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COMPENSATION OF SMALL DENOMINATORS AND RAMIFIED LINEARISATION OF LOCAL OBJECTS

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1. REMINDER ABOUT LOCAL OBJECTS. THE THREE MAIN ANALYTIC FACTS.

By *local objects* we will understand either *local analytic vector fields* (or *fields* for short) on $\mathbf{C}^\nu$ at 0 :

$$X = \sum_{i=1}^{\nu} X_i(x)\partial_{x_i} \qquad (X_i(x) \in \mathbf{C}\{x\};\ X_i(0) = 0) \tag{1.1}$$

or *local analytic selfmappings* (or *diffeos*, short for diffeomorphisms) of $\mathbf{C}^\nu$ with 0 as fixed point :

$$f : x_i \mapsto f_i(x) \qquad (i = 1, \dots, \nu) \qquad (f_i(x) \in \mathbf{C}\{x\};\ f(0) = 0) \tag{1.2}$$

or again, equivalently, the related *substitution operators* :

$$F : \varphi \mapsto F.\varphi \stackrel{\text{def}}{=} \varphi \circ f \qquad (\varphi(x)\,\text{and}\,\varphi \circ f(x) \in \mathbf{C}\{x\}) \tag{1.3}$$

Throughout, we will assume *diagonalisability* of the linear part and work with (analytic) *prepared forms* of the object on hand. That is to say, we will deal with vector fields given by :

$$X = X^{\text{lin}} + \sum \mathbb{B}_n \tag{1.4}$$

$$X^{\text{lin}} = \sum \lambda_i x_i \partial_{x_i} \tag{1.4'}$$

$$\mathbb{B}_n = \text{ homogeneous part of degree } n = (n_1, \dots, n_\nu) \qquad \text{with } n_i \geq -1 \tag{1.4''}$$

and with diffeos given by :

$$F = \{1 + \sum \mathbb{B}_n\} F^{\text{lin}} \tag{1.5}$$

$$F^{\text{lin}} \varphi(x_1, \dots, x_\nu) \stackrel{\text{def}}{=} \varphi(\ell_1 x_1, \dots, \ell_\nu x_\nu) \tag{1.5'}$$

$$\mathbb{B}_n = \text{ homogeneous part of degree } n = (n_1, \dots, n_\nu) \quad \text{with } n_i \geq -1 \tag{1.5''}$$

Of course, n-homogeneity means that for each monomial $x^m = x_1^{m_1} \dots x_\nu^{m_\nu}$:

$$\mathbf{B}_n.x^m = \beta_{n,m}x^{n+m} \qquad \text{with } \beta_{n,m} \in \mathbf{C}. \tag{1.6}$$

Note that, for any given $\mathbf{B}_n$, *at most* one component n_i may assume the value -1.

The scalars λ_i and ℓ_i are the object's *multipliers*. Together, they constitute its spectrum. If the spectrum is "random", the object turns out to be formally and even *analytically linearisable* (see below) and that about ends the matter, as far as the local study is concerned. For interesting problems to arise, at least one of three specific complications C_1, C_2, C_3 (see below) must come into play. Then numerous difficulties, mostly due to divergence, have to be sorted out. Yet the remarkable thing is that three easy, formal statements F_1, F_2, F_3 (F for formal) and three non-trivial, analytic theorems A_1, A_2, A_3 (A for analytic) suffice, between themselves, to give a fairly comprehensive picture of the whole situation. In this paper, we shall be mainly concerned with statement A_3 about the *effective, ramified linearisation of local objects*. Nonetheless, both for completeness and orientation, we shall begin with a brief review of all six statements. But first, we list the three "complications".

C_1. **Resonance.**

For a vector field, this means additive resonance of the λ_i :

$$\sum_{i=1}^{\nu} m_i\,\lambda_i = 0 \tag{1.7}$$

or

$$\sum_{i=1}^{\nu} m_i\lambda_i = \lambda_j \quad (m_i \in \mathbf{N}) \tag{1.7'}$$

and for a diffeo it means multiplicative resonance of the ℓ_i :

$$\prod_{i=1}^{\nu} (\ell_i)^{m_i} = 1 \tag{1.8}$$

or

$$\prod_{i=1}^{\nu} (\ell_i)^{m_i} = \ell_j \ (m_i \in \mathbf{N}) \tag{1.8'}$$

C_2. **Quasiresonance.**

This means that among all the non-vanishing expressions $\alpha(m) = < m, \lambda >$ or $\alpha(m) = \ell^m - 1$ (with all $m_i \geq 0$ except at most one that may be $\equiv -1$) there is a subinfinity that tends to 0 "abnormally" fast, thus violating the two equivalent diophantine conditions :

$$S \stackrel{\text{def}}{=} \sum 2^{-k} \log(1/\varpi(2^k)) \ < +\infty \qquad \text{(A.D. Bruno)} \tag{1.9}$$

$$S^* \stackrel{\text{def}}{=} \sum k^{-2} \log(1/\varpi(k)) \ < +\infty \qquad \text{(H. Rüssmann)} \tag{1.9'}$$

with $\varpi(k) = \inf|\alpha(m)|$ for $m_1 + \ldots m_\nu \leq k$. (Clearly, $1/2 \leq S^*/S \leq 2$).

C_3. Nihilence.

It amounts to the existence of a "first integral" in the form of a power series H :

$$X.H(x) \equiv 0 \text{ or } H(f(x)) \equiv H(x) \text{ with } H(x) \in \mathbb{C}[[x]] = \mathbb{C}[[x_1, \ldots, x_\nu]] \tag{1.10}$$

along with the existence of small denominators :

$$\inf |\alpha(m)| = 0 \text{ for } \alpha(m) \neq 0. \tag{1.11}$$

Note that conditions (1.10) presupposes *resonance*, but that condition (1.11) differs from (1.9) in that it involves no arithmetical condition.

Resonance is fairly common, if only because it includes all diffeos tangent to the identity map, for which indeed $\ell_1 = \ell_2 = \ldots \ell_\nu = 1$. *Quasiresonance* is decidedly exceptional in single objects, but becomes inescapable when one studies parameter-dependent families of objects. *Nihilence* is common with volume-preserving or symplectic objects, where it may occur, respectively, from dimension 3 and 4 onwards.

All three complications may coexist. They may even occur in layers. Indeed, whenever the ordinary or first-level multipliers of an object X or F are involved in *multiple resonance*, there is a natural notion of reduced object X^{red} or F^{red} acting on the algebra of resonant monomials, and endowed with its own multipliers (second-level multipliers), which may in turn give rise to second-level resonance, quasiresonance or nihilence; and so forth. This daunting multiplicity of cases and subcases makes the existence of universally valid statements like A_1, A_2, A_3 (infra) all the more remarkable. But first let us go through the formal statements F_1, F_2, F_3 which, though fairly trivial, will clear the ground for A_1, A_2, A_3 and settle some useful terminology.

F$_1$. *In the absence of resonance, a local object is formally linearisable.*

The proof is straightforward. Indeed, inductive coefficient identification yields formal, entire changes of coordinates :

$$h^{\text{ent}} : x \mapsto y \quad \text{with} \quad y_i = h_i^{\text{ent}}(x) = x_i\{1 + \ldots\} \tag{1.12}$$

$$k^{\text{ent}} : y \mapsto x \quad \text{with} \quad x_i = k_i^{\text{ent}}(y) = y_i\{1 + \ldots\} \tag{1.13}$$

which take us from the given analytic chart $x = (x_i)$ to a formal chart $y = (y_i)$ where the object reduces to its linear part X^{lin} or F^{lin}. But to pave the way for the forthcoming analytic study, we require explicit expansions for h^{ent} and k^{ent}, or rather for the corresponding formal substitution operators Θ_{ent} and Θ_{ent}^{-1}. We use the variables x_i throughout :

$$\Theta_{\text{ent}}\, \varphi(x) \overset{\text{def}}{\equiv} \varphi \circ h^{\text{ent}}(x) \qquad (\varphi(x), \varphi \circ h^{\text{ent}}(x) \in \mathbb{C}[[x]]) \tag{1.14}$$

$$\Theta_{\text{ent}}^{-1}\,\varphi(x) \overset{\text{def}}{\equiv} \varphi \circ k^{\text{ent}}(x) \qquad (\varphi(x), \varphi \circ k^{\text{ent}}(x) \in \mathbf{C}[[x]]) \tag{1.15}$$

Written down in compact form, those expansions read, for a vector field :

$$\Theta_{\text{ent}} = \sum S^{\bullet}\,\mathbf{B}_{\bullet} \overset{Def}{=} 1 + \sum_{r\geq 1}\sum_{\omega_i} S^{\omega_1,\ldots,\omega_r}\,\mathbf{B}_{\omega_1,\ldots,\omega_r} \tag{1.16}$$

$$\Theta_{\text{ent}}^{-1} = \sum \mathcal{S}^{\bullet}\,\mathbf{B}_{\bullet} = 1 + \ldots \tag{1.17}$$

and for a diffeo :

$$\Theta_{\text{ent}} = \sum \mathscr{S}^{\bullet}\,\mathbf{B}_{\bullet} \overset{Def}{=} 1 + \sum_{r\geq 1}\sum_{\omega_i} \mathscr{S}^{\omega_1,\ldots,\omega_r}\mathbf{B}_{\omega_1,\ldots,\omega_r} \tag{1.18}$$

$$\Theta_{\text{ent}}^{-1} = \sum \mathfrak{S}^{\bullet}\,\mathbf{B}_{\bullet} = 1 + \ldots \tag{1.19}$$

There being no resonance, the cumbersome $n = (n_1, \ldots, n_\nu)$ indexation can be replaced by the handier $\omega = \sum n_i \lambda_i$ indexation (using any determination $\lambda_i = \log \ell_i$ for a diffeo) and each of the above expansions reduces to the contraction of a *mould* $M^{\bullet}$ with a *comould* $\mathbf{B}_{\bullet}$ (see §12). Here, the relevant comould $\mathbf{B}_{\bullet}$ is defined by :

$$\mathbf{B}_{\phi} = 1 \ \text{ and } \ \mathbf{B}_{\omega_1,\ldots,\omega_r} = \mathbf{B}_{\omega_r} \ldots \mathbf{B}_{\omega_1} \tag{1.20}$$

from the homogeneous parts $\mathbf{B}_n$ of the local object (see (1.4), (1.15)) after reindexing from n to ω. The relevant moulds involve only the spectrum of the object and are defined as follows for each sequence $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$:

$$S^{\phi} = \mathcal{S}^{\phi} = \mathscr{S}^{\phi} = \mathfrak{S}^{\phi} = 1 \qquad (\phi = \text{ empty sequence}) \tag{1.21}$$

$$S^{\boldsymbol{\omega}} = (-1)^r \,(\check{\omega}_1\, \check{\omega}_2 \ldots \check{\omega}_r)^{-1} \qquad \text{with} \quad \check{\omega}_i = \omega_1 + \ldots \omega_i \tag{1.22}$$

$$\mathcal{S}^{\boldsymbol{\omega}} = (\hat{\omega}_1\, \hat{\omega}_2 \ldots \hat{\omega}_r)^{-1} \qquad \text{with} \quad \hat{\omega}_i = \omega_i + \ldots \omega_r \tag{1.23}$$

$$\mathscr{S}^{\boldsymbol{\omega}} = (-1)^r e^{-\|\omega\|}\,(1 - e^{-\check{\omega}_1})^{-1} \ldots (1 - e^{-\check{\omega}_r})^{-1} \ \text{ with } \ \|\boldsymbol{\omega}\| = \omega_1 + \ldots \omega_r \tag{1.24}$$

$$(1.25) \qquad \mathcal{S}^{\omega} = (e^{\hat{\omega}_1} - 1)^{-1} \dots (e^{\hat{\omega}_r} - 1)^{-1}$$

In the case of a field, the moulds $S^\bullet, \mathcal{S}^\bullet$ are *symmetral* and the comould $\mathbf{B}_\bullet$ is *cosymmetral* (see §12). In the case of a diffeo, the moulds $S^\bullet, \mathcal{S}^\bullet$ are *symmetrel* and the comould $\mathbf{B}_\bullet$ is *cosymmetrel* (see §12). Contraction of like-natured moulds and comoulds yields formal automorphisms :

$$(1.26) \qquad \Theta^{\pm 1}_{\mathrm{ent}}(\varphi.\psi) \equiv (\Theta^{\pm 1}_{\mathrm{ent}}\varphi)\,(\Theta^{\pm 1}_{\mathrm{ent}}\psi)$$

So the only thing left to check is that the substitution operators $\Theta^{\pm 1}_{\mathrm{ent}}$ just defined do satisfy the linearisation identities :

$$(1.27) \qquad X = \Theta X^{\mathrm{lin}} \Theta^{-1}$$

$$(1.28) \qquad F = \Theta X^{\mathrm{lin}} \Theta^{-1}$$

This in turn readily follows from the obvious identities :

$$(1.29) \qquad \|\omega\| S^{\omega} = -S^{\omega'}$$

$$(1.30) \qquad \|\omega\|\, \mathcal{S}^{\omega} = +S^{{}^\backprime\omega}$$

$$(1.31) \qquad e^{\|\omega\|} S^{\omega} = S^{\omega} - S^{\omega'} + S^{\omega''} - S^{\omega'''} + \dots$$

$$(1.32) \qquad e^{\|\omega\|}\, \mathcal{S}^{\omega} = \mathcal{S}^{\omega} + S^{{}^\backprime\omega}$$

with $\omega = (\omega_1, \dots, \omega_r)$; $\|\omega\| = \omega_1 + \dots \omega_r$; $\omega' = (\omega_1, \dots, \omega_{r-1})$; ${}^\backprime\omega = (\omega_2, \dots, \omega_r)$.

$\mathbf{F}_2$. *In the presence of resonance, there are formal changes of coordinates which bring the local object to a prenormal form (containing only resonant monomials) and even to a normal form (containing a "minimum number" of resonant monomials, with in front of them scalar coefficients which are formal invariants of the object).*

A *resonant monomial* is of couse any $x^m = x_1^{m_1} \dots x_\nu^{m_\nu}$ such that $< m, \lambda >= 0$ for a field or $\ell^m = 1$ for a diffeo. This case being of secondary concern to us here, we refer to [E.2] [E.3] [E.7] and also to §10 *infra*, where we shall investigate the overlap between resonance and ramified linearisation. We may note in passing the rule of thumb : one (resp. several) degree of resonance implies the existence of finitely (resp. infinitely) many independent formal invariants.

$\mathbf{F}_3$. ***Any non-degenerate local object (even if resonant) can be linearised by means of a formal-ramified change of coordinates.***

We now ***ramify*** the coordinates x_i (relative to a ***prepared form*** (1.4) or (1.5)), i.e. we regard them as elements of $\underset{\bullet}{\mathbf{C}} = \widetilde{\mathbf{C}\backslash\{0\}}$.

Non-degenerate means $\lambda_i \neq 0$ for a field and $\underset{\bullet}{\ell_i} \neq \underset{\bullet}{0}, \underset{\bullet}{1}$ for a diffeo ($\underset{\bullet}{\ell_i}, \underset{\bullet}{0}, \underset{\bullet}{1} \in \underset{\bullet}{\mathbf{C}}$).

A *formal-ramified change of coordinates* is any formal transformation of the form :

$$y_i = x_i\{1 + \tilde{\varphi}_i(x)\} \tag{1.33}$$

with formal series $\tilde{\varphi}_i(x)$ without constant term and of type :

$$\sum a_{\sigma_1,\ldots,\sigma_\nu} x_1^{\sigma_1} \ldots x_\nu^{\sigma_\nu} \qquad (a_\sigma \in \mathbf{C}) \tag{1.34}$$

or of type

$$\sum a_{\sigma_1,\ldots,\sigma_\nu;n_1,\ldots,n_\nu} x_1^{\sigma_1} \ldots x_\nu^{\sigma_\nu} (\log x_1)^{n_1} \ldots (\log x_\nu)^{n_\nu} \qquad (a_{\sigma,n} \in \mathbf{C}) \tag{1.34$'$}$$

with real positive powers σ_i ranging over some discrete subset of $(\mathbf{R}^+)^\nu$ and integers n_i constrained by :

$$\limsup\ (n_1 + \ldots n_\nu)/(\sigma_1 + \ldots \sigma_\nu) < +\infty \tag{1.35}$$

or by the stronger conditions :

$$\limsup\ n_i/\sigma_i < +\infty \qquad (\text{for } i = 1, \ldots, \nu) \tag{1.36}$$

Lastly, the transformation of ℓ_i into $\underset{\bullet}{\ell_i}$ means that, for a diffeo, we may replace each multiplier $\ell_i \in \mathbf{C}$ by some $\underset{\bullet}{\ell_i} \in \underset{\bullet}{\mathbf{C}}$ that lies over ℓ_i, subject only to $\underset{\bullet}{\ell_i} \neq \underset{\bullet}{0}, \underset{\bullet}{1}$. For this to be possible, ℓ_i must be $\neq 0$, but it may well be $= 1$. Thus, *non-degeneracy for a diffeo is not the "exponential" of non-degeneracy for a field.*

Strictly speaking, formal-ramified linearisability has to be established for resonant (or nihilent) objects only, because otherwise entire linearisation is available by proposition F_1. For explicit formulae relative to resonant objects, see §10 infra. But we will use ramified linearisation *also*, and even *mostly*, in the absence of resonance, to overcome the divergence caused by quasiresonance.

Now, let us come to the real thing - the three analytic statements A_1, A_2, A_3.

$\mathbf{A}_1$. ***The Siegel-Bruno-Rüssmann theorem about the innocuousness of diophantine small denominators : Any non-resonant and non-quasiresonant local object (i.e. any object free of C_1, C_2, C_3) can be analytically linearised.***

The first result in this direction was obtained by Siegel, but under an unnecessarily strong diophantine condition. The proof of A_1, under the probably optimal diophantine condition (1.9) (see §10) is due to A.D. Bruno [B] for fields and to H. Rüssmann [R] for diffeos. In the introductory paras of [E.8], we gave an essentially similar proof, but in a concise form which easily extends to a host of other situations. We must here give a sketch of this method, because we will require it in the sequel.

