-1}G,X) \arrow[ul] & \end{tikzcd} \tag{2.13}$$

which converges to $\pi_*\big(X^{hG}\big)$.

The G-homotopy equivalence $E_kG_+\wedge EG_+ \underset{G}{\simeq} E_kG_+$ induces the first equivalence in

$$\begin{aligned} \mathrm{Map}(E_kG_+,X) &\underset{G}{\simeq} \mathrm{Map}(E_kG_+,\ \mathrm{Map}(EG_+X)) \\ &\underset{G}{\simeq} D(E_kG_+)\wedge\ \mathrm{Map}(EG_+,X) \end{aligned}$$

and the second is a standard fact in Spanier-Whitehead duality theory, since $D(E_kG_+) = \mathrm{Map}\big(E_kG_+,S^0\big)$. There are cofibration sequences

$$S^0 \to \tilde{E}_kG \to S^1\wedge(E_{k-1}G_+)$$

with dual cofiberings

$$D(\tilde{E}_kG) \to S^0 \to D(E_{k-1}G_+)$$

With the notation from (2.8), $F_{-k} = D(\tilde{E}_kG)$, so we get

$$\begin{aligned} \Sigma^{-1}(\mathrm{Map}(E_{k-1}G_+,X)/\ \mathrm{Map}(EG_+,X)) &\underset{G}{\simeq} \Sigma^{-1}(D(E_{k-1}G_+))/S^0\wedge\ \mathrm{Map}(EG_+,X) \\ &\underset{G}{\simeq} D(\tilde{E}_kG)\wedge\ \mathrm{Map}(EG_+,X) \\ &\underset{G}{\simeq} F_{-k}\wedge\ \mathrm{Map}(EG_+,X) \end{aligned}$$

This shows that (2.13) is precisely the exact couple coming from the filtration $[F_{-*}\wedge\ \mathrm{Map}(EG_+,X)]^G$ of $\mathrm{Map}_G(EG_+\ ,\ X)$ and establishes the result. □

The skeleton spectral sequence for calculating the homotopy groups of

$$(EG_+\wedge X)^G \simeq EG_+\wedge_G X = X_{hG}$$

has

$$E^2_{s,t}(X_{hG}) = H_s(G;\pi_tX) \Rightarrow \pi_{s+t}(X_{hG})$$

Since $EG_+\wedge X \simeq_G EG_+\wedge F(EG_+,X)$ and $E_kG_+/E_{k-1}G_+ = \Sigma^{-1}F_{k+1}/F_k$ the map

$$F_{s+1}\wedge\ \mathrm{Map}(EG_+,X) \to \begin{cases} F_{s+1}/F_0\wedge\ \mathrm{Map}(EG_+,X) & ,\ s\geq 0 \\ * & ,\ s<0 \end{cases}$$

induces a map of spectral sequences

$$\partial_* : E^r_{s+1,t}(\hat{\mathbb{H}}(G,X)) \to E^r_{s,t}(X_{hG})$$

which is injective for $s \geq 0$ and $r \geq 2$. On the E^∞-level it is associated to the natural map from $\Sigma^{-1}\hat{\mathbb{H}}(G,X)$ to X_{hG}, and the injectivity is in agreement with exactnes in

$$\pi_* X^{hG} \xrightarrow{\Psi_*} \pi_*\hat{\mathbb{H}}(G,X) \xrightarrow{\partial_*} \pi_{*-1}X_{hG}$$

From Lemma 2.12 it follows that there is a map of spectral sequences

$$E^r_{s,t}(X^{hG}) \to E^r_{s,t}(\hat{\mathbb{H}}(G,X))$$

which is surjective for $s \leq 0$ and all r. An element $\alpha \in E^\infty_{-s,t}(X^{hG}), s \geq 0$ is in the kernel of

$$E^\infty_{-s,t}(\Psi) : E^\infty_{-s,t}(X^{hG}) \to E^\infty_{-s,t}(\hat{\mathbb{H}}(G,X)) \tag{2.14}$$

precisely if there exists an $r > s$ and an element

$$\beta \in E^r_{r-s,t-r+1}(\hat{\mathbb{H}}(G,X))$$

with $d^r(\beta) = \alpha$. Now

$$E^r_{r-s,t-r+1}(\hat{\mathbb{H}}(G,X)) \to E^r_{r-s-1,t-r+1}(X_{hG})$$

and we can consider β as an infinite cycle in the skeleton spectral sequence for X_{hG}. We argue below that β survives to $E^\infty(X_{hG})$, i.e. that $\beta \notin \operatorname{Im} d^m$ for any $m > r$, so that β represents an element of $\pi_{t-s}(X_{hG})$ which under $N^h_* : \pi_{t-s}(X_{hG}) \to \pi_{t-s}(X^{hG})$ is mapped to an element representing α. Thus we can summarize the situation in

Theorem 2.15. *Suppose*

$$\alpha \in \operatorname{Ker}\{E^\infty_{-s,t}(\Psi) : E^\infty_{-s,t}(X^{hG}) \to E^\infty_{-s,t}(\hat{\mathbb{H}}(G,X))\}$$

with $s \geq 0$. Then there exists an $r > s$ such that α belongs to the image of the differential

$$\hat{d}^r : E^r_{r-s,t-r+1}(\hat{\mathbb{H}}(G,X)) \to E^r_{-s,t}(\hat{\mathbb{H}}(G,X))$$

Moreover, if $\hat{d}^r(\beta) = \alpha$ then the element $\partial_*(\beta)$ associated to β under the injection

$$E^r_{r-s,t-r+1}(\hat{\mathbb{H}}(G,X)) \xrightarrow{\partial_*} E^r_{r-s-1,t-r+1}(X_{hG})$$

survives to $E^{\infty}_{r-s-1,t-r+1}(X_{hG})$ and is represented by an element of $\pi_{t-s}(X_{hG})$. The representative may be chosen such that it's image under N^h_* in $\pi_{t-s}(X^{hG})$ represents $\alpha \in E^{\infty}_{-s,t}(X^{hG})$.

Proof. Let us write

$$J_s = [F_s \wedge \text{Map}(EG_+, X)]^G$$

so that $E^r(\hat{\mathbb{H}}(G,X))$ is the spectral sequence of the exact couple associated with the string of cofibrations:

$$\dots \to J_{-s} \to J_{-s+1} \to \dots \to J_0 \to J_1 \to \dots \to J_s \to \dots$$

Then $E^r(X_{hG})$ is the spectral sequence associated with

$$\Sigma^{-1}J_1/J_0 \to \Sigma^{-1}J_2/J_0 \to \dots \to \Sigma^{-1}J_s/J_0 \to \dots$$

The differential $\hat{d}^r$ is equal to the additive relation

$$\pi_*\Sigma^{-1}J_{r-s}/J_{r-s-1} \to \pi_*J_{r-s-1} \leftarrow \pi_*J_{-s} \to \pi_*J_{-s}/J_{-s-1}$$

Since $s \geq 0, \beta \in \pi_*\Sigma^{-1}J_{r-s}/J_{r-s-1}$ maps to zero under the composition

$$\pi_*\Sigma^{-1}J_{r-s}/J_{r-s-1} \to \pi_*J_{r-s-1} \to \pi_*J_{r-s-1}/J_0$$

The exact homotopy sequence of the cofibration

$$\Sigma^{-1}J_{r-s}/J_0 \to \Sigma^{-1}J_{r-s}/J_{r-s-1} \to J_{r-s-1}/J_0$$

shows that there exist $\bar{\beta} \in \pi_*\Sigma^{-1}J_{r-s}/J_0$, i.e. an infinite cycle in the spectral sequence $E^r(X_{hG})$. We have left to show that $\bar{\beta}$ survives to $E^{\infty}_{r-s-1,t-r+1}(X_{hG})$. Suppose there is a non-trivial differential

$$d^m : E^m_{k+m,l-m+1}(X_{hG}) \to E^m_{k,l}(X_{hG}),\ k \geq 0$$

We show that the differential

$$\hat{d}^m : E^m_{k+m+1,l-m+1}(\hat{\mathbb{H}}(G,X)) \to E^m_{k+1,l}(\hat{\mathbb{H}}(G,X))$$

is then non-zero on the corresponding element.

The two differentials are represented by the additive relations

$$\partial : \pi_*\Sigma^{-2}J_{k+m}/J_{k+m-1} \to \pi_*\Sigma^{-1}J_{k+m}/J_0 \leftarrow \pi_*\Sigma^{-1}J_k/J_0 \to \pi_*\Sigma^{-1}J_k/J_{k-1}$$
$$\hat{\partial} : \pi_*\Sigma^{-2}J_{k+m}/J_{k+m-1} \to \pi_*\Sigma^{-1}J_{k+m} \leftarrow \pi_*\Sigma^{-1}J_k \to \pi_*\Sigma^{-1}J_k/J_{k-1}$$

respectively. Since

$$\begin{array}{ccc} J_k & \longrightarrow & J_{k+m} \\ \downarrow & & \downarrow \\ J_k/J_0 & \longrightarrow & J_{k+m}/J_0 \end{array}$$

is (co)Cartesian, the indeterminacy on ∂ agrees with the indeterminacy on $\hat{\partial}$. Finally, we consider the diagram

$$\begin{array}{ccccccc} & & \pi_*(\Sigma^{-1}J_0) & \stackrel{\cong}{\longleftarrow} & \pi_*\Sigma^{-1}J_0 & & \\ & & \downarrow & & \downarrow & & \\ & & \pi_*\Sigma^{-1}J_{k+m} & \longleftarrow & \pi_*\Sigma^{-1}J_{k+1} & & \\ & \nearrow & & & & \searrow & \\ \pi_*\Sigma^{-2}J_{k+m+1}/J_{k+m} & & \downarrow & & \downarrow & & \pi_*\Sigma^{-1}J_{k+1}/J_k \\ & \searrow & & & & \nearrow & \\ & & \pi_*\Sigma^{-1}J_{k+m}/J_0 & \longleftarrow & \pi_*\Sigma^{-1}J_{k+1}/J_0 & & \\ & & \downarrow & & \downarrow & & \\ & & \pi_*J_0 & \stackrel{\cong}{\longleftarrow} & \pi_*J_0 & & \end{array}$$

with exact columns. A simple diagram chase shows that the upper and lower additive relation have the same domains of definition, and thus in turn agree. □

To summarize, the Tate skeleton spectral sequence (2.9) contains both the spectral sequences in (2.6). In particular

$$E^2_{s,t}(\hat{\mathbb{H}}(G,X)) = E^2_{s,t}(X^{hG})\ ,\ s < 0$$
$$E^2_{s+1,t}(\hat{\mathbb{H}}(G,X)) = E^2_{s,t}(X_{hG})\ ,\ s \geq 1$$

The precise relationship for $s = 0, 1$ is governed by

$$0 \to \hat{H}^{-1}(G;\pi_t X) \to H_0(G;\pi_t X) \xrightarrow{\text{norm}} H^0(G;\pi_t X) \to \hat{H}(G;\pi_t X) \to 0$$

The differentials in $E^r_{s,t}(\hat{\mathbb{H}}(G;X))$ which cross over the fiber line $s = 0$ represent elements of $\pi_*(X_{hG})$ and $\pi_*\left(X^{hG}\right)$which are connected under $N^h : X_{hG} \to X^{hG}$, of course as always, up to filtration.

When X is an equivariant ring spectrum so is X^{hG}, and it is well-known that the spectral sequence

$$E^r_{*,*}(X^{hG}) \Rightarrow \pi_*(X^{hG})$$

has a product with d^r a derivation in the usual sense, compatible with the obvious products on $\pi_*\left(X^{hG}\right)$ and $H^*(G;\pi_* X)$. Similarly, the Tate spectrum associated to an equivariant ring spectrum is again a ring spectrum. The ring structure is defined to be the composition

$$\begin{aligned}
&\hat{\mathbb{H}}(G,X) \wedge \hat{\mathbb{H}}(G,X) \to \\
&[\tilde{E}G \wedge \tilde{E}G \wedge F(EG_+,X) \wedge F(EG_+,X)]^G \overset{(\psi\wedge 1)^{-1}\wedge\sigma}{\longrightarrow} \\
&[\tilde{E}G \wedge F(EG_+ \wedge EG_+, X \wedge X)]^G \overset{1\wedge F(d,\mu)}{\longrightarrow} \\
&[\tilde{E}G \wedge F(EG_+,X)]^G
\end{aligned}$$

Here $\psi : S^0 \to \tilde{E}G$, σ is the smash product, d is the diagonal and μ is the ring structure. The Tate skeleton spectral sequence inherits a product, compatible with the standard product on Tate cohomology groups, and the differentials become derivations.

We shall occasionally use also the norm cofibering when G is the circle group,

$$\Sigma X_{hS^1} \overset{N^h}{\to} X^{hS^1} \to \hat{\mathbb{H}}\left(S^1; X\right)$$

Apart for the changes caused by the extra suspension factor, there are obvious analogies of the above for $G = S^1$. The spectral sequences become

$$\begin{aligned}
E^2_{s,t}(\hat{\mathbb{H}}\left(S^1, X\right)) &= \hat{H}^{-s}\left(S^1; \pi_t X\right) \\
E^2_{s,t}(X^{hS^1}) &= H^{-s}\left(S^1; \pi_t X\right) \qquad (2.16) \\
E^2_{s,t}(X_{hS^1}) &= H_s\left(S^1; \pi_t X\right)
\end{aligned}$$

where $\hat{H}^*$, H^* and H_* denote group (co)homology i.e.

$$\begin{aligned}
\hat{H}^*\left(S^1; \pi_* X\right) &= P\left[t, t^{-1}\right] \otimes \pi_* X \ , \ \deg(t) = -2 \\
H^*\left(S^1; \pi_* X\right) &= H^*\left(BS^1\right) \otimes \pi_* X \\
H_*\left(S^1; \pi_* X\right) &= H_*\left(BS^1\right) \otimes \pi_* X
\end{aligned}$$

Let us finally notice that there are commutative diagrams

$$\begin{array}{ccccc}
\Sigma X_{hS^1} & \to & X^{hS^1} & \to & \hat{\mathbb{H}}(S^1; X) \\
\downarrow & & \downarrow & & \downarrow \\
X_{hC_{p^n}} & \to & X^{hC_{p^n}} & \to & \hat{\mathbb{H}}(C_{p^n}; X)
\end{array} \qquad (2.17)$$

which relate the norm sequence for $G = S^1$ with the norm sequences for the cyclic groups of order p^n. Moreover we have

Lemma 2.18. *After completion at p, the restriction maps in (2.17) define a weak homotopy equivalence*

$$\left(X^{hS^1}\right)_p^\wedge \xrightarrow{\simeq} \operatorname{holim}\left(X^{hC_{p^n}}\right)_p^\wedge$$

provided X has finite type.

Proof. We have

$$\pi_k(X^{hS^1})_p^\wedge = [S^k \wedge ES^1_+, X]^{S^1} \otimes \hat{\mathbb{Z}}_p = [S^k \wedge ES^1_+, X_p^\wedge]^{S^1}$$

and similarly with S^1 replaced by C_{p^n}, and must check that

$$[S^k \wedge ES^1_+, X_p^\wedge]^{S^1} \to \varprojlim\ [S^k \wedge ES^1_+, X_p^\wedge]^{C_{p^n}}$$

is an isomorphism. There is an equivariant Postnikov decomposition of $X_p^\wedge$,

$$X_p^\wedge \to \cdots \to Y_n \to Y_{n-1} \to \cdots \to Y_0$$

in which each fiber of $Y_n \to Y_{n-1}$ is an equivariant Eilenberg-MacLane spectrum $H_G(\pi_n(X_p^\wedge), n)$. The resulting spectral sequences are of the form

$$\begin{aligned} H^*(S^k \wedge BS^1_+; \pi_* X_p^\wedge) &\Rightarrow [S^k \wedge ES^1_+, X_p^\wedge]^{S^1} \\ H^*(S^k \wedge BC_{p^n+}; \pi_* X_p^\wedge) &\Rightarrow [S^k \wedge ES^1_+, X_p^\wedge]^{C_{p^n}} \end{aligned}$$

There are no $\lim^{(1)}$-terms in this situation because all groups are compact. The lemma now follows because

$$H^*\left(BS^1; M\right) \xrightarrow{\cong} \varprojlim\ H^*(BC_{p^n}; M)$$

for a p-complete compact abelian group M. □

§ 3 The skeleton spectral sequences for J

We denote by K the spectrum which represents p-complete periodic complex K-theory, p an odd prime; its $2n$-th space is $(BU \times \mathbb{Z})_p^\wedge$. Let g be an integer which generates the units of $\mathbb{Z}/p^2$ and let

$$\psi : K \to K$$

be the endomorphism of the spectrum K which on the $2n$-th space K_{2n} corresponds to $g^{-n}\psi^g$ where ψ^g is the Adams operation associated with a generator of $\left(\mathbb{Z}/p^2\right)^\times$.

The homotopy fixed set of ψ is the spectrum J,

$$J = \mathrm{hofib}(\psi - 1)$$

It is by [31, (8.12)] the $K\mathbb{Z}/p$ localization of the sphere spectrum. Thus it has a natural action of any compact Lie group G arising from the G-equivariant sphere spectrum S^0_G. It is a ring spectrum whose mod p homotopy ring is

$$\pi_*(J; \mathbb{F}_p) = E\{a\} \otimes P\left[b, b^{-1}\right], \tag{3.1}$$

the tensor product of an exterior algebra with generator a of degree $2p-3$ and the algebra of finite Laurant series in a generator b of degree $2p-2$. Moreover, the Bockstein operator maps b to a, $\beta_1(b) = a$.
In this section we shall calculate the differentials in the skeleton sequence with abutment $\pi_*\left(J^{hC_{p^n}}; \mathbb{F}_p\right)$. The E^2-term is

$$\begin{aligned} E^2_{*,*}\left(J^{hC_{p^n}}; \mathbb{F}_p\right) &= E\{u_n\} \otimes E\{a\} \otimes P[t] \otimes P\left[b, b^{-1}\right] \\ \deg u_n &= (-1, 0),\ \deg t = (-2, 0) \\ \deg a &= (0, 2p-3),\ \deg b = (0, 2p-2) \end{aligned} \tag{3.2}$$

The spectral sequence is situated in the 2. and 3. quadrant.

Theorem 3.3. *The non-zero differentials in $E^r_{*,*}\left(J^{hC_{p^n}}; \mathbb{F}_p\right)$ with source $E^r_{*,0}$ are*

$$\begin{aligned} (i) \quad & d^{2(p^{k+1}-1)}\left(t^{p^k}\right) = \gamma_k t^{p^{k+1}+p^k-1} a b^{p\left(\frac{p^k-1}{p-1}\right)} && , 0 \le k < n \\ (ii) \quad & d^{2p^n-1}(u_n) = \gamma t^{p^n} b^{\left(\frac{p^n-1}{p-1}\right)} && , n \ge 1 \end{aligned}$$

and the multiplicative consequences for $u_n^i t^j$, where $\gamma_k, \gamma \in \mathbb{F}_p^\times$.

The rest of the section is spent on the proof of this result. Since S^0_G is a split spectrum, in the sense that its fixed point spectrum contains S^0 itself, cf. [28], the same will be the case for its $K\mathbb{Z}/p$-localization J. Hence

$$F_G(EG_+, J) \simeq F(BG_+, J)$$

has homotopy groups equal to the J-cohomology of BG. The issue of differentials in the spectral sequence is then to determine, for a given k, the maximal l, so that the additive relation

$$J^*(B_kG, B_{k-1}G) \to \widetilde{J}^*(B_kG) \leftarrow \widetilde{J}^*(B_{l-1}G) \xrightarrow{\delta^*} J^*(B_lG, B_{l-1}G) \tag{3.4}$$

is non-zero (modulo indeterminacy). In the situation of (3.3), $G = C_{p^n}$ and we use $\mathbb{F}_p$ coefficients.

The inclusion $BC_{p^n} \to BS^1$ induces a map from X^{hS^1} to $X^{hC_{p^n}}$ and in fact a map of spectral sequences

$$E^r_{*,*}\left(X^{hS^1}; \mathbb{F}_p\right) \to E^r_{*,*}\left(X^{hC_{p^n}}; \mathbb{F}_p\right)$$

which is an injection on the E^2-level. The differentials in (3.3) (i) can be read of from the differentials in $E^r_{*,*}\left(J^{hS^1}; \mathbb{F}_p\right)$, as it turns out. We have

$$E^2_{*,*}\left(J^{hS^1}; \mathbb{F}_p\right) = P[t] \otimes E\{a\} \otimes P\left[b, b^{-1}\right]$$

Theorem 3.5. *The non-zero differentials in $E^r_{*,*}\left(J^{hS^1}; \mathbb{F}_p\right)$ are multiplicatively generated from*

$$d^{2(p^{k+1}-1)}\left(t^{p^k}\right) = \gamma_k t^{p^{k+1}+p^k-1} a b^{p\left(\frac{p^k-1}{p-1}\right)}$$

with $\gamma_k \in \mathbb{F}_p^\times$.

The skeletons $B_{2k}S^1$ and $B_{2k+1}S^1$ are both equal to $\mathbb{C}P^k$ which has torsion free K-groups. Hence there are exact sequences

$$(3.6)\quad 0 \to J^{2s}(\mathbb{C}P^n; \mathbb{F}_p) \to K^{2s}(\mathbb{C}P^n; \mathbb{F}_p) \xrightarrow{\psi-1} K^{2s}(\mathbb{C}P^n; \mathbb{F}_p) \to J^{2s+1}(\mathbb{C}P^n; \mathbb{F}_p) \to 0$$

We have

$$(3.7)\qquad K^*(\mathbb{C}P^n; \mathbb{F}_p) = \mathbb{F}_p\left[u, u^{-1}, \lambda\right] / \left\langle \lambda^{n+1} \right\rangle$$

where $u \in K^2(pt)$ is the Bott class and where $\lambda \in \tilde{K}^0(\mathbb{C}P^\infty)$ is the reduced Hopf bundle. Thus

$$\psi(u) = g^{-1}u, \quad \psi(\lambda) = (1+\lambda)^g - 1$$

The formula (3.7) works with any kind of coefficients. If we replace $\mathbb{F}_p$ by $\mathbb{Q}$ then $l(\lambda) = \log(1+\lambda)$ makes sense, and

$$\psi(u^s l(\lambda)^s) = u^s l(\lambda)^s$$

so that $u^s l(\lambda)^s \in \tilde{J}^{2s}(\mathbb{C}P^n; \mathbb{Q})$ for all n.

The next technical lemma is the basis for our proof of (3.5). Let

$$\psi : \mathbb{F}_p[[\lambda]] \to \mathbb{F}_p[[\lambda]]$$

be the ring homomorphism with $\psi(\lambda) = (1+\lambda)^g - 1$ where g generates $\left(\mathbb{Z}/p^2\right)^\times$. Let $v_\lambda : \mathbb{F}_p[[\lambda]] \to \mathbb{Z} \cup \{\infty\}$ be the valuation with

$$v_\lambda(f(\lambda)) = n \Leftrightarrow f(\lambda) = \sum_{i \geq n} a_i \lambda^i$$

and $a_n \neq 0$.