It is based on the process of ***arborification-coarborification***, expounded in [E.8], and recalled in §12. The idea is to replace contracted sums of type $\sum M^{\bullet}\mathbf{B}_{\bullet}$ by formally equivalent sums $\sum M^{\overset{<}{\bullet}}\mathbf{B}_{\overset{<}{\bullet}}$, using the dual operations of arborification :

$$M^{\overset{<}{\boldsymbol{\omega}}} = \sum \mathrm{sh}\begin{pmatrix}\overset{<}{\boldsymbol{\omega}}\\ \boldsymbol{\omega}\end{pmatrix} M^{\boldsymbol{\omega}} \text{ (fields) or} \sum \mathrm{ctsh}\begin{pmatrix}\overset{<}{\boldsymbol{\omega}}\\ \boldsymbol{\omega}\end{pmatrix} M^{\boldsymbol{\omega}} \text{ (diffeos)} \tag{1.37}$$

and coarborification :

$$\mathbf{B}_{\boldsymbol{\omega}} = \sum \mathrm{sh}\begin{pmatrix}\overset{<}{\boldsymbol{\omega}}\\ \boldsymbol{\omega}\end{pmatrix} \mathbf{B}_{\overset{<}{\boldsymbol{\omega}}} \text{ (fields) or} \sum \mathrm{ctsh}\begin{pmatrix}\overset{<}{\boldsymbol{\omega}}\\ \boldsymbol{\omega}\end{pmatrix} \mathbf{B}_{\overset{<}{\boldsymbol{\omega}}} \text{ (diffeos)} \tag{1.38}$$

Here, the boldfaced symbols $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$ (resp. $\overset{<}{\boldsymbol{\omega}} = (\omega_1, \ldots, \omega_r)^<$) denote sequences of ω_i with a *full* (resp. *arborescent*) order on them, and the coefficients $\mathrm{sh}\begin{pmatrix}\overset{<}{\boldsymbol{\omega}}\\ \boldsymbol{\omega}\end{pmatrix}$ (resp. $\mathrm{ctsh}\begin{pmatrix}\overset{<}{\boldsymbol{\omega}}\\ \boldsymbol{\omega}\end{pmatrix}$) are equal to the number of order-preserving *bijections* (resp. *surjections*) of $\overset{<}{\boldsymbol{\omega}}$ into $\boldsymbol{\omega}$. For details, see [E.8] and §12 infra.

Each sum (1.37) or (1.38) contains many terms - close to $r!$ on average. It is predictable therefore, and readily verified, that coarborification $\mathbf{B}_{\boldsymbol{\omega}} \mapsto \mathbf{B}_{\overset{<}{\boldsymbol{\omega}}}$ should entail a drastic ***reduction*** of norms :

$$\|\mathbf{B}_{\boldsymbol{\omega}}\|_{\mathcal{D},\mathcal{D}'} < C^r.r! \qquad (C = C_{\mathcal{D},\mathcal{D}'} = \mathrm{Cste}) \tag{1.39}$$

$$\|\mathbf{B}_{\overset{<}{\boldsymbol{\omega}}}\|_{\mathcal{D},\mathcal{D}'} < C^r \tag{1.40}$$

with norms relative to two bounded neighbourhoods $\mathcal{D}' \subset \mathcal{D}$ of 0 in $\mathbf{C}^\nu$:

$$\|\varphi\|_{\mathcal{D}} = \sup_x \varphi(x) \quad \text{for } x \in \mathcal{D} \qquad (\varphi \in \mathbf{C}\{x\}) \tag{1.41}$$

$$\|\mathbf{B}\|_{\mathcal{D},\mathcal{D}'} = \sup_\varphi \|\mathbf{B}\varphi\|_{\mathcal{D}'} \quad \text{for } \|\varphi\|_{\mathcal{D}} \leq 1 \qquad (\mathbf{B} \in \mathrm{End}\,\mathbf{C}\{x\}) \tag{1.42}$$

Moreover, for any $\epsilon > 0$, one may chose $\mathcal{D}, \mathcal{D}'$ so that $C \leq \epsilon$.

By the same token, one would expect arborification $M^{\boldsymbol{\omega}} \mapsto M^{\overset{<}{\boldsymbol{\omega}}}$ to bring about a corresponding *increase* in norm. Fortunately, however, this is seldom the case, due to massive cancellations in the sums (1.37). In the present case, for instance, we have the easy identities :

$$\mathcal{S}^{\overset{<}{\boldsymbol{\omega}}} = \prod_i (\hat{\omega}_i)^{-1} \tag{1.43}$$

$$\mathscr{S}^{\overset{<}{\boldsymbol{\omega}}} = \prod_i (e^{\hat{\omega}_i} - 1)^{-1} \tag{1.44}$$

the only difference with (1.23) and (1.25) being that now each sum $\hat{\omega}_i = \sum \omega_j$ extends to all j posterior (or equal) to i *relative to the arborescent order* of $\overset{<}{\boldsymbol{\omega}}$. The bounds for $\mathcal{S}^{\overset{<}{\boldsymbol{\omega}}}$ and $\mathscr{S}^{\overset{<}{\boldsymbol{\omega}}}$ are therefore almost as good as those for $\mathcal{S}^{\boldsymbol{\omega}}$ and $\mathscr{S}^{\boldsymbol{\omega}}$. Indeed, under Bruno's diophantine condition (1.9) we have :

$$|\mathcal{S}^{\boldsymbol{\omega}}| \leq C^r \quad ; \quad |\mathscr{S}^{\boldsymbol{\omega}}| \leq C^r \tag{1.45}$$

$$|\mathcal{S}^{\overset{<}{\boldsymbol{\omega}}}| \leq C^r \quad ; \quad |\mathscr{S}^{\overset{<}{\boldsymbol{\omega}}}| \leq C^r \tag{1.46}$$

$\omega = (\omega_1, \ldots, \omega_r)$ and a constant C depending on the spectrum λ or ℓ. The inequalities (1.46) are implicit in Bruno's paper [B]. The inequalities (1.45) are easy and could be had under *slightly weaker* diophantine conditions than (1.9). See also [Y].

Pairing the estimates (1.40) and (1.46) and using (12.22), we immediately deduce the normal convergence of the arborified expansions :

$$\Theta_{\text{ser}}^{-1} = \sum \mathcal{S}^{\overset{<}{\bullet}}\, \mathbf{B}_{\underset{\bullet}{<}} \ \text{(field)} \quad \text{or} \quad \sum \mathscr{S}^{\overset{<}{\bullet}}\, \mathbf{B}_{\underset{\bullet}{<}} \ \text{(diffeo)} \tag{1.47}$$

In contrast, the original, non-arborified expansions (1.17) and (1.19) do not converge in norm, at least in the presence of small denominators.

A$_2$. ***Theorem about the resurgent normalisation of resonant local objects : For resonant local objects, the normalising change of coordinates is usually divergent, but resurgent with respect to one or several "critical times" z. Moreover, the Bridge Equation holds :***

$$\overset{\bullet}{\Delta}_\omega\, x(z,u) = \mathbb{A}_\omega\, x(z,u) \tag{1.48}$$

with on the right-hand side ordinary differential operators $\mathbb{A}_\omega$ which, taken together, constitute a complete system of holomorphic invariants of the object. Lastly, the $x(z,u)$ are usually resummable, but only in sectors of the (z,u) space.

Since this whole topic of resonance-cum-resurgence has been dealt with at lenght in [E.2] and [E.3], and is only of incidental relevance to the present study (see §§9 and 10), we will limit ourselves to a few cursory indications. The ingredients of the Bridge Equation are three. First, a so-called *formal integral* :

$$x(z,u) = \{x_1(z,u_1,\ldots,u_{\nu-1}),\ldots,x_\nu(z,u_1,\ldots,u_{\nu-1})\} \tag{1.49}$$

which is a general (i.e. parameter-saturated) formal solution of the differential system associated with a field X

$$\partial_z\, x_i(z,u) = X_i(x(z,u)) \qquad (\partial_z = \frac{\partial}{\partial z};\ i = 1,\ldots,\nu) \tag{1.50}$$

or of the difference system associated with a diffeo f :

$$x_i(z+1,u) = f_i(x(z,u)) \qquad (i = 1,\ldots,\nu) \tag{1.51}$$

It is thus a formal, non-entire chart $(z,u_1,\ldots,u_{\nu-1})$ in which the object assumes the simplest conceivable form, namely :

$$X = \frac{\partial}{\partial z} \qquad \text{or} \qquad f : z \mapsto z+1 \tag{1.52}$$

The precise shape of $x(z,u)$ varies considerably from case to case. For a simple example, see below in §10. Many more are given in [E.2] [E.3] [E.5] [E.6] [E.7] [E.8].

The symbols $\overset{\bullet}{\Delta}_\omega$ on the left-hand side of (1.48) denote pointed *alien derivations.* (See §12). Their indexes ω range over an enumerable, and usually discrete, subset of $\underset{\bullet}{\mathbb{C}}$.

Lastly, the $\mathbb{A}_\omega$ on the right-hand side of (1.48) are ordinary differential operators in z and u, subject to no other a priori constraints than :

$$[\mathbb{A}_\omega, \partial] = 0 \quad \text{for a field} \qquad (\partial = \frac{\partial}{\partial z}) \tag{1.53}$$

$$[\mathbb{A}_\omega, \exp\partial] = 0 \quad \text{for a diffeo} \qquad (\exp\partial = \ \text{translation of step 1.}) \tag{1.54}$$

The $\mathbb{A}_\omega$ are both *analytic invariants* (an analytic change of coordinates leaves them unchanged) and *holomorphic invariants* (they are holomorphic functions of the object, when the latter ranges over a given formal class). For a full treatment, see [E.3] [E.8] and for the overlap with ramified linearisation, see [E.8] chap. 3 and §10 infra.

A$_3$. *Any non-degenerate local object (whether or not affected by C_1, C_2, C_3) can be linearised by means of a seriable-ramified change of coordinates.*

Non-degeneracy, we recall, is a very mild requirement, meaning simply $\lambda_i \neq 0$ for a field and $\underset{\bullet}{\ell_i} \neq \underset{\bullet}{0}, \underset{\bullet}{1}$ for a diffeo.

For the precise definition and properties of *seriability*, we will have to wait until §§2,3 but for the moment we may say that a seriable-ramified change of coordinates is a formal change of type (1.34) or (1.34′) with condition (1.35) or its strengthened variant (1.36), which furthermore can be resummed in a *unique way*, by means of a suitable Borel-Laplace procedure, to a sum that is defined, analytic in a ramified, spiralling neighbourhood of the origin, i.e. in a neighbourhood, not of $0 \in \mathbf{C}^{\nu}$, but of $\underset{\bullet}{0} \in \underset{\bullet}{\mathbf{C}}^{\nu}$. These linearisation domains are optimally large, and seriable-ramified linearisation, while less simple than analytic linearisation, retains the latter's essential feature, which is "formalisability", namely the quality for a function germ $\varphi(x)$ of being totally and constructively reducible to a formal object $\tilde{\varphi}(x)$.(*)

Before concluding this introductory section, let us introduce the idea of *compensation*, on which seriable-ramified linearisation rests. Assume non-resonance for a start. Then by statement F_1, the local object X or F can be reduced, by means of a formal, entire change of coordinates Θ_{ent}, to its linear part X^{lin} or F^{lin}. But if we allow for ramifications, we find a huge group INV of formal-ramified changes of coordinates Θ_{inv}, which involve only *resonant monomials* $y^{\sigma} = y_1^{\sigma_1} \dots y_{\nu}^{\sigma_{\nu}}$ and leave the linear part of the object invariant :

$$(1.55) \qquad y_i \mapsto y_i^* = y_i\{1 + \sum a_{\sigma} y^{\sigma}\} \ \text{ with } <\sigma, \lambda> = \sum \sigma_i \lambda_i = 0 \quad (\sigma_i \in \mathbf{R}^+)$$

The idea therefore is to look for ramified transformations of the form :

$$(1.56) \qquad \Theta_{\text{ser}} = \Theta_{\text{ent}} \Theta_{\text{inv}}^{-1} \ ; \ \Theta_{\text{ser}}^{-1} = \Theta_{\text{inv}} \Theta_{\text{ent}}^{-1}$$

(with *ser, ent, inv* for *seriable, entire, invariant) where the divergence present in* Θ_{ent} *(due to quasiresonance or nihilence) is offset by a similar divergence in* Θ_{inv}.

For instance, if we are dealing with a local diffeo of $\mathbf{C}^1$ of type :

$$(1.57) \qquad f : x \mapsto x\{\ell + \sum a_n x^n\} \qquad (\ell = e^{2\pi i \lambda^*}; \lambda^* > 0)$$

and if λ^* is strongly liouvillian, we are faced with complication C_1 (quasiresonance) and the entire linearisation Θ_{ent} usually involves an infinity of monomials $c_n x^n$ with nearly integral $n\lambda^*$ and very large coefficients c_n :

$$(1.58) \qquad c_n = a_n(\ell^n - 1)^{-1} + \dots; \ n\lambda^* = m + \epsilon_n; \ \epsilon_n \text{ exceptionnally small}$$

which hopefully may be neutralised by a ramified transformation Θ_{inv} involving monomials of type $c_n x^{m/\lambda^*}$.

(*) This is sometimes referred to as *germ quasianalyticity*, but the meaning of this term has been so over-stretched that we prefer to keep it for *function quasianalyticity* of Denjoy-Carleman or related types.

The procedure is not different in the case of ***resonant objects***, with factorisation (1.56) replaced by (10.2) or (10.3).

Now it so happens that Θ_{ser}^{-1} is simpler to calculate than Θ_{ser}. So we will look for explicit expansions :

$$\Theta_{\text{ser}}^{-1} = \sum \mathcal{S}_{\text{co}}^{\bullet}(x)\mathbf{B}_{\bullet} \qquad \text{(fields)} \tag{1.59}$$

$$\Theta_{\text{ser}}^{-1} = \sum \mathcal{S}_{\text{co}}^{\bullet}(x)\mathbf{B}_{\bullet} \qquad \text{(diffeos)} \tag{1.60}$$

which differ from (1.17) and (1.19) only in that the moulds $M^{\bullet}(x)$ being contracted with the comould $\mathbf{B}_{\bullet}$ are no longer constant, but functions $\mathcal{S}_{\text{co}}^{\bullet}(x)$ or $\mathcal{S}_{\text{co}}^{\bullet}(x)$ of $x = (x_1, \ldots, x_\nu)$ that must meet three main requirements :

First, they must be ***compensators*** in x, i.e. sums of monomials which remain small though their coefficients may be very large.

Second, they must satisfy the following equations :

$$(\|\boldsymbol{\omega}\| + X^{\text{lin}})\, \mathcal{S}_{\text{co}}^{\boldsymbol{\omega}}(x) = \mathcal{S}_{\text{co}}^{\grave{\boldsymbol{\omega}}}(x) \qquad \text{(fields)} \tag{1.61}$$

$$(e^{\|\boldsymbol{\omega}\|} F^{\text{lin}})\ \mathcal{S}_{\text{co}}^{\boldsymbol{\omega}}(x) = \mathcal{S}_{\text{co}}^{\grave{\boldsymbol{\omega}}}(x) + \mathcal{S}_{\text{co}}^{\boldsymbol{\omega}}(x) \qquad \text{(diffeo)} \tag{1.62}$$

(with $\boldsymbol{\omega}$, $\|\boldsymbol{\omega}\|$, $\grave{\boldsymbol{\omega}}$ as in (1.32)) which ensure that the operators Θ_{ser}^{-1} defined by (1.59), (1.60) are indeed solutions of the linearisation equations (1.27), (1.28).

Third, they have to be ***symmetral*** (for fields) or ***symmetrel*** (for diffeos) ***functions*** of the sequence $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$, to ensure that the operators Θ_{ser}^{-1} be formal automorphisms (like Θ_{ent}^{-1} in (1.26)). However, we don't have to worry about this last requirement, since it is an automatic consequence of equations (1.61), (1.62).

In the present paper, we shall prove the theorem about seriable-ramified linearisability, not for all local objects, but only for an important subclass : the ***girators*** (i.e. vector fields with purely imaginary eigenvalues λ_i) and ***girations*** (i.e. diffeos with eigenvalues ℓ_i of modulus 1). The reasons for this restriction are two :

Reason one : Linearisation is specially relevant for girators or girations because, in the presence of analytic, entire linearisation, we have in the y chart continuous or discrete orbits :

$$y_i(t) = y_i(0)\ e^{\lambda_i t} \qquad (t \in \mathbf{R};\ \text{for girators}) \tag{1.63}$$

$$y_i(n) = y_i(0)\ \ell_i^n \qquad (n \in \mathbf{Z};\ \text{for girations}) \tag{1.64}$$

which "girate" indefinitely at a fixed distance from the origin. In the absence of analytic linearisability, the dynamics become much more complicated, but ramified linearisation sheds some light on it.

Reason two : Only for girators and girations may one find simple and nearly "canonical" expansions of type (1.59) or (1.60) with truly explicit moulds $\mathcal{S}_{\mathrm{co}}^{\omega}(x)$ of $\mathcal{S}_{\mathrm{co}}^{\omega}(x)$. For other objects, such expansions still exist, but the construction is more arduous and involves a far higher degree of arbitrariness, so that we must postpone it to a future paper.

The main results are stated and proved in §§5,6,7,8,10. The next three paras (§§2,3,4) introduce the necessary machinery of *compensation* and *seriability* and §9 investigates the link with resurgence.

2. SYMMETRIC COMPENSATORS.

Compensators are usually finite sums $\sum a_i z^{\sigma_i}$ which, due to internal cancellations, remain small even though the a_i may be large. The most basic ones are the *symmetric compensators* :

2.1 Definition : *symmetric compensator of order r*

$$z^{\sigma_0,\sigma_1,\dots,\sigma_r} \stackrel{\text{def}}{=} z^{\sigma_0} \star z^{\sigma_1} \star \dots z^{\sigma_r} \qquad (z \in \underset{\bullet}{\mathbf{C}};\sigma_0,\sigma_1,\dots,\sigma_r \in \mathbf{R}^+) \tag{2.2}$$

Here, $\star$ denotes the multiplicative convolution of holomorphic functions on $\underset{\bullet}{\mathbf{C}}$, defined as follows :

$$\varphi_1 \star \varphi_2(z) = \int_{\underset{\bullet}{1}}^{z} \varphi_1(z_1)\varphi_2(z/z_1)\frac{dz_1}{z_1} \tag{2.3}$$

for z close to $\underset{\bullet}{1}$ and, in the large, by analytic continuation. Convolution being commutative, $z^{\sigma_0,\dots,\sigma_r}$ is a symmetric function of the σ_i, uniform for z ranging over $\underset{\bullet}{\mathbf{C}}$. For distinct σ_i we have :

$$z^{\sigma_0,\dots,\sigma_r} = \sum_{0\le i\le r} z^{\sigma_i}\prod_{j\neq i}(\sigma_i-\sigma_j)^{-1} \tag{2.4}$$

and in case of repetitions :

$$z^{\sigma_0^{(1+n_0)},\dots,\sigma_r^{(1+n_r)}} = (n_0!\dots n_r!)^{-1}(\partial_{\sigma_0})^{n_0}\dots(\partial_{\sigma_r})^{n_r} z^{\sigma_0,\dots,\sigma_r} \tag{2.5}$$

with $\sigma_i \neq \sigma_j$ for $i \neq j$; $\partial_{\sigma_i} = \partial/\partial\sigma_i$; and with $\sigma_i^{(1+n_i)}$ denoting a subsequence of $1+n_i$ identical σ_i. We have in particular :

$$z^{\sigma_0,\dots,\sigma_r} = (1/r!)z^{\sigma}(\log z)^r \ \text{ if } \sigma_0 = \sigma_1 = \dots \sigma_r = \sigma \tag{2.5'}$$

Compensators multiply according to the rule :

$$z^{\sigma_0,\dots,\sigma_r}.z^{\tau_0,\dots,\tau_s} = \sum z^{\sigma_{i_0}+\tau_{j_0},\dots,\sigma_{i_{r+s}}+\tau_{j_{r+s}}} \tag{2.6}$$

with a sum extending to all (i_n, j_n) such that :

$$i_n + j_n = n \ \text{ for } n = 0, 1, \dots, r+s \tag{2.6'}$$

$$0 = i_0 \leq i_1 \leq \dots \leq i_{r+s} = r \tag{2.6''}$$

$$0 = j_0 \leq j_1 \leq \dots \leq j_{r+s} = s \tag{2.6'''}$$

The quickest derivation of (2.6) is by formula (4.10) which relates *symmetric* and *symmetral* compensators, the latter obeying the universal multiplication rule (12.4). That same formula (4.10), combined with the integral representation (4.8) of symmetral compensators, also yields the following useful bounds :

$$|z^{\sigma_0,\dots,\sigma_r}| \leq (1/r!)\ |\log z|^r |z|^{\sigma_*} \tag{2.7}$$

$$\left|\frac{z^{\sigma_0,\dots,\sigma_r}}{x_0^{\sigma_1,\dots,\sigma_r}}\right| \leq \left|\frac{\log z}{\log x_0}\right|^r \left|\frac{z}{x_0}\right|^{\sigma_*} \tag{2.8}$$

valid for any $z \in \underset{\bullet}{\mathbb{C}}$ with $|z| < x_0 < 1$, and $\sigma_i > 0$, $\sigma_* = \inf \sigma_i$. Moreover, for $0 < x < 1$:

$$(-1)^r x^{\sigma_0,\dots,\sigma_r} > 0 \tag{2.9}$$

3. COMPENSATION AND SERIATION.

We are going to introduce germs of functions which, though defined on ramified neighbourhoods of $\underset{\bullet}{0} \in \underset{\bullet}{\mathbb{C}}$ and having generically divergent asymptotic series, yet retain most of the regularity proper to holomorphic germs at $0 \in \mathbb{C}$.

Fundamental spiralling domains.

For each real pair x_0, κ_0 such that :

$$0 < x_0 < e^{-\kappa_0} < 1 \tag{3.1}$$

let $\mathcal{D} = \mathcal{D}_{x_0,\kappa_0}$ denote the connected part of $\underset{\bullet}{\mathbb{C}}$ defined by :

$$\mathcal{D}_{x_0,\kappa_0} = \{z; |z||\log z|^{\kappa_0} \leq x_0|\log x_0|^{\kappa_0}\} \tag{3.2}$$

x_0 is the "radius" of $\mathcal{D}$ and κ_0 measures the speed at which its boundary $\partial\mathcal{D}$ coils in towards $\underset{\bullet}{0}$. The smaller κ_0, the more $\mathcal{D}$ resembles the covering space of a punctured disc.

The norms $\|\bullet\|^{\mathcal{D}}$ and $\|\bullet\|^{\mathcal{D}}_{\text{comp}}$ on ramified polynomials.

A ramified polynomial is a finite sum of the form :

$$\varphi(z) = \sum a_{\sigma,r}\, z^{\sigma}(\log z)^r \qquad (\sigma \text{ real } > 0;\ r \text{ integral } \geq 0) \tag{3.3}$$

On the algebra spanned by such polynomials we introduce the usual *uniform norms* :

$$\|\varphi\|^{\mathcal{D}} \overset{\text{def}}{=} \sup_{z\in\mathcal{D}} |\varphi(z)| \quad (\|\varphi\|^{\mathcal{D}} < +\infty \text{ iff } \sup(r/\sigma) \leq \kappa_0) \tag{3.4}$$

as well as the *compensation norms* defined by :

$$\|\varphi\|^{\mathcal{D}}_{\text{comp}} \overset{\text{def}}{=} \inf\left\{\sum |A_{\boldsymbol{\sigma}}|.\|z^{\boldsymbol{\sigma}}\|^{\mathcal{D}}\right\} \leq \|\varphi\|^{\mathcal{D}} \tag{3.5}$$

with an inf relative to all possible finite decompositions of $\varphi(z)$:

$$\varphi(z) \equiv \sum A_{\boldsymbol{\sigma}} z^{\boldsymbol{\sigma}} = \sum A_{\sigma_0,\ldots,\sigma_r} z^{\sigma_0,\ldots,\sigma_r} \tag{3.5$'$}$$

into sums of compensators of arbitrary order r, but with $r \leq \kappa_0\sigma_i$.

The norms $\|\bullet\|^{\mathcal{D}}$ are obviously multiplicative, but so are the norms $\|\bullet\|^{\mathcal{D}}_{\text{comp}}$:

$$\|\varphi_1\varphi_2\|^{\mathcal{D}}_{\text{comp}} \leq \|\varphi_1\|^{\mathcal{D}}_{\text{comp}}\|\varphi_2\|^{\mathcal{D}}_{\text{comp}} \tag{3.6}$$

because for any fundamental domain $\mathcal{D} = \mathcal{D}_{x_0,\kappa_0}$:

$$\|z^{\boldsymbol{\sigma}}\|^{\mathcal{D}} = |x_0^{\boldsymbol{\sigma}}| \qquad \text{(by (2.8)) if } r \leq \kappa_0\sigma_i. \tag{3.7}$$

$$|x_0^{\boldsymbol{\sigma}}.x_0^{\boldsymbol{\tau}}| = |x_0^{\boldsymbol{\sigma}}|\ |x_0^{\boldsymbol{\tau}}| \qquad \text{(by (2.6) and (2.9))} \tag{3.8}$$

The latter identity stems from the fact that for $0 < z < 1$ all terms on the right-hand side of (2.6) have the same sign, namely $(-1)^{r+s}$.

The algebras of seriable or compensable functions.

For any discrete, additive semigroup $T \in \mathbf{R}^+$ and any fundamental spiralling domain $\mathcal{D}$, we denote by $\mathrm{Ser}^{T,\mathcal{D}}(z)$ and $\mathrm{Comp}^{T,\mathcal{D}}(z)$ the closure under $\| \bullet \|^{\mathcal{D}}$ and $\| \bullet \|^{\mathcal{D}}_{\mathrm{comp}}$ of the algebra of ramified polynomials (3.3) with coefficients σ in T, and we put :

(3.9)
$$\mathrm{Ser}(z) \overset{\mathrm{def}}{=} \bigcup_{T,\mathcal{D}} \mathrm{Ser}^{T,\mathcal{D}}(z)$$

(3.10)
$$\mathrm{Comp}(z) \overset{\mathrm{def}}{=} \bigcup_{T,\mathcal{D}} \mathrm{Comp}^{T,\mathcal{D}}(z)$$

Clearly $\mathrm{Ser}(z)$ and $\mathrm{Comp}(z)$ are both seminormed algebras. Their elements are known as *seriable* and *compensable* (germs of) functions respectively. $\mathrm{Ser}(z)$ contains $\mathrm{Comp}(z)$ but it is unclear to us whether the inclusion is strict.

3.11 Proposition : Asymptotic expansions

Each seriable or compensable (germ of) function $\varphi(z)$ admits a unique asymptotic expansion of the form :

(3.12)
$$\tilde{\varphi}(z) = \sum a_{\sigma,r}\, z^{\sigma}(\log z)^r \quad \text{with } r/\sigma \leq \kappa$$

for some finite κ and z going radially to $\underset{\bullet}{0}$. Although the coefficients admit universal bounds in terms of T and $\mathcal{D}$:

(3.13)
$$|a_{\sigma,r}| < C^{T,\mathcal{D}}_{\sigma,r} . \|\varphi\|^{\mathcal{D}}$$

(3.14)
$$|a_{\sigma,r}| < D^{T,\mathcal{D}}_{\sigma,r} . \|\varphi\|^{\mathcal{D}}_{\mathrm{comp}}$$

the series $\tilde{\varphi}(z)$ is usually divergent.

3.15 Proposition : Resummation of $\tilde{\varphi}$.

Each $\tilde{\varphi}(z)$ is resummable to $\varphi(z)$ by an appropriate Borel-Laplace procedure :

(3.16)
$$\tilde{\varphi}(z) \overset{1}{\mapsto} \tilde{\varphi}(e^{-t}t^{-\rho}) = \tilde{\psi}(t) \overset{2}{\mapsto} \hat{\psi}(\tau) \overset{3}{\mapsto} \psi(t) \overset{4}{\mapsto} \varphi(z)$$

for $\rho > 0$ large enough.

Proof of Propositions 3.11 and (3.15).

The arrows 1 and 4 in (3.16) simply denote the change of variable $z = e^{-t}t^{-\rho}$. Arrow 2 denotes the *formal* or *term-wise* Borel transform (12.29) and arrow 3 the Laplace transform (12.28), which reverses it. Under arrow 1 the monomials of (3.12) become :

(3.17)
$$P_{\sigma,r}(z) = z^{\sigma}(\log z)^r \overset{1}{\mapsto} Q_{\sigma,r}(t) = (-1)^r e^{-\sigma t} t^{r-\sigma\rho}(1 + \rho t^{-1} \log t)^r$$

and under arrow 2 they become functions on $\mathbf{R}^+$:

$$\hat{Q}_{\sigma,r}(\tau) = \begin{cases} 0 \text{ if } \tau \leq \sigma \\ (-1)^r \frac{(\tau-\sigma)^{\sigma\rho-r-1}}{\Gamma(\sigma\rho-r)}. \{1+\ldots\} \text{ if } \tau > \sigma \end{cases} \tag{3.18}$$

which can be rendered as smooth as one wishes (at the singularity $\tau = \sigma$) by choosing ρ large enough. Finally, arrows 3 and 4 take $\hat{Q}_{\sigma,r}(\tau)$ back to $Q_{\sigma,r}(t)$ and $P_{\sigma,r}(z)$ Now, for t_0 and ρ large enough, any given $\mathcal{D} = \mathcal{D}_{x_0,\kappa_0}$ contains the image of the half-plane $\mathrm{Re}(t) \geq t_0$ under $t \mapsto z = e^{-t}t^{-\rho}$. Therefore, the summation procedure (3.16) clearly applies to all ramified polynomials from a given algebra $\mathrm{Ser}^{T,\mathcal{D}}(z)$ and yields the universal bounds (3.13) which remain valid under *closure*, thus ensuring the existence of an asymptotic expansion $\tilde{\varphi}(z)$ of type (3.12) for all elements $\varphi(z)$, polynomial or not, of $S^{T,\mathcal{D}}(z)$. It also ensures that for any such pair $\varphi, \tilde{\varphi}$ the function :

$$\hat{\psi}(\tau) \stackrel{\text{def}}{=} \sum a_{\sigma,r}\hat{Q}_{\sigma,r}(\tau) \tag{3.19}$$

obtained after step 2, and which for each finite τ contains only a finite number of non-vanishing terms, has at most exponential growth

$$|\hat{\psi}(\tau)| \leq \text{Const}\,.e^{\kappa\tau} \qquad (\tau \in \mathbf{R}^+) \tag{3.20}$$

although the coefficients $a_{\sigma,r}$ may themselves *fail to possess exponential bounds* :

$$\limsup |a_{\sigma,r}|^{1/\sigma} \leq +\infty \tag{3.21}$$

so that, for too small a choice of ρ, we might have ended up with a $\hat{\psi}(\tau)$ of faster-than-exponential growth.

The same results hold a fortiori for *compensable functions.*

Remark 1 : Stability under composition.

The algebra $\mathrm{Ser}(z)$ is clearly stable under composition :

$$\varphi, \psi \mapsto \varphi \circ \psi \equiv \varphi(\psi) \qquad (\text{if } \psi(0) = 0) \tag{3.22}$$

The same holds, though less obviously so, for $\mathrm{Comp}(z)$.

Remark 2 : Choice of a critical "slow time".

The auxiliary variable t used in (3.16) is known in general resummation theory as a *critical slow time*. Here, it depends on ρ only. If $\kappa \stackrel{\text{def}}{=} \limsup(r/\sigma) = 0$, then any positive ρ goes. If $\kappa > 0$, then any $\rho > \kappa$ goes. But κ may be unknown, in which case one may choose a still slower time, valid for all κ. For instance :

$$\log(1/z) = t + (\log t)(\log\log t) \tag{3.23}$$

$$\log(1/z) = t + (\log t)^2. \tag{3.24}$$

Remark 3 : Justification of the term "seriable".

Most of the time, the Borel-Laplace resummation of divergent series $\tilde{\varphi}(z) = \sum a_n \epsilon_n(z)$ involves *critical times* $t = h(z)$ relative to which the monomials $e_n(t) \equiv \epsilon_n(z)$ *decrease subexponentially* as t goes to ∞. Their Borel transforms $\hat{e}_n(\tau)$ are therefore analytic on $\underset{\bullet}{\mathbb{C}}$ and, for each value of τ, calculating $\sum a_n \hat{e}_n(\tau)$ involves summing an infinite series, usually for small values of τ, and then resorting to analytic or quasianalytic continuation (See for ex. [E.5], chap 2). In the present instance, however, each monomial $e_n(t)$ has a strictly exponential rate of decrease, leading to $\hat{e}_n(\tau)$ which vanish for small $\tau > 0$ and to finite sums $\sum a_n \hat{e}_n(\tau)$ (for each *given* τ). We have therefore a much simpler procedure, under which the contribution of each single monomial e_n remains clearly individualised and can be dealt with "serially".

In the sequel, we will apply the terms *seriable* and *compensable* not only to (germs of) functions $\varphi(z)$ but also to the corresponding asymptotic series $\tilde{\varphi}(z)$.

Examples of seriable functions $\varphi(x)$ with divergent series $\tilde{\varphi}(x)$.

Example 1 :

$$\tilde{\varphi}(x) \stackrel{\text{def}}{=} \sum_{n \geq 0} n!\; x^{\boldsymbol{\sigma}^n} = \sum_{n \geq 0} n!\; x^{\sigma_0^n, \sigma_1^n, \ldots, \sigma_n^n} \tag{3.25}$$

with fixed α, β $(0 < \alpha < \beta)$ and

$$\alpha + n = \sigma_0^n \leq \sigma_1^n \leq \ldots \leq \sigma_n^n = \beta + n. \tag{3.25'}$$

The *seriability* of $\tilde{\varphi}(x)$ follows from the estimates (2.7), while its *divergence* follows from direct coefficient calculation : see the next examples.

Example 2 :

We take $\tilde{\varphi}(x)$ as above, but with $\alpha = 0,\ \beta = 1$ and with coefficients σ_i^n equally spaced in $\boldsymbol{\sigma}^n$. An immediate calculation yields the obviously divergent, "decompensated " series :

$$\tilde{\varphi}(x) = \sum_{p,q \geq 0} x^{p+q+p(p+q)^{-1}} (-1)^p (p+q)^{p+q} (p+q)! (p!)^{-1} (q!)^{-1} \tag{3.26}$$

Example 3 :

Again, we take $\tilde{\varphi}(x)$ as in (3.25) but with :

$$\alpha = 0, \beta = 1;\ \sigma_0^n = n;\ \sigma_1^n = \sigma_2^n = \ldots \sigma_n^n = 1 + n \tag{3.27}$$

Using formula (2.5) we get :

$$\tilde{\varphi}(x) = \sum (-1)^{n-1} \left[\partial_{\sigma_1}^{n-1} \left(\frac{x^{\sigma_0} - x^{\sigma_1}}{\sigma_0 - \sigma_1} \right) \right]_{\substack{\sigma_0 = n \\ \sigma_1 = n+1}} \tag{3.28}$$

and after some regrouping of terms this becomes :

$$\tilde{\varphi}(x) = \tilde{v}(x) - x\tilde{v}(x(1 + x \log x)^{-1}) \in \mathbf{C}[[x, x \log x]] \tag{3.29}$$

with

$$\tilde{v}(x) = -\sum n!\ x^{n+1} \tag{3.29'}$$

The divergent series $\tilde{v}(x)$ is ***resurgent*** in $z = x^{-1}$. It has ***sectorial sums***, which are regular only in sectorial neighbourhoods of $x = \underset{\bullet}{0}$, with finite apertures. However, the two terms on the right-hand side of (3.29) balance each other in such a way that, although $\tilde{\varphi}(x)$ is left to diverge, the resurgence in it is "compensated away", so that the sum $\varphi(x)$ is now regular in a fundamental neighbourhood $\mathcal{D}$ of $\underset{\bullet}{0}$, with infinite aperture.