Lemma 3.8 *There are elements $f_{p^n}(\lambda) \in \mathbb{F}_p[[\lambda]]$ for $n \geq 0$ which satisfy*

$$\begin{array}{ll} (i) & v_\lambda(f_{p^n}(\lambda)) = p^n \\ (ii) & v_\lambda(\psi(f_{p^n}(\lambda)) - g f_{p^n}(\lambda)) = p^{n+1} + p^n - 1 \end{array}$$

Proof. The proof is basically induction over n but with $n = 0, 1, 2$ as special cases. We begin by explaining these. Consider the following elements of $\mathbb{Q}[[\lambda]]$,

$$(3.9) \qquad \begin{aligned} F_1(\lambda) &= l(\lambda) \qquad\qquad (l(\lambda) = \log(1+\lambda)) \\ F_p(\lambda) &= pl(\lambda) - l(\lambda)^{p^2-p+1} \\ F_{p^2}(\lambda) &= p^2 l(\lambda) - (pl(\lambda))^{p^2-p+1} - \frac{1}{2} l(\lambda)^{p^3+p^2-2p+1} \end{aligned}$$

We shall see below that

$$(3.10) \qquad \begin{aligned} F_1(\lambda) &= f_1(\lambda) + \frac{1}{p}\lambda^p + \ldots \\ F_p(\lambda) &= f_p(\lambda) - \frac{1}{2p}\lambda^{p^2+p-1} + \ldots \\ F_{p^2}(\lambda) &= f_{p^2}(\lambda) - \frac{1}{2p}\lambda^{p^3+p^2-1} + \ldots \end{aligned}$$

with $f_{p^j}(\lambda) \in \mathbb{Z}[\lambda]$ of degree $p^{j+1} + p^j - 1$; the dots indicate terms of higher degree. Since

$$\psi(\lambda^n) = g^n \lambda^n + \text{higher terms}$$

and since $\psi(F_{p^n}(\lambda)) = gF_{p^n}(\lambda)$ we have

$$v_\lambda(\psi(f_{p^n}(\lambda)) - g f_{p^n}(\lambda)) = p^{n+1} + p^n - 1$$

when $n \leq 2$, and in fact

$$\psi(f_{p^n}(\lambda)) - g f_{p^n}(\lambda) = a_{p^n} \lambda^{p^{n+1}+p^n-1} + \text{higher terms}$$

for integers a_{p^n} which can be calculated to be

$$
\begin{aligned}
a_1 &= (g - g^p)/p = (-g)\left(g^{p-1} - 1\right)/p \\
a_p &= \left(g^{p^2+p-1} - g\right)/p = 2g\left(g^{p-1} - 1\right)/p \qquad (3.11)\\
a_{p^2} &= \left(g^{p^3+p^2-1} - g\right)/2p = g\left(g^{p-1} - 1\right)/p
\end{aligned}
$$

Consider the ring homomorphism ψ^p of $\mathbb{Z}[[\lambda]]$ with $\psi^p(\lambda) = (1+\lambda)^p - 1$. Modulo terms in $p\mathbb{Z}[[\lambda]]$ the first non-trivial coefficient of $\psi^p(F_p(\lambda) - f_p(\lambda))$ is in degree $p^3 + p^2 - p > p^3 + (p-1)^2$. Note that $v_\lambda(F_{p^2}(\lambda) - f_{p^2}(\lambda)) = p^3 + p^2 - 1$. $v_\lambda\left(F_{p^2}(\lambda) - \psi^p(F_p(\lambda))\right) = p^3 + (p-1)^2$ by (3.9), and since with $\mathbb{F}_p$ coefficients $\psi^p(\lambda) = \lambda^p$, we have in $\mathbb{F}_p[[\lambda]]$ that

$$\varphi_2(\lambda) = f_{p^2}(\lambda) - f_p(\lambda^p)$$

has valuation $v_\lambda(\varphi_2(\lambda)) > \mathrm{p}^3$. Moreover,

$$\psi(\varphi_2(\lambda)) - g\varphi_2(\lambda) = a_{p^2}\lambda^{p^3+p^2-1} - a_p\lambda^{p^3+p^2-p}$$

We now attempt for $n \geq 3$ the inductive definition

$$f_{p^n}(\lambda) = f_{p^{n-1}}(\lambda^p) + \varphi_n(\lambda)$$

where $\varphi_n(\lambda)$ is still to be determined. Suppose we have

$$\psi\left(f_{p^{n-1}}(\lambda)\right) - g f_{p^{n-1}}(\lambda) \equiv a_{p^{n-1}} \cdot \lambda^{p^n+p^{n-1}-1}$$

modulo $\left(\lambda^{p^n+p^{n-1}}, p\right)$. In order to have the similar equation for $f_{p^n}(\lambda)$ the unknown $\varphi_n(\lambda)$ must satisfy

$$\psi(\varphi_n(\lambda)) - g\varphi_n(\lambda) \equiv a_{p^n}\lambda^{p^{n+1}-p^n-1} - a_{p^{n-1}}\lambda^{p^{n+1}+p^n-p} \qquad (*)$$

modulo $\left(\lambda^{p^{n+1}+p^n}, p\right)$. But this can be achieved by defining

$$\varphi_n(\lambda) = \frac{a_{p^{n-1}}}{a_p}\lambda^{p^{n+1}+p^n-p^3-p^2} \cdot \varphi_2(\lambda)$$

We just use that

$$\psi\left(\lambda^{p^2k}\right) \equiv g^k\lambda^{p^2k} \quad \text{mod} \quad \left(\lambda^{\mathrm{p}^2(\mathrm{k}+1)}, p\right)$$

together with $v_\lambda(\varphi_2(\lambda)) > p^3$ to show that $(*)$ is satisfied. This gives the inductive determination

$$a_{p^n} = a_{p^{n-1}}a_{p^2}/a_p \quad \in \mathbb{F}_p$$

so that $a_{p^n} = a_{p^2}^{n-1}/a_p^{n-2}$, $n \geq 2$. Finally, a_p and a_{p^2} in (3.11) are non-zero in $\mathbb{F}_p$ because g is a generator of $\left(\mathbb{Z}/p^2\right)^\times$, so all a_{p^n} are units mod p. This completes the proof. □

Let us comment briefly on (3.10). The first formula is obvious. To prove the second, consider the integral power series $h(\lambda) = l(\lambda) - \frac{1}{p}l(\lambda^p)$. We calculate modulo $\left(\lambda^{p^2+p}\right)$ and have

$$\begin{aligned} l(\lambda)^{p^2-p+1} &= (h(\lambda) + l(\lambda^p)/p)^{p^2-p+1} \\ &\equiv h(\lambda)^{p^2-p+1} + (p^2-p+1)h(\lambda)^{p^2-p}l(\lambda^p)/p + \binom{p^2-p+1}{2}h(\lambda)^{p^2-p-1}(l(\lambda^p)/p)^2 \\ &\equiv h_1(\lambda) + 1/p\lambda^{p^2} - 1/2p\lambda^{p^2+p-1} \end{aligned}$$

with $h_1(\lambda) \in \mathbb{Z}[[\lambda]]$. Now the second formula in (3.10) follows. To prove the third formula we calculate modulo $\left(\lambda^{p^3+p^2}\right)$ and use that

$$\begin{aligned} (pl(\lambda))^{p^2-p+1} &\equiv \left(ph(\lambda) + h(\lambda^p) + 1/p\, h\left(\lambda^{p^2}\right) + 1/p^2 h\left(\lambda^{p^3}\right)\right)^{p^2-p+1} \\ &\equiv h_1(\lambda) + \binom{p^2-p+1}{1}h(\lambda^p)^{p^2-p}\left(1/p\, h\left(\lambda^{p^2}\right)\right) + \\ &\binom{p^2-p+1}{2}(h(\lambda^p))^{p^2-p-1}\left(1/p\, h\left(\lambda^{p^2}\right)\right)^2 \end{aligned}$$

The second congruence comes from expanding by the multinominal formula with terms

$$\binom{p^2+p-1}{a_1, a_0, a_{-1}, a_{-2}}(ph(\lambda))^{a_1}(h(\lambda^p))^{a_0}\left(1/ph\left(\lambda^{p^2}\right)\right)^{a_{-1}}\left(1/p^2h(\lambda^p)\right)^{a_{-2}}$$

Calculating modulo $\left(\mathbb{Z}[[\lambda]], \lambda^{p^3+p^2}\right)$ there are only three terms of interest, namely

$$(a_1, a_0, a_{-1}, a_{-2}) = \left(0, p^2-p, 1, 0\right), \left(0, p^2-p-1, 2, 0\right), \left(1, p^2-p-2, 2, 0\right)$$

The multinomial coefficient in the third case is divisible by p, so does not count. We see that

$$(pl(\lambda))^{p^2-p+1} \equiv h_2(\lambda) + 1/p\lambda^{p^3} - 1/2p\, \lambda^{p^3+p^2-p}$$

with $h_2(\lambda)$ integral. Finally, one checks that

$$l(\lambda)^{p^3+p^2-2p+1} \equiv h_3(\lambda) + 1/p\lambda^{p^3+p^2-p} - 1/p\lambda^{p^3+p^2-1}$$

similar to the above evaluation of $l(\lambda)^{p^2-p+1}$. Now the third case of (3.10) follows.

Proof of Theorem 3.5. Let us first remark that given the differentials $d^{2(p^{k+1}-1)}(t^{p^k})$ for $k \leq n-1$, then the formula

$$(*) \qquad d^{2(p^{n+1}-1)}\left(t^{p^n}\right) = (\text{unit}) \cdot t^{p^{n+1}+p^n-1} a b^{p\left(\frac{p^n-1}{p-1}\right)}$$

is the first possible non-zero differential on t^{p^n}. This requires a little calculation, which we safely leave to the reader (cf. also sect.4 below). To prove $(*)$, given the differentials on t^{p^k} when $k < n$, is then equivalent to showing that the additive relation d in the diagram

$$\begin{array}{ccc} \tilde{J}^{2p^n}\left(\mathbb{C}P^{p^n}, \mathbb{C}P^{p^{n-1}}; \mathbb{F}_p\right) & \xrightarrow{d} & \tilde{J}^{2p^n+1}\left(\mathbb{C}P^{p^{n+1}+p^n-1}, \mathbb{C}P^{p^{n+1}+p^n-2}; \mathbb{F}_p\right) \\ \downarrow & & \uparrow \\ \tilde{J}^{2p^n}\left(\mathbb{C}P^{p^n}; \mathbb{F}_p\right) & \longleftarrow & \tilde{J}^{2p^n}\left(\mathbb{C}P^{p^{n+1}+p^n-1}; \mathbb{F}_p\right) \end{array}$$

is non-trivial; here the lower horizontal arrow is induced from restriction. We use (3.6) and (3.7) to evaluate the groups. With notation from (3.8), consider

$$f_{p^n}(\lambda) \cdot u^{p^n} \in \tilde{K}^{2p^n}\left(\mathbb{C}P^{p^n+s}; \mathbb{F}_p\right)$$

From (3.8),

$$\psi\left(f_{p^n}(\lambda)u^{p^n}\right) = g^{1-p^n} f_{p^n}(\lambda) \cdot u^{p^n} + a_{p^n} \cdot \lambda^{p^{n+1}+p^n-1} u^{p^n} + \ldots$$

and since $g^{p^n} = g$ in $\mathbb{F}_p$,

$$f_{p^n}(\lambda)u^{p^n} \in \tilde{J}^{2p^n}\left(\mathbb{C}P^{p^n+s}; \mathbb{F}_p\right)$$

for $s < p^{n+1}-1$, but not for $s = p^{n+1}-1$. In fact it is easy to see from (2.6), applied to the terms in

$$0 \to \tilde{K}^{2p^n}\left(S^{2q}; \mathbb{F}_p\right) \to \tilde{K}^{2p^n}(\mathbb{C}P^q; \mathbb{F}_p) \to \tilde{K}^{2p^n}\left(\mathbb{C}P^{q-1}; \mathbb{F}_p\right) \to 0$$

with $q = p^{n+1}+p^n-1$ that

$$\tilde{J}^{2p^n}\left(\mathbb{C}P^{q-1}; \mathbb{F}_p\right) \xrightarrow{\delta} \tilde{J}^{2p^n+1}\left(S^{2q}\right) = \mathbb{F}_p$$

maps $f_{p^n}(\lambda)u^{p^n}$ into $a_{p^n} \in \mathbb{F}_p^\times$. Since $f_{p^n}(\lambda) = \lambda^{p^n} + \ldots$, this gives the claim. □

Remark 3.12. It follows from the proof above that the units γ_k are the numbers a_{p^k}/g given in (3.11) and towards the end of the proof of (3.8). Concretely,

$$\gamma_0 = \left(1 - g^{p-1}\right)/p \ , \quad \gamma_1 = 2 \cdot \frac{g^{p-1}}{p} \ , \ \gamma_2 = \frac{g^{p-1} - 1}{p}$$

and in general

$$\gamma_n = \frac{g^{p-1} - 1}{2^{n-2}p} \qquad , n \geq 1$$

Proof of Theorem 3.3. The differentials on t^{p^k} follow from (3.5) upon using the natural map $J^{hS^1} \longrightarrow J^{hC_{p^n}}$ which maps the E^2-terms injectively because $H^2(BS^1; \mathbb{F}_p) \cong H^2(BC_{p^n}; \mathbb{F}_p)$, so we have only left to determine the differential on u_n.

The lens space $L^{2m-1} = S(\mathbb{C}^m)/C_{p^n}$ has K-theory

$$K^*\left(L^{2m-1}\right) = \mathbb{Z}\left[u, u^{-1}, \lambda\right] / \left((1+\lambda)^{p^n} - 1, \lambda^m\right)$$

This follows e.g. from the identification of $K(L^{2m-1})$ with $K_{C_{p^n}}(S(\mathbb{C}^m))$ which can be evaluated from the long exact sequence in $K_{C_{p^n}}$-theory of the pair $(D(\mathbb{C}^m), S(\mathbb{C}^m))$.

It suffices to show that the additive relation

$$\tilde{J}^1\left(L^1; \mathbb{F}_p\right) \leftarrow \tilde{J}^1\left(L^{2p^n-1}; \mathbb{F}_p\right) \xrightarrow{\delta^*} \tilde{J}^2\left(L^{2p^n+1}, L^{2p^n-1}; \mathbb{F}_p\right) = \tilde{J}^2\left(S^{2p^n}; \mathbb{F}_p\right)$$

is non-zero. There is zero indeterminacy, since the element $t^{p^n} b^{\frac{p^n-1}{p-1}}$ is not an earlier differential in the spectral sequence.

There are isomorphisms

$$\tilde{J}^1\left(L^1; \mathbb{F}_p\right) \cong \tilde{J}^1\left(L^3; \mathbb{F}_p\right) \xrightarrow{\beta} \tilde{J}^2\left(L^3\right)[p]$$

where the bracket $[p]$ indicates elements of order p. Consider the polynomial $f(\lambda) \in \mathbb{Z}[\lambda]$ with

$$f(\lambda) + \lambda^{p^n}/p \equiv p^{n-1}\log(1+\lambda) \qquad \bmod\left(\lambda^{p^n+1}\right)$$

Then $\psi(f(\lambda)u) = f(\lambda u) + \frac{1-g^{p-1}}{p}\lambda^{p^n}$ so that $f(\lambda)u \in \operatorname{Ker}(\psi - 1) = \tilde{J}^2(L^{2p^n})[p]$. Moreover,

$$pf(\lambda) = p^n\log(1+\lambda) = \log\left((1+\lambda)^{p^n}\right) = 0$$

as $(1+\lambda)^{p^n} = 1$. Thus $f(\lambda)u$ belongs to the kernel of

$$\psi - 1 : \tilde{K}^2\left(L^{2p^n-1}\right)[p] \to \tilde{K}^2\left(L^{2p^n-1}\right)[p]$$

and so to $\tilde{J}^2\left(L^{2p^n-1}\right)[p]$. The image of $f(\lambda)u$ in $J^3\left(L^{2p^n+1}, L^{2p^n-1}\right)[p] = \tilde{J}^3\left(S^{2p^n}\right)[p]$ is $\left(1-g^{p-1}\right)/p$ times the image of the generator under

$$\beta : \tilde{J}^2\left(S^{2p^n}; \mathbb{F}_p\right) \to \tilde{J}^3\left(S^{2p^n}\right)[p]$$

This completes the proof. $\square$

§ 4 The skeleton spectral sequences for THH($\mathbb{Z}$)

Let F_R be the functor with smash product associated to the ring R, that is

$$F_R(S) = |R\Delta_\bullet(S)/R\Delta_\bullet(*)|$$

for a based space S. Then $F_R(S^n)$ is the Eilenberg-MacLane space of type (R, n). The K-theory of F_R, see e.g [11], sect.5, is homotopy equivalent with $BGL(R)^+ \times \mathbb{Z}$. We write $T(R)$ for the cyclotomic spectrum $T(F_R)$ defined in sect.1, and THH(R) instead of THH(F_R) for its underlying infinite loop space. If R is finite over $\mathbb{Z}$ then

$$T(R)_p^\wedge \simeq T(R \otimes \mathbb{Z}_p)_p^\wedge \tag{4.1}$$

In particular, $T(\mathbb{Z})_p^\wedge \simeq T(\mathbb{Z}_p)_p^\wedge$. When R is commutative then $T(R)$ and all its fixed point spectra are ring spectra.

The mod p homotopy groups of $T(\mathbb{Z}_p)$ were calculated in [9] to be

$$\pi_*(T(\mathbb{Z}_p); \mathbb{F}_p) = E\{e\} \otimes P[f] \tag{4.2}$$

with $\deg(e) = 2p-1$ and $\deg(f) = 2p$. Moreover, the Bockstein operator β_1 maps f to e.

The following conjectural structure of the differentials in the skeleton spectral sequence for $T(\mathbb{Z}_p)$ with $\mathbb{F}_p$ coefficients will be discussed in considerable detail in the next section, but we can note right away that the suggested differentials are derived from the differentials in Theorem 3.3 by the substitutions $a = te$ and $b = tf$.

Conjecture 4.3. *The non-zero differentials in $E^r_{*,*}\left(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p\right)$ with source on the base line $E^r_{*,0}$ are multiplicatively generated from*

$$(i) \qquad d^{2p(k+1)}\left(t^{p^k}\right) = \lambda_k t^{p^k + p(k+1)} e f^{p(k)}, \quad 0 \le k < n$$

$$(ii) \qquad d^{2p(n)+1}(u_n) = \lambda t^{p(n)+1} f^{p(n-1)+1}, \quad n \ge 1$$

where $p(k) = p\binom{\frac{p^k-1}{p-1}}{}, p(0) = 0$ *and* $\lambda_k, \lambda \in \mathbb{F}_p^\times$.
The non-zero differentials from the fiber line are generated multiplicatively from (i) and the fact that te *and* tf *are permanent cycles.*

In the rest of this section we assume (4.3) and derive the resulting mod p homotopy groups of $T(\mathbb{Z}_p)^{hC_{p^n}}$. The E^2-term is

$$\begin{aligned} E^2_{*,*} &= E\{u_n\} \otimes P[t] \otimes E\{e\} \otimes P[f] \\ \deg u_n &= (-1,0), \deg t = (-2,0) \\ \deg e &= (0, 2p-1), \deg f = (0,2p) \end{aligned}$$

The first differentials are d^{2p}-differentials where we have (when neglecting units in $\mathbb{F}_p$):

$$d^{2p}\left(t^i f^j\right) = (i-j)t^{i+p} e f^j$$

This follows from (3.3), (i) and the statement that tf is a permanent cycle. It gives

$$\begin{aligned} E^{2p+1} =& \left\langle u_n^\epsilon t^i f^j | v_p(i-j) \geq 1\right\rangle \oplus \\ & \left\langle u_n^\epsilon t^i e f^j | p \leq i, v_p(i-j) \geq 1\right\rangle \oplus \\ & \left\langle u_n^\epsilon t^i e f^j | 0 \leq i < p\right\rangle \end{aligned}$$

Here $< >$ means the $\mathbb{F}_p$ vector space generated by the listed elements; $v_p(x)$ is the p-adic valuation and $\epsilon \in \{0,1\}$. If $n = 1$ then

$$d^{2p+1}\left(u_n t^i e^\epsilon f^j\right) = t^{i+p+1} e^\epsilon f^{j+1}$$

and we can list the E^{2p+2}-term as follows:

$$\begin{aligned} E^{2p+2} =& \left\langle t^{i+1} f^{j+1} | 0 \leq i < p, v_p(i-j) \geq 1\right\rangle \\ & \oplus \left\langle u_1 t^i e f^j | 0 \leq i < p, v_p(i-j) = 0\right\rangle \\ & \oplus \left\langle t^i e f^j \mid 0 \leq i < p\right\rangle \\ & \oplus \left\langle t^p e f^j | v_p(j) \geq 1\right\rangle \\ & \oplus \left\langle f^j | v_p(j) \geq 1\right\rangle \\ & \oplus \langle t^{ip} \mid i \geq 1\rangle \\ & \oplus \langle t^{ip} e \mid i \geq 1\rangle \end{aligned}$$

For filtration reasons there are no further differentials and hence an isomorphism of $\mathbb{F}_p$ vector spaces

$$\pi_*\Big(T(\mathbb{Z}_p)^{hC_p}; \mathbb{F}_p\Big) \cong E^{2p+2}_{*,*}$$

If $n > 1$, a similar calculation gives

$$
\begin{aligned}
E^{2p(n)+2} = & \sum_{k=0}^{n-1}{}^{\oplus} \langle t^{i+1} f^{j+1} \mid p(k) \le i < p(k+1), v_p(i-j) \ge n \rangle \\
& \oplus \sum_{k=0}^{n-1}{}^{\oplus} \langle u_n t^i e f^j \mid p(k) \le i < p(k+1), n > v_p(i-j) \ge k \rangle \\
& \oplus \sum_{k=0}^{n-1}{}^{\oplus} \langle t^i e f^j | p(k) \le i < p(k+1),\ v_p(i-j) \ge k \rangle \\
& \oplus \Big\langle f^j \mid v_p(j) \ge n \Big\rangle \oplus \Big\langle t^{p(n)} e f^j \mid v_p(j - p(n)) \ge n \Big\rangle \\
& \oplus \sum_{k=0}^{n-1}{}^{\oplus} \Big\langle u_n^{\varepsilon} t^i e f^j \mid 0 \le j < p(k), i \ge p(k+1), v_p(i-j) = k \Big\rangle \\
& \oplus \langle t^i e f^j \mid 0 \le j \le p(n-1), i \ge p(n), v_p(i-j) > n \rangle \\
& \oplus \langle t^i f^j \mid i > p(n), 0 < j \le p(n-1), v_p(i-j) = n \rangle \\
& \oplus \langle t^i \mid v_p(i) \ge n \rangle
\end{aligned}
\tag{4.4}
$$

Again for obvious reasons there can be no further differentials, so

$$\pi_*\Big(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p\Big) \cong E^{2p(n)+2}_{*,*}$$

The last 4 summands in (4.4) are concentrated in negative degrees.
Some extra comments are in order: The summands in row 4 can be rewritten as

$$E\{s^{-1}(f^{p^n})\} \otimes P\{f^{p^n}\}$$

where $s^{-1}(\ )$ indicates shift down in degrees by 1. Indeed, if $j = p(n-1) + p^n\nu, \nu \ge 0$ then $t^{p(n)} e f^j = (t^p e)(t^p f)^{p(n-1)} f^{p^n \nu}$ has degree $2p^{n+1}\nu - 1$. The elements $u_n^{\epsilon} t^i e f^j$ with $0 \le j < p(k), i \ge p(k+1)$ and $v_p(i-j) = k$ all have negative degrees. The terms $t^{i+1} f^{j+1}$ and $u_n e t^i f^j$ from the first two vector spaces in (3.4) have the same degree, and degree one less than the term $t^i e f^j$ so for each element $t^i e f^j$ in the third term in (3.4) there is one element of degree one less in the sum of the two first terms.

In order to get a better hold of the homotopy groups, we introduce the following notation. Write

$$\begin{aligned}\frac{r}{p-1} &= r_1 + \frac{r_2}{1-p}\ , 0 \le r_2 < p-1 \\ &= \sum_{i=0}^{\infty} \alpha_i p^i\ ,\ 0 \le \alpha_i < p\end{aligned}$$

Then

$$\frac{r-p}{p-1} = \sum_{i=0}^{\infty} a_i p^i \text{ with } 0 \le a_0 < p,\ 0 < a_i \le p \text{ for } i \ge 1 \text{ and } a_0 = \alpha_0.$$

Note that $a_i = r_2 + 1$ for i sufficiently large. Define classes in $\pi_{2r-1}\Big(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p\Big)$ by

$$x_r(0) = t^{a_0} e f^{1/p(r-p+a_0)}, \dots, x_r(k) = x_r(k-1)(t^p f)^{p^{k-1} a_k}, \dots \tag{4.5}$$

Of course, the classes are only well-defined modulo terms of lower filtration, i.e. in $E^{\infty}_{*,*}\Big(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p\Big)$. In positive degrees one then has:

$$\begin{aligned}\pi_{2r-1}\Big(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p\Big) &\cong \begin{cases} \langle x_r(0), \dots, x_r(n-1)\rangle, & v_p(r) \le n \\ \langle x_r(0), \dots, x_r(n-1), x_r(n)\rangle, & v_p(r) > n \end{cases} \\ \pi_{2r-2}\Big(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p\Big) &\cong \begin{cases} \langle s^{-1}x_r(0), \dots, s^{-1}x_r(n-1)\rangle & v_p(r-1) \le n \\ \Big\langle s^{-1}x_r(0), \dots, s^{-1}x_r(n-1), f^{(r-1)/p}\Big\rangle, & v_p(r-1) > n \end{cases}\end{aligned} \tag{4.6}$$

Indeed, we have only left to check that $s^{-1}\Big(f^{r/p}\Big) = x_r(n)$ when $v_p(r) > n$. So suppose $r = p^{n+1}\nu$. Then

$$\frac{r-p}{p-1} = \frac{p^{n+1}-p}{p-1} + \frac{(\nu-1)p^{n+1}}{p-1} = p(n) + a_{n+1}p^{n+1} + a_{n+2}p^{n+2} + \dots$$

which gives

$$x_r(n) = t^{p(n)} e f^{(r-p+p(n))/p}$$

This is precisely the element denoted $s^{-1}\Big(f^{r/p}\Big)$ above.

The skeleton spectral sequence with abutment $\pi_*\left(\mathrm{T}(\mathbb{Z}_p)^{hS^1};\mathbb{F}_p\right)$ is slightly simpler than the above. Its E^2–term is $\mathrm{E}\{e\}\otimes\mathrm{P}[t]\otimes\mathrm{P}[f]$ and the differentials are multiplicatively generated by 4.3(i), as one sees by using the restriction map from $\mathrm{T}(\mathbb{Z}_p)^{hS^1}$ to $\mathrm{T}(\mathbb{Z}_p)^{hC_{p^n}}$.

The calculations leading to (4.6) then also calculate the homotopy groups of $T(\mathbb{Z}_p)^{hS^1}$; the result is:

$$\begin{aligned}\pi_*\left(T(\mathbb{Z}_p)^{hS^1};\mathbb{F}_p\right) &= \left\langle e^\varepsilon (tf)^i \mid i\geq 0, \varepsilon=0,1\right\rangle\\ &\oplus \prod_{k=0}^{\infty}\left\langle t^i e f^j \mid p(k)\leq i< p(k+1)\ ,\ v_p(i-j)\geq k\right\rangle\end{aligned}\tag{4.7}$$

where the first vector space is $E\{e\}\otimes P[tf]$ and where the second vector space is an infinite product of $\mathbb{F}_p$ in each odd dimension $2r-1$, with generators $x_r(0), x_r(1),\ldots$

The spectral sequence (2.9) with abutment $\pi_*(\hat{\mathbb{H}}(C_{p^n},T(\hat{\mathbb{Z}}_p))$ has E^2–term

$$\hat{E}^2_{*,*}=E\{u_n\}\otimes P\left[t,t^{-1}\right]\otimes E\{e\}\otimes P[f]$$

The differentials are derived from (4.3) by the evident rule: $d^r\left(t^it^{-i}\right)=0$;

$$\begin{aligned}\hat{E}^{2p+1} &=\left\langle u_n^\varepsilon t^{pi+j}f^j \mid i\in\mathbb{Z}\ ;\ j\geq 0\ ;\ \varepsilon=0,1\right\rangle\\ &\oplus\left\langle u_n^\varepsilon t^{pi+j}ef^j \mid i\in\mathbb{Z}\ ;\ j\geq 0\ ;\ \varepsilon=0,1\right\rangle\end{aligned}$$

For $n=1$, the next differential is $d^{2p+1}(u_1)=t^{p+1}f$ and we are left with

$$\hat{E}^{2p+2}=\left\langle e^\varepsilon t^{pi} \mid i\in\mathbb{Z}\ ;\ \varepsilon=0,1\right\rangle$$

This is also the E^∞-term, so

$$\pi_*\hat{\mathbb{H}}(C_p;T(\mathbb{Z}_p))=E\{e\}\otimes P\left[t^p,t^{-p}\right]\tag{4.8}$$

For $n>1$ the next differential is $d^{2p(2)}\left(t^{pi+j}f^j\right)=it^{pi+p(2)+j}ef^{j+p}$, and then $d^{2p(3)}$ etc. We get

$$\begin{aligned}\hat{E}^{2p(n)+1} &=\left\langle u_n^\varepsilon t^{p^ni+j}f^j \mid i\in\mathbb{Z}\ ;\ j\geq 0\ ;\ \varepsilon=0,1\right\rangle\\ &\oplus\left\langle u_n^\varepsilon t^{p^ni+j}ef^j \mid i\in\mathbb{Z}\ ;\ j\geq 0\ ;\ \varepsilon=0,1\right\rangle\\ &\oplus\sum_{k=1}^{n-1\oplus}\left\langle u_n^\varepsilon t^{p^ki+j}ef^j \mid v_p(i)\leq n-1, p(k-1)\leq j<p(k)\ ;\ \varepsilon=0,1\right\rangle\end{aligned}$$

The differential $d^{2p(n)+1}(u_n) = t^{p(n)+1}f^{p(n-1)+1}$ gives

$$
(4.9)\qquad
\begin{aligned}
\hat{E}^{2p(n)+2} = & \sum_{k=1}^{n-1\oplus} \left\langle t^{p^k i+j} e f^j \mid i \in \mathbb{Z}\ ;\ p(k-1) \le j < p(k) \right\rangle \\
& \oplus \sum_{k=1}^{n-1\oplus} \left\langle u_n e t^{p^k i+j} f^j \mid v_p(i) \le n-1, p(k-1) \le j < p(k) \right\rangle \\
& \oplus \sum_{k=1}^{n-1\oplus} \left\langle t f t^{p^k i+j} f^j \mid v_p(i) \ge n, p(k-1) \le j < p(k) \right\rangle \\
& \oplus \left\langle t^{p^n i} \mid i \in \mathbb{Z} \right\rangle \\
& \oplus \left\langle t^{p^n i+p(n-1)} e f^{p(n-1)} \mid i \in \mathbb{Z} \right\rangle
\end{aligned}
$$

There are no further differentials since for $r \ge 2p(n)+2$, d^r will leave the strip $0 \le j \le p(n-1)$. Thus we have

$$\pi_*\left(\hat{\mathbb{H}}(C_{p^n}, T(\mathbb{Z}_p)); \mathbb{F}_p\right) = \hat{E}^{2p(n)+2}$$

The two last summands in (4.9) can be rewritten as

$$E\{s^{-1}(t^{p^n})\} \otimes P[t^{p^n}, t^{-p^n}]$$

We introduce new names for the generators in degree $2r-1$, similar to (4.5), namely

$$y_r(0) = t^{p-r+pa_0} e f^{a_0}, \dots, y_r(k) = y_r(k-1)(t^p f)^{p^k a_k}, \dots$$

and note that $s^{-1}\left(t^{-p^n \nu}\right) = y_r(n-1) = t^{-p^n(\nu-1)+p(n-1)} e f^{p(n-1)}$. Then we have for $r \in \mathbb{Z}$:

$$
(4.10)\qquad
\begin{aligned}
\pi_{2r-1}\left(\hat{\mathbb{H}}(C_{p^n}; T(\mathbb{Z}_p)); \mathbb{F}_p\right) &\cong \begin{cases} \langle y_r(0), \dots, y_r(n-2) \rangle & , v_p(r) < n \\ \langle y_r(0), \dots, y_r(n-2), y_r(n-1) \rangle & , v_p(r) \ge n \end{cases} \\
\pi_{2r-2}\left(\hat{\mathbb{H}}(C_{p^n}; T(\mathbb{Z}_p)); \mathbb{F}_p\right) &\cong \begin{cases} \left\langle s^{-1} y_r(0), \dots, s^{-1} y_r(n-2) \right\rangle & , v_p(r-1) < n \\ \left\langle s^{-1} y_r(0), \dots, s^{-1} y_r(n-2), t^{r-1} \right\rangle & , v_p(r-1) \ge n \end{cases}
\end{aligned}
$$

In particular we note by comparing (4.6) with (4.10) that

$$(4.12)\qquad \pi_i\left(\hat{\mathbb{H}}(C_{p^n}; T(\mathbb{Z}_p)); \mathbb{F}_p\right) \cong \pi_i\left(T(\mathbb{Z}_p)^{hC_{p^{n-1}}}; \mathbb{F}_p\right) \text{ for } i \ge 0$$

We shall see in the next section that the (-1)-connected coverings of $\hat{\mathbb{H}}(C_{p^n}; T(\mathbb{Z}_p))$ and $T(\mathbb{Z}_p)^{hC_{p^{n-1}}}$ are indeed homotopy equivalent.

Remark 4.13. Assuming Conjecture 4.3 for $n = m$, the differentials in the spectral sequence for $\mathrm{T}(\mathbb{Z}_p)^{hC_{p^{m+1}}}$ are the stated $d^{2p(k+1)}(t^{p^k})$ as long as $k < m$. This follows easily from the restriction mapping $\mathrm{T}(\mathbb{Z}_p)^{hC_{p^{m+1}}} \longrightarrow \mathrm{T}(\mathbb{Z}_p)^{hC_{p^m}}$. Moreover, the calculations above show that

$$d^{2p(m+1)}\left(t^{p^m}\right) = \lambda_m t^{p^m+p(m+1)} e f^{p(m)}$$
$$d^{2p(m+1)+1}(\overline{u}_{m+1}) = \lambda t^{p(m+1)} f^{p(m)+1}$$

for some $\lambda_m, \lambda \in \mathbb{F}_p$. Thus the proposed differentials are the first possible ones. The question is if $\lambda_m \neq 0$ and $\lambda \neq 0$. In this respect, Conjecture 4.3 gives the simplest possible differentiable structure.

§ 5 Discussion of conjecture 4.3

The basic idea is to compare the topological Hochschild homology spectra for the identity FSP and for the linear functor $F_{\mathbb{Z}_p}(S) = |\mathbb{Z}_p\Delta_\bullet(S)/\mathbb{Z}_p\Delta_\bullet(*)|$. The resulting cyclotomic spectra will be denoted $T(*)$ and $T(\mathbb{Z}_p)$, respectively. One knows from [11], Proposition 4.25, that $T(*)$ is G-equivalent to the equivariant sphere spectrum S^0_G for every finite subgroup G of S^1. More precisely, there is an S^1-equivalent map from $S^0_{S^1}$ to $T(*)$ which is an $\mathcal{F}$-equivalence, where $\mathcal{F}$ is the family of finite subgroups. The affirmed Segal conjecture, proved in [12], then gives

$$T(*)^{hG} \simeq \left(S^0_G\right)^{hG} \simeq \left(S^0_G\right)^G$$

for $G \subseteq S^1$. The fixed point spectrum $\left(S^0_G\right)^G$ is known by [14] or [35] to be

$$\left(S^0_G\right)^G \simeq \bigvee_{H \subseteq G} \Sigma^\infty B(G/H)_+$$

Any functor F with smash product gives rise to a ring (pre)spectrum $\{F(S^n)\}_n$, whose associated spectrum will be denoted F^s, and there is a map of spectra

$$\sigma : S^1_+ \wedge F^s \longrightarrow T(F)$$

This is an S^1-equivariant map when we give the source the extended S^1-structure, i.e action in the first factor only. For the functor F_R, associated to a ring, F^s_R is the Eilenberg-MacLane spectrum $H(R)$, so σ becomes a map from $S^1_+ \wedge H(R)$ to $T(R)$. For $R = \mathbb{Z}$, one knows from [9] that

$$T(\mathbb{Z}) \simeq H(\mathbb{Z}) \vee \bigvee_{n=1}^{\infty} \Sigma^{2n-1} H(\mathbb{Z}/n)$$

and that the composition

$$S^1_+ \wedge H(\mathbb{Z}) \to T(\mathbb{Z}) \to \Sigma^{2p-1} H(\mathbb{Z}/p)$$

represents the suspension of $P^1(\iota_0)$, the first mod p Steenrod operation applied to ι_0.
Let us now fix an odd prime p.
In the rest of this section we shall assume that all spectra are completed at p usually without further indication in notation.
Since $T(\mathbb{Z})^\wedge_p \simeq T(\mathbb{Z}_p)^\wedge_p$,

$$T(\mathbb{Z}_p) = H(\mathbb{Z}_p) \vee \bigvee_{n=1}^{\infty} \Sigma^{2n-1} H(\mathbb{Z}/n \otimes \mathbb{Z}_p)$$

and the composition

$$S^1_+ \wedge H(\mathbb{Z}_p) \to T(\mathbb{Z}_p) \to \Sigma^{2p-1} H(\mathbb{Z}/p) \tag{5.1}$$

represents the suspension of $P^1(\iota_0)$.
Consider the fiber $\overline{S}^0$ of the map from S^0 to $H(\mathbb{Z}_p)$ which represents the generator. There is a diagram of cofibrations

$$\begin{array}{ccccc} C_1 & \longrightarrow & S^2 & \longrightarrow & C \\ \uparrow & & \uparrow & & \uparrow \\ T(\mathbb{Z}_p, *) & \longrightarrow & T(*) & \longrightarrow & T(\mathbb{Z}_p) \\ \uparrow \sigma_1 & & \uparrow \sigma_0 & & \uparrow \sigma \\ S^1_+ \wedge \overline{S}^0 & \longrightarrow & S^1_+ \wedge S^0 & \longrightarrow & S^1_+ \wedge H(\mathbb{Z}_p) \end{array} \tag{5.2}$$

Lemma 5.3. *The mapping*

$$l : \mathrm{Map}_{S^1}\Big(S^3_+, T(*)\Big) \to \mathrm{Map}_{S^1}\Big(S^3_+, T(\mathbb{Z}_p)\Big)$$

induced from the component map from S^0 to $\mathbb{Z}_p$ *defines an isomorphism on* $\pi_i(\ ;\mathbb{F}_p)$ for $i = 2p-3$ *and for* $i = 2p-2$.

Proof. Both range and domain for $\pi_{2p-3}(l;\mathbb{F}_p)$ is a single copy of $\mathbb{F}_p$. In (5.2), $\pi_2(C) = \mathbb{Z}_p$ and $\pi_{2p-1}(C) = \mathbb{Z}/p$, and it follows from (5.1) that $H_{2p-1}(C) = 0$. Thus the first k-invariant is non-trivial. Consequently C is homotopy equivalent to S^2 through dimension $4p-2$, and it follows from the diagram that C_1 is $(4p-2)$-connected. This gives that

$$\pi_i \mathrm{Map}_{S^1}\Big(S^3_+, S^1_+ \wedge \overline{S}^0\Big) \to \pi_i \mathrm{Map}_{S^1}\Big(S^3_+, T(\mathbb{Z}, *)\Big)$$

is an isomorphism for $i < 4p - 5$. From [23], §2, Theorem 6.2 we know that

$$S^1_+ \wedge \overline{S}^0 \simeq {}_{S^1}F\left(S^1_+, \Sigma\overline{S}^0\right)$$

Thus

$$\begin{aligned}\mathrm{Map}_{S^1}\left(S^3_+, S^1_+ \wedge \overline{S}^0\right) &\simeq \mathrm{Map}_{S^1}\left(S^1_+ \wedge S^3_+, \Sigma\overline{S}^0\right)\\ &\simeq \mathrm{Map}\left(S^3_+, \Sigma\overline{S}^0\right)\\ &\simeq \Sigma^{-2}\overline{S}^0 \vee \Sigma\,\overline{S}^0\end{aligned}$$

Now $\overline{S}^0 \simeq \Sigma^{2p-3}H(\mathbb{Z}/p)$ in the relevant range of dimensions, and we get

$$\pi_i \mathrm{Map}_{S^1}\left(S^3; T(\mathbb{Z}_p, *)\right) = \begin{cases} \mathbb{Z}/p & i = 2p-5, 2p-2 \\ 0 & \text{otherwise, } i < 4p-3 \end{cases}$$

It follows that $\pi_{2p-3}(l; \mathbb{F}_p)$ and $\pi_{2p-2}(l; \mathbb{F}_p)$ are isomorphisms. □

In the skeleton spectral sequence

$$H^*\left(BS^1, \pi_*(T(\mathbb{Z}_p), \mathbb{F}_p)\right) \Rightarrow \pi_*\left(T(\mathbb{Z}_p)^{hS^1}; \mathbb{F}_p\right)$$

the E^2-term is $E\{e\} \otimes P[f] \otimes P[t]$. As a consequence of the previous lemma we have:

Corollary 5.4 *The classes te and tf are permanent cycles in the skeleton spectral sequence for* $T(\mathbb{Z}_p)^{hS^1}$.

Proof. We compare with the spectral sequence for $T(*)$. Consider the diagram

$$\begin{array}{ccc} \mathrm{Map}_{S^1}(ES^1_+, T(*)) & \xrightarrow{L} & \mathrm{Map}_{S^1}(ES^1_+, T(\mathbb{Z}_p)) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{S^1}(S^3_+, T(*)) & \xrightarrow{l} & \mathrm{Map}_{S^1}(S^3_+, T(\mathbb{Z}_p)) \\ \downarrow & & \downarrow \\ T(*) & \longrightarrow & T(\mathbb{Z}_p) \end{array}$$

where the vertical maps are induced from restrictions with respect to

$$S^1_+ \subset S^3_+ \subset ES^1_+$$

The S^1-equivariant map from the sphere spectrum to $T(*)$ is a non-equivariant homotopy equivalence, so induces a homotopy equivalence of homotopy fixed sets. The equivariant sphere spectrum is a *split* spectrum in the sense that the inclusion of its fixed points has a right inverse (up to homotopy). It follows that for every k-skeleton,

$$\mathrm{Map}_{S^1}\left(E_k S^1_+, T(*)\right) \simeq \mathrm{Map}\left(B_k S^1_+, T(*)\right)$$

In particular, the restriction

$$\mathrm{Map}_{S^1}\left(E_k S^1_+, T(*)\right) \to T(*)$$

is split, and

$$\mathrm{Map}_{S^1}\left(S^3_+, T(*)\right) = T(*) \vee \Omega^2 T(*)$$

The element $\alpha \in \pi_{2p-3}(S^0) = \mathbb{Z}/p$ and its preimage $v_1 \in \pi_{2p-2}(S^0; \mathbb{F}_p)$ under the Bockstein operator give elements in $\pi_*(\mathrm{Map}_{S^1}(ES^1_+, T(*)); \mathbb{F}_p)$ which map non-trivially to $\pi_*(\mathrm{Map}_{S^1}(S^3_+, T(\mathbb{Z}_p)); \mathbb{F}_p)$. The skeleton spectral sequence calculating this group has only two non-trivial lines corresponding to filtration degrees $s = 0$ and $s = -2$. The only classes in the E^2-term in these filtrations and in the given degrees are *te* and *tf* which must then be the images of α and v_1, respectively. Thus *te* and *tf* are permanent cycles as claimed. $\square$

Consider the cofibration

$$\mathrm{Cok} J_p \to S^0 \to \mathrm{Im}\, J_p$$

where $\mathrm{Im}\, J_p$ is the (-1)-connected cover of the (p-completed) J-space, i.e. $\mathrm{Im}\, J$ is the fiber of

$$\psi^g - 1 : (BU \times \mathbb{Z}) \to BU$$

where g is a prime which generates $(\mathbb{Z}/p^2)^\times$. Alternatively $\mathrm{Im}\, J_p \simeq K(\mathbb{F}_g)^\wedge_p$ by [30]. We have

$$\mathrm{Map}\,(BC_{p^n+}, S^0) \simeq \mathrm{Map}_{C_{p^n}}(EC_{p^n+}, T(*))$$

and linearization can be considered as a map

$$L : \mathrm{Map}\,(BC_{p^n+}, S^0) \to T(\mathbb{Z}_p)^{hC_{p^n}}$$

There is a natural inclusion of S^0 into $\mathrm{Map}\,(BC_{p^n+}, S^0)$ and thus also an inclusion

$$\mathrm{Cok}\, J_p \to \mathrm{Map}\,(BC_{p^n+}, S^0)$$

The arguments in the remainder of this section will be based upon the following assertion, which we at the time of writing have not been able to prove.

Conjecture 5.5. *For every $n \geq 0$ the composition*

$$\operatorname{Cok} J_p \to \operatorname{Map}(BC_{p^n+}, S^0) \xrightarrow{L} \operatorname{TC}(\mathbb{Z}_p) \to \operatorname{holim}_{\leftarrow} T(\mathbb{Z}_p)^{hC_{p^n}}$$

is null-homotopic.

The reader should note that this is a weak form of the standard conjecture that

$$\operatorname{Cok} J_p \to S^0 \to K(\mathbb{Z}_p) \tag{5.6}$$

is null-homotopic (as a mapping of spectra). The triviality of (5.6) on the level of spaces is known by results from [29]. The composition of (5.6) with the cyclotomic trace is the map which we in (5.5) assert is null.
Given (5.5), the composition

$$\operatorname{Map}(BC_{p^n+}, \operatorname{Cok} J_p) \to \operatorname{Map}(BC_{p^n+}, S^0) \to T(\mathbb{Z}_p)^{hC_{p^n}}$$

is null-homotopic, since the first map is homotopic to

$$\operatorname{Map}(BC_{p^n+}, \operatorname{Cok} J_p) \to \operatorname{Map}(BC_{p^n} \times BC_{p^n}, S^0) \xrightarrow{\Delta^*} \operatorname{Map}(BC_{p^n+}, S^0)$$

Thus there results a mapping

$$L : \operatorname{Map}(BC_{p^n+}, \operatorname{Im} J_p) \to T(\mathbb{Z}_p)^{hC_{p^n}} \tag{5.7}$$

The connected covering map $\operatorname{Im} J_p \to J_p$ induces

$$\operatorname{Map}(BC_{p^n+}, \operatorname{Im} J_p) \to \operatorname{Map}(BC_{p^n+}, J_p)$$

which becomes a homotopy equivalence of (-1)-connected covers. The skeleton spectral sequence of the domain can then be read off from the calculations of sect.3. More precisely,

$$E^r_{i,j}(\operatorname{Map}(BC_{p^n+}, \operatorname{Im} J_p); \mathbb{F}_p) = E^r_{i,j}(J^{hC_{p^n}}; \mathbb{F}_p) \quad \text{for } i+j \geq 0$$

We may replace C_{p^n} by S^1 everywhere to get

$$L : \operatorname{Map}(BS^1_+, \operatorname{Im} J_p) \to [T(\mathbb{Z}_p)^{hS^1}]$$
$$E^r_{i,j}(\operatorname{Map}(BS^1, \operatorname{Im} J_p); \mathbb{F}_p) = E^r_{i,j}(J^{hS^1}; \mathbb{F}_p)$$

Theorem 5.8

(i) *Conjecture 4.3 is true for* $n = 1$
(ii) *Conjecture 5.5 implies Conjecture 4.3 for all* n

Before presenting the argument, we introduce some notation. The free S^1-space ES^1 is filtered by its S^1-skeleta $E_kS^1 = S^{2k+1}$ with quotient $B_kS^1 = \mathbb{CP}^k$. For $j \leq i$ we define

$$F_j^i(\mathbb{Z}_p) = \pi_*\Big(\operatorname{Map}_{S^1}\Big(S^{2i+1}/S^{2j-1}, T(\mathbb{Z}_p)\Big); \mathbb{F}_p\Big) \tag{5.9}$$

with a similar notation $F_j^i(*)$ and $F_j^i(\mathrm{Im} J_p)$ to indicate the mapping space with $T(\mathbb{Z}_p)$ replaced by $T(*) = S^0$ and by $\mathrm{Im}\ J_p$. For $j = 0$ we interpret $S^{-1} = \emptyset$, so that $F_0^\infty(\mathbb{Z}_p) = \pi_* T(\mathbb{Z}_p)^{hS^1}$ etc. The map $T(*) \to T(\mathbb{Z}_p)$ induces a homomorphism L from $F_j^i(*)$ to $F_j^i(\mathbb{Z}_p)$, and under the assumption of Conjecture 5.5 this factors as

$$L : F_j^i(*) \longrightarrow F_j^i(\mathrm{Im}\ J_p) \xrightarrow{\overline{L}} F_j^i(\mathbb{Z}_p). \tag{5.10}$$

The spectra $T(*)$, $\mathrm{Im}\ L_p$ and $T(\mathbb{Z}_p)$ are ring spectra and there are induced graded multiplications on $F_0^\infty(*)$, $F_0^\infty(\mathrm{Im} J_p)$ and $F_0^\infty(\mathbb{Z}_p)$, and in fact multiplications

$$F_j^i \times F_l^k \longrightarrow F_{j+l}^{\min(i+l,j+k)} \tag{5.11}$$

in all three cases, compatible with the maps in (5.10).
Since $S^{2i+1}/S^{2i-1} \simeq S_+^1 \wedge S^{2i}$ as S^1-spaces,

$$F_i^i(\mathbb{Z}_p) = \pi_*\Big(\operatorname{Map}_{S^1}\Big(S_+^1 \wedge S^{2i}, T(\mathbb{Z}_p)\Big); \mathbb{F}_p\Big) = \pi_{*+2i}(T(\mathbb{Z}_p); \mathbb{F}_p)$$

and similarly in the other cases. This is the $E^2_{-2i,*}$-term of the skeleton spectral sequence of sect. 4. More generally, the S^1-skeleton filtration of S^{2i+1} induces a spectral sequence with abutment $F_j^i(\mathbb{Z}_p)$ and

$$\begin{aligned} E^2_{-2k,*}(\mathbb{Z}_p) &= F_k^k(\mathbb{Z}_p) \quad \text{for} \quad j \leq k \leq i \\ E^2_{-2k+1,*}(\mathbb{Z}_p) &= 0, \end{aligned}$$

and similarly in the other cases. This spectral sequence is precisely the part of the skeleton spectral sequence of sect. 4 lying in the strip between filtration degree $-2i$ and $-2j$. There are homomorphisms for $r \geq 0$, $0 \leq s \leq j$

$$\mathrm{res}_* : F_j^{i+r} \longrightarrow F_j^i, \ \mathrm{inc}_* : F_j^i \longrightarrow F_{j-s}^i, \ \partial_* : F_j^{i-1} \longrightarrow F_i^\infty$$

induces from the S^1-maps

$$S^{2i+1}/S^{2j-1} \longrightarrow S^{2(i+r)+1}/S^{2j-1}, \ S^{2i+1}/S^{2j-1} \longrightarrow S^{2i+1}/S^{2(j+s)-1}$$

and from the connected homomorphism in the fibration

$$\mathrm{Map}_{S^1}\left(S^\infty/S^{2i-1},T\right) \overset{\mathrm{inc}}{\to} \mathrm{Map}_{S^1}\left(S^\infty/S^{2j-1},T\right) \overset{\mathrm{res}}{\to} \mathrm{Map}_{S^1}\left(S^{2i-1}/S^{2j-1},T\right).$$

The d^{2r}-differentials in the skeleton spectral sequences are induced from the additive relations

$$F_j^j \overset{\mathrm{res}_*}{\leftarrow} F_j^{j+r-1} \overset{\partial_*}{\to} F_{j+r}^{\infty} \overset{\mathrm{res}_*}{\to} F_{j+r}^{j+r}.$$

With these notions, Lemma 5.3 and its corollary can be rephrased as follows: the composition

$$\pi_*(T(*);\mathbb{F}_p) \overset{I_*}{\to} F_0^\infty(*) \overset{\mathrm{res}_*}{\to} F_0^1(*) \overset{L_*}{\to} F_0^i(\mathbb{Z}_p) \overset{\mathrm{inc}_*}{\leftarrow} F_1^i(\mathbb{Z}_p) \tag{5.12}$$

maps α and v_1 non-trivially (into te and tf); the images are infinite cycles because $L_* \circ \mathrm{res}_* = \mathrm{res}_* \circ L_*$. Here I is induced from the natural inclusion

$$T(*) \to T(*)^{S^1} \to T(*)^{hS^1}$$

where the first map is a component of the fixed set, or alternatively the inclusion into the mapping space $\mathrm{Map}(BS^1_+,T(*))$ when one uses the identification $T(*)^{hS^1} \simeq \mathrm{Map}\left(BS^1_+,T(*)\right)$. We can then use the product (5.11) to see that the elements αv_1^{k-1} are mapped non-trivially into $t^k e f^{k-1} \in F_k^k(\mathbb{Z}_p)$ under the composition

$$\pi_*(T(*);\mathbb{F}_p) \to F_0^\infty(*) \overset{L_*}{\to} F_0^1(*) \overset{\mathrm{res}_*}{\to} F_0^k(\mathbb{Z}_p) \overset{\mathrm{inc}_*}{\leftarrow} F_k^k(\mathbb{Z}_p) \tag{5.13}$$

The commutative diagram

$$\begin{array}{ccc} S^0 & \to & H\mathbb{Z}_p \\ \downarrow\simeq & & \downarrow \\ T(*) & \overset{L}{\to} & T(\mathbb{Z}_p), \end{array}$$

with the vertical maps being inclusions of the "0–sceleta" in the simplicial spaces defining $T(*)$ and $T(\mathbb{Z}_p)$, shows that $\overline{S}^0 \to T(*) \to T(\mathbb{Z}_p)$ is null-homotopic. This gives a factorization

$$\begin{array}{ccc} T(*) & \overset{I}{\longrightarrow} & T(*)^{hS^1} \to T(\mathbb{Z}_p)^{hS^1} \\ \uparrow & & \uparrow \\ \overline{S}^0 & \longrightarrow & \mathrm{Map}_{S^1}\left(S^\infty/S^1,T(\mathbb{Z}_p)\right) \end{array}$$

and hence a natural homomorphism

$$\pi_*\left(\overline{S}^0;\mathbb{F}_p\right) \to F_1^\infty(\mathbb{Z}_p).$$

This explains the filtration shift in (5.12) and (5.13) for homotopy classes in positive degrees: k-fold product must map into $F_1^\infty(\mathbb{Z}_p)^k \subset F_k^\infty(\mathbb{Z}_p)$.