This phenomenon is relevant for all ***resonant local objects*** and will be investigated in §§9,10. Indeed, putting $x = z^{-1}$ and anticipating on the notations of §9, we find :

$$\tilde{v}(x) \equiv \tilde{\mathcal{V}}^{\eta}(z) \qquad \text{with } \eta = \binom{\omega}{\sigma} = \binom{+1}{-1} \text{ with } \rho = -1 \tag{3.30}$$

$$\tilde{\varphi}(x) \equiv \tilde{\mathcal{V}}^{\eta}_{\mathrm{co}}(z) \text{ with } \rho = -1 \tag{3.31}$$

where $\tilde{\mathcal{V}}^{\eta}(z)$ is the "simplest " ***resurgent monomial***, and $\tilde{\mathcal{V}}^{\eta}_{\mathrm{co}}(z)$ the "simplest " ***compensated-resurgent monomial.***

Example 4 : The Riemann zeta function $\zeta(s)$:

Denote by $\mu(n)$ the arithmetical Möbius function and put :

$$\varphi(z) \overset{\mathrm{def}}{=} \zeta^{-1}(2^{-1} \log(z^{-1})) = \sum_{n \geq 1} \mu(n) z^{(\log n)/2} \qquad (z \in \underset{\bullet}{\mathbf{C}}) \tag{3.32}$$

$$\psi(t) \overset{\mathrm{def}}{=} \varphi(e^{-t-\rho \log t}) = \zeta^{-1}(2^{-1}(t + \rho \log t)) = \sum \mu(n) n^{-t/2-\rho(\log t)/2} \tag{3.33}$$

Clearly, the Riemann hypothesis is true ***iff*** the series $\tilde{\psi}(t)$ is Borel-Laplace summable to the ***function*** $\psi(t)$, for each given $\rho > 0$, with summation abscissa $t = 1$. The question, of course, is whether, underneath this "overconvergence", there lies hidden an expansion of $\varphi(z)$ into a series of compensators :

$$\sum_{n \geq 1} \mu(n) z^{(\log n)/2} = \sum a_{\sigma_0, \ldots, \sigma_r}\ z^{\sigma_0, \ldots, \sigma_r} \tag{3.34}$$

with the desired domain of absolute convergence, namely $|z| < e^{-1}$.

Seriable functions of several variables. Rigid algebras.

A function $\varphi(z_1,\dots,z_\nu)$ defined in a neighbourhood of $\underset{\bullet}{0} \in \underset{\bullet}{\mathbb{C}}^\nu$ respectively of the form :

$$(3.35) \qquad (\sup_i |z_i|)(\sup_i |\log z_i|)^{\kappa_0} < x_0 \qquad (0 < x_0; 0 < \kappa_0)$$

or of the form :

$$(3.36) \qquad \sup_i \,(|z_i|\; |\log z_i|)^{\kappa_i} < x_0 \qquad (0 < x_0; 0 < \kappa_i)$$

is said to be *weakly* (resp. *strongly*) seriable if for each *positive* (resp. *non-negative*) vector $\alpha = (\alpha_1,\dots,\alpha_\nu)$ with $\alpha_i > 0$ (resp $\alpha_i \geq 0$) and for any fixed $z = (z_1,\dots,z_\nu)$ in $\underset{\bullet}{\mathbb{C}}^\nu$, the function :

$$(3.37) \qquad \varphi_{\alpha,z}(t) \overset{\text{def}}{=} \varphi(z_1 t^{\alpha_1},\dots,z_\nu t^{\alpha_\nu}) \qquad (t \in \underset{\bullet}{\mathbb{C}})$$

is a seriable function of its one variable t.

A subalgebra RIG of seriable functions $\varphi(z)$ with asymptotic expansions $\tilde{\varphi}(z)$ is said to be *rigid* if all $\tilde{\varphi}(z)$ are convergent. For instance, a formal entire power series $\tilde{\varphi}(z)$ of one or several variables is automatically *convergent* if *seriable.*

4. SYMMETRAL AND SYMMETREL COMPENSATORS.

The *symmetric compensators* of the preceding sections are simplest from the theoretical point of view, but for applications we shall require *six new compensators* which, when viewed as *moulds* (see §12), turn out to be either *symmetral* or *symmetrel* (see §12) ***functions*** of the sequence $\omega = (\omega_1,\dots,\omega_r)$. Those six compensators go in pairs of mutually inverse moulds :

$$(4.1) \qquad 1^\bullet = S^\bullet_{\text{co}}(z) \times \mathcal{S}^\bullet_{\text{co}}(z) = \mathcal{S}^\bullet_{\text{core}}(z) \times \mathcal{S}^\bullet_{\text{core}}(z) = \mathcal{S}^\bullet_{\text{coim}}(z) \times \mathcal{S}^\bullet_{\text{coim}}(z)$$

relative to mould multiplication (see §12). Here *co* stands for *compensation, re* for *real, im* for *imaginary.*

The compensators from the first pair are *symmetral*; easily deducible from the symmetric compensators; and of use in the study of differential equations or vector fields.

The compensators from the second and third pairs are *symmetrel*; not reducible to finite combinations of symmetric compensators; and of use in the study of difference equations or diffeomorphisms.

A. The symmetral compensators $S^\bullet_{\mathrm{co}}(z)$ and $\mathcal{S}^\bullet_{\mathrm{co}}(z)$.

Their simplest definition is by factorisation :

$$S^\bullet_{\mathrm{co}}(z) = \mathcal{S}^\bullet(z) \times S^\bullet \qquad \text{with } \mathcal{S}^{\omega_1,\ldots,\omega_r}(z) = z^{\omega_1+\ldots\omega_r}\,\mathcal{S}^{\omega_1,\ldots,\omega_r} \tag{4.2}$$

$$\mathcal{S}^\bullet_{\mathrm{co}}(z) = \mathcal{S}^\bullet \times S^\bullet(z) \qquad \text{with } S^{\omega_1,\ldots,\omega_r}(z) = z^{\omega_1+\ldots\omega_r}\,S^{\omega_1,\ldots,\omega_r} \tag{4.3}$$

from the symmetral moulds $S^\bullet$ and $\mathcal{S}^\bullet$ introduced in (1.22), (1.23).

We also have the inductive characterisation by :

$$\begin{cases} (\|\bullet\| - z\partial_z)S^\bullet_{\mathrm{co}}(z) = -S^\bullet_{\mathrm{co}}(z) \times I^\bullet & \|\bullet\| = \|\bullet\| = \omega_1 + \ldots\omega_r \quad (4.4) \\ (\|\bullet\| - z\partial_z)\,\mathcal{S}^\bullet_{\mathrm{co}}(z) = +I^\bullet \times \mathcal{S}^\bullet_{\mathrm{co}}(z) & I^\bullet \text{ as in (12.9)} \quad (4.5) \end{cases}$$

with the initial conditions :

$$S^\bullet_{\mathrm{co}}(1) = \mathcal{S}^\bullet_{\mathrm{co}}(1) = 1^\bullet \qquad (1^\bullet \text{ as in (12.8)}) \tag{4.6}$$

or alternatively :

$$S^\bullet_{\mathrm{co}}(0) = S^\bullet ; \mathcal{S}^\bullet_{\mathrm{co}}(0) = \mathcal{S}^\bullet \qquad (\text{if } \mathrm{Re}(\omega_i) > 0) \tag{4.7}$$

The symmetral compensators also admit useful integral representations :

$$S^{\omega_1,\ldots,\omega_r}_{\mathrm{co}}(z) = (-1)^r \int z_1^{\omega_1-1}\ldots z_r^{\omega_r-1} dz_1 \ldots dz_r \tag{4.8}$$

with integration along the multipath $\{z < z_1 < \ldots < z_r < 1\}$, and

$$\mathcal{S}^{\omega_1,\ldots,\omega_r}_{\mathrm{co}}(z) = \int z_1^{\omega_1-1}\ldots z_r^{\omega_r-1} dz_1 \ldots dz_r \tag{4.9}$$

with integration along the multipath $\{z < z_r < \ldots < z_1 < 1\}$. Lastly, they are linked to the symmetric compensator of §2 by the following formulae :

$$S^{\omega_1,\ldots,\omega_r}_{\mathrm{co}}(z) = z^{0,\,\check{\omega}_1,\,\check{\omega}_2,\ldots,\,\check{\omega}_r} = z^{0,\omega_1,\omega_1+\omega_2,\ldots,\omega_1+\ldots\omega_r} \tag{4.10}$$

$$\mathcal{S}^{\omega_1,\ldots,\omega_r}_{\mathrm{co}}(z) = (-1)^r z^{\hat{\omega}_1,\,\hat{\omega}_2,\ldots,\,\hat{\omega}_r,0} = z^{\omega_1+\ldots\omega_r,\omega_2+\ldots\omega_r,\ldots,\omega_r,0} \tag{4.11}$$

B. The symmetrel compensators $\mathcal{S}^\bullet_{\text{core}}(z)$ and $\mathcal{S}\!\!\!\!-^\bullet_{\text{core}}(z)$.

Their simplest definition is by factorisation :

$$\mathcal{S}^\bullet_{\text{core}}(z) = \mathcal{S}\!\!\!\!-^\bullet_{\text{re}}(z) \times \mathcal{S}^\bullet \qquad \text{with } \mathcal{S}\!\!\!\!-^{\omega_1,\dots,\omega_r}_{\text{re}}(z) = z^{\omega_1+\dots\omega_r}\, \mathcal{S}\!\!\!\!-^{\omega_1,\dots,\omega_r} \tag{4.12}$$

$$\mathcal{S}\!\!\!\!-^\bullet_{\text{core}}(z) = \mathcal{S}\!\!\!\!-^\bullet \times \mathcal{S}^\bullet_{\text{re}}(z) \qquad \text{with } \mathcal{S}^{\omega_1,\dots,\omega_r}_{\text{re}}(z) = z^{\omega_1+\dots\omega_r}\mathcal{S}^{\omega_1,\dots,\omega_r} \tag{4.13}$$

from the symmetrel moulds $\mathcal{S}^\bullet$ and $\mathcal{S}\!\!\!\!-^\bullet$ introduced in (1.24) and (1.25).

We also have the inductive characterisation by :

$$e^{\|\bullet\|}\mathcal{S}^\bullet_{\text{core}}(z/e) = \mathcal{S}^\bullet_{\text{core}}(z) \times (1^\bullet + I^\bullet)^{-1} \tag{4.14}$$

$$e^{\|\bullet\|}\, \mathcal{S}\!\!\!\!-^\bullet_{\text{core}}(z/e) = (1^\bullet + I^\bullet) \times \mathcal{S}\!\!\!\!-^\bullet_{\text{core}}(z) \tag{4.15}$$

with the initial conditions :

$$\mathcal{S}^\bullet_{\text{core}}(1) = \mathcal{S}\!\!\!\!-^\bullet_{\text{core}}(1) = 1^\bullet \tag{4.16}$$

or alternatively

$$\mathcal{S}^\bullet_{\text{core}}(0) = \mathcal{S}^\bullet; \mathcal{S}\!\!\!\!-^\bullet_{\text{core}}(0) = \mathcal{S}\!\!\!\!-^\bullet \qquad (\text{if } \operatorname{Re}(\omega_i) > 0) \tag{4.17}$$

The preceding definitions give rise to factors $(1-\exp\ \check{\omega}_i)^{-1}$ and $(1-\exp\ \hat{\omega}_i)^{-1}$. Yet, due to internal cancellations, for any given z on $\underset{\bullet}{\mathbf{C}}$ the scalars $\mathcal{S}^{\omega_1,\dots,\omega_r}_{\text{core}}(z)$ and $\mathcal{S}\!\!\!\!-^{\omega_1,\dots,\omega_r}_{\text{core}}(z)$, as functions of the sequence $\boldsymbol{\omega} = (\omega_1,\dots,\omega_r)$, are free of singularities on $\mathbf{R}^r$ (though not on $\mathbf{C}^r$) and therefore deserve the label of "compensators". That fact is easiest to derive from the following "integral" representations (4.19) (4.20), which involve the *compensation measures* $\delta_{\boldsymbol{\sigma}}$:

$$\delta_{\boldsymbol{\sigma}}(t) = \delta_{\sigma_0,\dots,\sigma_r}(t) = \sum_{i=0}^{r} \delta_{\sigma_i}(t) \prod_{j\neq i}(\sigma_i - \sigma_j)^{-1} \qquad (\delta_{\sigma_i} = \text{ Dirac}) \tag{4.18}$$

and mirror the integral formulae (4.8) (4.9) for the symmetral compensators.

$$\mathcal{S}^{\omega_1,\dots,\omega_r}_{\text{core}}(z) = \sigma_0\sigma_1\dots\sigma_{r-1} \int t^{\log z}\delta_{\sigma_0,\dots,\sigma_r}(t) \tag{4.19}$$

with $\sigma_0 = 1$ and $\sigma_i = \exp\ \check{\omega}_i = \exp(\omega_1 + \dots\omega_i)$

(4.20)
$$\mathcal{S}_{\mathrm{core}}^{\omega_1,\ldots,\omega_r}(z) = (-1)^r \int t^{r-1+\log z}\delta_{\sigma_0,\ldots,\sigma_r}(t)$$

with $\sigma_0 = 1$ and $\sigma_i = \exp\ \overset{\wedge}{\omega}_i = \exp(\omega_i + \ldots\omega_r)$.

C. The symmetrel compensators $\mathcal{S}^{\bullet}_{\mathrm{coim}}(z)$ and $\mathcal{S}^{\bullet}_{\mathrm{coim}}(z)$.

After the symmetrel real-compensators (*core*) indexed by real sequences $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$, we shall require symmetrel imaginary-compensators (*coim*) indexed by pure imaginary sequences :

(4.21)
$$\omega = (\omega_1, \ldots, \omega_r) = (2\pi i\omega_1^*, \ldots, 2\pi i\omega_r^*) \text{ with } \omega_j^* \in \mathbf{R}^+$$

The new compensators still admit remarkable factorisations :

(4.22)
$$\mathcal{S}^{\bullet}_{\mathrm{coim}}(z) = \mathcal{S}^{\bullet}_{\mathrm{im}}(z) \times \mathcal{S}^{\bullet}$$

(4.23)
$$\mathcal{S}^{\bullet}_{\mathrm{coim}}(z) = \mathcal{S}^{\bullet} \times \mathcal{S}^{\bullet}_{\mathrm{im}}(z)$$

with the elementary, scalar-valued moulds $\mathcal{S}^{\bullet}$ and $\mathcal{S}^{\bullet}$ as in (1.24), (1.25) and with non-elementary moulds $\mathcal{S}^{\bullet}_{\mathrm{im}}(z)$ and $\mathcal{S}^{\bullet}_{\mathrm{im}}(z)$ which, up to a trivial factor, are convergent power series of z :

(4.24)
$$\mathcal{S}_{\mathrm{im}}^{\omega_1,\ldots,\omega_r}(z) \text{ and } \mathcal{S}_{\mathrm{im}}^{\omega_1,\ldots,\omega_r}(z) \in z^{-(\omega_1^*+\ldots\omega_r^*)}\mathbb{C}\{z\}$$

The moulds $\mathcal{S}^{\bullet}_{\mathrm{coim}}(z)$ and $\mathcal{S}^{\bullet}_{\mathrm{coim}}(z)$ are therefore *mixed* power series, with components drawn from several spaces $z^{-\alpha}\mathbb{C}\{z\}$. However, despite this composite character, they admit an easy, direct definition (see (4.29) below) so that equations (4.22) (4.23) should be viewed as characterising $\mathcal{S}^{\bullet}_{\mathrm{im}}(z)$ and $\mathcal{S}^{\bullet}_{\mathrm{im}}(z)$ rather than $\mathcal{S}^{\bullet}_{\mathrm{coim}}(z)$ or $\mathcal{S}^{\bullet}_{\mathrm{coim}}(z)$. To define the latter moulds, the quickest way is to introduce a linear operator θ acting on all positive powers of z according to :

(4.25)
$$\theta.z^\sigma \overset{\mathrm{def}}{=} \frac{z^\sigma}{e^{2\pi i\sigma}-1} + \frac{1}{2\pi i}\sum_{n\geq 0}\frac{z^n}{n-\sigma} \quad \text{if } \sigma \in \mathbf{R}^+ - \mathbf{N}$$

(4.26)
$$\theta.z^\sigma \overset{\mathrm{def}}{=} \frac{1}{2\pi i}\, z^\sigma \log z + \frac{1}{2\pi i}\sum_{n\geq 0; n\neq\sigma}\frac{z^n}{n-\sigma} \quad \text{if } \sigma \in \mathbf{N}$$

For any z on $\underset{\bullet}{\mathbb{C}}$ and $|z| < 1$, the series (4.25) and (4.26) converge towards a ramified function $J(z)$ equal in both cases to the integral

(4.27)
$$J(z) = -\frac{1}{2\pi i}\int_0^1 \frac{z_1^\sigma}{z_1 - z}\, dz_1 \qquad (\text{when } 0 < \arg z < 2\pi)$$

so that, for any z on $\underset{\bullet}{\mathbb{C}}$ and close to $\underset{\bullet}{0}$, we have :

$$J(e^{2\pi i}z) - J(z) \equiv z^{\sigma}. \tag{4.28}$$

Next, for imaginary ω_j and real ω_j^* as in (4.21), we put :

$$\mathscr{S}_{\mathrm{coim}}^{\omega_1,\dots,\omega_r}(z) \overset{\mathrm{def}}{\equiv} z^{-(\omega_1^*+\dots\omega_r^*)}\theta z^{\omega_1^*}\theta z^{\omega_2^*}\dots\theta z^{\omega_r^*} \tag{4.29}$$

where each θ acts on all terms to its right, and not just on its immediate neighbour. Thus $\theta z^a \theta z^b$ should really read $\theta(z^a\theta(z^b))$.