Proof of Theorem 5.8 (i) We begin by showing that in the skeleton spectral sequence

$$\widehat{H}^*\left(S^1;\pi_*(T(\mathbb{Z}_p);\mathbb{F}_p)\right) \Rightarrow \pi_*\left(T(\mathbb{Z}_p)^{hS^1};\mathbb{F}_p\right),$$

the d^{2p}-differential sends t into $t^{p+1}e$. As explained in sect. 4 this implies the corresponding result for the skeleton spectral sequence for $T(\mathbb{Z}_p)^{hC_p}$.
We know from Theorem 3.3 that $d^{2p-2}t = \alpha t^p$ in the spectral sequence for $T(*)^{hS^1}$, or in other words that the additive relation

$$F_1^1(*) \overset{\text{res}_*}{\longleftarrow} F_1^{p-1}(*) \overset{\partial_*}{\longrightarrow} F_p^{\infty}(*) \overset{\text{res}_*}{\longrightarrow} F_p^p(*)$$

is non-trivial on $\pi_{-2}(\ ;\mathbb{F}_p)$, modulo indeterminacy. In fact there is no indeterminacy in this degree: both res$*$ and inc$*$ are injective, since $\alpha \in \pi_{2p-3}(T(*);\mathbb{F}_p)$ is the first non-trivial homotopy class.
Let $(L_p^1)_*$ be the additive relation

$$\left(L_p^1\right)_* : F_p^p(*) \overset{\text{res}_*}{\longleftarrow} F_p^{p+1}(*) \overset{L_*}{\longrightarrow} F_p^{p+1}(\mathbb{Z}_p) \overset{\text{inc}_*}{\longleftarrow} F_{p+1}^{p+1}(\mathbb{Z}_p).$$

Then there is a commutative diagram for comparison of the differentials in $T(*)^{hS^1}$ and $T(\mathbb{Z}_p)^{hS^1}$:

$$\begin{array}{ccccccc}
F_1^1(*) & \overset{\text{res}_*}{\longleftarrow} & F_1^{p-1}(*) & \overset{\partial_*}{\longrightarrow} & F_p^{p+1}(*) & \overset{\text{res}_*}{\longrightarrow} & F_p^p(*) \\
\downarrow L_* & & \downarrow L_* & & \downarrow L_* & & \downarrow \left(L_p^1\right)_* \\
F_1^1(\mathbb{Z}_p) & \overset{\text{res}_*}{\longleftarrow} & F_1^{p-1}(\mathbb{Z}_p) & \overset{\partial_*}{\longrightarrow} & F_p^{p+1}(\mathbb{Z}_p) & \overset{\text{inc}_*}{\longleftarrow} & F_{p+1}^{p+1}(\mathbb{Z}_p)
\end{array}$$

Thus it suffices to show that $(L_p^1)_*\ (\alpha t^p) = t^{p+1}e$. To this end, let $(t^p)^\vee \in F_p^{p+1}(*)$ be an element with $\text{res}_*(t^p)^\vee = t^p \in F_p^p(*)$; it exists since $d^2(t^p) = 0$ in the spectral sequence for $T(*)^{hS^1}$, and it is unique. Viewing $\alpha \in F_0^{\infty}(*)$, the product $\alpha(t^p)^\vee \in F_p^{p+1}(*)$ restricts to $\alpha t^p \in F_p^p(*)$. But L_* is multiplicative, so

$$L_*\left((t^p)^\vee_\alpha\right) = L_*\left((t^p)^\vee\right) \cdot L_*(\alpha) \in F_p^{p+1}(\mathbb{Z}_p) \cdot F_1^{\infty}(\mathbb{Z}_p) \subset F_{p+1}^{p+1}(\mathbb{Z}_p).$$

Since $L_*(\alpha)$ restricts to $te \in F_1^1(\mathbb{Z}_p)$ and since $F_p^{p+1}(\mathbb{Z}_p) \cdot F_1^1(\mathbb{Z}_p) \subset F_{p+1}^{p+1}(\mathbb{Z}_p)$ this proves that $(L_p^1)_*$ maps αt^p to et^{p+1} as required. hence $d^{2p}(t) = t^{p+1}e$. Finally to prove that $d^{2p+1}u_1 = t^{p+1}f$, we can simply use that d^{2p+1} commutes with the Bockstein operator, and that $\beta u_1 = t$, $\beta f = e$. □

Proof of Theorem 5.8 (ii). The argument is similar in spirit to the argument above. We may assume inductively the result for $n < m$. Then the first possible differential on t^{p^m} in the spectral sequence for $T(\mathbb{Z}_p)^{hS^1}$ is

$$d^{2p(m+1)}\left(t^{p^m}\right) = t^{p^m+p(m+1)}ef^{p(m)}$$

where $p(m) = p(\frac{p^m-1}{p-1})$. Let $i(m) = p^{m+1} + p^m - 1$ and set $q(m) = i(m) + p(m) + 1 = p^m + p(m+1)$. Theorem 3.3 tells us that the spectral sequence for $(\mathrm{Im}\ J)^{hS^1}$,

$$(*) \qquad d^{2(p^{m+1}-1)}t^{p^m} = t^{i(m)}\alpha \vee_1 p(m),$$

and we must study the commutative diagram

$$\begin{array}{ccccccc}
F^{p^m}_{p^m}(\mathrm{Im}\ J_p) & \overset{\mathrm{res}_*}{\longleftarrow} & F^{i(m)-1}_{p^m}(\mathrm{Im}\ J_p) & \overset{\partial_*}{\longrightarrow} & F^{q(m)}_{i(m)}(\mathrm{Im}\ J_p) & \overset{\mathrm{res}_*}{\longleftarrow} & F^{i(m)}_{i(m)}(\mathrm{Im}\ J_p) \\
\downarrow \overline{L}_* & & \downarrow \overline{L}_* & & \downarrow \overline{L}_* & & \downarrow L^{q(m)}_{i(m)} \\
F^{p^m}_{p^m}(\mathbb{Z}_p) & \overset{\mathrm{res}_*}{\longleftarrow} & F^{i(m)-1}_{p^m}(\mathbb{Z}_p) & \overset{\partial_*}{\longrightarrow} & F^{q(m)}_{i(m)}(\mathbb{Z}_p) & \overset{\mathrm{res}_*}{\longleftarrow} & F^{q(m)}_{q(m)}(\mathbb{Z}_p).
\end{array}$$

Here L^q_i is the additive relation which makes right hand box commutative. In degree $-2p^m$ the upper row represents $d^{2(p^{m+1}-1)}t^{p^m}$ in the spectral sequence for $(\mathrm{Im}\ J)^{hS^1}$, and the lower row represents $d^{2p(m+1)}(t^{p^m})$ in the spectral sequence for $T(\mathbb{Z}_p)^{hS^1}$. Thus we must show that

$$\left(L^{q(m)}_{i(m)}\right)_*\left(t^{i(m)}\alpha v_1^{p(m)}\right) = t^{q(m)}ef^{p(m)}.$$

Since by Theorem 3.3 $t^{i(m)}\alpha$ is an infinite cycle in the spectral sequence for $(\mathrm{Im}\ J_p)^{hS^1}$, there is an element $A_i \in F^\infty_i(\mathrm{Im}\ J_p)$ with $\mathrm{res}_*(a_i) = t^i_\alpha$. Its image $L_*(A_i) \in F^\infty_i(\mathbb{Z}_p)$ lifts to $F^\infty_{i+1}(\mathbb{Z}_p)$. Moreover,

$$L_*\left(A_i \cdot v_i^{k-1}\right) = L_*(A_i) \cdot L_*\left(v_i^{k-1}\right) \in F^\infty_{i+1}(\mathbb{Z}_p) \cdot F^\infty_1(\mathbb{Z}_p)^{k-1} \subset F^\infty_{i+k}(\mathbb{Z}_p).$$

This has restriction $t^{i+k}ef^{k-1} \in F^{i+k}_{i+k}(\mathbb{Z}_p)$. Apply this for $i = i(m)$ and $k = p(m) + 1$ to get the wanted result, and hence the differential on t^{p^m}. To settle the differential on u_m one proceeds analogously, but now using the spectral sequence for $T(*)^{hC_{p^m}}$ and $T(\mathbb{Z}_p)^{hC_{p^m}}$. Details are left for the reader. □

Remark 5.14 In the spectral sequence for $T(*)^{hS^1}$ the E^2-term contains as a direct summand the E^2-term for $(\mathrm{Im}\ J)^{hS^1}$ but the differentials are different. Do to the affirmed Segal conjecture there are no elements of negative degree in the E^∞-term. For example, $t^{p(p+1)-1}\alpha$ cannot survive to E^∞. Indeed, one knows that

$$d^{2(p-1)^2}\left(t^{p(p+1)-1}\alpha\right) = t^{2p(p-1)+p^2}\beta_1$$

where β_1 is the first element in $\pi_*(S^0;\mathbb{F}_p)$ outside $\pi_*(\mathrm{Im}\ J_p;\mathbb{F}_p)$. This influence from $\pi_*(\mathrm{Cok}\ J;\mathbb{F}_p)$ makes it impossible to generalize the proof of Theorem 5.8 (i) to obtain a proof of Conjecture 4.3 in general. Once one gets out of the range where $\pi_*(\mathrm{Cok}\ J,\mathbb{F}_p)$ is known one looses control over the spectral sequence for $T(*)^{hS^1}$. Enough however is known to prove Conjecture 4.3 for $n = 2$ in this fashion, and one could probably push the argument to affirm also $n = 3$.

§ 6 The equivalence between $\hat{\mathbb{H}}(C_{p^n}, T(\mathbb{Z}_p))$ and $T(\mathbb{Z}_p)^{hC_{p^{n-1}}}$

Assuming Conjecture 4.3, we shall compare the norm cofibration of (2.3)

$$T(F)_{hC_{p^n}} \xrightarrow{N^h} T(F)^{hC_{p^n}} \xrightarrow{\Psi} \hat{\mathbb{H}}(C_{p^n}, T(F))$$

with the cofibration of theorem 1.10. We recall from its proof that we have the $C_{p^{n-1}}$ equivariant homotopy equivalence

$$T(F) \xleftarrow{\simeq} [T(F)\wedge \Sigma EC_{p^n}]^{C_p}$$

The inclusion γ of $T(F)$ into $\mathrm{Map}\,(EC_{p^n+}, T(F))$ as the constant map is C_{p^n}-equivariant, so induces a $C_{p^{n-1}}$-equivariant mapping

$$[T(F)\wedge \Sigma EC_{p^n}]^{C_p} \longrightarrow [\mathrm{Map}\,(EC_{p^n+}, T(F))\wedge \Sigma EC_{p^n}]^{C_p}$$

The range is $\hat{\mathbb{H}}(C_p, T(F))$ with fixed set

$$\hat{\mathbb{H}}(C_p, T(F))^{C_{p^{n-1}}} = \hat{\mathbb{H}}(C_{p^n}, T(F))$$

We obtain a $C_{p^{n-1}}$ equivariant map

$$\hat{\gamma} : T(F) \longrightarrow \hat{\mathbb{H}}(C_p, T(F))$$

Let $\hat{\Gamma} = \hat{\gamma}^{C_{p^{n-1}}}$, $\Gamma = \gamma^{C_{p^n}}$

$$\hat{\Gamma} : T(F)^{C_{p^{n-1}}} \longrightarrow \hat{\mathbb{H}}(C_{p^n}, T(F)) \quad , \quad \Gamma : T(F)^{C_{p^n}} \longrightarrow T(F)^{hC_{p^n}}$$

They fit together in a homotopy commutative diagram of cofibrations of spectra

$$(6.1)\qquad \begin{array}{ccccc} T(F)_{hC_{p^n}} & \xrightarrow{N} & T(F)^{C_{p^n}} & \xrightarrow{\Phi} & T(F)^{C_{p^{n-1}}} \\ \| & & \downarrow \Gamma & & \downarrow \hat{\Gamma} \\ T(F)_{hC_{p^n}} & \xrightarrow{N^h} & T(F)^{hC_{p^n}} & \xrightarrow{\Psi} & \hat{\mathbb{H}}(C_{p^n}, T(F)) \end{array}$$

For the functor F with $T(F) = T(\mathbb{Z}_p)$ we have some control over the homotopy groups of the lower cofibration of (6.1) by the spectral sequences of Sect.2. This will be our main tool for evaluating $\pi_*\left(T(\mathbb{Z}_p)^{C_{p^n}}; \mathbb{F}_p\right)$ and then in turn $\pi_*((TC(\mathbb{Z}_p), p); \mathbb{F}_p)$.

Remark 6.2. We can note from (6.1) that N^h factors over Γ. This is quite generally true for any G-spectrum T (G finite). Indeed, there is a commutative diagram of G-spectra

$$\begin{array}{ccc} \mathrm{Map}\,(EG_+, T) \wedge EG_+ & \xrightarrow{1\wedge\pi} & \mathrm{Map}\,(EG_+, T) \wedge S^0 \\ \uparrow\ \pi^* \wedge 1 & & \uparrow\ \gamma \wedge 1 \\ \mathrm{Map}\,(S^0, T) \wedge EG_+ & \longrightarrow & T \wedge S^0 \end{array}$$

where $\pi : EG_+ \longrightarrow S^0$ is the projection. On fixed sets we get upon identifying T with $\mathrm{Map}\,(S^0, T)$, respectively $T \wedge S^0$, the commutative diagram

$$\begin{array}{ccc} [\mathrm{Map}\,(EG_+, T) \wedge EG_+]^G & \xrightarrow{1\wedge\pi} & T^{hG} \\ \uparrow\ (\gamma \wedge 1)^G & & \uparrow\ \Gamma \\ (T \wedge EG_+)^G & \xrightarrow{1\wedge\pi} & T^G \end{array}$$

The norm map is $(1 \wedge \pi) \circ (\gamma \wedge 1)^G$ composed with the equivalence

$$|T| \wedge_G EG_+ \simeq (T \wedge EG_+)^G$$

and we get the claimed factorization

$$\begin{array}{ccc} T_{hG} & \xrightarrow{N} & T^G \\ \| & & \downarrow \Gamma \\ T_{hG} & \xrightarrow{N^h} & T^{hG} \end{array}$$

Lemma 6.3. *Let T be a G-spectrum (indexed on trivial G-representations) with $\pi_i T = 0$ for $i \geq m$. Then $\pi_i(T^{hG}) = 0$ for $i \geq m$ and the restriction map induces an isomorphism from $\pi_{m-1}(T^{hG})$ to $\pi_{m-1}(T)$.*

Proof. By definition $\pi_i T^{hG} = \left[S^i \wedge EG_+, T\right]^G$, and the result follows by elementary equivariant obstruction theory. Indeed, the obstructions to deform a G-mapping equivariantly to zero lie in the Bredon cohomology,

$$H^j_G\left(S^i \wedge EG_+, \pi_j T\right) = H^j\left(S^i \wedge BG_+, \pi_j T\right)$$

cf. [6]. These groups vanish when $i \geq m$. For $i = m-1$ there is one non-zero obstruction group, equal to $\pi_{m-1}T$, and the isomorphism

$$\pi_{m-1}T^{hG} \xrightarrow{\cong} \pi_{m-1}T$$

is induced from the evaluation at $G_+ \subset EG_+$. □

Corollary 6.4. *Suppose $f : T_1 \to T_2$ is a G-map between two equivariant spectra, which induces isomorphisms in homotopy groups in dimensions larger than or equal to m. Then the induced map $f^{hG} : T_1^{hG} \to T_2^{hG}$ has the same property.*

Proof. Let T be the homotopy fiber of f. Then $\pi_i T = 0$ for $i \geq m$ and we can apply (6.3). Since taking homotopy fixed sets preserves fibrations, the result follows easily. □

Lemma 6.5. $\hat{\gamma}_* : \pi_i(T(\mathbb{Z}_p); \mathbb{F}_p) \to \pi_i\left(\hat{\mathbb{H}}(C_p, T(\mathbb{Z}_p)); \mathbb{F}_p\right)$ *is an isomorphism for* $i \geq 0$.

Proof. We already know the homotopy groups involved, namely

$$\pi_*(T(\mathbb{Z}_p); \mathbb{F}_p) = E\{e\} \otimes P[f]$$
$$\pi_*\left(\hat{\mathbb{H}}(C_p, T(\mathbb{Z}_p)); \mathbb{F}_p\right) = E\{e\} \otimes P\left[t^p, t^{-p}\right]$$

with $\deg e = 2p-1$ and $\deg f = \deg t^{-p} = 2p$, cf. sect. 4 and (5.11). Thus it suffices to check that $\hat{\gamma}_*$ induces an isomorphism at π_0 and that

$$\hat{\gamma}_*(f) = t^{-p}, \quad \hat{\gamma}_*(e) = e$$

To check on π_0 we can use the diagram

$$\begin{array}{ccc} T(*) & \xrightarrow{\hat{\gamma}(*)} & \hat{\mathbb{H}}(C_p, T(*)) \\ \downarrow l & & \downarrow \hat{l} \\ T(\mathbb{Z}_p) & \xrightarrow{\hat{\gamma}(\mathbb{Z}_p)} & \hat{\mathbb{H}}(C_p, T(\mathbb{Z}_p)) \end{array}$$

The map $\pi_0(l)$ is an isomorphism, and the same is then the case for $\pi_0(\hat{l})$. But $T(*) \simeq {}_{C_p}S^0_{C_p}$ and the affirmed Segal conjecture for S^0 asserts that γ^{C_p} and hence that $\hat{\gamma}(*)$ is a homotopy equivalence. It follows that $\pi_0(\hat{\gamma}(\mathbb{Z}_p))$ is an isomorphism.
We prove in Theorem 10.14 below that the topological Dennis trace map

$$\mathrm{Tr} : K(\mathbb{Z}_p) \to T(\mathbb{Z}_p)$$

induces a surjection on homotopy groups in degree $2p-1$ where the range is a copy of $\mathbb{F}_p$, generated by the element e. Let $e_K \in K_{2p-1}(\mathbb{Z}_p)$ be an element with $\mathrm{Tr}(e_K) = e$.
The cyclotomic trace is a factorization of the Dennis trace and we have the homotopy commutative square

$$\begin{array}{ccc} K(\mathbb{Z}_p) & \stackrel{\mathrm{Trc}}{\rightarrow} & \mathrm{TC}(\mathbb{Z}_p, p) \\ \downarrow \mathrm{Tr} & & \downarrow \alpha \\ T(\mathbb{Z}_p) & \stackrel{D}{\leftarrow} & T(\mathbb{Z}_p)^{C_p} \end{array}$$

One knows from the definition of $\mathrm{TC}(\mathbb{Z}_p, p)$ that $D \circ \alpha \simeq \Phi \circ \alpha$. Here D is the inclusion of the fixed set, and $\Phi : T(\mathbb{Z}_p)^{C_p} \to T(\mathbb{Z}_p)$ is the map from (1.9). We further have the triangle

$$\begin{array}{ccc} T(\mathbb{Z}_p)^{C_p} & \stackrel{\Gamma}{\rightarrow} & T(\mathbb{Z}_p)^{hC_p} \\ & D \searrow \quad \swarrow D^h & \\ & T(\mathbb{Z}_p) & \end{array}$$

The homotopy groups $\pi_*\left(T(\mathbb{Z}_p)^{hC_p}; \mathbb{F}_p\right)$ were evaluated in (4.6), this used only the part of Conjecture 4.3, which is easily proved as explained at the end of sect.5. In particular

$$\pi_{2p-1}\left(T(\mathbb{Z}_p)^{hC_p}; \mathbb{F}_p\right) = \mathbb{F}_p$$

generated by $x_p(0)$, and $D^h_*(x_p(0)) = e$ by (4.5).
The map $\Psi : T(\mathbb{Z}_p)^{hC_p} \to \hat{\mathbb{H}}(C_p, T(\mathbb{Z}_p))$ induces an isomorphism on the E^∞ of the skeleton spectral sequences, so with the notation of (4.10) $\Psi_*(x_p(0)) = y_p(0)$ in $\pi_{2p-1}\left(\hat{\mathbb{H}}(C_p, T(\mathbb{Z}_p)); \mathbb{F}_p\right)$. One has $\Gamma_*\alpha_*\mathrm{Trc}_*(e_K) = x_p(0)$, since the left hand side is mapped to the non-zero element $\mathrm{Tr}_*(e_K)$ under D^h_*. Hence $\Psi_*\Gamma_*\alpha_*\mathrm{Trc}_*(e_K) \neq 0$ and as $\Psi \circ \Gamma \simeq \hat{\Gamma} \circ \Phi$ by (6.1), $\hat{\Gamma}_*$ maps $\Phi_*\alpha_*\mathrm{Trc}_*(e_K)$ non-zero. This shows that

$\hat{\Gamma}_*(e) = y_p(0)$. Finally $\hat{\Gamma}_*$ commutes with the first Bockstein operator, and since $\beta_1(f) = e$, $\hat{\Gamma}_*(f) \neq 0$. □

The model for $\hat{\mathbb{H}}(C_p, T(\mathbb{Z}_p))$ used in the beginning of this section has an action of $C_{p^{n-1}}$ whose fixed set is $\hat{\mathbb{H}}(C_{p^n}, T(\mathbb{Z}_p))$.

Lemma 6.6. *Assuming (4.3),* $\hat{\mathbb{H}}\Big(C_{p^{n-1}}, \hat{\mathbb{H}}(C_p, T(\mathbb{Z}_p))\Big) = 0$ *for all* n.

Proof. Since the spectrum in question is p-complete, it suffices to show that

$$\pi_*\Big(\hat{\mathbb{H}}\Big(C_{p^{n-1}}, \hat{\mathbb{H}}(C_p, T(\mathbb{Z}_p))\Big); \mathbb{F}_p\Big) = 0$$

We use the skeleton spectral sequence with

$$E^2_{p,q} = \hat{\mathbb{H}}^{-p}\Big(C_{p^{n-1}}; \pi_q \hat{\mathbb{H}}(C_p, T(\mathbb{Z}_p))\Big)$$

It converges to the mod$p$ homotopy groups of the spectrum under discussion by the criteria of Boardman, cf. [3], [21]. If we write

$$\pi_*\Big(\hat{\mathbb{H}}(C_p, T(\mathbb{Z}_p)); \mathbb{F}_p\Big) = E\{e\} \otimes P\Big[f, f^{-1}\Big]$$

then the E^2-term is

$$E^2_{*,*} = E\{u_{n-1}\} \otimes P\Big[t, t^{-1}\Big] \otimes E\{e\} \otimes P\Big[f, f^{-1}\Big]$$

We can use the $C_{p^{n-1}}$-equivariant map $\hat{\gamma} : T(\mathbb{Z}_p) \longrightarrow \hat{\mathbb{H}}(C_p, T(\mathbb{Z}_p))$ to compare the spectral sequence above with the skeleton spectral sequence with abutment $\pi_*\Big(\hat{\mathbb{H}}\big(C_{p^{n-1}}, T(\mathbb{Z}_p)\big); \mathbb{F}_p\Big)$. In particular, we get from (4.3) that the non-zero differentials are generated multiplicatively from the formulas

$$\begin{aligned} d^{2p(k+1)-1}(t^{p^k}) &= t^{p^k+p(k+1)-1} e f^{p(k)}, \qquad k \leq n-1 \\ d^{2p(n)-1}(u_{n-1}) &= t^{p(n)} f^{p(n-1)} \end{aligned}$$

together with the claim that te and tf are infinite cycles; cf. (4.3) for notation. Now it is easy to see that

$$E^{p(n)}_{p,q} = 0 \qquad\qquad \text{for all } (p, q)$$

Indeed, one argues as in sect.4 and uses that

$$d^{2p(n)-1}\Big(t^{p^n} u_{n-1} (tf)^{-p(n-1)}\Big) = 1$$

This completes the proof. □

Theorem 6.7. *Assuming conjecture (4.3), the vertical arrows in (6.4) induce isomorphisms*

$$\Gamma_* : \pi_i\Big(T(\mathbb{Z}_p)^{C_{p^n}}; \mathbb{F}_p\Big) \to \pi_i\Big(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p\Big)$$
$$\hat{\Gamma}_* : \pi_i\Big(T(\mathbb{Z}_p)^{C_{p^{n-1}}}; \mathbb{F}_p\Big) \to \pi_i\Big(\hat{\mathbb{H}}\Big(C_{p^n}, T(\mathbb{Z}_p)^{hC_{p^n}}\Big); \mathbb{F}_p\Big)$$

for all $i \geq 0$.

Proof. We use induction over n. For $n = 1$, it follows from (6.5) and a 5-lemma argument. Assuming the result for $n < m$ we proceed to prove it for $n = m$. We use the diagram

$$(6.8)_{\mathrm{n}} \qquad \begin{array}{ccc} T(\mathbb{Z}_p)^{C_{p^{n-1}}} & \xrightarrow{\Gamma} & T(\mathbb{Z}_p)^{hC_{p^{n-1}}} \\ \downarrow \hat{\gamma}^{C_{p^{n-1}}} & & \downarrow \hat{\gamma}^{hC_{p^{n-1}}} \\ \hat{\mathbb{H}}(C_p, T(\mathbb{Z}_p))^{C_{p^{n-1}}} & \xrightarrow{G} & \hat{\mathbb{H}}(C_p, T(\mathbb{Z}_p))^{hC_{p^{n-1}}} \end{array} \qquad (n \geq 2)$$

where the horizontal maps are inclusions of fixed sets into homotopy fixed sets. We want to prove that $\hat{\Gamma} = \hat{\gamma}^{C_{p^{m-1}}}$ is a homotopy equivalence in non-negative degrees. Since $\hat{\gamma}$ is an equivalence in non-negative degrees, so is $\hat{\gamma}^{hC_{p^{m-1}}}$ according to (6.4). The upper horizontal map in $(6.8)_{\mathrm{m}}$ induces an isomorphism on $\pi_i(-; \mathbb{F}_p)$ for $i \geq 0$ by the inductive assumption, so we conclude that $\pi_i(G; \mathbb{F}_p)$ is an epimorphism in the same dimensions. We know from (4.11) that $\pi_i\Big(\hat{\mathbb{H}}(C_{p^m}; T(\mathbb{Z}_p)); \mathbb{F}_p\Big)$ and $\pi_i\Big(T(\mathbb{Z}_p)^{hC_{p^{m-1}}}; \mathbb{F}_p\Big)$ are abstractly isomorphic finite dimensional $\mathbb{F}_p$ vector spaces. Hence the source and target of $\pi_i(G; \mathbb{F}_p)$ are isomorphic, and the epimorphism must be an isomorphism. By $(6.8)_{\mathrm{m}}$, $\pi_i\Big(\hat{\Gamma}; \mathbb{F}_p\Big)$ is an isomorphism. We use (6.1) to show that also $\pi_i(\Gamma; \mathbb{F}_p)$ is an isomorphism for $n = m$ and $i \geq 0$. □

Remark 6.8 For any ring R which is a module over $\mathbb{Z}p$, $T(R)$ is a module spectrum over $T(\mathbb{Z}_p)$ and hence $\widehat{\mathbb{H}}\Big(C_{p^{n-1}}, \widehat{\mathbb{H}}(C_p, T(R))\Big)$ is a module spectrum over the corresponding spectrum for $R = \mathbb{Z}_p$. It follows then from Lemma 6.6 that $\widehat{\mathbb{H}}\Big(C_{p^{n-1}}, \widehat{\mathbb{H}}(C_p, T(R))\Big) = 0$. This is an interesting special feature for rings which are not shared by $T(F)$ for general functors with smach product; $F = \mathrm{Id}$ is a counter example

Remark 6.9 (added in January 1994). Stavros Tsalidis proves in his 1994 thesis that the map

$$\Gamma_n \ : \ T(\mathbb{Z}_p)^{C_{p^n}} \to T(\mathbb{Z}_p)^{hC_{p^n}}$$

induces an isomorphism on mod p homotopy groups in non-negative degrees for all n provided this is the case for $n = 1$. Our Remark 5.12 and Lemma 6.5 give the induction start $n = 1$, cf. the proof of Theorem 6.7. It follows by a five lemma argument that

$$\widehat{\Gamma}_n \ : \ T(\mathbb{Z}_p)^{C_{p^{n-1}}} \to \widehat{\mathbb{H}}(C_{p^n}, T(\mathbb{Z}_p))[0,\infty)$$

is also a homotopy equivalence. In particular

$$(*) \qquad \pi_*\Big(T(\mathbb{Z}_p)^{hC_{p^{n-1}}}; \mathbb{F}_p\Big) \cong \pi_*\Big(\widehat{\mathbb{H}}(C_{p^n}, T(\mathbb{Z}_p)); \mathbb{F}_p\Big)$$

for all n and $* \geq 0$. Tsalidis then uses our Remark 4.13 to show that if the differentials are not as claimed in Conjecture 4.3 then $(*)$ cannot be satisfied.

§ 7 The modulo p homotopy groups of TC$(\mathbb{Z}_p, p)$

The results of this section are based upon Conjecture 4.3, and the subsequent calculations of the skeleton spectral sequences as summarized in (4.6) and (4.10), and upon
Theorem 6.7. Let $\mathrm{TC}^{(n)}(\mathbb{Z}_p, p)$ be the spectrum which fits into the cofibration

$$(7.1) \qquad \mathrm{TC}^{(n)}(\mathbb{Z}_p, p) \to T(\mathbb{Z}_p, p)^{C_{p^n}} \xrightarrow{D-\Phi} T(\mathbb{Z}_p)^{C_{p^{n-1}}}.$$

Then we have

$$(7.2) \qquad \mathrm{TC}(\mathbb{Z}_p, p) = \operatorname{holim}_{\leftarrow} \mathrm{TC}^{(n)}(\mathbb{Z}_p, p)$$

and thus $\pi_*(\mathrm{TC}(\mathbb{Z}_p, p); \mathbb{F}_p) = \lim_{\leftarrow} \pi_*\Big(\mathrm{TC}^{(n)}(\mathbb{Z}_p, p); \mathbb{F}_p\Big)$.

Lemma 7.3. *The group* $\lim_{\leftarrow} \pi_{2r}(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p)$ *vanishes when* $r \not\equiv 0 \bmod (p-1)$ *and is a single copy of* $\mathbb{F}_p$ *when* $r \equiv 0 \bmod (p-1)$. *The generator in the latter case is named* $(tf)^i$ *in* $\lim_{\leftarrow} E^\infty \pi_{2i(p-1)}(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p)$.

Proof. From (2.18),

$$\lim_{\leftarrow} \pi_{2r}(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p) = \pi_{2r}(T(\mathbb{Z}_p)^{hS^1}; \mathbb{F}_p)$$

so it suffices to look at the skeleton spectral sequence:

$$H^*(BS^1; \pi_*(T(\mathbb{Z}_p); \mathbb{F}_p)) \Rightarrow \pi_*(T(\mathbb{Z}_p)^{hS^1}; \mathbb{F}_p).$$

By (4.7) only the polynomial algebra $P[tf]$ survives to E^∞. □

We now restrict attention to odd dimensions. Here we can begin by observing that

$$\begin{aligned} &D^h : T(\mathbb{Z}_p)^{hC_{p^{n+1}}} \longrightarrow T(\mathbb{Z}_p)^{hC_{p^n}} \\ &\hat{D} : \hat{\mathbb{H}}(C_{p^{n+1}}, T(\mathbb{Z}_p)) \longrightarrow \hat{\mathbb{H}}(C_{p^n}, T(\mathbb{Z}_p)) \end{aligned}$$

on the E^∞-terms are tabulated as

$$(7.4) \qquad \begin{aligned} E^\infty D^h_*(x_r(i)) &= \begin{cases} x_r(i) & \text{for } 0 \le i < n \\ 0 & \text{for } i = n \end{cases} \\ E^\infty \hat{D}(y_r(i)) &= \begin{cases} y_r(i) & \text{for } 0 \le i < n-1 \\ 0 & \text{for } i = n-1 \end{cases} \end{aligned}$$

with the notation from (4.6) and (4.10). The map

$$(7.5) \qquad E^\infty \Psi_* : E^\infty \pi_*(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p) \longrightarrow E^\infty \pi_*(\hat{\mathbb{H}}(C_{p^n}, T(\mathbb{Z}_p)); \mathbb{F}_p)$$

can be calculated from the information of sect.4; this simply amounts to determining the coincidence relations between the $x_r(i)$'s and the $y_r(j)$'s when one thinks of them as monomials in t, u_n, e and f. To this end, write

$$r - p/p - 1 = a_0 + a_1 p + a_2 p^2 + \dots, 0 \le a_0 < p, \quad 0 < a_i \le p \text{ for } i \ge 1.$$

We call this the p-series for $r - p/p - 1$. The coefficients are unique and, since $r - p/p - 1$ is a rational number, constant for large i. Let $l(r)$ be the smallest number with a_i constant for $i > l(r)$. The linear map $E^\infty \Psi_*$ maps each basis element $x_r(v)$ from (4.6) into a multiplum of a basis element $y_r(v)$ from (4.10).

Lemma 7.6. (i) *$E^\infty \Psi_*(x_r(v)) = y_r(v-1)$ if and only if $v > l(r)$*

(ii) *$E^\infty \Psi_*(x_r(v)) = y_r(v)$ if and only if $r \equiv p(p-1)$ with $r \ne 1$ and $v = l(r) - 1$.*

(iii) *$E^\infty \Psi_*(x_r(v)) = 0$ in all other cases except that $E^\infty \Psi_*(x_r(v)) = y(v+k+1)$ when $r = p + p^{v+k+1} + (p-1)a$ with $p(v) \le a < p(v)$ and $k \gtrsim 2$.*

Proof. Suppose $E^\infty\Psi_*(x_r(v+k)) = y_r(v)$ with $k \geq 1$. Looking at the power of t which appears in the monomials $x_r(v+k)$ and $y_r(v)$ we have

$$a_0 + a_1 p + \ldots + a_{v+k}p^{v+k} = p - r + a_0 p + a_1 p^2 + \ldots + a_v p^{v+1}$$

and inserting the p-series for $(r-p)/p-1$ we get

$$(*) \qquad \sum_{i \geq v+k+1} a_i p^i = \sum_{i \geq v+1} a_i p^{i+1}$$

But this gives $a_{v+1} = p, a_{v+2} = p-1, \ldots, a_{v+k-1} = p-1$ and $a_{v+1} = p, a_{v+2} = p-1, \ldots, a_{v+k-1} = p-1, a_{v+k}+1 = a_{v+2+1} = a_{v+k+2} = \ldots$ Thus $l(r) = v+k$ and $a_i \geq 1$ for $i > l(r)$. It follows that for some $a > 1$,

$$\begin{aligned} \frac{r-p}{p-1} &= a_0 + \ldots + a_v p^v + a\frac{p^{v+k}}{1-p} \\ &< p + \ldots + p^{v+1} + a\frac{p^{v+k}}{1-p} \end{aligned}$$

and hence that $r < p^{v+2} - ap^{v+k}$. Since $r > 0$ we must have $k \leq 1$. For $k = 1, (*)$ gives $a_{v+1} = a_{v+2} = \ldots$ so that $v+1 > l(r)$. This proves (i) and part of (iii).
If $E^\infty\Psi_*(x_r(v)) = y_r(v)$ then $(*)$ is satisfied for $k=0$ and must represent 0. Thus $r \equiv p(p-1)$ and

$$a_{v+1} = p, p-1 = a_{v+2} = a_{v+3} = \ldots$$

so that $l(r) = v+1$. This proves (ii). We leave the rest of (iii) to the reader.□

For a given positive number r and for all n $E^\infty\pi_{2r-1}(T(\mathbb{Z}_p)^{hC_{p^n}};\mathbb{F}_p)$ and $E^\infty\pi_{2r-1}(\hat{\mathbb{H}}(C_{p^{n+1}}, T(\mathbb{Z}_p));\mathbb{F}_p)$ have dimension n with basis $\{x_r(i)\}_{i=0}^n$ and $\{y_r(i)\}_{i=0}^{n-1}$, except if the p-adic valuation $v_p(r) \geq n$ where the dimension is $n+1$. Suppose $v_p(r) < n$. We choose elements

$$\begin{gathered} x_r^h(0), \ldots, x_r^h(n) \in \pi_{2r-1}(T(\mathbb{Z}_p)^{hC_{p^{n+1}}};\mathbb{F}_p) \\ \hat{y}_r(0), \ldots, \hat{y}_r(n-1) \in \pi_{2r-1}(\hat{\mathbb{H}}(C_{p^{n+1}}, T(\mathbb{Z}_p));\mathbb{F}_p) \end{gathered}$$

which represent $x_r(0), \ldots, x_r(n)$ and $y_r(0), \ldots y_r(n-1)$ in the E^∞-terms, and such that

$$(7.7) \qquad \begin{aligned} D_*^h(x_r^h(n)) = 0, &\quad D_*^h(x_r^h(i)) = x_r^h \text{ for } i < n \\ \hat{D}_*(\hat{y}_r(n-1)) = 0, &\quad \hat{D}_*(\hat{y}_r(i)) = \hat{y}_r(i) \text{ for } i < n-1 \end{aligned}$$

If $v_p(r) = n$ then

$$D_* : \pi_{2r-1}(T(\mathbb{Z}_p)^{hC_{p^{n+1}}}; \mathbb{F}_p) \to \pi_{2r-1}(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p)$$

is an isomorphism. If $v_p(r) > n$ then we have one more basis element both in the source and in the target for D_* and $\hat{D}_*$, and

$$D_*^h(x_r^h(n+1)) = 0, \quad \hat{D}_*(\hat{y}_r(n)) = 0 \tag{7.8}$$

whereas D_*^h and $\hat{D}_*$ map the other basis elements by the identity.

The claims (7.7) and (7.8) follow by comparing (4.6) and (4.10) for various values of n. Indeed $E^\infty D_*^h$ and $E^\infty \hat{D}_*$ do satisfy the analogous claims in $E^\infty \pi_{2r-1}(-; \mathbb{F}_p)$, and we can inductively pick the $x_r^h(i)$ and $\hat{y}_r(i)$ with the listed properties. We are interested in large n, and assume from now on that $v_p(r) < n$. Let

$$\xi_r(0), \ldots, \xi_r(n) \in \pi_{2r-1}(T(\mathbb{Z}_p)^{C_{p^{n+1}}}; \mathbb{F}_p)$$
$$\eta_r(0), \ldots, \eta_r(n-1) \in \pi_{2r-1}(T(\mathbb{Z}_p)^{C_{p^n}}; \mathbb{F}_p)$$

be the preimages of the $x_r^h(i)$ and $\hat{y}_r(i)$ under the isomorphisms Γ_* and $\hat{\Gamma}_*$ from (6.1). Since $D^h \circ \Gamma = \Gamma \circ D$ and $\hat{D} \circ \hat{\Gamma} = \hat{\Gamma} \circ D$

$$D_*(\xi_r(n)) = 0\ ,\ D_*(\xi_r(i) = \xi_r(i) \ \text{ for } i < n$$
$$D_*(\eta_r(n-1)) = 0\ ,\ D_*(\eta_r(i)) = \eta_r(i) \ \text{ for } i < n-1$$

The two bases $\{\xi_r(0), \ldots, \xi_r(n-1)\}$ and $\{\eta_r(0), \ldots, \eta_r(n-1)\}$ for $\pi_{2r-1}(T(\mathbb{Z}_p)^{C_{p^n}}; \mathbb{F}_p)$ are associated by a triangular matrix $\Lambda = (\lambda_{ij})$, that is,

$$\eta_r(i) = \lambda_{ii}\xi_r(i) + \lambda_{i+1,i}\xi_r(i+1) + \ldots + \lambda_{n-1,i}\xi_v(n-1) \tag{7.9}$$

with $\lambda_{ii} \in \mathbb{F}_p^\times$. This follows from (7.7).

Suppose now that v is an index so that $x_r(v)$ is in the kernel of $E^\infty(\Psi_*)$, i.e. that $x_r(v)$ is a boundary in the skeleton spectral sequence for $\hat{\mathbb{H}}(C_{p^{n+1}}, T(\mathbb{Z}_p))$, $x_r(v) = \hat{d}^s(a_r(v))$. By Theorem 1.15 there is an element

$$a_{r,h}(v) \in \pi_{2r-1}(T(\mathbb{Z}_p)_{hC_{p^{n+1}}}; \mathbb{F}_p)$$

so that the norm map N^h from (6.1) has

$$N_*^h(a_{r,h}(v)) = x_r^h(v) + \sum_{i=v+1}^{n} \mu_i x_r^h(i) \tag{7.10}$$

for some $\mu_i \in \mathbb{F}_p$. Since $\Psi_* \circ N_*^h = 0$ the $\mu_i = 0$ unless $E^\infty\Psi_*(x_r(i)) = 0$. Suppose $v_0 < v_1$ and that both $x_r(v_0)$ and $x_r(v_1)$ are in the kernel of $E^\infty\Psi_*$. It is easy to

check from the structure of the skeleton spectral sequence for $\hat{\mathbb{H}}(C_{p^{n+1}}, T(\mathbb{Z}_p))$ given in sect.4, that the differential killing $x_r(v_0)$ is shorter than the differential killing $x_r(v_1)$ so that $a_r(v_0)$ has strictly lower filtration than $a_r(v_1)$.
It follows from Lemma 7.6 that

$$(7.11) \quad \begin{aligned} &(a) \quad \mathrm{Im}(N_*^h) = \langle x_r^h(0), ..., x_r^h(e(r)) \rangle \\ &(b) \quad \mathrm{Im}(N_*^h) = \langle x_r^h(0), ..., \widehat{x_r^h(i+1)}, ..., x_r^h(l(r)-1) \rangle \end{aligned}$$

Here (b) corresponds to 7.5 (ii) (with $i = l(r) - 1$) and to 7.6 (iii).
We are now ready to evaluate the map

$$D_* - \Phi_* : \pi_{2r-1}(T(\mathbb{Z}_p)^{C_{p^n}}; \mathbb{F}_p) \to \pi_{2r-1}(T(\mathbb{Z}_p)^{C_{p^{n-1}}}; \mathbb{F}_p)$$

where we have:

Theorem 7.12. *Suppose $n \geq v_p(r)$.*

(i) *If $r \not\equiv 1$ mod $(p-1)$ or $r = 1$ then $D_* - \Phi_*$ is surjective and its fiber is a single copy of $\mathbb{F}_p$.*

(ii) *If $r \equiv 1$ mod $(p-1)$ and $r \neq 1$ then $D_* - \Phi_*$ has kernel $\mathbb{F}_p \oplus \mathbb{F}_p$ and cokernel $\mathbb{F}_p$.*

Proof. We begin with (i). There are two cases to consider according to Lemma 7.6, corresponding to r being of the exceptional form $r = p^{i+1} + p^{i+j+2} + (p-1)r_0$ or not. In the former case

$$\frac{r-p}{p-1} = \sum_{i=0}^{i} a_i p^i + p - p^{i+1} + (p-1)p^{i+1} + ... + (p-1)p^{i+j+1} + (p-2)p^{i+j+2} + ...$$

so $l(r) = i + j + 1$. We have (7.6) and (7.11):

$$\begin{aligned} &\Phi_*(\xi_r(v)) = \mu_{v-1,v-1}\eta_r(v-1) + ... + \mu_{v-1,n-1}\eta_r(n-1) \text{ for } v \geq i+j+2 \\ (*) \quad &\Phi_*(\xi_r(i)) = \mu_{i+j,i+j}\eta_r(i+j) + ... + \mu_{i+j,n-1}\eta_r(n-1) \\ &\Phi_*(\xi_r(v)) = 0 \text{ for } v \leq i+j+1 \text{ and } v \neq i. \end{aligned}$$

with $\mu_{k,k} \neq 0$. We can change basis by (7.9) and replace the $\eta_r(\nu)$ in (*) to $\xi_r(\nu)$. This will alter the coefficients $\mu_{k,l}$, but the new diagonal coefficients $\mu'_{k,k}$ will still be non-zero. Since $D_*\xi_r(\nu) = \xi_r(\nu)$ for $\nu < n$ and D_* annihilates $\xi_r(n)$, the obvious inductive argument shows that $D_* - \Phi_*$ is surjective, and hence also that its kernel is $\mathbb{F}_p$. For later use we note that

$$(7.13) \qquad \mathrm{Ker}(D_* - \Phi_*) \subseteq \langle \xi_r(i+j+1), ..., \xi_r(n) \rangle$$

If r is not of the exceptional form, then

$$(**) \quad \begin{aligned} &\Phi_*(\xi_r(v)) = \mu_{v-1,v-1}\xi_r(v-1) + ... + \mu_{v-1,n-1}\xi_r(n-1) \text{ for } v > l(r) \\ &\Phi_*(\xi_r(v)) = 0 \text{ otherwise} \end{aligned}$$

Again it is clear that $D_* - \Phi_*$ is surjective, that the kernel is $\mathbb{F}_p$ and that

$$(7.14) \qquad \operatorname{Ker}(D_* - \Phi_*) \subseteq \langle \xi_r(l(r)), \dots, \xi_r(n) \rangle$$

Finally, the case $r = 1$ corresponds to $l(r) = 0$. This prove (i).
In case (ii) we can argue as above that

$$\operatorname{Ker}(D_* - \Phi_*) \cap \langle \xi_r(l(r)), \dots, \xi_r(n) \rangle = \mathbb{F}_p.$$

and we must find a second $\mathbb{F}_p$ in the kernel. Write $l = l(r) - 1$ and note that

$$x_r(l) = e(tf)^{\frac{r-p}{p-1}} \in E^\infty \pi_{2r-1}(T(\mathbb{Z}_p)^{hC_{p^{n+1}}}; \mathbb{F}_p)$$

so that the natural candidate for a second $\mathbb{F}_p$ is a good choice of $\xi_r(l)$.
Let $i : S^0 \to A(*)$ be the unit of the ring structure, split by the topological Dennis trace

$$Tr : A(*) \to T(*)$$

(cf. [35]). Consider the diagram

$$\begin{array}{ccccccc}
 & & A(*) & \to & T(*)^{C_{p^{n+1}}} & \xrightarrow{\simeq} & T(*)^{hC_{p^{n+1}}} \\
 & \overset{i}{\nearrow} & & & & & \\
S^0 & & \downarrow & & \downarrow & & \downarrow \\
 & \underset{i}{\searrow} & & & & & \\
 & & K(\mathbb{Z}) & \to & T(\mathbb{Z})^{C_{p^{n+1}}} & \xrightarrow{\Gamma} & T(\mathbb{Z})^{hC_{p^{n+1}}}
\end{array}$$

By (5.3), $v_1 \in \pi_{2p-2}(S^0, \mathbb{F}_p)$ is mapped into $tf \in E^\infty \pi_{2p-2}(T(\mathbb{Z})^{hC_{p^{n+1}}}; \mathbb{F}_p)$ for $n \geq 0$. Let $e_K \in \pi_{2p-1}(K(\mathbb{Z}); \mathbb{F}_p)$ be the element with $T_r(e_K) = e$ in $\pi_{2p-1}(T(\mathbb{Z}); \mathbb{F}_p)$, cf. sect.6, and consider the product

$$e_K \cdot v_1^{\frac{r-p}{p-1}} \in \pi_{2r-1}(K(\mathbb{Z}); \mathbb{F}_p)$$

induced from the module structure $S^0 \wedge K(\mathbb{Z}) \to K(\mathbb{Z})$. We have

$$\operatorname{Trc}(e_K \cdot v_1^{\overline{r-p}p-1}) = \operatorname{Trc}(e_K) \cdot v_1^{\frac{r-p}{p-1}} \in \pi_{2r-1}(T(\mathbb{Z}_p)^{C_{p^{n+1}}}; \mathbb{F}_p)$$

and in $E^\infty \pi_{2r-1}(T(\mathbb{Z}_p)^{hC_{p^{n+1}}}; \mathbb{F}_p)$,

$$\Gamma_*(\operatorname{Trc}(e_K) \cdot v_1^{\frac{r-p}{p-1}}) = e(tf)^{\frac{r-p}{p-1}} = x_r(l)$$

We may choose $\xi_r(l) = \operatorname{Trc}(e_K \cdot v_1^{\frac{r-1}{p-1}})$. By definition D_* is equal to Φ_* on any $\operatorname{Trc}(\mu)$, so $\xi_r(l) \in \operatorname{Ker}(D_* - \Phi_*)$. $\square$

The elements $(tf)^i \in E^\infty\pi_{2i(p-1)}(T(\mathbb{Z}_p)^{C_{p^{n+1}}};\mathbb{F}_p)$ can be represented by $\mathrm{Trc}(i_*(v_1^i))$ so lie in the kernel of $(D-\Phi)_*$. We now list the main conclusion of this section in

Theorem 7.15.

$$\pi_{2r-1}(\mathrm{TC}(\mathbb{Z}_p,p);\mathbb{F}_p) = \begin{cases} \mathbb{F}_p \text{ if } r \not\equiv 0,1 \bmod (p-1) \text{ or if } r=1 \\ \mathbb{F}_p\oplus\mathbb{F}_p \text{ otherwise}\end{cases}$$

$$\pi_{2r}(\mathrm{TC}(\mathbb{Z}_p;p);\mathbb{F}_p) = \begin{cases} \mathbb{F}_p\oplus\mathbb{F}_p \text{ if } r\equiv 0 & \bmod (p-1), r\neq 0 \\ \mathbb{F}_p & , r=0 \\ 0 & , \text{otherwise}\end{cases}$$

Proof. Since we are working with finite coefficients there are no $\varprojlim^{(1)}$-terms, and

$$\pi_*(\mathrm{TC}(\mathbb{Z}_p,p);\mathbb{F}_p) \cong \varprojlim\ \pi_*(\mathrm{TC}^{(n)}(\mathbb{Z}_p,p);\mathbb{F}_p)$$

Thus we have the exact sequence

$$0 \to \varprojlim_{D_*} \frac{\pi_{2r}(T(\mathbb{Z}_p)^{C_{p^n}};\mathbb{F}_p)}{\mathrm{Im}(D_*-\Phi_*)} \to \pi_{2r-1}(\mathrm{TC}(\mathbb{Z}_p,p);\mathbb{F}_p) \to \varprojlim_{D_*}\ \mathrm{Ker}(D_*-\Phi_*) \to 0.$$

From Theorem 7.12 we see that

$$\mathrm{Ker}(D_*-\Phi_* : \pi_{2r-1}(T(\mathbb{Z}_p)^{C_{p^{n+1}}};\mathbb{F}_p) \to \pi_{2r-1}(T(\mathbb{Z}_p)^{C_{p^n}};\mathbb{F}_p))$$

is equal $\mathbb{F}_p$ if $r=1$ or if $r\not\equiv 1 \bmod (p-1)$, and is otherwise $\mathbb{F}_p\oplus\mathbb{F}_p$. Moreover, D_* maps the kernel isomorphically when $n\geq v_p(r)$. Finally we can use (7.3) to see that

$$\varprojlim_{D_*}\ \frac{\pi_{2r}(T(\mathbb{Z}_p)^{C_{p^n}};\mathbb{F}_p)}{\mathrm{Im}(D_*-\Phi_*)} = \begin{cases}\mathbb{F}_p \text{ for } r\equiv 0 \bmod (p-1)\\ 0 \text{ otherwise}\end{cases}$$

The even dimensional groups are calculated in a similar fashion □

§ 8 Periodicity for TC $(\mathbb{Z}_p,p)$

Every spectrum is a module over the sphere spectrum and every stable map is a homomorphism of modules. In particular, letting v_1 be the non-trivial element of $\pi_{2(p-1)}(S^0;\mathbb{F}_p)$, we have an operator

(8.1) $$v_1 : \pi_k(\mathrm{TC}\,(\mathbb{Z}_p,p);\mathbb{F}_p) \to \pi_{k+2(p-1)}(\mathrm{TC}\,(\mathbb{Z}_p,p);\mathbb{F}_p)$$

which commutes with homomorphisms induced from stable maps. In this section we show that v_1 is an isomorphism. The proof is calculational and does not, at the moment, shed much light on the phenomenon. In fact, the result appears somewhat miraculous in that our proof also shows that

$$v_1 : \pi_k(T(\mathbb{Z}_p)^{C_{p^n}}; \mathbb{F}_p) \to \pi_{k+2(p-1)}(T(\mathbb{Z}_p)^{C_{p^n}}; \mathbb{F}_p) \tag{8.2}$$

is *not* an isomorphism for any specific value of n. Thus $T(\mathbb{Z}_p)^{C_{p^n}}$ is not a K-local spectrum, or more precisely, is not the connected cover of a K-local spectrum. Nevertheless we derive (8.1) by calculation of (8.2) which in turn we study via the isomorphism

$$\Gamma_* : \pi_k(T(\mathbb{Z}_p)^{C_{p^n}}; \mathbb{F}_p) \xrightarrow{\cong} \pi_k(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p)$$

and our calculations in the previous section.

Lemma 8.3. *In even dimensions*

$$v_1 : \varprojlim \pi_{2r}(T(\mathbb{Z}_p)^{C_{p^n}}; \mathbb{F}_p) \to \varprojlim \pi_{2r+2(p-1)}(T(\mathbb{Z}_p)^{C_{p^n}}; \mathbb{F}_p)$$

is an isomorphism.