This defines the mould $\mathscr{S}_{\mathrm{coim}}^{\bullet}(z)$ and its multiplicative inverse $\mathcal{S}_{\mathrm{coim}}^{\bullet}(z)$, according to (4.1).

For the coming applications, we require the induction formulae :

$$e^{\|\bullet\|}\mathcal{S}_{\mathrm{coim}}^{\bullet}(e^{2\pi i}z) = \mathcal{S}_{\mathrm{coim}}^{\bullet}(z) \times (1^{\bullet} + I^{\bullet})^{-1} \tag{4.30}$$

$$e^{\|\bullet\|}\,\mathscr{S}_{\mathrm{coim}}^{\bullet}(e^{2\pi i}z) = (1^{\bullet} + I^{\bullet}) \times \mathscr{S}_{\mathrm{coim}}^{\bullet}(z) \tag{4.31}$$

with $\| \bullet \| = \|\omega\| = \omega_1 + \dots\omega_r$ and $1^{\bullet} + I^{\bullet}$ as in §12. Unlike in the earlier examples, however, the present induction falls far short of characterizing the mould $\mathcal{S}_{\mathrm{coim}}^{\bullet}(z)$ and $\mathscr{S}_{\mathrm{coim}}^{\bullet}(z)$, even in combination with initial conditions involving the values of those moulds for $z = 0$ or $z = 1$.

Lastly, from (4.27) and (4.29) we deduce the crucially important integral representations :

$$\mathcal{S}_{\mathrm{coim}}^{\omega_1,\dots,\omega_r}(z) = \left(\frac{1}{2\pi i}\right)^r \int_0^1 \frac{(z_1/z)^{\omega_1^*}(z_2/z)^{\omega_2^*}\dots(z_r/z)^{\omega_r^*}}{(z_1-z_2)(z_2-z_3)\dots(z_r-z)}\,dz_1\dots dz_r \tag{4.32}$$

$$\mathscr{S}_{\mathrm{coim}}^{\omega_1,\dots,\omega_r}(z) = \left(\frac{1}{2\pi i}\right)^r \int_0^1 \frac{(z_1/z)^{\omega_1^*}(z_2/z)^{\omega_2^*}\dots(z_r/z)^{\omega_r^*}}{(z-z_1)(z_1-z_2)\dots(z_{r-1}-z_r)}\,dz_1\dots dz_r \tag{4.33}$$

In (4.32) we integrate successively in $z_1, z_2, \dots, z_r$ and let each z_i bypass the *following* variables $z_{i+1}, z_{i+2}, \dots, z_r, z$ *to the right.* In (4.33) we integrate successively in $z_r, z_{r-1}, \dots, z_1$ and let each z_i bypass the *preceding* variables $z_{i-1}, z_{i-2}, \dots, z_1, z$ also *to the right.* We may take the variable z first on the segment $[\underset{\bullet}{0}, \underset{\bullet}{1}]$ and then let it range freely over the universal covering of $\underset{\bullet}{\mathbb{C}} \setminus \{\underset{\bullet}{0}, \underset{\bullet}{1}, \underset{\bullet}{\infty}\}$.

We now have at our disposal all the *compensators* which we shall require in the sequel, but we still lack the main information about them, namely their *bounds*, both *before* and *after* arborification. These bounds will be established singly, when each will first be needed, and then a general synopsis will be appended to the concluding section §11.

5. RAMIFIED LINEARISATION OF GIRATORS (SEMI-MIXED SPECTRUM).

Recall that *girators* are non-degenerate vector fields $2\pi i X$ with purely imaginary spectrum. Their study is obviously equivalent to that of vector fields X with real spectrum $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ and with the non-degeneracy condition, meaning :

(*) diagonalisable linear part

(**) no vanishing eigenvalue $(\lambda_i \neq 0)$

Three cases may occur :

(i) all λ_i have the same sign.

(ii) there is one positive (or negative) λ_i and all others have the opposite sign.

(iii) there are at least two positive and two negative λ_i.

Case (i) is entirely trivial, since it ensures analytic linearisability. Indeed, it rules out small denominators, even diophantine ones, so that the expansions (1.16) and (1.17) of the linearisators $\Theta_{\text{ent}}^{\pm 1}$ become *normally* convergent, even prior to arborification, due to the obvious bounds :

$$|S^{\boldsymbol{\omega}}| < c/r! \; ; \; |\mathcal{S}^{\boldsymbol{\omega}}| < c/r! \qquad (C = \text{Const}; \; \boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)) \tag{5.1}$$

which in case (i) improve upon (1.45), (1.46) and match with the bounds (1.39) for $\mathbb{B}_\omega$.

That leaves only case (ii), which will be dealt with in the present section, and the significantly more complex case (iii), to be studied in the next section.

So, let us for the moment consider a girator $2\pi i X$, with X of semi-mixed real spectrum $(\lambda_1, \ldots, \lambda_\nu)$. We may assume :

$$\lambda_1 < 0; \; \lambda_2 > 0, \; \lambda_3 > 0, \ldots, \; \lambda_r > 0 \tag{5.2}$$

and we easily check that there always exist analytic prepared forms (1.14) with homogeneous components $\mathbb{B}_n$ such that :

$$n_1 \geq 1/\kappa_0 > 0 \tag{5.3}$$

$$n_1 - \omega/\lambda_1 = -(n_2\lambda_2 + \ldots n_\nu\lambda_\nu)/\lambda_1 \geq 1/\kappa_0 > 0 \tag{5.4}$$

for some suitable finite constant κ_0. As usual, $n = (n_1, \ldots, n_\nu)$ and $\omega = < n, \lambda > = \Sigma n_i \lambda_i$.

Proposition 5.1. (Seriable linearisation of semi-mixed girators.)

The following expansion

$$\Theta_{\text{ser}}^{-1} = \sum_{\bullet} \mathcal{S}_{\text{co}}^{\bullet}(z_1)\mathbb{B}_{\bullet} \qquad (\text{with } z_1 = x_1^{-1/\lambda_1}) \tag{5.5}$$

relative to a prepared form of type (5.3), (5.4), defines an inverse linearisator Θ_{ser}^{-1} and is normally convergent on a spiral-like domain $\underset{\bullet}{\mathbb{C}} \times \mathbb{C}^{\nu-1}$ of the form

$$\{|x_1|\,|\log x_1|^{\kappa_0} \le x_0; \|x_2| \le x_0; \ldots, |x_\nu| \le x_0\} \tag{5.6}$$

Proof. In agreement with the general contraction rule, the sum (5.5) stands for :

$$\sum \mathcal{S}_{\text{co}}^{\omega_1,\ldots,\omega_r}(z_1)\mathbf{B}_{n_r}\ldots\mathbf{B}_{n_1} \tag{5.7}$$

which in the non-resonant case may be re-indexed to :

$$\sum \mathcal{S}_{\text{co}}^{\omega_1,\ldots,\omega_r}(z_1)\mathbf{B}_{\omega_r}\ldots\mathbf{B}_{\omega_1} \tag{5.8}$$

with $\omega_i =< n_i, \lambda >= n_{i,1}\lambda_1 + \ldots + n_{i,\nu}\lambda_\nu$.

Using the derivation rule (4.5) of symmetral compensators, one checks (1.61), so that the series (5.5) formally satisfies the conjugacy equation (1.27) and therefore defines a formal linearisator. Next, we must establish the convergence of the expansion (5.5). Reverting to the definition (4.3) of $\mathcal{S}_{\text{co}}^{\bullet}(z)$ (or its counterpart in the case of vanishing indexes ω_i) we may count the powers of x_1 in each term :

$$\mathcal{S}_{\text{co}}^{\omega_1,\ldots,\omega_r}(z_1)\; x_1^{n_{1,1}+n_{2,1}+\ldots n_{r,1}} \qquad (\text{with } n_i = (n_{i,1},\ldots,n_{i,\nu})) \tag{5.9}$$

Using $z_1 = x_1^{-1/\lambda_1}$ and discounting possible logarithmic factors $(\log x_1)^{r'}$ (with $r' \le r$) we find a finite number of terms of the form :

$$x_1^{\sigma_i} = x_1^{\sigma_i'+\sigma_i''} \qquad (i = 0,1,\ldots,\nu) \tag{5.10}$$

with

$$\sigma_i' = n_{1,1} + n_{2,1} + \ldots n_{i-1,1} \tag{5.11}$$

$$\sigma_i'' = (n_{i,1} - \omega_i/\lambda_1) + (n_{i+1,1} - \omega_{i+1}/\lambda_1) + \ldots (n_{r,1} - \omega_r/\lambda_1) \tag{5.12}$$

Using (5.3), (5.4) we find :

$$\sigma_i' \ge i/\kappa_0;\; \sigma_i'' \ge (r-i)/\kappa_0;\; \sigma_i \ge r/\kappa_0 \tag{5.13}$$

Each expression (5.9) is therefore, due to (4.11), of the form :

$$x_1^{\sigma_0,\ldots,\sigma_r} \text{ with } \sigma_i \ge r/\kappa_0 \text{ for } i = 0,1,\ldots,\nu. \tag{5.14}$$

Using the estimates (2.7), (2.8) for symmetric compensators, we find for (5.9) bounds of the form :

$$\frac{1}{r!}\left(|x_1|\left|\log\frac{1}{x_1}\right|^{\kappa_0}\right)^{r/\kappa_0} \tag{5.15}$$

which, paired with the obvious estimates

$$\left\| x_1^{-(n_{1,1}+n_{2,1}+\ldots n_{r,1})} \mathbf{B}_{n_r} \ldots \mathbf{B}_{n_2} \mathbf{B}_{n_1} \right\|_{\mathcal{D},\mathcal{D}'} \leq r!\,\mathrm{Const} \tag{5.16}$$

(for suitable polydiscs $\mathcal{D}, \mathcal{D}'$) ensure the normal convergence of the expansion (5.5) on a domain of the form (5.6).

Remark 1. If $\mathbf{B}_n.x_1 \equiv 0$ for each n (and one may always achieve this by premultiplying X by an analytic unit), the direct linearisator admits the normal convergent expansion :

$$\Theta_{\mathrm{ser}} = \sum_{\bullet} S^{\bullet}_{\mathrm{co}}(z_1) \mathbf{B}_{\bullet} \qquad (\text{with } z_1 = x_1^{-1/\lambda_1}) \tag{5.17}$$

Even without the assumption $\mathbf{B}_n.x_1 \equiv 0$, a similar expansion holds, with a slightly modified comoulds $\mathbf{B}_{\bullet}$.

Remark 2. For a non-resonant spectrum, the constant κ_0 in (5.3), (5.4) may be chosen as small as we wish. This is a favourable circumstance, since κ_0 turns out to measure the "spiralling speed" of the ramified domain (3.2) where seriable linearisation holds.

Remark 3. For a resonant spectrum, with one or several degrees of resonance, there is a limit to the smallness of the acceptable constants κ_0, due to the generic presence of *unremovable resonant monomials* x^m (unremovable, that is, under formal, entire changes of coordinates). In that case, there is usually *one* optimal domain (3.2), which cannot be improved upon, and where seriable linearisation holds.

Remark 4. For a canonical factorisation of the seriable linearisator Θ_{ser}, see §8 in the non-resonant case and §10 in the resonant case.

6. RAMIFIED LINEARISATION OF GIRATORS (GENERAL SPECTRUM).

Consider as before a girator $2\pi i X$, but assume now that X has more than one positive λ_i and more than one negative λ_i. Formula (5.5) for $\Theta_{\mathrm{ser}}^{-1}$ continues to make formal sense, but becomes unsatisfactory from the point of view of analysis, because it entails *negative* powers of x_1, irrespective of whether λ_1 is positive or negative. In order to end up with positive powers only, we must "compensate" or "seriate" not in one, but in at least two variables.

Let λ_1 be one of the *negative* λ_i and λ_2 one of the *positive* λ_i and put :

$$z_1 = x_1^{-1/\lambda_1} \; z_2 = x_2^{-1/\lambda_2} \qquad (\lambda_1 < 0 < \lambda_2) \tag{6.1}$$

Instead of considering compensators $\mathcal{S}^{\bullet}_{\mathrm{co}}(z_1)$ of z_1 only, we shall use *mixed compensators* $\mathcal{S}^{\bullet}_{\mathrm{co}}(z_1, z_2)$ which will involve only positive powers of z_1 and negative powers of z_2, that is to say only ***positive powers of both*** x_1 ***and*** x_2, but which will otherwise retain all the useful properties of $\mathcal{S}^\bullet_{\text{co}}(z_1)$ and $\mathcal{S}^\bullet_{\text{co}}(z_2)$. The passage from $\mathcal{S}^\bullet_{\text{co}}(z_1)$ and $\mathcal{S}^\bullet_{\text{co}}(z_2)$ to $\mathcal{S}^\bullet_{\text{co}}(z_1,z_2)$ is achieved by an important mould-theoretical operation, known as *mixing*.

Definition 6.1. (Mould mixing)

To any pair $A^\bullet, B^\bullet$ of moulds with real indices ω_i, the mixing associates a mould $C^\bullet$ defined by :

$$C^{\omega_1,\dots,\omega_r} \stackrel{\text{def}}{=} \sum_{\left\{\substack{0\le m\le r\\ \pi\in\Sigma_r}\right.} H^{\omega_1,\dots,\omega_r}_{\pi,m} \tilde{B}^{\omega_{\pi(1)},\dots,\pi_{\pi(m)}} A^{\omega_{\pi(m+1)},\dots,\pi_{\pi(r)}} \tag{6.2}$$

with a sum extending to all permutations π of the set $\{1,\dots,r\}$ and involving the conjugate mould $\tilde{B}$:

$$\tilde{B}^{\omega_1,\dots,\omega_r} \stackrel{\text{def}}{=} (-1)^r B^{\omega_r,\dots,\omega_1} \tag{6.3}$$

as well as universal ***disorder coefficients*** *:*

$$H^{\omega_1,\dots,\omega_r}_{\pi,m} = s_{\epsilon_1}(\hat{\omega}_1) s_{\epsilon_2}(\hat{\omega}_2)\dots s_{\epsilon_r}(\hat{\omega}_r) \tag{6.4}$$

which assume only the values $0, +1, -1$ and are made up of the following ingredients :

$$\hat{\omega}_i \stackrel{\text{def}}{=} \omega_i + \omega_{i+1} + \dots \omega_r \tag{6.5}$$

$$s_+(\omega) \stackrel{\text{def}}{=} \begin{cases} 1 & if\ \omega \ge 0 \\ 0 & if\ \omega < 0 \end{cases} \qquad s_-(t) \stackrel{\text{def}}{=} \begin{cases} 0 & if\ \omega > 0 \\ -1 & if\ \ \omega \le 0 \end{cases} \tag{6.6}$$

$$\epsilon_1 \stackrel{\text{def}}{=} \begin{cases} + & if\ \ m < \pi^{-1}(1) \\ - & if\ \ \pi^{-1}(1) \le m \end{cases} \tag{6.7}$$

$$\epsilon_i \stackrel{\text{def}}{=} \begin{cases} + & if\ \ \pi^{-1}(i-1) < \pi^{-1}(i) \\ - & if\ \ \pi^{-1}(i) < \pi^{-1}(i-1) \end{cases} \qquad (for\ i > 1) \tag{6.8}$$

$C^\bullet$ is called the "mixture" of $A^\bullet$ and $B^\bullet$, and denoted by ($A^\bullet$ mix $B^\bullet$).

Lemma 6.1 (Main properties of mould mixing)

$$\{A^\bullet \text{ and } B^\bullet \text{ symmetral}\} \Rightarrow \{A^\bullet \text{ mix } B^\bullet \text{ symmetral}\} \tag{6.9}$$

$$\{A^\bullet \text{ symmetral}\} \Rightarrow \{A^\bullet \text{ mix} A^\bullet \ = A^\bullet\} \tag{6.10}$$

$$(C^\bullet \times A^\bullet) \text{ mix } (C^\bullet \times B^\bullet) = C^\bullet \times (A^\bullet \text{ mix } B^\bullet) \tag{6.11}$$

Moreover, the mixing "separates signs" in the sense that the definition (6.2) of $C^{\omega_1,\ldots,\omega_r}$ involves only terms $A^{\alpha_1,\ldots,\alpha_{r_1}}$ and $B^{\beta_1,\ldots,\beta_{r_2}}$ such that :

$$\overset{\wedge}{\alpha}_i \overset{\text{def}}{=} \alpha_i + \ldots \alpha_{r_1} \geq 0 \quad (\forall i \leq r_1) \text{ and } \overset{\wedge}{\beta}_i \overset{\text{def}}{=} \beta_i + \ldots \beta_{r_2} < 0 \quad (\forall i \leq r_2) \tag{6.12}$$

The shortest way to prove the above lemma is by introducing on the free commutative algebra generated by the symmetral symbols A^ω and B^ω a *derivation operator D* and an infinity of *"integration" operators $I_\pm^{\omega_0}$* acting as follows :

$$DA^\bullet \overset{\text{def}}{=} I^\bullet \times A^\bullet;\ DB^\bullet \overset{\text{def}}{=} I^\bullet \times B^\bullet;\ D\tilde{B}^\bullet \overset{\text{def}}{=} -\tilde{B}^\bullet \times I^\bullet$$

$$D(\tilde{B}^{\beta_1,\ldots,\beta_r}A^{\alpha_1,\ldots,\alpha_s}) \overset{\text{def}}{=} \tilde{B}^{\beta_1,\ldots,\beta_r}A^{\alpha_2,\ldots,\alpha_s} - \tilde{B}^{\beta_1,\ldots,\beta_{r-1}}A^{\alpha_1,\ldots,\alpha_s}$$

$$I_+^{\omega_0}(\tilde{B}^{\beta_1,\ldots,\beta_r}A^{\alpha_1,\ldots,\alpha_s}) \overset{\text{def}}{=} \sum_{0\leq i\leq r} \tilde{B}^{\beta_1,\ldots,\beta_i}A^{\beta_{i+1},\ldots,\beta_r,\omega_0,\alpha_1,\ldots,\alpha_s}$$

$$I_-^{\omega_0}(\tilde{B}^{\beta_1,\ldots,\beta_r}A^{\alpha_1,\ldots,\alpha_s}) \overset{\text{def}}{=} \sum_{0\leq i\leq s} \tilde{B}^{\beta_1,\ldots,\beta_r,\omega_0,\alpha_1,\ldots,\alpha_i}A^{\alpha_{i+1},\ldots,\alpha_s}$$

and then to observe that $I_+^{\omega_0}$ and $I_-^{\omega_0}$ reverse D :

$$DI_+^{\omega_0} = DI_-^{\omega_0} = 1 \qquad (\forall \omega_0)$$

and that (6.2) may be written :

$$C^{\omega_1,\ldots,\omega_r} = \sum_{\epsilon_i=\pm} s_{\epsilon_1}(\overset{\wedge}{\omega}_1)\ldots s_{\epsilon_r}(\overset{\wedge}{\omega}_r)C^{\omega_1,\ldots,\omega_r}_{\epsilon_1,\ldots,\epsilon_r}$$

with the following induction :

$$DC^{\omega_1,\ldots,\omega_r}_{\epsilon_1,\ldots,\epsilon_r} = \epsilon_1 C^{\omega_2,\ldots,\omega_r}_{\epsilon_2,\ldots,\epsilon_r}$$

$$C^{\omega_1,\ldots,\omega_r}_{\epsilon_1,\ldots,\epsilon_r} = \epsilon_1 I^{\omega_1}_{\epsilon_1} C^{\omega_2,\ldots,\omega_r}_{\epsilon_2,\ldots,\epsilon_r}$$

The details are formal and left to the reader.