Proof. Both range and domain are zero unless $r \equiv 0 \bmod (p-1)$ and are in this case a single copy of $\mathbb{F}_p$. This follows from (7.3). Moreover, the non-trivial $\mathbb{F}_p$ is, upon applying Γ_*, represented by

$$(tf)^{r/p-1} \in E^\infty \pi_{2r}(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p)$$

It is a consequence of (5.3) that multiplication by v_1 on $\pi_{2r}(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p)$ corresponds to multiplication by $t.f$ in $E^\infty \pi_*(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p)$. This is visibly an isomorphism on $\varprojlim E^\infty \pi_*(T(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p)$, and the lemma follows. □

In odd dimensions we study the diagram

$$\begin{array}{ccccccc}
0 & \to & \mathrm{Ker}(D_* - \Phi_*) & \to & \pi_{2r-1}(T(\mathbb{Z}_p)^{C_{p^{n+1}}}; \mathbb{F}_p) & \to & \pi_{2r-1}(T(\mathbb{Z}_p)^{C_{p^n}}; \mathbb{F}_p) \\
 & & \downarrow v_1 & & \downarrow v_1 & & \downarrow v_1 \\
0 & \to & \mathrm{Ker}(D_* - \Phi_*) & \to & \pi_{2s-1}(T(\mathbb{Z}_p)^{C_{p^{n+1}}}; \mathbb{F}_p) & \to & \pi_{2s-1}(T(\mathbb{Z}_p)^{C_{p^n}}; \mathbb{F}_p)
\end{array} \tag{8.4}$$

with $s = r + (p-1)$. Suppose we have the p-series

$$r - p/p - 1 = \sum_{i=0}^{l} a_i p^i + qp^{l+1} + qp^{l+2} + \ldots \tag{8.5}$$

where $0 \leq a_0 < p, 0 < a_i \leq p$ and $0 < q \leq p$. Actually q must be strictly less than p since for $q = p$ we would have

$$r - p/p - 1 = \sum_{i=0}^{l} a_i p^i + p^{l+2}/1 - p < p(\frac{p^{l+1} - 1}{p - 1}) + p^{l+2}/1 - p$$

and hence $r - p < p(p^{l+1} - 1) - p^{l+2}$ or $r < 0$. In sect.7 we introduced generators $\xi_r(i)$, $0 \leq i \leq n$, for $\pi_{2r-1}(T(\mathbb{Z}_p)^{C_{p^{n+1}}}; \mathbb{F}_p)$ such that in all cases

$$\begin{aligned} &\operatorname{Ker}(D_* - \Phi_*) \cap \langle \xi_r(l+1), \ldots, \xi_r(n) \rangle = 0 \\ &\operatorname{Ker}(D_* - \Phi_*) \cap \langle \xi_r(l), \ldots, \xi_r(n) \rangle = \mathbb{F}_p \end{aligned} \tag{8.6}$$

given the p-series (8.5), cf. (7.13) and (7.14). If $r \equiv p \bmod (p-1)$ and $r > 1$ there is a further generator of $\operatorname{Ker}(D_* - \Phi_*)$, namely

$$\xi_r(l-1) = \operatorname{Trc}(e_K v_1^{r-p/p-1})$$

In this case the p-series (8.5) has $a_l = p$ and $q = p - 1$ and (8.5) gives

$$r - p/p - 1 = \sum_{i=0}^{l-1} a_i p^i$$

Let us introduce the notation $a = a(r)$ for the sum $\sum_{i=0}^{l} a_i p^i$ in (8.5) and note that $l + 1 = l(r)$. In $E^\infty \pi_{2r-1}(T(\mathbb{Z}_p)^{hC_{p^{n+1}}}; \mathbb{F}_p)$,

$$\begin{aligned} &\Gamma_* \xi_r(l) = t^{a(r)} e f^{a(r) - qp^l} = x_r(l) \\ &\Gamma_* \xi_r(l+i) = t^{a(r) + qp^{l+1}(\frac{p^{i+1}-1}{p-1})} e f^{a(r) + qp^{l+1}(\frac{p^i-1}{p-1})} = x_r(l+i) \end{aligned} \tag{8.7}$$

and when $r \equiv p (\bmod p - 1)$ and $r > 1$,

$$\Gamma_* \xi_r(l-1) = t^{r-p/p-1} e f^{r-p/p-1} \tag{8.8}$$

If $s = r + (p-1)$ then $l(s) = l(r)$ and $a(s) = a(r) + 1$ except if (8.5) has $a_0 = p - 1$ and $a_i = p$ for $0 < i \leq l$. In this exceptional case

$$r = 1 - p + p^{l+1}(p - q)$$

and the p-series for $s - p/p - 1$ is

$$s - p/p - 1 = 0 + 1 \cdot p + \ldots + 1 \cdot p^l + (q+1) p^{l+1} q p^{l+2} + \ldots \tag{8.5}$$

so in the exceptional case:

$$l(s) = l(r) + 1\ ,\ a(s) = a(r) + qp^{l+1} + 1$$

Lemma 8.9. *Suppose* $s = r + (p-1)$ and $l(s) = l(r)$. *Then multiplication with* v_1 *maps* $\langle \xi_r(l), ..., \xi_r(n) \rangle$ *isomorphically to* $\langle \xi_s(l), ..., \xi_s(n) \rangle$.

Proof. We can apply the isomorphism Γ_* and use (8.6). Since $a(s) = a(r) + 1$ and v_1 corresponds to multiplication with $t.f$ in $E^\infty \pi_*(T(\mathbb{Z}_p)^{hC_{p^{n+1}}}; \mathbb{F}_p)$ we see that $(t.f)x_r(l+i) = x_s(l+i)$, so that multiplication by $t.f$ maps $\langle x_r(l), ..., x_r(n) \rangle$ isomorphically to $\langle x_s(l), ..., x_s(n) \rangle$. The same statement then follows when $x_r(\nu), x_s(\nu)$ are replaced by $\xi_r(\nu), \xi_s(\nu)$. □

Lemma 8.10. *Suppose that* $l(s) = l(r) + 1$ *and thus* $r = 1 - p + p^{l(r)}(p - q)$. *Then* $v_1 \cdot \xi_r(l(r) - 1) = 0$

Proof. In $E^\infty \pi_{2r-1}(T(\mathbb{Z}_p)^{hC_{p^{n+1}}}; \mathbb{F}_p)$,

$$x_r(l) = t^{p(l+1)-1} e f^{p(l+1)-qp^l-1}$$

This is a boundary in the Tate skeleton spectral sequence converging to $\pi_*(\hat{\mathbb{H}}(C_{p^{n+1}}, T(\mathbb{Z}_p); \mathbb{F}_p))$. Indeed from sect.4,

$$\hat{d}^{p(l+1)}(t^{-1} f^{p^{l+1}-qp^l-1}) = x_r(l), \quad p(l+1) = p(\frac{p^{l+1}-1}{p-1})$$

The element $t^{-1} f^{p^{l+1}-qp^l-1} \in E^{p(l+1)}(\hat{\mathbb{H}}(C_{p^{n+1}}, T(\mathbb{Z}_p); \mathbb{F}_p))$ represents a non-trivial element in $E^\infty \pi_{2r-1}(T(\mathbb{Z}_p)_{hC_{p^{n+1}}}; \mathbb{F}_p)$ by (2.15). Moreover, as it has minimal filtration degree in the given dimension there is a unique element, say

$$a_{r,h}(l) \in \pi_{2r-1}(T(\mathbb{Z}_p)_{hC_{p^{n+1}}}; \mathbb{F}_p)$$

represented by $t^{-1} f^{p^{l+1}-qp^l-1}$, and

$$N_*^h(a_{r,h}(l)) = x_r^h(l) \in \pi_{2r-1}(T(\mathbb{Z}_p)^{hC_{p^{n+1}}}; \mathbb{F}_p)$$

represents $x_r(l)$ in $E^\infty \pi_{2r-1}(T(\mathbb{Z}_p)^{hC_{p^{n+1}}}; \mathbb{F}_p)$, see also (7.11). Since by definition, $\Gamma_* \xi_r(l) = x_r^h(l)$ we can write this equation as

$$N_*\big(a_{r,h}(l)\big) = \xi_r(l)$$

where N is the norm map from $T(\mathbb{Z}_p)_{hC_{p^{n+1}}}$ to $T(\mathbb{Z}_p)^{C_{p^{n+1}}}$. Thus $v_1\xi_r(l) = N_*(v_1, a_{r,h}(l))$. But in $E^\infty\pi_{2r-1}(T(\mathbb{Z}_p)_{hC_{p^{n+1}}}; \mathbb{F}_p)$, $v_1 a_{r,h}(l) = f^{p^{l+1}-qp^l-1}$ is equal to zero, since it lies to the left of the filtration line $s = 1$ in the Tate skeleton spectral sequence for $\hat{\mathbb{H}}(C_{p^{n+1}}, T(\mathbb{Z}_p))$, cf. the discussion in sect.2. □

Proposition 8.11. *The map*

$$v_1 : \pi_{2r-1}(T(\mathbb{Z}_p)^{C_{p^{n+1}}}; \mathbb{F}_p) \longrightarrow \pi_{2s-1}(T(\mathbb{Z}_p)^{C_{p^{n+1}}}; \mathbb{F}_p)$$

induces an isomorphism of $\mathrm{Ker}(D_* - \Phi_*)$.

Proof. If $r \equiv 1 (\mathrm{mod}\, p-1)$ and $r > 1$ then the element $\mathrm{Trc}(e_K \cdot v_1^{r-p/1-p})$ maps to $\mathrm{Trc}(e_K v_1^{s-p/1-p})$, so we have left to check for all r that v_1 maps $\langle \xi_r(l), ..., \xi_r(n)\rangle \cap \mathrm{Ker}(D_* - \Phi_*)$ isomorphically. This follows from (8.9) when $l(s) = l(r)$. If $l(s) = l(r) + 1$ then $v_1 \cdot \xi_r(l(r) - 1) = 0$ by (8.10), so

$$v_1 : \langle \xi_r(l(r) - 1), ..., \xi_r(n)\rangle \longrightarrow \langle \xi_s(l(s) - 1), ..., \xi_s(n)\rangle$$

We recall from sect.7 that with $l = l(r) - 1$:

$$\Phi_*(\xi_r(l)) = 0,\ \Phi_*(\xi_r(l + i)) = \lambda_i \xi_r(l + i - 1) + ...$$

for $i > 0$. Here λ_i is non-zero in $\mathbb{F}_p$ and the dots indicate a linear combination of the elements $\xi_r(l + i), ..., \xi_r(n)$. Also, $D_*(\xi_r(\nu)) = \xi_r(\nu)$ except that $D_*\xi_r(n) = 0$. To show that v_1 maps $\mathrm{Ker}(D_* - \Phi_*) \cap \langle \xi_r(l), ..., \xi_r(n)\rangle$ monomorphically, we can compose with $D_* \circ ... \circ D_*$ ($n - l - 1$ factors), or in other words reduce to the case $n = l + 1 \quad (= l(s) - 1)$. Then

$$\mathrm{Ker}(D_* - \Phi_*) \cap \langle \xi_r(l), \xi_r(l + 1)\rangle = \langle \lambda_0\xi_r(l) + \xi_r(l + 1)\rangle$$
$$\mathrm{Ker}(D_* - \Phi_*) \cap \langle \xi_s(l + 1)\rangle = \langle \xi_s(l + 1)\rangle$$

and

$$v_1(\lambda_0\xi_r(l) + \xi_r(l + 1)) = v_1 \cdot \xi_r(l + 1) = \xi_s(l + 1)$$

The first equality is (8.10), the second follows from (8.7):

$$\Gamma_*\xi_r(l + 1) = t^{a(r)+qp^{l+1}} e f^{a(r)}$$
$$\Gamma_*(\xi_s(l + 1)) = t^{a(s)} e f^{a(s)-qp^{l(s)-1}}$$

and $a(s) = a(r) + q \cdot p^{l+1} + 1$, $l(s) = l + 2$. □

Theorem 8.12. *Multiplication by* v_1

$$v_1 : \pi_k(\mathrm{TC}(\mathbb{Z}_p, p); \mathbb{F}_p) \to \pi_{k+2(p-1)}(\mathrm{TC}(\mathbb{Z}_p, p); \mathbb{F}_p)$$

is an isomorphism for all k.

Proof. We use the cofibration

$$\mathrm{TC}^{(n)}(\mathbb{Z}_p, p) \to T(\mathbb{Z}_p)^{C_{p^n}} \xrightarrow{D-\Phi} T(\mathbb{Z}_p)^{C_{p^{n-1}}}$$

and that

$$\mathrm{TC}(\mathbb{Z}_p, p) = \operatorname{holim}_{\leftarrow} \mathrm{TC}^{(n)}(\mathbb{Z}_p, p)$$

In $\pi_{2i(p-1)}(T(\mathbb{Z}_p)^{C_{p^n}}; \mathbb{F}_p)$ we have the element $\mathrm{Trc}(\iota_*(v_1^i))$ where $\iota : S^0 \to K(\mathbb{Z}_p)$ is the unit, and by (8.3) these are the only elements which contribute to the limit over n. Clearly

$$v_1 \cdot \mathrm{Trc}(\iota_*(v_1^i)) = \mathrm{Trc}(\iota_*(v_1^{i+1}))$$

and as $D_*\mathrm{Trc}(\iota_*(v_1^i)) = \Phi_*\mathrm{Trc}(\iota_*(v_1^i))$, the elements

$$\mathrm{Trc}(\iota_*(v_1^i)) \in \lim_{\leftarrow} \pi_{2i(p-1)}(\mathrm{TC}^{(n)}(\mathbb{Z}_p, p); \mathbb{F}_p)$$
$$\partial_*\mathrm{Trc}(\iota_*(v_1^i)) \in \lim_{\leftarrow} \pi_{2i(p-1)-1}(\mathrm{TC}^{(n)}(\mathbb{Z}_p, p; \mathbb{F}_p)$$

maps to each other under multiplication by v_1. For $r \equiv 1 \bmod (p-1), r > 1$ we similarly have

$$\mathrm{Trc}(e_K v_1^{i-1}) \in \pi_{2i(p-1)+1}(T(\mathbb{Z}_p)^{C_{p^n}}; \mathbb{F}_p)$$

with

$$D_*\mathrm{Trc}(e_K v_1^{i-1}) = \Phi_*\mathrm{Trc}(e_K v_1^{i-1}) = 0$$

Moreover, from the proof of (7.12)(ii) we know that

$$\mathrm{Trc}(e_K v_1^{i-1}) \notin \mathrm{Im}(D_* - \Phi_*)$$

so that we have non-zero elements

$$\mathrm{Trc}(e_K v_1^{i-1}) \in \lim_{\leftarrow} \pi_{2i(p-1)+1}(\mathrm{TC}^{(n)}(\mathbb{Z}_p; p); \mathbb{F}_p)$$
$$\partial_*\mathrm{Trc}(e_K v_1^{i-1}) \in \lim_{\leftarrow} \pi_{2i(p-1)}(\mathrm{TC}^{(n)}(\mathbb{Z}_p, p); \mathbb{F}_p)$$

which map to each other by multiplication with v_1. Finally, we have the elements in

$$\mathbb{F}_p \cong \mathrm{Ker}(D_* - \Phi_*) \cap \langle \xi_r(l(r)-1), \ldots, \xi_r(n-1) \rangle \subsetneq \pi_{2r-1}(T(\mathbb{Z}_p)^{C_{p^n}}; \mathbb{F}_p)$$

which account for the remaining elements $\lim_{\leftarrow} \pi_{2r-1}(\mathrm{TC}^{(n)}(\mathbb{Z}_p, p); \mathbb{F}_p)$. They correspond under multiplication by v_1 according to (8.11). □

§ 9 The p-integral homotopy type of TC $(\mathbb{Z}_p, p)$

Let TC $(*, p)$be the topological cyclic homology for the identity FSP, cf.[11], sect.5. Linearization defines a homotopy commutative diagram

$$
\begin{array}{ccc}
A(*) & \rightarrow & \mathrm{TC}\,(*, p) \\
\downarrow & & \downarrow \\
K(\mathbb{Z}_p) & \rightarrow & \mathrm{TC}\,(\mathbb{Z}_p, p)
\end{array}
\tag{9.1}
$$

The p-adic completion of TC $(*, p)$ is known; there is a homotopy Cartesian diagram of spectra

$$
\begin{array}{ccc}
\mathrm{TC}\,(*, p)_p^\wedge & \xrightarrow{\beta} & \Sigma^\infty(\Sigma_+(\mathbb{C}P^\infty))_p^\wedge \\
\downarrow \alpha & & \downarrow \mathrm{trf} \\
\Sigma^\infty(S^0)_p^\wedge & \xrightarrow{0} & \Sigma^\infty(S^0)_p^\wedge
\end{array}
\tag{9.2}
$$

where trf denotes the S^1-transfer of the classifying fibration $ES^1 \rightarrow BS^1$, $\Sigma_+(X) = \Sigma(X \cup \{\infty\})$, and $\Sigma^\infty(Y)$ is the suspension spectrum of Y. When there is no possibility for confusions we sometimes drop Σ^∞ from the notation. For example S^0 will always denote the sphere spectrum. We have

$$\Sigma^\infty(\Sigma_+(\mathbb{C}P^\infty)) = \Sigma^\infty S^1 \vee \Sigma^\infty(\Sigma\mathbb{C}P^\infty)$$

and trf : $\Sigma^\infty S^1 \rightarrow \Sigma^\infty S^0$ is trivial because it is the S^1-transfer for the restriction of the classifying S^1-bundle to a base point. Thus

$$TC(*, p)_p^\wedge \simeq \Sigma^\infty(S^0)_p^\wedge \vee \Sigma^\infty(S^1)_p^\wedge \vee hF(\Sigma^\infty(\Sigma\mathbb{C}P^\infty) \xrightarrow{\mathrm{trf}} S^0)_p^\wedge \tag{9.3}$$

It will be illuminating in our comparison of TC $(*, p)$ to TC $(\mathbb{Z}_p, p)$ to make use of localization at (topological) K-theory, and we begin by recalling some facts about the localization of (9.3). Let $L_{K\mathbb{F}_p}(-)$ denote the Bousfield localization at mod p K-theory, [4]. One has

$$L_{K\mathbb{F}_p}(\Sigma^\infty(S^0)_p^\wedge) = J_p^\wedge, L_{K\mathbb{F}_p}(\Sigma^\infty(S^1)_p^\wedge = \Sigma J_p^\wedge \tag{9.4}$$

where J is the non-connective image of J space, i.e. the homotopy fiber in

$$J \rightarrow K_p^\wedge \xrightarrow{\psi-1} K_p^\wedge$$

Here K is the periodic K-theory spectrum with $2n$-th space equal to $BU \times \mathbb{Z}$. The (-1)-connected cover of J is the spectrum whose bottom space is $\operatorname{Im} J \times \mathbb{Z}$. This spectrum fits into the fibration

$$\operatorname{Im} J \times \mathbb{Z}_p \to (BU \times \mathbb{Z})_p^\wedge \xrightarrow{\psi-1} BU_p^\wedge$$

and the localization map $(S^0)_p^\wedge \to J_p^\wedge$ factors over a map $e : (S^0)_p^\wedge \to (\operatorname{Im} J \times \mathbb{Z})_p^\wedge$. The localization of (the suspension spectrum of) $\Sigma\mathbb{C}P^\infty$ is more involved. We have the mapping of spaces

$$\varepsilon : \Sigma\mathbb{C}P^\infty \to SU$$

whose adjoint is the Hopf bundle $\mathbb{C}P^\infty \to BU$. It induces a map of spectra

$$\Sigma^\infty\Sigma\mathbb{C}P^\infty \to \Sigma bu = su$$

where we have written bu for the 1-connected cover of K - its 0^{th} space is BU since $\Omega SU = BU$. Let $\operatorname{Im} J$ be the 0-connected cover of J (or of $\operatorname{Im} J \times \mathbb{Z}$). Then we have the (co)fibration sequence of spectra

$$\Sigma bu \to \Sigma bu \xrightarrow{T} \operatorname{Im} J \to bu \xrightarrow{\psi-1} bu$$

where ψ is the stable map which on the 0^{th} space is $\psi^g : BU \to BU$. It follows from the appendix of [32] that there is a homotopy commutative diagram of spectra

$$\begin{array}{ccc} \Sigma(\mathbb{C}P^\infty) & \xrightarrow{\varepsilon} & \Sigma bu \\ \downarrow \mathrm{trf} & & \downarrow T \\ \Sigma^\infty(S^0) & \xrightarrow{e} & (\operatorname{Im} J \times \mathbb{Z}) \end{array} \tag{9.4}$$

It is in order to point out that ε is not the connected cover of the $K\mathbb{F}_p$-localization of the spectrum $\Sigma^\infty(\Sigma\mathbb{C}P^\infty)$. In fact $L_{K\mathbb{F}_p}(\mathbb{C}P^\infty)[0,\infty]$ contains an infinite wedge of spectra $\Sigma^{-1}bu\mathbb{Z}_{p^\infty}$ along with Σbu, cf. [21] or [31]. Hence

$$L_{K\mathbb{F}_p}(\Sigma\mathbb{C}P^\infty)[0,\infty) = \Sigma bu \vee \bigvee^{\infty} bu\mathbb{Z}_{p^\infty}$$

where $\mathbb{Z}_{p^\infty} = (\mathbb{Q}/\mathbb{Z})_{(p)}$ and $bu\mathbb{Z}_{p^\infty} = bu \wedge \operatorname{holim}_{\rightarrow} S^0/p^k$.
Localization at $K\mathbb{F}_p$ is related to v_1-periodicity by the following fundamental result

$$L_{K\mathbb{F}_p}\left(S^0\right)/p \simeq S^0/p[1/v_1] \tag{9.5}$$

where S^0/p is the mod p Moore spectrum, and

$$S^0/p[1/v_1] = \operatorname{holim}_{\rightarrow} \ (S^0/p \xrightarrow{v_1} S^0/p \xrightarrow{v_1} \ldots)$$

Since for any spectrum, $L_{K\mathbb{F}_p}(X)/p = L_{K\mathbb{F}_p}(S^0) \wedge X/p$, the mod p homotopy groups of the localization are v_1-periodic. Conversely, if say a (-1)-connected spectrum has v_1 periodic homotopy groups then $l : X \rightarrow L_{K\mathbb{F}_p}(X)$ induces an isomorphism on mod p-homotopy groups in positive degrees. If X is further p-adically complete, $X \simeq X_p^\wedge$, and of finite p-type, then the localization map

$$X \rightarrow L_{K\mathbb{F}_p}(X)[0, \infty)$$

is a homotopy equivalence. In particular we can reinterpret the main result of the previous section to give

Corollary 9.6. *There is a homotopy equivalence*

$$\mathrm{TC}\,(\mathbb{Z}_p, p)_p^\wedge \simeq L_{K\mathbb{F}_p}(\mathrm{TC}\,(\mathbb{Z}_p, p)_p^\wedge)[0, \infty)$$

This result implies that the linearization map in (9.1) after p-adic completion factors over $L_{K\mathbb{F}_p}(\mathrm{TC}\,(*, p)_p^\wedge)[0, \infty)$. In particular we have stable maps

$$\text{(9.7)} \qquad \begin{array}{ccc} (\operatorname{Im} J \times \mathbb{Z})_p^\wedge & \xrightarrow{l_0} & \mathrm{TC}\,(\mathbb{Z}, p)_p^\wedge \\ B(\operatorname{Im} J \times \mathbb{Z})_p^\wedge & \xrightarrow{l_1} & \mathrm{TC}\,(\mathbb{Z}, p)_p^\wedge \end{array}$$

The composition of $\mathrm{Trc} : A(*) \rightarrow \mathrm{TC}\,(*, p)$ with α in (9.2) is the topological Dennis trace map, and

$$\Sigma^\infty\left(S^0\right) \xrightarrow{\iota} A(*) \xrightarrow{\mathrm{Trc}} \mathrm{TC}\,(*, p) \xrightarrow{\alpha} \Sigma^\infty\left(S^0\right)$$

is homotopic to the identity [8], [39]. We saw in the proof of (8.12) that

$$\iota_*(v_1^i) \in \pi_{2i(p-1)}(\mathrm{TC}\,(\mathbb{Z}_p, p); \mathbb{F}_p)$$

is non-zero and conclude that

$$(l_0)_* : \pi_{2i(p-1)}(\operatorname{Im} J \times \mathbb{Z}_p; \mathbb{F}_p) \rightarrow \pi_{2i(p-1)}(\mathrm{TC}\,(\mathbb{Z}_p, p); \mathbb{F}_p)$$

is injective. This proves the first half of

Lemma 9.8. (i) *In degrees $2i(p-1), \pi_*(l_0; \mathbb{F}_p)$ is a monomorphism.*
(ii) In degrees $2i(p-1)+1, \pi_*(l_1; \mathbb{F}_p)$ is a monomorphism.