Lemma 6.2. (Definition and properties of mixed compensators)

$$S^\bullet_{\text{co}}(z_1,z_2) \times \mathcal{S}^\bullet_{\text{co}}(z_1,z_2) = 1^\bullet \tag{6.13}$$

$$\mathcal{S}^\bullet_{\text{co}}(z_1,z_2) \overset{\text{def}}{=} \mathcal{S}^\bullet_{\text{co}}(z_1) \text{ mix } \mathcal{S}^\bullet_{\text{co}}(z_2) = \mathcal{S}^\bullet \times (S^\bullet(z_1) \text{ mix } S^\bullet(z_2)) \tag{6.14}$$

$$(6.15) \qquad (\omega_1 + \ldots \omega_r - z_1\partial_{z_1} - z_2\partial_{z_2}) S_{co}^{\omega_1,\ldots,\omega_r}(z_1,z_2) = -S_{co}^{\omega_1,\ldots,\omega_{r-1}}(z_1,z_2)$$

$$(6.16) \qquad (\omega_1 + \ldots \omega_r - z_1\partial_{z_1} - z_2\partial_{z_2}) \mathcal{S}_{co}^{\omega_1,\ldots,\omega_r}(z_1,z_2) = \mathcal{S}_{co}^{\omega_2,\ldots,\omega_r}(z_1,z_2)$$

For any sequence $\omega = (\omega_1, \ldots, \omega_r)$, of whatever signs, the mixed compensators $S_{co}^{\omega}(z_1,z_2)$ and $\mathcal{S}_{co}^{\omega}(z_1,z_2)$ involve only positive powers of z_1 and negative powers of z_2.

Proof : (6.13) defines $S_{co}^{\bullet}(z_1,z_2)$ indirectly in terms of $\mathcal{S}_{co}^{\bullet}(z_1,z_2)$ and (6.14) defines $\mathcal{S}_{co}^{\bullet}(z_1,z_2)$ directly in terms of the one-variable compensators $\mathcal{S}_{co}^{\bullet}(z_i)$ or the even more elementary moulds $S^{\bullet}(z_i)$, defined as in (4.3). The equivalence between the last two terms of (6.14) follows from (6.11) and (4.3). The derivation rules (6.15) and (6.16) follow from the derivation rules (4.4) and (4.5) applied to the identity (6.2) with the symmetral moulds :

$$(6.17) \qquad A^{\bullet}, B^{\bullet}, \tilde{B}^{\bullet}$$

replaced by

$$(6.18). \qquad \mathcal{S}_{co}^{\bullet}(z_1),\ \mathcal{S}_{co}^{\bullet}(z_2),\ S_{co}^{\bullet}(z_2)$$

Lastly, the concluding remark in Lemma 6.2 about the "separation of signs" in mixed compensators follows from (6.12) with (6.18) in place of (6.17).

Now, let us return to a general *girator* $2\pi i X$ with truly mixed spectrum. As in the preceding section, one establishes the existence of prepared analytic forms (1.14) with homogeneous components $\mathbf{B}_n$ such that :

$$(6.19) \qquad \sum_{\lambda_i<0} n_i \geq -\lambda_1/\kappa_1 > 0 \qquad (\lambda_1 < 0)$$

$$(6.20) \qquad \sum_{\lambda_i>0} n_i \geq +\lambda_2/\kappa_2 > 0 \qquad (\lambda_2 > 0)$$

for some suitable finite constants κ_1, κ_2. The reason for this is that one may go from any analytic form (1.14) to one satisfying (6.19), (6.20) by means of a formal change of variables which involves no small denominators (not even diophantine) and is therefore not merely formal, but analytic.

Proposition 6.1. (Seriable linearisation of general girators)

The following expansion

$$(6.21) \qquad \Theta_{ser}^{-1} = \sum \mathcal{S}_{co}^{\bullet}(z_1,z_2)\mathbf{B}_{\bullet}$$

with z_1, z_2 as in (6.1), with a mixed compensator $\mathcal{S}^{\bullet}_{\mathrm{co}}(z_1, z_2)$ as in (6.14) and with a comould $\mathbf{B}_{\bullet}$ relative to a prepared form of type (6.19)(6.20), defines an inverse linearisator $\Theta^{-1}_{\mathrm{ser}}$ which involves only positive powers of all variables x_i (and of $\log x_1, \log x_2$ in case of resonance). Such as it stands, the expansion (6.21) generally fails to converge in norm, but its arborified counterpart :

$$\Theta^{-1}_{\mathrm{ser}} = \sum \mathcal{S}^{\overset{<}{\bullet}}_{\mathrm{co}}(z_1, z_2)\mathbf{B}_{\underset{\bullet}{<}} \tag{6.22}$$

is always normally convergent on a spiral-like domain of $\underset{\bullet}{\mathbf{C}}^2 \times \mathbf{C}^{\nu-2}$ of the form

$$\left\{ |\log x_1|^{\kappa_1} \sup_{\lambda_i<0} |x_i| \leq x_0 \right\} \times \left\{ |\log x_2|^{\kappa_2} \sup_{\lambda_i>0} |x_i| \leq x_0 \right\} \tag{6.23}$$

Proof of Proposition 6.1 :

As usual, the contracted sum (6.21) stands for :

$$\sum \mathcal{S}^{\omega_1,\ldots,\omega_r}_{\mathrm{co}}(z_1, z_2)\mathbf{B}_{n_r} \ldots \mathbf{B}_{n_1} \tag{6.24}$$

with $\omega_i =< n_i, \lambda >$. Recalling the definition (6.1) of z_1, z_2, we see that :

$$X^{\mathrm{lin}}\, \mathcal{S}^{\bullet}_{\mathrm{co}}(z_1, z_2) = -(z_1\partial_{z_1} + z_2\partial_{z_2})\, \mathcal{S}^{\bullet}_{\mathrm{co}}(z_1, z_2) \tag{6.25}$$

Using the derivation rule (6.16), we check (1.61), so that the series (6.24) formally satisfies the conjugacy equation (1.27) and therefore defines a formal linearisator.

But when we turn to the question of convergence, we find that mixed compensators no longer admit good estimates of types (5.15) with $r!$ as a denominator. Indeed, for $|x_1|$ and $|x_2|$ small enough, we have :

$$\mathcal{S}^{\omega_1,\ldots,\omega_r}_{\mathrm{co}}(z_1, z_2) = \sum_{\pi,m} H^{\omega_1,\ldots,\omega_r}_{\pi,m}\, \mathcal{S}^{\omega_{\pi(1)},\ldots,\omega_{\pi(m)}}_{\mathrm{co}}(z_2)\, \mathcal{S}^{\omega_{\pi(m+1)},\ldots,\omega_{\pi(r)}}_{\mathrm{co}}(z_1) \tag{6.26}$$

$$|\mathcal{S}^{\omega_1,\ldots,\omega_r}_{\mathrm{co}}(z_1, z_2)| \leq \sum_{\pi,m} \left|\mathcal{S}^{\omega_{\pi(1)},\ldots,\omega_{\pi(m)}}_{\mathrm{co}}(z_2)\right| \cdot \left|\mathcal{S}^{\omega_{\pi(m+1)},\ldots,\omega_{\pi(r)}}_{\mathrm{co}}(z_1)\right| \tag{6.27}$$

$$|\mathcal{S}^{\omega_1,\ldots,\omega_r}_{\mathrm{co}}(z_1, z_2)| \leq \sum_{\pi,m} (|\log x_2|^m/m!)\ \ (|\log x_1|^{r-m}/(r-m)!)) \tag{6.27bis}$$

For badly mixed sequence ω_i (the worst being the alternate sequence $(-1)^i\omega_i > 0$) the sum (6.27) may contain more than $r!2^{-r}$ terms, but of course less than $(r+1)!$, so that we get bounds of type :

$$|\mathcal{S}_{\text{co}}^{\omega_1,\ldots,\omega_r}(z_1,z_2)| \leq C_0 C_1^r(|\log x_1| + |\log x_2|)^r \tag{6.28}$$

Using the special case when the indices ω_i are very small, plus some combinatorics, one verifies that the estimates (6.28) are essentially sharp. In view of the equally sharp bounds (1.39) for $\mathbf{B}_\bullet$, this means that the expansion (6.21) is, generally speaking, divergent in norm. So we must take recourse to arborification and use the following lemma :

Lemma 6.3 (Mixing and arborification)

Take $A^\bullet, B^\bullet$ and their mixture $C^\bullet$ as defined in (6.2). Let $\overset{<}{C}{}^\bullet$ denote the arborification of $C^\bullet$ as defined in (12.17). Then, for any arborescent sequence $\overset{<}{\omega} = (\omega_1,\ldots,\omega_r)^<$ we have

$$C^{\overset{<}{\omega}} = \sum_{\left\{\substack{0\leq m\leq r\\ \pi\in\Sigma_r}\right.} H_{\pi,m}^{\overset{<}{\omega}} \tilde{B}^{\omega_{\pi(1)},\ldots,\omega_{\pi(m)}}\ A^{\omega_{\pi(m+1)},\ldots,\omega_{\pi(r)}} \tag{6.29}$$

with coefficients $H_{\pi,m}^{\overset{<}{\omega}}$ still defined by :

$$H_{\pi,m}^{\overset{<}{\omega}} = s_{\epsilon_1}(\hat{\omega}_1) s_{\epsilon_2}(\hat{\omega}_2)\ldots s_{\epsilon_r}(\hat{\omega}_r) \tag{6.30}$$

but with partial sums $\hat{\omega}_i$ now relative to the arborescent order of $\overset{<}{\omega}$:

$$\hat{\omega}_i \overset{\text{def}}{=} \sum \omega_j \ \ for\ all\ j\ equal\ or\ posterior\ to\ i\ in\ \overset{<}{\omega}. \tag{6.31}$$

and with signs ϵ_i reflecting the compatibility, or otherwise, of the substitution π with the arborescent order $\overset{<}{\omega}$

$$\epsilon_i \overset{\text{def}}{=} \begin{cases} + & if\ \ m < \pi^{-1}(i) \\ - & if\ \ \pi^{-1}(i) \leq m \end{cases} \qquad (i\ root\ of\ \overset{<}{\omega}\ ) \tag{6.32}$$

$$\epsilon_i \overset{\text{def}}{=} \begin{cases} + & if\ \pi^{-1}(i_-) < \pi^{-1}(i) \\ - & if\ \pi^{-1}(i) < \pi^{-1}(i_-) \end{cases} \qquad (i_-\ antecedent\ of\ i) \tag{6.33}$$

*(6.33) holds when i has a (necessarily unique) antecedent i_- in $\overset{<}{\omega}$ and (6.32) holds when i is a **least element**, or **root**, of $\overset{<}{\omega}$.*

The verification of the above lemma is straightforward, by induction on the length r of $\overset{<}{\omega}$, but its significance is startling. Indeed, the right-hand sides of (6.2) and (6.29) contain exactly the same number of terms, namely $(r+1)!$, and both sums involve coefficients $H^{\boldsymbol{\omega}}_{\pi,m}$ or $H^{\overset{<}{\boldsymbol{\omega}}}_{\pi,m}$ which assume only the values $0,+1,-1$. However, the arborification construction (12.17) implies :

$$C^{\overset{<}{\boldsymbol{\omega}}} = \sum \mathrm{sh}\begin{pmatrix} \overset{<}{\boldsymbol{\omega}} \\ \boldsymbol{\omega} \end{pmatrix} C^{\boldsymbol{\omega}} \tag{6.34}$$

$$H^{\overset{<}{\boldsymbol{\omega}}}_{\pi,m} = \sum \mathrm{sh}\begin{pmatrix} \overset{<}{\boldsymbol{\omega}} \\ \boldsymbol{\omega} \end{pmatrix} H^{\boldsymbol{\omega}}_{\pi,m} \tag{6.35}$$

with sums Σ involving on average $r!2^{-r}$ terms. Now, what the lemma 6.3 says, in essence, is that in spite of this the $C^{\overset{<}{\boldsymbol{\omega}}}$ and $H^{\overset{<}{\boldsymbol{\omega}}}_{\pi,m}$ are not significantly larger than the unarborified $C^{\boldsymbol{\omega}}$ and $H^{\boldsymbol{\omega}}_{\pi,m}$

If we now apply (6.29) to the triplet (6.18) in place of (6.17), we get for the mixed compensators the following estimates :

Lemma 6.4. *For $|z_1| \ll 1 \ll |z_2|$ and real indices ω_i :*

$$|S^{\overset{<}{\boldsymbol{\omega}}}_{co}(z_1,z_2)| \leq C_0 C_1^r (|\log z_1| + |\log z_2|)^r \tag{6.36}$$

$$|\mathcal{S}^{\overset{<}{\boldsymbol{\omega}}}_{co}(z_1,z_2)| \leq C_0 C_1^r (|\log z_1| + |\log z_2|)^r \tag{6.37}$$

which in view of (6.1) are ***no worse*** than the estimates (6.28). But these have to be paired with the estimates for $\mathbf{B}_{\overset{<}{\bullet}}$:

$$\|x^{-(n_1+\dots n_r)}\mathbf{B}_{(n_1,\dots,n_r)^<}\|_{\mathcal{D}_1,\mathcal{D}_2} < C^r \tag{6.38}$$

which are much better than the corresponding estimates for $\mathbf{B}_\bullet$. Taking the "preparation" (6.19), (6.20) into account, we immediately obtain the normal convergence of the arborified expansion (6.22), and therefore of the seriable linearisators $\Theta^{\pm 1}_{\mathrm{ser}}$, on domains of the form (6.23), which ends the proof of Proposition 6.1.

Remark 1. By regrouping in (6.21) all terms $\mathcal{S}^\bullet_{\mathrm{co}}\,\mathbf{B}_\bullet$ relative to sequences $\boldsymbol{\omega} = (\omega_1,\dots,\omega_r)$ that differ only by the order of the ω_i, we can restore ***normal convergence*** in a seemingly simpler way than through arborification. Such regroupings are indeed useful when it comes to actually ***calculating*** the seriable linearisators, but for the purpose of ***proving*** normal convergence, they are "too large" for an easy identification of cancellations, and there is no convenient alternative to arborification.

Remark 2. If there exists an analytic ***prepared form*** such that :

$$\mathbf{B}_n.x_1 = \mathbf{B}_n.x_2 = 0 \qquad (\forall x) \tag{6.39}$$

(which is exceptional, even after premultiplication of X by an analytic unit), then the direct linearisator admits the expansion :

$$\Theta_{\text{ser}} = \sum S^{\bullet}_{\text{co}}(z_1, z_2)\mathbf{B}_{\bullet} = \sum \quad S^{\overset{<}{\bullet}}_{\text{co}}(z_1, z_2)\mathbf{B}_{\underset{\bullet}{<}} \tag{6.40}$$

with normal convergence of the ***arborified*** sum (to the right). Even without assuming (6.39), there exists a variant of (6.40), with a slightly modified comould $\mathbf{B}_{\bullet}$.

Remark 3. For a non-resonant spectrum, seriable linearisation involves positive real powers of x_1, x_2 and positive integral powers of the remaining x_i. The constants κ_1 and κ_2 in (6.19) (6.20) may be chosen as small as we wish, leading to excellent domains of seriable linearisation.

Remark 4. For a resonant spectrum, seriable linearisation also involves logarithms of x_1 and x_2. The constants κ_1 and κ_2 cannot, generally speaking, be chosen smaller than a certain optimal size, which depends on the resonance relations.

Remark 5. For a canonical factorisation of the seriable linearisator Θ_{ser}, see §8 in the non-resonant case and §10 in the resonant case.

Remark 6. There exist other explicit seriable linearisations, with expansions of type (6.21), but with better domains of convergence, namely domains of type (3.36) instead of type (3.35) as in (6.23). However, these fine-tuned linearisations involve more than two ramified variables $z_i = x_i^{-1/\lambda_i}$ and rely on higher-order compensators $S^{\bullet}_{\text{co}}(z_1, z_2, z_3, \ldots)$ obtained by repeated use of "mould mixing" (see lemma 6.1).

7. RAMIFIED LINEARISATION OF GIRATIONS.

Unlike *girators*, whose study reduces to that of vector fields with real spectrum, *girations* (that is to say diffeomorphisms with multipliers of modulus 1) differ markedly from diffeomorphisms with real eigenvalues.