Proof. We have already proved (i). For (ii), consider the cofibration diagram

$$\begin{array}{ccccc} T(*)_{hC_{p^n}} & \xrightarrow{N} & T(*)^{C_{p^n}} & \xrightarrow{\Phi} & T(*)^{C_{p^{n-1}}} \\ \downarrow L_n & & \downarrow L^n & & \downarrow L^{n-1} \\ T(\mathbb{Z}_p)_{hC_{p^n}} & \xrightarrow{N} & T(\mathbb{Z}_p)^{C_{p^n}} & \xrightarrow{\Phi} & T(\mathbb{Z}_p)^{C_{p^{n-1}}} \end{array}$$

where $T(*)$ is the topological Hochschild homology spectrum of the identity FSP. Since $T(*) = S^0, L^0$ is $(2p-3)$-connected after p-completion. Induction over n shows that L^n is $(2p-3)$-connected for all n, and it follows that

$$\mathrm{TC}^{(n)}(*,p)_p^\wedge \to \mathrm{TC}^{(n)}(\mathbb{Z}_p,p)_p^\wedge$$

is $(2p-4)$-connected for all n. The same is then true for

$$L : \mathrm{TC}(*,p)_p^\wedge \to \mathrm{TC}(\mathbb{Z}_p,p)_p^\wedge$$

In particular,

$$\pi_1(\mathrm{TC}(*,p);\mathbb{F}_p) \xrightarrow{\cong} \pi_1(\mathrm{TC}(\mathbb{Z}_p,p);\mathbb{F}_p)$$

But

$$\pi_1(\mathrm{TC}(*,p);\mathbb{F}_p) = \pi_1(B(\operatorname{Im} J \times \mathbb{Z});\mathbb{F}_p) = \mathbb{F}_p$$

so we can conclude that $\pi_1(l_1;\mathbb{F}_p)$ is an isomorphism. Since multiplication by v_1 induces isomorphism,

$$v_1 : \pi_{1+2i(p-1)}(B(\operatorname{Im} J \times \mathbb{Z});\mathbb{F}_p) \to \pi_{1+2(i+1)(p-1)}(B(\operatorname{Im} J \times \mathbb{Z});\mathbb{F}_p)$$

and since

$$v_1 : \pi_{1+2i(p-1)}(\mathrm{TC}(\mathbb{Z}_p,p);\mathbb{F}_p) \to \pi_{1+2(i+1)(p-1)}(\mathrm{TC}(\mathbb{Z}_p,p);\mathbb{F}_p)$$

is a monomorphism, (ii) follows. □

The stable maps of (9.7) together define a map of (-1)-connected spectra

$$l_{01} : (\operatorname{Im} J \times \mathbb{Z}_p) \times B(\operatorname{Im} J \times \mathbb{Z}_p) \to \mathrm{TC}(\mathbb{Z}_p,p)_p^\wedge$$

We want to determine the cofibre of l_{01}. To this end we first show that $\pi_*(l_{01})$ is an injection (on integral homotopy groups). Given (9.8) this amounts to showing that the (primary) Bockstein operator of the classes $(l_0)_*(v_1)$ and $v_1(l_1)_*(l_1)$ give non-zero elements of $\pi_{2p-3}(\mathrm{TC}\,(\mathbb{Z}_p,p);\mathbb{F}_p)$ and $\pi_{2p-2}(\mathrm{TC}\,(\mathbb{Z}_p,p);\mathbb{F}_p)$, respectively. General properties of the Bockstein spectral sequence will then determine all the higher Bocksteins because

$$\beta_{p^k}(l_{0_*}(v_1^{p^k})) = \beta_1(l_{0*}(v_1))\cdot v_1^{p^k-1}$$
$$(9.9)\qquad \beta_{p^k}(v_1^{p^k}l_{1*}(\iota_1)) = \beta_1(v_1 l_{1*}(\iota_1))\cdot v_1^{p^k-1}$$

We etablish the non-triviality of $\beta_1(l_{0*}(v_1))$ and $\beta_1(v_1l_{1*}(\iota_1))$ by evaluating the integral homotopy groups of the homotopy fibre $\mathrm{TC}\,(\mathbb{Z}_p,*,p)$ of

$$\mathrm{TC}\,(*,p)^\wedge_p \to \mathrm{TC}\,(\mathbb{Z}_p,p)^\wedge_p$$

through the range $2p-2$. The idea is due to T. Goodwillie. In fact, the proof of the following result was shown to us by Goodwillie during the special year in Topology at MSRI, Berkeley in 1989. It was instrumental for arriving at the correct conjecture about the structure of $\mathrm{TC}\,(\mathbb{Z}_p,p)$

Theorem 9.10. (Goodwillie). *The fiber* $\mathrm{TC}\,(\mathbb{Z}_p,*,p)$ *of the linearization map* $\mathrm{TC}\,(*,p)^\wedge_p \to \mathrm{TC}\,(\mathbb{Z}_p,p)^\wedge_p$ *is* $(2p-3)$*-connected and* $\pi_{2p-2}\mathrm{TC}\,(\mathbb{Z}_p,*,p) = \mathbb{Z}/p$.

Proof. Let SG be the component of $Q(S^0)$ consisting of maps of degree 1. Consider the diagram of connected spectra

$$(*)\qquad \begin{array}{ccc} \mathrm{Q}(\mathrm{SG}_+)^\wedge_\mathrm{p} & \overset{Q(\varepsilon)}{\longrightarrow} & (QS^0)^\wedge_p \\ \downarrow \theta_+ & & \downarrow d \\ (QS^0)^\wedge_p & \overset{d}{\longrightarrow} & H\mathbb{Z}_p, \quad d = \text{ degree} \end{array}$$

In $(*), \varepsilon : SG_+ \to S^0$ collapses SG to a point and $\theta_+ : Q(SG_+) \to QS^0$ is essentially the action map of SG considered as an infinite loop space:

$$Q(SG_+) = Q(SG)\times QS^0 \overset{\theta\times id}{\longrightarrow} SG\times QS^0 \overset{mult}{\longrightarrow} QS^0$$

After p-completion (*) becomes approximately homotopy Cartesian in the sense that the map from $Q(SG_+)$ to the homotopy pull-back $\mathrm{hol\underline{im}}\,(QS^0 \overset{d}{\to} H\mathbb{Z} \overset{d}{\leftarrow} QS^0)^\wedge_p$ is $(4p-6)$-connected.

We can of course reinterpret (*) as a diagram of functors with smash products with the off-diagonal corners corresponding to the identity and, the left hand upper corner corresponding to the functor $X \mapsto X \wedge SG_+$ and the lower right-hand corner to $X \mapsto \mathbb{Z}X$. Thus we get a diagram

$$\begin{array}{ccc} \mathrm{TC}\,(BSG,p)^\wedge_p & \to & \mathrm{TC}\,(*,p)^\wedge_p \\ \downarrow\ \mathrm{TC}\,(\theta_+) & & \downarrow \\ \mathrm{TC}\,(*,p)^\wedge_p & \to & \mathrm{TC}\,(\mathbb{Z}_p,p)^\wedge_p \end{array}$$

This diagram is also $(4p-6)$-Cartesian. This uses that $\mathrm{TC}(-,p)$ is a *1*-analytic functor in the sense of Goodwillie [16]. We conclude that

$$\pi_i \mathrm{TC}(\mathbb{Z}_p,*,p) \simeq \pi_{i+1}(\mathrm{TC}(\theta_+,p))$$

for $i \leq 2p-2$. We can now use the calculation of $\mathrm{TC}(X,p)^\wedge_p$ from [11], sect.5, i.e. the homotopy Cartesian diagram

$$\begin{array}{ccc} \mathrm{TC}(BSG,p)^\wedge_p & \to & \Sigma^\infty(\Sigma_+(ES^1 \times_{S^1} \Lambda BSG))^\wedge_p \\ \downarrow & & \downarrow\ \mathrm{trf} \\ \Sigma^\infty(\Lambda BSG_+)^\wedge_p & \stackrel{1-\Delta_p}{\to} & \Sigma^\infty(\Lambda BSG_+)^\wedge_p \end{array}$$

where Λ indicates the free loop space. Also, since BSG is simply connected Proposition 3.9 of [10] shows that

$$\begin{array}{ccc} \Sigma^\infty(\Lambda BSG_+) & \stackrel{1-\Delta_p}{\to} & \Sigma^\infty(\Lambda BSG_+) \\ \downarrow\ \mathrm{ev} & & \downarrow\ \mathrm{ev} \\ \Sigma^\infty(BSG_+) & \stackrel{0}{\to} & \Sigma^\infty(BSG_+) \end{array}$$

is homotopy Cartesian where ev $: \Lambda BSG \to BSG$ is the map which evaluates a free loop in $1 \in S^1$. Thus

$$\mathrm{TC}\,(BSG,p)^\wedge_p \simeq \Sigma^\infty(BSG_+)^\wedge_p \times hF(\Sigma^\infty(\Sigma_+(ES^1 \times_{S^1} \Lambda BSG)) \to \Sigma^\infty(BSG_+))^\wedge_p$$

This can be compared with (9.2) which we restate as

$$\mathrm{TC}\,(*,p)^\wedge_p \simeq \Sigma^\infty(S^0)^\wedge_p \times \text{ hofib } (\Sigma^\infty(\Sigma_+ BS^1) \to \Sigma^\infty\Big(S^0\Big))^\wedge_p$$

It follows that the homotopy fiber of $\mathrm{TC}\,(\theta_+,p)$ is the product of $\Sigma^\infty(BSG)$ and the homotopy fiber X of

$$\text{hofib } (\Sigma^\infty(\Sigma_+(ES^1 \times_{S^1} \Lambda BSG)) \to \Sigma^\infty(BSG_+)) \to hF(\Sigma^\infty(\Sigma_+ BS^1) \to \Sigma^\infty\Big(S^0\Big))$$

We claim that this homotopy fiber is $(2p-2)$-connected. Let F_1 and F_2 be the homotopy fibers of the two stable maps

$$\Sigma^\infty\Big(\Sigma_+\big(ES^1 \times_{S^1} \Lambda BSG\big)\Big)^\wedge_p \to \Sigma^\infty\big(\Sigma_+ BS^1\big)^\wedge_p, \Sigma^\infty(BSG_+)^\wedge_p \to \Sigma^\infty\big(S^0\big)^\wedge_p$$

induced from the projection map of the Borel construction and from the map which collapses BSG to one point. Then F_1 and F_2 are $(2p-3)$-connected and

$$\pi_{2p-2}F_1 = \mathbb{F}_p \quad , \quad \pi_{2p-2}(F_2) = \mathbb{F}_p \quad , \quad \pi_{2p-1}(F_2) = 0$$

Since X is the homotopy fiber of the map from F_1 to F_2 induced from the S^1-transfer, we just have to prove that $\pi_{2p-2}(F_1) \to \pi_{2p-2}(F_2)$ is non-trivial or equivalently that the composition

$$T : \Sigma^\infty(\Sigma\Lambda BSG) \xrightarrow{i} \Sigma^\infty(\Sigma(ES^1 \times_{S^1} \Lambda BSG)) \xrightarrow{\mathrm{trf}} \Sigma_+(\Lambda BSG) \to \Sigma^\infty \Lambda BSG$$

maps non-trivially on homotopy in degree $2p-2$. Here i is induced from the inclusion of the fiber $\Lambda BSG \to ES^1 \times_{S^1} \Lambda BSG$. We first remark that the map

$$E : \Sigma(\Lambda BSG^\wedge_p) \to BSG^\wedge_p \quad , \quad E(t, \lambda) = \lambda(t)$$

induces an isomorphism on $\pi_{2p-2}(-)$. Hence we must show that

$$\mathrm{ev} \circ T : \Sigma^\infty(\Sigma\Lambda BSG) \to \Sigma^\infty(BSG)$$

is homotopic to $\Sigma^\infty(E)$. Consider the pull-back diagram of S^1-fibrations

$$(**) \qquad \begin{array}{ccc} S^1 \times \Lambda BSG & \xrightarrow{\hat{i}} & ES^1 \times \Lambda BSG \\ \downarrow & & \downarrow \\ S^1 \times_{S^1} \Lambda BSG & \xrightarrow{i} & ES^1 \times_{S^1} \Lambda BSG \end{array}$$

There is an induced homotopy commutative diagram relating the S^1-transfers

$$\begin{array}{ccc} \Sigma^\infty_+(S^1 \times \Lambda BSG) & \xrightarrow{\hat{i}} & \Sigma^\infty_+(ES^1 \times \Lambda BSG) \\ \uparrow \mathrm{trf}_0 & & \uparrow \mathrm{trf} \\ \Sigma^\infty\Sigma_+(S^1 \times_{S^1} \Lambda BSG) & \xrightarrow{i} & \Sigma^\infty\Sigma_+(ES^1 \times_{S^1} \Lambda BSG) \end{array}$$

so it suffices to study the transfer on the left. The circle S^1 acts diagonally on $S^1 \times \Lambda BSG$. This action is homeomorphic to the action completely concentrated on the first factor via the homeomorphism

$$f : S^1 \times \Lambda BSG \to S^1 \times \Lambda BSG, \quad f(z, \lambda) = (z, z\lambda)$$

The bundle $S^1 \times \Lambda BSG \longrightarrow S^1 \times_{S^1} \Lambda BSG$ is therefore homeomorphic to the product bundle of $S^1 \longrightarrow *$ with ΛBSG with trivial S^1-action. The S^1-transfer for $S^1 \longrightarrow *$ is just the Thom collapse map associated with the embedding $S^1 \subseteq \mathbb{C}$ and gives upon stabilization the transfer

$$\mathrm{trf}_1 : \Sigma^\infty(S^1)^\wedge_p \to \Sigma^\infty(S^1_+)^\wedge_p = \Sigma^\infty(S^1)^\wedge_p \vee \Sigma^\infty(S^0)^\wedge_p$$

which is the identity on the first factor and trivial on the second (as p is odd). From the commutative diagram

$$\begin{array}{ccc} \Sigma^\infty(S^1 \wedge \Lambda BSG_+) & \xrightarrow{\mathrm{trf}_1 \wedge \mathrm{id}} & \Sigma^\infty_+(S^1 \times \Lambda BSG) \\ \downarrow \Sigma^\infty(f) & & \downarrow f \\ \Sigma^\infty(S^1 \wedge \Lambda BSG_+) & \xrightarrow{\mathrm{trf}_0} & \Sigma^\infty_+(S^1 \times \Lambda BSG) \end{array}$$

it then follows that $pr_1 \circ \mathrm{trf}_0$ is precisely

$$\Sigma^\infty f : \Sigma^\infty(S^1 \wedge \Lambda BSG_+) \to \Sigma^\infty(\Lambda BSG_+)$$

Thus

$$\Sigma^\infty(S^1 \wedge \Omega BSG_+) \to \Sigma^\infty(S^1 \wedge \Lambda BSG_+) \xrightarrow{T} \Sigma^\infty(\Lambda BSG_+) \to \Sigma^\infty(BSG_+)$$

is homotopic to the map induced from evaluation $f : S^1 \wedge \Omega BSG_+ \longrightarrow BSG$ and this, as was already pointed out, induces isomorphism on $\pi_{2p-2}(-)$. □

Let us also notice that since the spectra $S^0 \wedge X$ and $H\mathbb{Z} \wedge X$ are rationally equivalent we have:

Lemma 9.11. *The spectrum* $\mathrm{TC}(\mathbb{Z}_p, *, p)$ *is rationally trivial.* □

We next examine $\pi_i \mathrm{TC}(*, p)^\wedge_p$ for $i \leq 2p - 1$. The usual map $(S^0)^\wedge_p \longrightarrow H\mathbb{Z}_p$ is $(2p-2)$-connected, so induces a $(2p-2) + c$ connected map

$$\Sigma^\infty(X)^\wedge_p \to H\mathbb{Z}_p \wedge X$$

when X is a c-connected spectrum. It follows that

$$\pi_i(\Sigma^\infty(\Sigma BS^1)^\wedge_p) = \tilde{H}_{i-1}(BS^1; \mathbb{Z}_p)$$

for $i \leq 2p - 1$. The S^1-transfer

$$\pi_i(\Sigma^\infty(\Sigma_+ BS^1)^\wedge_p) \to \pi_i(S^0)^\wedge_p$$

is surjective for $i = 2p-3$, e.g. by the Kahn-Priddy theorem, so (9.2) implies that

$$\pi_i \mathrm{TC}\,(*,p)^\wedge_p = \begin{cases} \tilde{H}_{i-1}(BS^1;\mathbb{Z}_p) & ,i<2p-3 \\ \mathbb{Z}/p \oplus p\mathbb{Z}_p, & ,i=2p-3 \\ \mathbb{Z}/p & ,i=2p-2 \\ \mathbb{Z}_p & ,i=2p-1 \end{cases} \tag{9.11}$$

We can then use (9.10) to get the low dimensional homotopy groups of $TC(\mathbb{Z}_p,p)^\wedge_p$, namely:

Lemma 9.13. *The linearization*

$$l:\mathrm{TC}\,(*,p)^\wedge_p \to \mathrm{TC}\,(\mathbb{Z}_p,p)^\wedge_p$$

induces an isomorphism on homotopy groups in dimensions less than $2p-1$. In dimension $2p-1$ both homotopy groups are $\mathbb{Z}_p$ and $\pi_{2p-1}(l)$ is multiplication by p.

Proof. The spectrum $\mathrm{TC}\,(\mathbb{Z}_p,*,p)$ is $(2p-3)$-connected, has $\pi_{2p-2}\mathrm{TC}\,(\mathbb{Z}_p,*,p) = \mathbb{Z}/p$, and $\pi_*\mathrm{TC}\,(\mathbb{Z}_p,*,p)$ is torsion. Thus we have left to show that in the homotopy exact sequence

$$\ldots \to \pi_{2p-2}\mathrm{TC}\,(\mathbb{Z}_p,*,p) \xrightarrow{i_*} \pi_{2p-2}(*,p)^\wedge_p \xrightarrow{j_*} \pi_{2p-2}\mathrm{TC}\,(\mathbb{Z}_p,p)^\wedge_p \to \ldots$$

i_* is the zero homomorphism. Since $\pi_{2p-3}\mathrm{TC}\,(\mathbb{Z}_p,*,p) = 0$, j_* is surjective. By (9.11), $\pi_{2p-2}\mathrm{TC}\,(*,p)^\wedge_p = \mathbb{Z}/p$ so the triviality of i_* is equivalent to $\pi_{2p-2}\mathrm{TC}\,(\mathbb{Z}_p,p)^\wedge_p \neq 0$. Consider the Bockstein sequence

$$\pi_{2p-2}\mathrm{TC}\,(\mathbb{Z}_p,p) \xrightarrow{\rho} \pi_{2p-2}(\mathrm{TC}\,(\mathbb{Z}_p,p);\mathbb{F}_p) \xrightarrow{\beta} \pi_{2p-3}\mathrm{TC}\,(\mathbb{Z}_p,p)$$

Since $\pi_{2p-3}TC(\mathbb{Z}_p,p)^\wedge_p \cong \pi_{2p-3}TC(*,p)^\wedge_p$ has torsion subgroup $\mathbb{Z}/p$ and since $\pi_{2p-2}(\mathrm{TC}\,(\mathbb{Z}_p,p);\mathbb{F}_p) = \mathbb{Z}/p \oplus \mathbb{Z}/p$ the image of $\pi_{2p-2}\mathrm{TC}\,(\mathbb{Z}_p,p)$ under ρ is non-trivial. □

Corollary 9.14. *The mapping*

$$l_{01}:(\mathrm{Im}\,J\times\mathbb{Z}_p)\times B(\mathrm{Im}\,J\times\mathbb{Z}_p)\to \mathrm{TC}\,(\mathbb{Z}_p,p)^\wedge_p$$

induced from linearization is injective on both mod p and on integral homotopy groups.

Proof. We conclude from (9.13) that

$$\beta_1((l_{01})_*(v_1)) \neq 0 \ , \beta_1(v_1(l_{01})_*(\iota_1)) \neq 0$$

and then by (9.9) that

$$\beta_k\Big((l_{01})_*\Big(v_1^{p^k}\Big)\Big) \neq 0 \ , \ \beta_k\Big(v_1^{p^k}(l_{01})_*(\iota_1)\Big) \neq 0.$$

This is precisely the higher Bockstein structure in the domain. Thus $(l_{01})_*$ is injective on both mod p and integral homotopy groups. □

Let $\overline{\mathrm{TC}}(\mathbb{Z}_p, p)$ be the cofiber (in the category of spectra) of

$$l_{01} : \mathrm{Im}\, J \times \mathbb{Z}_p \times B(\mathrm{Im}\, J \times \mathbb{Z}_p) \longrightarrow \mathrm{TC}(\mathbb{Z}_p, p)_p^\wedge$$

The combination of (7.15) and (9.14) implies that

$$\pi_i(\overline{\mathrm{TC}}(\mathbb{Z}_p, p); \mathbb{F}_p) = \begin{cases} 0 & \text{for } i \equiv 0 (\mathrm{mod}\ 2) \\ \mathbb{F}_p & \text{for } i \equiv 1 (\mathrm{mod}\ 2), i > 1 \end{cases}$$

By (9.11) the rational types of $\overline{\mathrm{TC}}(\mathbb{Z}_p, p)$ and $\Sigma^\infty(\Sigma BS^1)$ agree,

$$\overline{\mathrm{TC}}(\mathbb{Z}_p, p)_{(0)} \simeq (\Sigma bu)_{(0)}.$$

Hence

$$\pi_i(\overline{\mathrm{TC}}(\mathbb{Z}_p, p)_p^\wedge) = \begin{cases} 0 & \text{for } i \equiv 0 (\mathrm{mod}\ 2) \\ \mathbb{Z}_p & \text{for } i \equiv 1 (\mathrm{mod}\ 2), i > 1 \end{cases}$$

We know from (8.12) that $\pi_*(\overline{\mathrm{TC}}(\mathbb{Z}_p, p); \mathbb{F}_p)$ is v_1-periodic. Thus $\overline{\mathrm{TC}}(\mathbb{Z}_p, p)_p^\wedge$ is a spectrum with the same integral homotopy groups as $\Sigma bu_p^\wedge$ and as it is v_1-periodic,

$$\overline{\mathrm{TC}}(\mathbb{Z}_p, p)_p^\wedge \simeq L_{K\mathbb{F}_p}(\mathrm{TC}(\mathbb{Z}_p, p)_p^\wedge)[0, \infty)$$

From [33] we conclude that

(9.15) $$\overline{\mathrm{TC}}(\mathbb{Z}_p, p)_p^\wedge \simeq (\Sigma bu)_p^\wedge$$

Hence we have a cofibration of spectra

(9.16) $$(\mathrm{Im} J \times \mathbb{Z}_p) \times B(\mathrm{Im}\, J \times \mathbb{Z}_p) \xrightarrow{l_{01}} \mathrm{TC}(\mathbb{Z}_p, p)_p^\wedge \longrightarrow (\Sigma bu)_p^\wedge$$

Theorem 9.17. For odd primes p

$$\mathrm{TC}(\mathbb{Z}_p, p)_p^\wedge \simeq (\Sigma bu)_p^\wedge \times (\mathrm{Im}\, J \times \mathbb{Z}_p) \times B(\mathrm{Im}\, J \times \mathbb{Z}_p)$$

Proof. We must show that (9.16) is a split cofibration or equivalently that the induced stable map

$$\sigma : bu_p^\wedge \to (\mathrm{Im}\, J \times \mathbb{Z}_p) \times B(\mathrm{Im}\, J \times \mathbb{Z}_p)$$

is homotopically trivial. In [1], 6.4.8 Adams proves that the mapping

$$[bu, K_p^\wedge] \to \mathrm{Hom}\big(\pi_*(bu), \pi_*(K_p^\wedge)\big)$$

is injective. Since

$$\psi^g - 1 : K_p^\wedge \to K_p^\wedge$$

induces a non-trivial map on all homotopy groups, namely multiplication with $g^n - 1$ in dimension $2n$, no non-trivial stable map from $bu_p^\wedge$ to $K_p^\wedge$ lifts to $J_p^\wedge$. Thus $[bu, J_p^\wedge] = 0$. Standard connectivity arguments show that

$$[bu, J_p^\wedge] \to [bu, \mathrm{Im}\, J \times \mathbb{Z}_p]$$

is an isomorphism. In contrast the cofibration

$$bu_p^\wedge \xrightarrow{\psi^g - 1} bu_p^\wedge \to B(\mathrm{Im}\, J)$$

shows that $[bu, B(\mathrm{Im}\, J \times \mathbb{Z}_p)] \neq 0$. However every non-trivial stable map

$$f : bu_p^\wedge \to B(\mathrm{Im}\, J \times \mathbb{Z}_p)$$

induces a non-trivial map on some homotopy group, cf. [26], [33]. Since (9.14) tells us that the possible

$$\sigma : bu_p^\wedge \to (\mathrm{Im}\, J \times \mathbb{Z}_p) \times B(\mathrm{Im}\, J \times \mathbb{Z}_p)$$

is trivial on homotopy groups we conclude that σ is homotopically trivial. □

§ 10 Appendix: The relative trace

In this appendix we examine the relative topological Dennis trace, and derive as a consequence that

$$\mathrm{Tr} :\ K(\mathbb{Z}) \to T(\mathbb{Z})$$

is a surjection on homotopy groups in dimension $2p-1$. This was used in Lemma 6.5 above. We shall consider K-theory and topological Hochschild homology of general

FSP's; the reader is referred to sect. 1, [11], sect. 5 and [22], sect. 1 for background material. We only consider FSP's with $F(S^i)$ $(i-1)$-connected.

An FSP F gives rise to a connected ring spectrum F^s, namely the spectrum associated with the prespectrum $F(S^i)$; the 0'th space of F^s is denoted $M_1(F)$. The components $\pi_0 M_1(F) = \pi_0(F^s)$ is a ring, and $GL_1(F)$ is the union of components which are units in $\pi_0 M_1(F)$. Given F and an integer $k \geq 1$ we have the associated FSP $F_{(k)}$ with $F_{(k)}(X) = \mathrm{Map}([\mathrm{k}], [\mathrm{k}] \wedge \mathrm{F(X)})$ where $[k] = \{0, 1, ..., k\}$ with 0 as base point, and where Map(,) denotes based maps. We write $M_k(F) = M_1(F_{(k)})$ and note that $\pi_0 M_k(F)$ is the ring of $k \times k$ matrices over $\pi_0 F^s$.