Let us first consider a one-dimensional giration :

$$f : x \mapsto \ell x + \sum_{n \geq 1} a_n x^{n+1}; \qquad (\mathbb{C}, 0) \to (\mathbb{C}, 0) \tag{7.1}$$

and let us lift f to a giration $\underset{\bullet}{f}$ of $(\underset{\bullet}{\mathbb{C}},\underset{\bullet}{0})$:

$$(7.2) \qquad \underset{\bullet}{f}\colon x \mapsto \underset{\bullet}{\ell}\, x + \sum_{n\geq 1} a_n x^{n+1}; \qquad (\underset{\bullet}{\mathbb{C}},\underset{\bullet}{0}) \to (\underset{\bullet}{\mathbb{C}},\underset{\bullet}{0})$$

by choosing for $\ell \in \mathbb{C}$ a determination $\underset{\bullet}{\ell} \in \underset{\bullet}{\mathbb{C}}$:

$$(7.3) \qquad \underset{\bullet}{\ell} = e^{\lambda} = e^{2\pi i \lambda^*} \qquad (\lambda^* \in \mathbf{R}^+)$$

It is essential for our purpose that λ^* should be $\neq 0$. Let us assume for definiteness that it is positive. With f and $\underset{\bullet}{f}$ we associate the substitution operators F and $\underset{\bullet}{F}$ with their (common) homogeneous parts $\mathbf{B}_n$ as in (1.5).

Of course, linearisation (resp. normalisation) difficulties arise for f only if ℓ is liouvillian (resp. a root of unity), but for the lifted $\underset{\bullet}{f}$ seriable linearisation becomes possible as soon as $\underset{\bullet}{\ell} \neq \underset{\bullet}{1}$, which can always be achieved, even if $\ell = 1$.

Proposition 7.1 (Seriable linearisation of one-dimensional girations)

The following expansions :

$$(7.4) \qquad \Theta_{\mathrm{ser}}^{-1} = \sum \mathcal{S}^{\bullet}_{\mathrm{coim}}(z/u)\, \mathbf{B}_{\bullet} = \sum \mathcal{S}^{\overset{<}{\bullet}}_{\mathrm{coim}}(z/u)\mathbf{B}_{\underset{\bullet}{<}}$$

with the symmetrel compensators introduced in (4.23) and :

$$(7.4bis) \qquad z = x^{1/\lambda^*};\ \lambda_* > 0;\ u \textit{ fixed on } \underset{\bullet}{\mathbb{C}} \textit{ but close enough to } \underset{\bullet}{0}$$

$$(7.4ter) \qquad \mathcal{S}^{\bullet}_{\mathrm{coim}} = \mathcal{S}^{\omega_1,\ldots,\omega_r}_{\mathrm{coim}};\ \mathbf{B}_{\bullet} = \mathbf{B}_{n_r}\ldots\mathbf{B}_{n_1};\ \omega_j = \lambda n_j = 2\pi i \lambda^* n_j$$

define an inverse linearisator of the giration $\underset{\bullet}{f}$. The unarborified expansion (in the middle) is almost never normally convergent, but its arborified counterpart (to the right) always is, on spiral-like domains of the form :

$$(7.5) \qquad \{|x|\cdot|\log x|^{\kappa_0} \leq x_0|\log x_0|^{\kappa_0}\} \qquad (x \in \underset{\bullet}{\mathbb{C}},\ 0 < x_0 < e^{-\kappa_0} < 1)$$

Proof of Proposition 7.1.

That $\Theta_{\mathrm{ser}}^{-1}$ as defined by (7.4) satisfies the linearisation equation (1.28) directly follows from (1.62) and (4.15). So the real issue is normal convergence. For this, we need :

Lemma 7.1. *Consider indices $\omega_j^* > 0$ and $\omega_j = 2\pi i \omega_j^*$, and define a mould $Q^\bullet$ by*

$$Q^{\omega_1,\ldots,\omega_r}(z) = z^{\omega_1^*+\ldots\omega_r^*}\,\mathcal{S}^{\omega_1,\ldots,\omega_r}_{\mathrm{coim}}(z) = \theta z^{\omega_1^*}\ldots\theta z^{\omega_r^*} \tag{7.6}$$

with θ as in (4.27) and $\mathcal{S}^\bullet_{\mathrm{coim}}$ as in (4.29). The mould $Q^\bullet$ and its symmetrel arborification $Q^{\overset{<}{\bullet}}$ (see (12.19)) admit essentially the same bounds, namely :

$$|Q^{\boldsymbol{\omega}}(z)| < (C(\epsilon))^r(|z\log z| + |\log(1-\overset{\bullet}{z})|)^r \tag{7.7}$$

$$|Q^{\overset{<}{\boldsymbol{\omega}}}(z)| < (C(\epsilon))^r(|z\log z| + |\log(1-\overset{\bullet}{z})|)^r \tag{7.8}$$

for $|z| < 1$ and $0 < \epsilon < \omega_j^*$ $\quad(z \in \underset{\bullet}{\mathbf{C}};\ \overset{\bullet}{z} \in \mathbf{C})$.

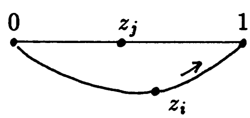
This is unexpected, since in view of (12.19), the $Q^{\overset{<}{\boldsymbol{\omega}}}$ are sums of many terms $Q^{\boldsymbol{\omega}}$, sometimes as many as $r!$ or even more, due to possible sequence-contractions (12.7). However, as usual with mould arborification, there happen to exist handy identities which explain why $Q^{\overset{<}{\boldsymbol{\omega}}}$ is no larger than $Q^{\boldsymbol{\omega}}$. They read :

$$Q^{\boldsymbol{\omega}}(z) = \int_0^1 \prod_{i=1}^{r} (z_i^{\omega_i^*}(z_{i-1} - z_i)^{-1}dz_i) \qquad (z_0 \overset{\mathrm{def}}{=} z) \tag{7.9}$$

$$Q^{\overset{<}{\boldsymbol{\omega}}}(z) = \int_0^1 \prod_{i=1}^{r} (z_i^{\omega_i^*}(z_{i_-} - z_i)^{-1}dz_i) \qquad (z_0 \overset{\mathrm{def}}{=} z) \tag{7.10}$$

In (7.10) i_- denotes the unique antecedent of i within the arborescent order $\overset{<}{\boldsymbol{\omega}}$ or, if i has no antecedent (if it is a *root* or *least element*), we put $i_- \overset{\mathrm{def}}{=} 0$ and $z_0 \overset{\mathrm{def}}{=} z$. In both (7.9) and (7.10) we integrate first in those variables z_i whose index i is largest and we systematically circumvent the still unused variables z_j (j prior to i within $\boldsymbol{\omega}$ or $\overset{<}{\boldsymbol{\omega}}$) to the right, like this :

(7.11)



Identity (7.9) is merely a rewriting of identity (4.33) in terms of $Q^{\boldsymbol{\omega}}$. Identity (7.10) follows from (7.9) by putting :

$$M^{z_1,\ldots,z_r}(z) \overset{\mathrm{def}}{=} (z-z_1)^{-1}(z_1-z_2)^{-1}\ldots(z_{r-1}-z_r)^{-1} = M^{z_1-z,\ldots,z_r-z}(0) \tag{7.12}$$

and by observing that the coefficients $M^\bullet$, regarded as *functions*, define a ***symmetral mould*** (see §12) but when regarded as *functionals* on the space of analytic functions on $[0,1]^r$:

$$M^\bullet : \varphi \mapsto \int_0^1 \varphi(z_1,\dots,z_r) M^{z_1,\dots,z_r}(z)dz_1\dots dz_r \tag{7.13}$$

with integration rule (7.11)), those same coefficients define a ***symmetrel mould*** (see §12). In other words, the *function* $M^\bullet$ satisfies identities of type (12.6) whereas the *functional* $M^\bullet$ satisfies identities of type (12.7). This difference stems from the fact that the integration rule (7.11) is not preserved under reshufflings of the sequence $(z_1,\dots,z_r)$, which calls for corrective terms corresponding precisely to sequence-contractions (12.7). Using (7.9) and (7.10) we now find the induction formulae :

$$Q^{\omega_1\bullet\boldsymbol{\omega}}(z) = \int_0^1 z_1^{\omega_1^*}\cdot(z-z_1)^{-1}Q^{\boldsymbol{\omega}}(z_1)dz_1 \tag{7.14}$$

$$Q^{\omega_1\bullet\overset{<}{\boldsymbol{\omega}}}(z) = \int_0^1 z_1^{\omega_1^*}\cdot(z-z_1)^{-1}Q^{\overset{<}{\boldsymbol{\omega}}}(z_1)dz_1 \tag{7.15}$$

$$Q^{\overset{<}{\boldsymbol{\omega}'}\oplus\overset{<}{\boldsymbol{\omega}''}\oplus\dots}(z) = Q^{\overset{<}{\boldsymbol{\omega}'}}(z)Q^{\overset{<}{\boldsymbol{\omega}''}}(z)\dots \tag{7.16}$$

with right-side circumvention of z by z_1 (for z on $[\dot{0},\dot{1}]$). Here, $\omega_1\bullet\boldsymbol{\omega}$ (resp. $\omega_1\bullet\overset{<}{\boldsymbol{\omega}}$) denotes the fully ordered sequence (resp. the arborescent sequence) consisting of ω_1 followed by the fully ordered sequence $\boldsymbol{\omega}$ (resp. the arborescent sequence $\overset{<}{\boldsymbol{\omega}}$); and $\overset{<}{\boldsymbol{\omega}'}\oplus\overset{<}{\boldsymbol{\omega}''}\dots$ denotes the arborescent sequence obtained by juxtaposition of $\overset{<}{\boldsymbol{\omega}'},\overset{<}{\boldsymbol{\omega}''}\dots$. From (7.14) (7.15) (7.16) we infer (7.7) and (7.8) by induction on r, which proves lemma 7.1.

We may now revert to the expansions (7.4) and rewrite them as :

$$\sum(u^{\|\boldsymbol{\omega}^*\|}Q^{\boldsymbol{\omega}}(z/u))(x^{-\|n\|}\mathbf{B}_n) = \sum(u^{\|\boldsymbol{\omega}^*\|}Q^{\overset{<}{\boldsymbol{\omega}}}(z/u))(x^{-\|n\|}\mathbf{B}_{\overset{<}{n}}) \tag{7.17}$$

with the usual bounds on $\mathbf{B}_\bullet$ and $\mathbf{B}_{\overset{<}{\bullet}}$ which, combined with (7.7) and (7.8), yield the *normal convergence* of the arborified expansion (7.4), along with the generic normal divergence of the unarborified expansion.

Remark 1 : There exist for Θ_{ser} expansions analogous to (7.4), but with a slightly redefined comould $\mathbf{B}_\bullet$.

Remark 2 : In the non-resonant case (i.e. for an irrational λ^*) we have a seriable linearisation of the form :

$$x = k^{\text{ser}}(y) \in y\,\mathbf{C}[[y, y^{1/\lambda^*}]] \tag{7.18}$$

and the spiralling index κ_0 may be chosen as small as one wishes. The series $k^{\text{ser}}(y)$ is convergent if λ^* is diophantine, and generically divergent if λ^* is liouvillian. Yet $k^{\text{ser}}(y)$ is always seriable and therefore summable by the methods of §3.

Remark 3 : In the resonant case (i.e. for a rational λ^*) we have :

$$x = k^{\text{ser}}(y) \in y\, \mathbb{C}[[y, y^{1/m}(\log y)^{\kappa/m}]] \qquad (m \in \mathbb{N};\ \kappa \in Q^+) \tag{7.19}$$

which makes it plain that the spiralling constant κ_0 in (4.5) cannot be taken smaller than κ.

Remark 4 : For the canonical factorisation of Θ_{ser} and k^{ser}, see §8 in the non-resonant case and §10 in the resonant case.

Remark 5. Seriability and conformal maps. Connection with the Douady-Ghys lemma.

We shall be pairing local diffeos f, g of $\mathbb{C}$ with "opposite" multipliers ℓ, ℓ', or rather lifted diffeos $\underset{\bullet}{f}, \underset{\bullet}{g}$ of $\underset{\bullet}{\mathbb{C}}$ with multipliers $\underset{\bullet}{\ell}, \underset{\bullet}{\ell'}$ such that :

$$\log \ell + \log \ell' = 0 \ (\operatorname{mod} 2\pi i);\ \log \underset{\bullet}{\ell} + \log \underset{\bullet}{\ell'} = 0 \ \ \text{(exactly)} \tag{7.20}$$

$$\underset{\bullet}{f}: x \mapsto \underset{\bullet}{\ell}\, x + \dots \ \text{ with } \underset{\bullet}{\ell} = e^{2\pi i \lambda^*},\ \lambda^* > 0,\ x \in \underset{\bullet}{\mathbb{C}} \tag{7.21}$$

$$\underset{\bullet}{g}: x \mapsto \underset{\bullet}{\ell'}\, x + \dots \ \text{ with } \underset{\bullet}{\ell'} = e^{2\pi i/\lambda^*},\ \lambda^* > 0,\ x \in \underset{\bullet}{\mathbb{C}} \tag{7.22}$$

For the moment we forget about $\underset{\bullet}{g}$ and start with a given $\underset{\bullet}{f}$. We fix $\underset{\bullet}{\epsilon}$ on $\underset{\bullet}{\mathbb{C}}$ small enough ($\underset{\bullet}{\epsilon}$ close to $\underset{\bullet}{0}$) and real ($\arg \underset{\bullet}{\epsilon} = 0$) and introduce on $\underset{\bullet}{\mathbb{C}}$ three sectorial neighbourhoods D^1, D^2, D^3 of $\underset{\bullet}{0}$, with apertures respectively $2\pi, 2\pi\lambda^*, 2\pi\lambda^*$:

$$D^1 \overset{\text{def}}{=} \{x \in \underset{\bullet}{\mathbb{C}};\ 0 < \arg x < 2\pi,\ |x| < \epsilon\} \tag{7.23}$$

$$D^2 \overset{\text{def}}{=} \{x \in \underset{\bullet}{\mathbb{C}};\ 0 < \arg x < 2\pi\lambda^*,\ |x| < \epsilon^{\lambda^*}\} \tag{7.24}$$

(7.25) $D^3 \overset{\text{def}}{=}$ domain of $\underset{\bullet}{\mathbb{C}}$ enclosed by the ray I from $\underset{\bullet}{0}$ to $\underset{\bullet}{\epsilon}^{\lambda^*}$, by its image $\underset{\bullet}{f}\,(I)$, and by any given curve Γ joining $\underset{\bullet}{\epsilon}^{\lambda^*}$ to $\underset{\bullet}{f}\,(\underset{\bullet}{\epsilon}^{\lambda^*})$ without crossing itself, nor I, nor $\underset{\bullet}{f}\,(I)$.

Lastly, we denote by $k^{j,i}$ *the* conformal mapping from D^i to D^j that preserves $\underset{\bullet}{0}$ $(i,j=1,2,3)$. Obviously, $k^{2,1}(x) \equiv x^{\lambda^*}$ and $k^{1,2}(x) \equiv x^{1/\lambda^*}$.

The Douady-Ghys lemma

$k^{1,3}$ conjugates $\underset{\bullet}{f}$ with the one-turn rotation $\underset{\bullet}{r}$:

$$(7.26) \qquad \underset{\bullet}{f} = k^{3,1} \circ \underset{\bullet}{r} \circ k^{1,3} \qquad (\textit{with } \underset{\bullet}{r}(x) \stackrel{\text{def}}{=} e^{2\pi i}x)$$

and the correspondence :

$$(7.27) \qquad \underset{\bullet}{f} \mapsto \underset{\bullet}{g} \stackrel{\text{def}}{=} k^{1,3} \circ \underset{\bullet}{r} \circ k^{3,1} \qquad (\textit{with same } \underset{\bullet}{r})$$

though dependent on the choice of ϵ and Γ, induces an intrinsic, one-to-one correspondence between analytic conjugacy classes of diffeos $\underset{\bullet}{f}$ and $\underset{\bullet}{g}$ of type (7.21) and (7.22).

Let us now reinterpret this result in the light of *seriability*. We replace the domain D^1 by D^2, which has the same aperture $2\pi\lambda^*$ as D^3, and we observe that :

$$(7.28) \qquad k^{2,3} \circ \underset{\bullet}{f} \circ k^{3,2}(x) \equiv e^{2\pi i \lambda^*} x \qquad (\forall x \in D^2)$$

since $k^{2,3} \circ \underset{\bullet}{f} \circ k^{3,2}$ leaves invariant $\underset{\bullet}{0}$ and the circle $|x| = \epsilon$. If we now introduce the seriable linearisation k^{ser} and put :

$$(7.29) \qquad k^{3,2} = k^{\text{ser}} \circ k \qquad (k^{\text{ser}}(x) \stackrel{\text{def}}{=} \Theta_{\text{ser}}^{-1}.x)$$

that *defines* a map k which leaves $\underset{\bullet}{0}$ invariant as well as the radii $\{\arg x = 0\}$ and $\{\arg x = 2\pi\lambda^*\}$, so that $k(x)$ must be the sum of a real, convergent power series $\tilde{k}(x)$ of the form :

$$(7.30) \qquad \tilde{k}(x) = x\{1 + \sum d_n x^{n/\lambda^*}\} \in x.\mathbf{R}\{x^{1/\lambda^*}\}$$

But Proposition 7.1 shows that k^{ser} is a *seriable* function with asymptotic series $\tilde{k}^{\text{ser}}$. Therefore the map $k^{3,2}$ in (7.29), as indeed all conformal maps $k^{i,j}$ of this section, are also *seriable* at $\underset{\bullet}{0}$.