Let $l : F_1 \to F_2$ be an r-connected morphism of FSP's with $r > 0$, so that

$$l_* : \pi_i F_1^s \to \pi_i F_2^s$$

is an isomorphism for $i \leq r-1$ and an epimorphism for $i = r$. In particular $\pi_0 F_1^s \cong \pi_0 F_2^s$. Define $F_0(X)$ to be the homotopy fiber of $l : F_1(X) \to F_2(X)$,

$$F_0(X) = \Big\{ (x_1, \lambda_2) \in F_1(X) \times F_2(X)^I \mid l(x_1) = \lambda_2(1),\ \lambda_2(0) = * \Big\}.$$

There are obvious maps

$$F_1(X) \wedge F_0(Y) \to F_0(X \wedge Y)$$
$$F_0(X) \wedge F_1(Y) \to F_0(X \wedge Y)$$

which makes F_0 into a "2–sided F_1-module". The set of *all* functions from $[k]$ to $[1]$ is denoted $\mathcal{P}[k]$. It has a partial ordering where $s \leq t$ if $s(i) \leq t(i)$ for all i. Let $\mathbf{1}$ be the constant function with value 1 and let $\mathcal{P}_1[k] = \mathcal{P}[k] - \{\mathbf{1}\}$. For $s \in \mathcal{P}[n]$ and spaces $X_0, ..., X_k$ we define

$$F_s(X_0, \cdots, X_k) = F_{s(0)}(X_0) \wedge \cdots \wedge F_{s(k)}(X_k)$$

Let I be the category of finite sets and injective maps, i.e. $\mathrm{ob}(I) = \mathbb{N}$, $I(n, m) = \Sigma_m$ as in [11], p. 476. Set

$$G_k(s; Y) = \operatorname{holim}_{\to} \mathrm{Map}(S^{n_0} \wedge ... \wedge S^{n_k}, F_s(S^{n_0}, ..., S^{n_k}) \wedge Y) \tag{10.1}$$

where the homotopy limit runs over $(n_0, ..., n_k) \in I^{k+1}$. For $s \leq t$, the map from F_0 to F_1 induces a map $G_k(s; y) \to G_k(t; Y)$, and we set

$$G_k(Y) = \operatorname{holim}_{\to} G_k(s; Y) \quad , \quad s \in \mathcal{P}_1[k].$$

We see that $G_k(\mathbf{1};Y) = \mathrm{THH}_k(F_1;Y)$ and that each $G_k(s;Y)$, and hence also $G_k(Y)$, maps to $G_k(\mathbf{1};Y)$ by using the map from $F_0(X)$ to $F_1(X)$. The spaces $G_k(Y)$, $k = 0, 1, \ldots$ form a simplicial (in fact cyclic) space. We specify the face operators and leave for the reader to define the degeneracy (and cyclic) operators. Let

$$d_i \;:\; \mathcal{P}_1[k] \to \mathcal{P}_1[k-1], \qquad 0 \leq i \leq k$$

be the function

$$d_i(s)(\nu) = \begin{cases} s(\nu), & 0 \leq \nu < i \\ s(i)s(i+1), & \nu = i \\ s(\nu+1), & i < \nu \leq k+1, \end{cases}$$
$$d_k(s)(\nu) = \begin{cases} s(k)s(0), & \nu = 0 \\ s(\nu), & 0 < \nu \leq k-1. \end{cases}$$

Since F_0 is a 2–sided F_1-module there are maps

$$\begin{aligned} d_i \;&:\; F_s(X_0, \ldots, X_k) \to F_{d_i(s)}(X_0, \ldots, X_i \wedge X_{i+1}, \ldots, X_k), \quad i < k \\ d_k \;&:\; F_s(X_0, \ldots, X_k) \to F_{d_k(s)}(X_k \wedge X_0, \ldots, X_{k-1}) \end{aligned}$$

which induce operators

$$d_i \;:\; G_k(s;Y) \to G_{k-1}(d_i s;Y)$$

quite similar to the face operators from $\mathrm{THH}_k(F_1;Y)$ to $\mathrm{THH}_{k-1}(F_1;Y)$. If $s \leq t$ then the diagram

$$\begin{array}{ccc} G_k(s;Y) & \xrightarrow{d_i} & G_{k-1}(d_i s;Y) \\ \downarrow & & \downarrow \\ G_k(t;Y) & \xrightarrow{d_i} & G_{k-1}(d_i t;Y) \end{array}$$

is commutative, so one gets the required face operator

$$d_i \;:\; G_k(Y) \to G_{k-1}(Y).$$

The topological relation of the cyclic space $G_\bullet(Y)$ will be denoted

$$\mathrm{THH}(F_1 \to F_2;Y) = |G_\bullet(Y)|.$$

We showed in sect. 1 (see also [22]) how $\mathrm{THH}(F;Y)$ leads to an equivalent S^1-spectrum $T(F)$, and quite similarly we obtain an equivariant S^1-spectrum $T(F_1 \to F_2)$ with an S^1-map to a $T(F_1)$.

Proposition 10.2 *There is an S^1-equivariant cofibration*

$$T(F_1 \to F_2) \longrightarrow T(F_1) \longrightarrow T(F_2).$$

Proof Let $F_i(X)/F_0(X)$ denote the cofiber of $F_0(X) \to F_1(X)$. For $(N-1)$-connected X, the induced map from $F_1(X)/F_0(X)$ to $F_2(X)$ is $(2N+r)$-connected with r being the connectivity of $l : F_1(X) \to F_2(X)$.

In general, for a based map $f : Z_0 \to Z_1$ with cofiber Z_2, we have a homeomorphism

$$\operatorname{cofib}\left(\underset{s\in\mathcal{P}_1[k]}{\operatorname{holim}_{\to}} Z_s \to Z_1^{\wedge(k+1)}\right) \cong Z_2^{\wedge(k+1)}$$

where $Z_s = Z_{s(0)} \wedge \ldots \wedge Z_{s(k)}$. Indeed this is easy to check for $k=1$ and follows in general by induction, based on the fact that homotopy colimits can be described iteratively. In particular for $Z_0 = F_0(X)$, $Z_1 = F_1(X)$ we get a cofibration

$$\underset{s\in\mathcal{P}_1[k]}{\operatorname{holim}_{\to}} F_s\left(S^N, \ldots, S^N\right) \wedge Y \to F_1\left(S^N\right)^{\wedge(k+1)} \wedge Y \to F_1/F_0\left(S^N\right)^{\wedge(k+1)} \wedge Y.$$

In a stable range we may replace $F_1/F_0(S^N)$ with $F_2(S^N)$ in the above. The homotopy colimit over I^{k+1} in (10.1) was shown in [8] to be well-behaved in the sense that a finite stage approximates the limit. In particular

$$\underset{s\in\mathcal{P}_1[k]}{\operatorname{holim}_{\to}} \Omega^{(k+1)N}\left(F_s\left(S^N, \ldots, S^N\right) \wedge Y\right)$$

approximates $G_k(Y)$ for $N \to \infty$, so we have a cofibration

$$G_k(S^m) \to \mathrm{THH}_k(F_1; S^m) \to \mathrm{THH}_k(F_2; S^m)$$

This proves the non-equivalent part of the proposition, which is all which we shall need in the following. We leave for the reader to show, using the subdivision functor, and arguments as in sect. 1 and [22], sect. 1 that

$$T(F_1 \to F_2)^C \to T(F_1)^C \to T(F_2)^C$$

is a cofibration for all $C \subseteq S^1$ $\square$

We next consider the simplicial subspace

$$\mathrm{THH}_k(F_1, F_0; Y) = \operatorname{holim}_{\overrightarrow{I^{k+1}}} \operatorname{Map}(S^{n_0} \wedge \ldots \wedge S^{n_k}; F_0(S^{n_0}) \wedge F_1(S^{n_1}) \wedge \ldots \wedge F_1(S^{n_k}))$$

of $G_k(Y)$. Using the cyclic structure on $G_k(Y)$ the inclusion extends to a map

$$C_{k+1_+} \wedge \mathrm{THH}_k(F_1; F_0; Y) \to G_k(Y) \tag{10.3}$$

which becomes a map of cyclic spaces when we give the left hand side the simplicial operators

$$d_i(\tau_k^s, x) = \begin{cases} (\tau_{k-1}^s, d_{i+s}x), & i+s \le k \\ \left(\tau_{k-1}^{s-1}, d_{i+s-k}x\right), & i+s > k \end{cases}$$

$$s_i(\tau_k^s, x) = \begin{cases} (\tau_{k+1}^s, s_{i+s}x), & i+s \le k \\ \left(\tau_{k+1}^{s+1}, s_{i+s}x\right), & i+s > k \end{cases}$$

$$t_k(\tau_k^s, x) = \left(\tau_k^{s-1}, x\right).$$

Here τ_k is the (chosen) generator of C_{k+1} and $0 \le s \le k$, cf. [20], sect. 3. The topological realization of (10.3) is an S^1-map

$$S^1_+ \wedge \mathrm{THH}(F_1, F_0; Y) \to \mathrm{THH}(F_1 \to F_2; Y)$$

by Lemma 3.2. In particular we get a map of S^1-spectra

$$S^1_+ \wedge T(F_1, F_2) \to T(F_1 \to F_2) \tag{10.4}$$

Lemma 10.5
If $l : F_1 \to F_2$ is r-connected then

$$S^1_+ \wedge T(F_1; F_0) \to T(F_1 F_2)$$

is $2r$-connected.

Proof With our assumptions we have $\pi_i F_0(S^N) = 0$ for $i < N + r$, so that $\pi_i G_k(S; S^N) = 0$ for $i < N + 2r$ and all $s \in \mathcal{P}_1[k]$, except the ones with the property that $s^{-1}(0)$ contains precisely one element. There are $(k+1)$ such, and the associated $G_k(s; S^N)$ is of the form $(\tau_k^i)_+ \wedge \mathrm{THH}_k(F_1, F_0; S^N)$. It follows that

$$C_{k+1_+} \wedge \mathrm{THH}_k\left(F_1, F_0; S^N\right) \to G_k\left(S^N\right)$$

is $(N + 2r)$-connected, and hence the same is the case for the map from $S^1_+ \wedge \mathrm{THH}(F_1, F_0; S^N)$ to $\mathrm{THH}(F_1 \to F_2; S^N)$. □

For any simplicial space $Z_\bullet$, the skeleton filtration of $|Z_\bullet|$ gives a spectral sequence with

$$E^2_{i,j} = H_j\Big(H_i(Z_\bullet), \sum(-1)^\nu H_i(d_\nu)\Big)$$

and abutment $H_{i+j}(|Z_\bullet)|)$. If the individual spaces Z_k are all r-connected then in total degree r only $E^2_{r,0} \neq 0$, and $E^2_{i,j} = 0$ for $i + j < r$, so

$$H_r(|Z_\bullet|) \cong E^\infty_{r,0} \cong E^2_{r,0} = H_r(Z_0)/d(H_r(Z_1)).$$

In the case of $Z_\bullet = \mathrm{THH}_\bullet(F_1, F_0)$ we get that $T(F_1, F_0)$ is $(r-1)$-connected; and using the Hurewicz theorem,

$$\pi_r T(F_1, F_0) \cong \pi_r F_0^s/[\pi_0 F_1, \pi_r F_0^s] \tag{10.7}$$

We next discuss relative K-theory $K(F_1 \to F_2)$. There is a diagram of homotopy fibrations

$$\begin{array}{ccccc} \mathrm{BGL}(F_1 \to F_2) & \to & \mathrm{BGL}(F_1) & \to & \mathrm{BGL}(F_2) \\ \downarrow & & \downarrow & & \downarrow \\ K(F_1 \to F_2) & \to & K(F_1) & \to & K(F_2) \end{array} \tag{10.8}$$

By definition the two vertical maps to the right are homology isomorphisms. Here $GL(F_i)$ is the limit of $GL_k(F_i)$ for $k \to \infty$. The homotopy fiber $GL_k(F_1 \to F_2)$ is homotopy equivalent to $M_k(F_0)$, by the map which adds (in the loop sum) the identity matrix to an element of $M_k(F_0) = \varinjlim_m \mathrm{Map}([k] \wedge S^m, [k] \wedge F_0(S^m))$. In particular

$$\pi_{r+1}\mathrm{BGL}_k(F_1 \to F_2) \cong M_k(\pi_r F_0^s)$$

The next result is a slight variation of [34], proposition 1.2.

Proposition 10.9 *If $l : F_1 \to F_2$ is r-connected with $r > 0$, then*

$$\pi_{r+1}K(F_1 \to F_2) \cong \pi_r F_0^s/[\pi_0 F_1^s, \pi_r F_0^s].$$

Proof For the convenience of the reader we repeat Waldhausen's argument. He studies the Hochschild-Serre spectral sequence of the upper homotopy fibration in (10.8) which has E^2-term

$$\begin{aligned} E^2_{p,q} &= H_p(\mathrm{BGL}(F_2); H_q\mathrm{BGL}(F_1 \to F_2)) \\ &\cong H_p\big(\mathrm{BGL}(F_2); H_{q-1}(M(F_0))\big), \qquad q \le 2r \\ &\cong \varinjlim_k H_p\big(\mathrm{BGL}_k(F_2); H_{q-1}(M_k(F_0))\big) \end{aligned}$$

Here $H_p(;)$ denotes homology with twisted coefficients. We note that $H_{q-1}(M_k(F_0)) \cong M_k(H_{q-1}(F_0^s))$ for $q \leq 2r$, and that

$$\pi_1 \mathrm{BGL}_k(F_2) \cong \pi_0 GL_k(F_2) \cong GL_k(\pi_0 F_2^s) \cong GL_k(\pi_0 F_1^s)$$

acts by conjugation.

Clearly $E^2_{p+q} = 0$ for $p + q < r + 1$, and in total degree $r + 1$ only $E^2_{r+1,0}$ and $E^2_{0,r+1}$ can be non-zero. We get

$$\begin{aligned} E^2_{0,r+1} &= \varinjlim_k H_0(GL_k(\pi_0 F_1^s); M_k(\pi_r F_0^s)) \\ &\cong \varinjlim_k M_k(\pi_r F_0^s)/[GL_k(\pi_0 F_1^s), M_k(\pi_r F_0^s)] \end{aligned}$$

For any ring R and bimodule P the trace induces an isomorphism

$$M_k(\mathrm{P})/[GL_k(\mathrm{R}), M_k(\mathrm{P})] \to \mathrm{P}/[\mathrm{R}, \mathrm{P}]$$

cf. [41], so

$$E^2_{0,r+1} \cong \pi_r F_0^s/[\pi_0 F_1^s, \pi_r F_0^s]$$

Let $\widehat{E}^r_{p+q}$ be the spectral sequence of the bottom fibration (10.8). Since this is a fibration of infinite loop spaces the coefficients are this time untwisted. The spectral sequences agree on the base line and on the E^∞-terms since $K(F)$ is homology equivalent to $\mathrm{BGL}(F)$. Hence in the diagram

$$\begin{array}{ccccccccc} 0 & \to & E^\infty_{r+2,0} & \to & E^2_{r+2,0} & \xrightarrow{d^{r+2}} & E^2_{0,r+1} & \to & E^\infty_{0,r+1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \widehat{E}^\infty_{r+2,0} & \to & \widehat{E}^2_{r+2,0} & \xrightarrow{\widehat{d}^{r+2}} & \widehat{E}^2_{0,r+1} & \to & \widehat{E}^\infty_{0,r+1} & \to & 0 \end{array}$$

the outer vertical maps are isomorphisms, and so is the map from $E^2_{r+2,0}$ to $\widehat{E}^2_{r+2,0}$. Hence

$$\pi_{r+1} K(F_1 \to F_2) \cong H_{r+1} K(F_1 \to F_2) \cong \widehat{E}^2_{0,r+2} \cong E^2_{0,r+1}$$

□

Let $F_1 \ltimes F_2$ be the 'semi-direct' FSP induced from the two-sided F_1-module structure on F_0, i.e. $F_1 \ltimes F_0(X) = F_1(X) \vee F_0(X)$, cf. [20], Definition 1.5. It has $GL_k(F_1 \ltimes F_2) \cong GL_k(F_1) \times M_k(F_0)$, and from [20], Proposition 2.1 we have the decomposition of S^1-spectra

$$T(F_1 \ltimes F_0) \cong \bigvee_{a=0}^{\infty} T_a(F_1 \ltimes F_0) \tag{10.10}$$

with $T_0(F_1 \ltimes F_0) = T(F_1)$ and $T_1(F_1 \ltimes F_0) \cong S^1_+ \wedge T(F_1; F_0)$. Roughly speaking $T_a(F_1 \ltimes F_0)$ is the part of $T(F_1 \ltimes F_0)$ with a factors F_0. There is a diagram

$$\begin{array}{ccccc} K(F_1 \ltimes F_0 \to F_1) & \to & K(F_1 \ltimes F_0) & \to & K(F_1) \\ \downarrow \widehat{f} & & \downarrow & & \downarrow \\ K(F_1 \to F_2) & \to & K(F_1) & \to & K(F_2) \end{array}$$

and a similar diagram for $T(-)$. It follows from the proofs of Lemma 10.5 and Proposition 10.9 that we have

Corollary 10.11 *The maps*

$$\begin{aligned} \widehat{f} &: K(F_1 \ltimes F_0 \to F_1) \to K(F_1 \to F_2) \\ \widehat{f} &: T(F_1 \ltimes F_0 \to F_1) \to T(F_1 \to F_2) \end{aligned}$$

are $2r$-connected when $l : F_1 \to F_2$ is r-connected. □

Lemma 10.5 shows that

$$\pi_{r+1}T(F_1 \to F_2) \cong \pi_{r+1}\Big(S^1_+ \wedge T(F_1, F_0)\Big) \cong \pi_{r+1}\Sigma T(F_1, F_0) \oplus \pi_{r+1}T(F_1, F_0)$$

since $S^1_+ \wedge T(F_1, F_0) = \Sigma T(F_1, F_0) \vee T(F_1, F_0)$. Writing $K_r(-)$ and $T_r(-)$ for $\pi_r K(-)$ and $\pi_r T(-)$ as usual, here is the main result of the appendix:

Theorem 10.12 *For an r-connected map $l : F_1 \to F_2$ with $r > 0$,*

$$K_{r+1}(F_1 \to F_2) \xrightarrow{\mathrm{Tr}_*} T_{r+1}(F_1 \to F_2) \xrightarrow{\mathrm{proj}} T_r(F_1, F_0)$$

is an isomorphism.

Proof It suffices to consider the relative situation $F_1 \ltimes F_0 \to F_0$. We can further pass to the setting of simplicial or, as we prefer, topological rings. Let $R_1 = \pi_0 F_1^s$ and $R_0 = \pi_r F_0^s$, and let $R_0(r)$ be the topological bimodule with $\pi_r R_0(r) = R_0$ and $\pi_i R_0(r) = 0$ for $i \neq r$. We can for example take $R_0(r)$ to be the topological realization of $R_0[S^r_\bullet]/R_0[*_\bullet]$ where $S^r_\bullet$ is the simplicial r-sphere. Since $R_0(r)$ is an R_1-bimodule we can form the semidirect product ring $R_1 \ltimes R_0(r) = R_1 \oplus R_0(r)$. In this linear situation there is an isomorphism

$$GL_k(R_1 \ltimes R_0(r)) \cong GL_k(R_1) \ltimes M_k(R_0(r)),$$

where the right hand side is the semi-direct product of groups. The proof of Proposition 10.9 shows that

$$K_{r+1}(F_1 \ltimes F_0 \to F_0) \cong K_{r+1}(R_1 \ltimes R_0(r) \to R_1).$$

The 'linear version' of $T(F_1 \ltimes F_0)$ is the cyclic complex $N^{cy}_{\otimes}(R_1 \ltimes R_0(r))$ with n-cimplices

$$N^{cy}_{\otimes}(R_1 \ltimes R_0(r))_n = (R_1 \ltimes R_0(r))^{\otimes(n+1)}.$$

This decomposes into a sum of cyclic sets, similar to the decomposition of $T(F_1 \ltimes F_0)$ in (10.10), and we have

$$\begin{aligned} T_{r+1}(F_1 \ltimes F_0 \to F_1) &\cong \pi_{r+1}\Big(S^1_+ \wedge T(F_1, F_0)\Big) \\ &\cong \pi_{r+1}\Big(S^1_+ \wedge N^{cy}_{\otimes}(R_1, R_0(r))\Big). \end{aligned}$$

Here $S^1_+ \wedge N^{cy}_{\otimes}(R_1, R_0(r))$ is the realization of the cimplicial object with n-simplices $C_{n+1} \ltimes R_0(r) \otimes R_1^{\otimes n}$. It injects into the n-simplices of $N^{cy}_{\otimes}(R_1, R_0(r))$ by the map

$$(10.13) \qquad (\tau_n^s, a_0 \otimes a_1 \otimes \ldots \otimes a_n) \mapsto a_s \otimes \ldots \otimes a_n \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_{s-1}$$

The subset of simplices with $s = 0$ form a sub complex, and we use the quotient simplicial set as a model for $S^1 \wedge N^{cy}_{\otimes}(R_1, R_0(r))$ and similarly for suspensions of the other simplicial objects.

The simplicial map $f_\bullet$ which in degree n is the map

$$C_{n+1} \wedge M_k(R_0(r)) \to (GL_k(R_1) \ltimes M_k(R_0(r)))^n$$

given by

$$(\tau_n^s, A_0) \mapsto \Big(\mathbf{1}_k^{(n-s)}, \mathbf{1}_k + A_0, \mathbf{1}_k^{(s-1)}\Big),$$

where $\mathbf{1}_k \in GL_k(R_1)$ the identity matrix, gives a map of realizations

$$S^1 \wedge M_k(R_0(r)) \to |N_\bullet(GL_k(R_1) \ltimes M_k(R_0(r)) \to GL_k(R_1))|$$

which can be composed with the map into $K(R_1 \ltimes R_0 \to R_0)$. The composed map becomes the surjection on π_{r+1} given by the canonical projection

$$M_k(R_0) \to M_k(R_0)/[GL_k(R_1), M_k(R_0)].$$

This follows from the proof of Proposition 10.9.

The Dennis trace is induced from the following map of simplicial objects:

$$N_\bullet(GL_k(R_1 \ltimes R_0(r))) \xrightarrow{I_\bullet} N^{cy}_\bullet(GL_k(R_1 \ltimes R_0(r))) \xrightarrow{S_\bullet} N^{cy}_{\otimes}(M_k(R_1 \ltimes R_0(r)))_\bullet$$

with

$$\begin{aligned} I_n(A_1, \ldots, A_n) &= \Big((\Pi A_i)^{-1}, A_1, \ldots A_n\Big) \\ S_n(A_0, \cdots, A_n) &= A_0 \otimes \cdots \otimes A_n. \end{aligned}$$

We are only interested in the composition with the projection

$$N_{\otimes}^{\mathrm{cy}}(M_k(R_1) \ltimes M_k(R_0)) \overset{\mathrm{proj}}{\longrightarrow} S^1 \wedge N_{\otimes}^{\mathrm{cy}}(M_k(R_1), M_k(R_0))$$

cf. (10.10).
In simplicial degree n, the composition $I_\bullet \cdot S_\bullet \cdot f_\bullet$ maps $(\tau_n^s, A_0) \in C_{n+1} \wedge M_k(R_0(r))$ into

$$(\mathbf{1} - A_0) \otimes \mathbf{1}_k^{\otimes(n-s)} \otimes (\mathbf{1} + A_0) \otimes \mathbf{1}_k^{\otimes(s-1)}$$

This expression decomposes into a sum of 4 terms and only the term $\mathbf{1}_k^{\otimes(n-s+1)} \otimes A_0 \otimes \mathbf{1}_k^{\otimes(n-1)}$ has a non-zero projection into the relevant component $C_{n+1} \wedge N_{\otimes}^{\mathrm{cy}}(M_k(R_1), M_k(R_0(r)))_n$, where it becomes $\tau_n^s \otimes A_0 \otimes \mathbf{1}_k^{(n+1)}$ according to (10.13). This shows that $\mathrm{proj} \circ S \circ I \circ f$ is the topological realization of $\mathrm{id} \wedge h \;:\; C_{n+1} \wedge M_k(R_0(r)) \to C_{n+1} \wedge N_{\otimes}^{\mathrm{cy}}(M_k(R_1), M_k(R_0(r)))_n$ where h is the map from the constant simplicial object into $N_{\otimes}^{\mathrm{cy}}(M_k(R_1), M_k(R_0(r)))_\bullet$ which is the identity in simplicial degree 0.
In conclusion, the Dennis trace maps $\pi_{r+1}(S^1 \wedge M_k(R_0(r)))$ surjectively onto $\pi_{r+1}(S^1 \wedge T(F_1, F_0))$. Indeed it is the canonical surjection

$$M_k(R_0) \to M_k(R_0)/[M_k(R_1), M_k(R_0)] \cong R_0/[R_1, R_0].$$

The theorem now follows from (10.7) and Proposition 10.9 □

We specialize in Theorem 10.12 to the case where $F_1 = I$ is the identity FSP and $F_2 = \mathbb{Z}$ is the FSP associated to the integers $\mathbb{Z}$ as in sect. 5 above. Then $K(F_1) = A(*)$ and $K(F_2) = K(\mathbb{Z})$. Let G be the monoid self homotopy equivalences of S^n for $n \to \infty$ and SG the component of maps of degree 1. There is a commutative diagram

$$\begin{array}{ccc} BG & \to & B\mathbb{Z}^* \\ \downarrow & & \downarrow \\ A(*) & \to & K(\mathbb{Z}) \end{array}$$

where the vertical maps correspond to the inclusions of $GL(F_i)$ into $K(F_i)$, and hence a map

$$\mathrm{BSG} \overset{w}{\to} K(I \to \mathbb{Z}) \overset{\mathrm{Tr}}{\to} T(I \to \mathbb{Z}).$$

We are only interested in p-local homotopy groups and can replace I and $\mathbb{Z}$ by their p-local counterparts. Then $I \to \mathbb{Z}$ becomes $2p-3$ connected, and the previous theorem applies in dimension $2p-2$.

In the special case at hand,

$$\pi_{2p-2}\Big(S^1 \wedge T(F_1, F_0); \mathbb{Z}_{(p)}\Big) \cong \pi_{2p-3} F_0^s = \mathbb{Z}/p$$

and the composition

$$\pi_{2p-1}\Big(T(\mathbb{Z}); \mathbb{Z}_{(p)}\Big) \xrightarrow{\partial_*} \pi_{2p-2}\Big(T(F_1 \to F_2); \mathbb{Z}_{(p)}\Big) \xrightarrow{\text{proj}} \pi_{2p-2}\Big(S^1 \wedge T(F_1, F_0); \mathbb{Z}_{(p)}\Big)$$

is an isomorphism. If p is odd then proj is an isomorphism. In [40], Corollary 3.7, Waldhausen proves

(a) $w_{2p-2} : \pi_{2p-2}\Big(\mathrm{BSG}; \mathbb{Z}_{(p)}\Big) \to \pi_{2p-2}\Big(K(I \to \mathbb{Z}); \mathbb{Z}_{(p)}\Big)$ is surjective

(b) $\partial_* : K_{2p-1}(\mathbb{Z}) \to \pi_{2p-2}(K(I \to \mathbb{Z}); \mathbb{Z}_p)$ maps onto Im w_{2p-2}.

The first statement is easy, and follows readily from the proof of Proposition 10.9. The second is not: it is a consequence of one of Waldhausen's main theorems: the composition

$$\mathrm{BSO} \xrightarrow{J} \mathrm{BSG} \to A(*) \to Wh^{\mathrm{diff}}(*)$$

is null-homotopic; the proof requires the manifold approach to $A(*)$, the vanishing of $\mu(*)$ and the well-known surjection $\pi_{2p-2}\mathrm{BSO} \to \pi_{2p-2}\mathrm{BSG}$.

The above discussion implies the following unpublished result of the first named author

Theorem 10.14 *The trace map* $\mathrm{Tr}_{2p-1} : K_{2p-1}(\mathbb{Z}) \to T_{2p-1}(\mathbb{Z})$ *is surjective.* □

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