Moreover, in the non-resonant case (λ^* irrational) we have the obvious factorisations (see also §8) between asymptotic series :

$$(7.31) \qquad \tilde{k}^{3,1} = \tilde{a}^{-1} \circ P_{\lambda^*} \circ \tilde{b} \qquad (\tilde{k}^{3,1}(x) = x^{\lambda^*} + \ldots)$$

$$\tilde{k}^{1,3} = \tilde{b}^{-1} \circ P_{1/\lambda^*} \circ \tilde{a} \qquad (\tilde{k}^{1,3}(x) = x^{1/\lambda^*} + \ldots) \tag{7.32}$$

$$\underset{\bullet}{\tilde{f}} = \tilde{a}^{-1} \circ \underset{\bullet}{r}^{\lambda^*} \circ \tilde{a} \qquad (\underset{\bullet}{\tilde{f}}(x) = e^{2\pi i \lambda^*} x + \ldots) \tag{7.33}$$

$$\underset{\bullet}{\tilde{g}} = \tilde{b}^{-1} \circ \underset{\bullet}{r}^{1/\lambda^*} \circ \tilde{b} \qquad (\underset{\bullet}{\tilde{g}}(x) = e^{2\pi i/\lambda^*} x + \ldots) \tag{7.34}$$

with

$$P_{\lambda^*}(x) \stackrel{\text{def}}{=} x^{\lambda^*};\ P_{1/\lambda^*}(x) \stackrel{\text{def}}{=} x^{1/\lambda^*} \tag{7.35}$$

$$\underset{\bullet}{r}^{\lambda^*}(x) \stackrel{\text{def}}{=} e^{2\pi i \lambda^*} x;\ \underset{\bullet}{r}^{1/\lambda^*}(x) \stackrel{\text{def}}{=} e^{2\pi i/\lambda^*} x \tag{7.36}$$

and with entire, but usually divergent power series $\tilde{a}, \tilde{b}$ of the form :

$$\tilde{a}(x) = x + \sum a_n x^{n+1};\ \tilde{b}(x) = x + \sum b_n x^{n+1} \tag{7.37}$$

Indeed, due to the small denominators present in their coefficients a_n, b_n, the series $\tilde{a}$ and $\tilde{b}$ are not only generically divergent, but hopelessly so : when divergent, there is no canonical way of summing them *separately*, even in sectorial neighbourhoods of $\underset{\bullet}{0}$. When *combined*, however, $\tilde{a}$ and $\tilde{b}$ undergo *compensation* and yield either *seriable series* $\tilde{k}^{3,1}, \tilde{k}^{1,3}$ as in (7.31), (7.32) or plain *convergent series* $\underset{\bullet}{\tilde{f}}, \underset{\bullet}{\tilde{g}}$ as in (7.33), (7.34).

Remark 6. Extension to Carleman classes $\mathcal{C}(M)$:

The alternative approach outlined in Remark 5, being based on conformal mappings, doesn't extend smoothly to higher dimensions, and even in dimension 1, like all geometrical methods, it fails in the face of classes $\mathcal{C}(M)$ of power series larger than the convergent or "analytic" class $\mathcal{A} = \mathcal{C}(1)$. Indeed, let $M_\sigma = (m_\sigma)^\sigma$ be defined for $\sigma > 0$, with $\log m_\sigma$ positive, convex and growing to $+\infty$ as σ goes to $+\infty$. Then the linear operator :

$$M : x^\sigma \mapsto M.x^\sigma \stackrel{\text{def}}{=} M_\sigma.x^\sigma \qquad (\forall \sigma \in \mathbf{R}^+) \tag{7.38}$$

turns the convergent class $\mathcal{A}$ into the so-called *Carleman class* $\mathcal{C}(M) \stackrel{\text{def}}{=} M.\mathcal{A}$ which, just like $\mathcal{A}$, enjoys stability under all useful operations, including postcomposition by elements of the form $\ell x + o(x)$. However, for every Carleman class $\mathcal{C}(M)$, no matter how large, the conjugacy equation :

$$\tilde{h} \circ \tilde{f}_1 = \tilde{f}_2 \circ \tilde{h} \qquad (\tilde{f}_1(x) = e^{2\pi i\lambda^*} x + \dots, \tilde{f}_2(x) = e^{2\pi i\lambda^*} x + \dots) \tag{7.39}$$

for a given pair $\tilde{f}_1, \tilde{f}_2$ in $\mathcal{C}(M)$, may fail to have solutions $\tilde{h}$ in $\mathcal{C}(M)$, due to small denominators which, for strongly liouvillian multipliers $\ell = \exp(2\pi i\lambda^*)$, can be as small as one wishes. Within a class $\mathcal{C}(M)$, therefore, neither the conjugacy problem, nor the linearisation problem, nor the correspondence between conjugacy classes $\mathrm{cl}(\underset{\bullet}{\tilde{f}})$ and $\mathrm{cl}(\underset{\bullet}{\tilde{g}})$ as in (7.21), (7.22), can be tackled by conformal mapping, but the methods based on seriability still apply with only slight changes. Actually, all it takes is to redefine the symmetrel compensators $\mathcal{S}^{\bullet}_{\mathrm{coim}}$, still using formula (4.29) but with the operator $M^{-1}.\theta.M$ in place of θ, and to define M-seriability of a series $\tilde{\varphi}(z) = \Sigma a_\sigma z^\sigma$ as meaning that, after changing variables :

$$\tilde{\varphi}(z) \equiv \tilde{\psi}(t) \quad \text{with } z = e^{-t}.t^{-\rho} \text{ and } \rho \text{ positive large} \tag{7.40}$$

and Borel-transforming $\tilde{\psi}(t)$ into $\hat{\psi}(\tau)$, we have the bounds :

$$|\hat{\psi}(\tau)| < \text{Const}.\ M_\tau e^{\kappa\tau} \qquad (\tau \in \mathbf{R}^+) \tag{7.41}$$

instead of (3.20). If we do this, expansion (7.4) still gives us a M-seriable linearisator $\Theta_{\mathrm{ser}}^{-1}$ of $\underset{\bullet}{\tilde{f}}$, while formula (7.27) becomes :

$$\underset{\bullet}{\tilde{f}} \mapsto \underset{\bullet}{\tilde{g}} \overset{\text{def}}{=} \tilde{k}^{\mathrm{ser}} \circ \underset{\bullet}{r} \circ \tilde{h}^{\mathrm{ser}} \tag{7.41}$$

with

$$\tilde{h}^{\mathrm{ser}}(x) = \Theta_{\mathrm{ser}}.x, \tilde{k}^{\mathrm{ser}}(x) = \Theta_{\mathrm{ser}}^{-1}.x;\ \underset{\bullet}{r}\,(x) = e^{2\pi i}.x \tag{7.41bis}$$

and still yields an intrinsic, one-to-one correspondence $\mathrm{cl}(\underset{\bullet}{\tilde{f}}) \to \mathrm{cl}(\underset{\bullet}{\tilde{g}})$ between conjugacy classes of diffeos of types (7.21) and (7.22) but in the class $\mathcal{C}(M)$.

Remark 7.Seriable linearisation of girators in higher dimension.

In dimension larger than one, the study of *girations* doesn't run exactly parallel to that of *girators*. Given a collection of ν multipliers ℓ_j on the unit circle of $\mathbf{C}$, we face the choice of lifting them into multipliers $\underset{\bullet}{\ell_j} = \exp(2\pi i\lambda_j^*)$ on $\underset{\bullet}{\mathbf{C}}$ with real numbers λ_j^* of one sign only, or of mixed signs. The first choice leads to linearisators that are close in form to (7.4), yet involve an infinity of independent ramified monomials $x_i^{\sigma_i}$. The second choice avoids this infinity, but at the cost of less explicit formulae. See the concluding section §11.

8. GENERALISED HOLONOMY. CORRESPONDENCES BETWEEN CONJUGACY CLASSES OF LOCAL OBJECTS.

In this section, we shall have two concerns. *First*, factorising (canonically and explicitly) the seriable linearisators of local objects. *Second*, generalising the notion of holonomy; that is to say, associating with a local object Ob (field or diffeo) a local diffeo $\underset{\bullet}{f}^{\mathrm{inv}}$ of $\underset{\bullet}{\mathbf{C}}^{\nu}$ in such a way that the analytic conjugacy class of Ob should determine that of $\underset{\bullet}{f}^{\mathrm{inv}}$.

For our first purpose, we will provisionally assume Ob to be non-resonant (that restriction will be lifted in §10), but our second construction will apply right away to all cases, resonant and non-resonant.

We recall the definition of the invariant algebra INV introduced in §1. For any given local object X or $\underset{\bullet}{F}$ with diagonal linear part X^{lin} or $\underset{\bullet}{F}^{\mathrm{lin}}$, INV is the algebra of formal power series spanned by all monomials :

$$x^{\sigma} = x_1^{\sigma_1} \dots x_\nu^{\sigma_\nu} \qquad (\sigma_i \in \mathbf{R}^+,\ x_i \in \underset{\bullet}{\mathbf{C}}) \tag{8.1}$$

invariant under X^{lin} or $\underset{\bullet}{F}^{\mathrm{lin}}$:

$$X^{\mathrm{lin}}.x^{\sigma} = 0; \qquad \underset{\bullet}{F}^{\mathrm{lin}}.x^{\sigma} = x^{\sigma} \tag{8.1bis}$$

Proposition 8.1. Factorisation of seriable linearisators.

Assuming non-resonance, all seriable linearisators hitherto defined factor as follows :

$$\Theta_{\mathrm{ser}} = \Theta_{\mathrm{ent}} \cdot \Theta_{\mathrm{inv}}^{-1}; \qquad \Theta_{\mathrm{ser}}^{-1} = \Theta_{\mathrm{inv}} \cdot \Theta_{\mathrm{ent}}^{-1} \tag{8.2}$$

or scalarly (mark the reversals) :

$$h^{\mathrm{ser}} = k^{\mathrm{inv}} \circ h^{\mathrm{ent}}; \qquad k^{\mathrm{ser}} = k^{\mathrm{ent}} \circ h^{\mathrm{inv}} \tag{8.2bis}$$

*into a usually divergent "**integral part**" Θ_{ent}, which preserves entire series, and a usually divergent "**invariant part**" Θ_{inv}, which preserves INV and commutes with the linear part of the girator or giration. The substitution operators Θ_{ent} and Θ_{inv} (we also recall Θ_{ser} for completeness) possess the following expansions in the case of **girators** :*

(8.3)* $\Theta_{\mathrm{ser}} = \sum S^{\bullet}_{\mathrm{co}}(z)\mathbf{B}_{\bullet}$ (8.3′) $\Theta_{\mathrm{ser}}^{-1} = \sum \mathcal{S}^{\bullet}_{\mathrm{co}}(z)\mathbf{B}_{\bullet}$

(8.4) $\Theta_{\mathrm{ent}} = \sum S^{\bullet}\mathbf{B}_{\bullet}$ (8.4′) $\Theta_{\mathrm{ent}}^{-1} = \sum \mathcal{S}^{\bullet}\,\mathbf{B}_{\bullet}$

(8.5) $\Theta_{\mathrm{inv}} = \sum S^{\bullet}(z)\mathbf{B}_{\bullet}$ (8.5′)* $\Theta_{\mathrm{inv}}^{-1} = \sum \mathcal{S}^{\bullet}(z)\mathbf{B}_{\bullet}$

and in the case of girations :

(8.6)* $\Theta_{\mathrm{ser}} = \sum S^{\bullet}_{\mathrm{coim}}(z)\mathbf{B}_{\bullet}$ (8.6′) $\Theta^{-1}_{\mathrm{ser}} = \sum \mathcal{S}^{\bullet}_{\mathrm{coim}}(z)\mathbf{B}_{\bullet}$

(8.7) $\Theta_{\mathrm{ent}} = \sum S^{\bullet}\mathbf{B}_{\bullet}$ (8.7′) $\Theta^{-1}_{\mathrm{ent}} = \sum \mathcal{S}^{\bullet}\,\mathbf{B}_{\bullet}$

(8.8) $\Theta_{\mathrm{inv}} = \sum S^{\bullet}_{\mathrm{im}}(z)\mathbf{B}_{\bullet}$ (8.8′)* $\Theta^{-1}_{\mathrm{inv}} = \sum \mathcal{S}^{\bullet}_{\mathrm{im}}(z)\mathbf{B}_{\bullet}$

Proof and comments :

The 12 above expansions involve standard mould-comould contractions of type $\Sigma A^{\bullet}\mathbf{B}_{\bullet}$ and the way to deduce the expansions of Θ_{ent} and Θ_{inv} from those of Θ_{ser} is by using the canonical factorisations $A^{\bullet} = A^{\bullet}_1 \times A^{\bullet}_2$ of compensators along with the identity

$$\sum(A^{\bullet}_1 \times A^{\bullet}_2)\mathbf{B}_{\bullet} = (\sum A^{\bullet}_2\mathbf{B}_{\bullet})(\sum A^{\bullet}_1\mathbf{B}_{\bullet}) \tag{8.9}$$

which is valid as soon as the differential operators $\mathbf{B}_n$ which go into the making of the comould $\mathbf{B}_{\bullet}$ commute with $A^{\bullet}_1$, that is to say, do not act on the variable or variables (if any) present in the mould $A^{\bullet}_1$. Note that there is no restriction here on the mould $A^{\bullet}_2$. This explains why the four star-marked formulae $(\ldots)^*$ in Proposition 8.1 hold only when the $\mathbf{B}_n$ don't act on the variable(s) carried by the corresponding mould. But even when this is not the case, as with diffeos, the formulae $(\ldots)^*$ retain their validity after a slight redefinition of the comould $\mathbf{B}_{\bullet}$. Observe that the scalar moulds $S^{\bullet}, \mathcal{S}^{\bullet}, S^{\bullet}, \mathcal{S}^{\bullet}$ appearing in the formulae, are defined as in §1. The one or two variable compensators $S^{\bullet}_{\mathrm{co}}(z), \mathcal{S}^{\bullet}_{\mathrm{co}}(z), S^{\bullet}_{\mathrm{coim}}(z), \mathcal{S}^{\bullet}_{\mathrm{coim}}(z)$ are the same as in sections 5,6,7. Lastly, the moulds $S^{\bullet}(z), \mathcal{S}^{\bullet}(z), S^{\bullet}_{\mathrm{im}}(z), \mathcal{S}^{\bullet}_{\mathrm{im}}(z)$ are defined as in (4.2) (4.3) (4.22) (4.23) in the case of *one* ramified variable, and by using mould mixing as in (6.14) in the case of *two* ramified variables.

Now, let us address the construction of the "generalised holonomy" operator $F_{\bullet\,\mathrm{inv}}$. Denote by $\underset{\bullet}{R}$ any rotation of r_i turns in each component $\underset{\bullet}{\mathbb{C}}$ of $\underset{\bullet}{\mathbb{C}}^{\nu}$:

$$\underset{\bullet}{R}\,\varphi(x_1,\ldots,x_\nu) \overset{\mathrm{def}}{=} \varphi(e^{2\pi i r_1}x_1,\ldots,e^{2\pi i r_\nu}x_\nu) \quad (r_i \in \mathbb{Z}) \tag{8.10}$$

Then, for any girator X or any lifted giration $\underset{\bullet}{F}$ with seriable linearisator Θ_{ser}, put

$$F_{\bullet\,\mathrm{inv}} \overset{\mathrm{def}}{=} \Theta^{-1}_{\mathrm{ser}}\,\underset{\bullet}{R}\,\Theta_{\mathrm{ser}} = \Theta_{\mathrm{inv}}\,\underset{\bullet}{R}\,\Theta^{-1}_{\mathrm{inv}} \tag{8.11}$$

or scalarly :

$$\underset{\bullet}{f}^{\mathrm{inv}} = h^{\mathrm{ser}} \circ \underset{\bullet}{r} \circ k^{\mathrm{ser}} = k^{\mathrm{inv}} \circ \underset{\bullet}{r} \circ h^{\mathrm{inv}} \tag{8.11bis}$$

The middle and right terms in (8.11) and (8.11bis) are indeed equal,because the factors $\Theta^{\pm 1}_{\mathrm{ent}}$ in $\Theta^{\pm 1}_{\mathrm{ser}}$ (see (8.2)) commute with $\underset{\bullet}{R}$ and cancel out. Moreover, as a product of two seriable factors $\Theta^{\pm 1}_{\mathrm{ser}}$ with an elementary $\underset{\bullet}{R}$ in between, $F_{\bullet\,\mathrm{inv}}$ is automatically seriable.

Now, the above construction of $\underset{\bullet}{F}_{\text{inv}}$ is not intrinsic. Suppose indeed that we calculate, according to the formulae of sections 5,6,7, different seriable linearisators Θ^i_{ser} relative to different charts $i = 1, 2, \ldots$. These linearisators will factor into :

$$\Theta^i_{\text{ser}} = (\Theta_{\text{ent}}) \cdot (\Theta^i_{\text{inv}})^{-1} \qquad (i = 1, 2, \ldots) \tag{8.12}$$

with identical Θ_{ent} but chart-dependent factors Θ^i_{inv} connected by :

$$\Theta^i_{\text{inv}} = \Theta^{ij}_{\text{inv}} \Theta^j_{\text{inv}} \qquad (i, j = 1, 2, 3 \ldots) \tag{8.13}$$

Consequently, we also get chart-dependent holonomies F^i_{inv} mutually connected by :

$$\underset{\bullet}{F}^i_{\text{inv}} = \Theta^{ij}_{\text{inv}} \underset{\bullet}{F}^j_{\text{inv}} \Theta^{ji}_{\text{inv}} \qquad (i, j = 1, 2 \ldots) \tag{8.14}$$

with

$$\Theta^{ij}_{\text{inv}} = (\Theta^i_{\text{inv}})(\Theta^j_{\text{inv}})^{-1} = (\Theta^i_{\text{ser}})^{-1}(\Theta^j_{\text{ser}}) \tag{8.15}$$

The last term shows that $\Theta^{i,j}_{\text{inv}}$ is seriable and the middle term shows that it preserves INV. Therefore, if the factors Θ^i_{inv} implicit in the Θ^i_{ser} *preserve a rigid subalgebra* RIG of INV (see end of §3), so do the connection factors $\Theta^{i,j}_{\text{inv}}$, which means that they correspond to power series $\tilde{k}^{ij}_{\text{inv}} \in$ RIG. *But being also seriable elements of a rigid algebra, the series $\tilde{k}^{ij}_{\text{inv}}$ are in fact convergent.* Therefore we have proved :

Proposition 8.2. Invariance of the holonomy class $\underset{\bullet}{F}_{\text{inv}}$

Whenever the seriable linearisators Θ_{ser} of a local object Ob (girator or giration) draw only on a rigid subalgebra RIG of the invariant algebra INV (and such is the case for all linearisators constructed in sections 5,6,7) the various (generalised) holonomies $\underset{\bullet}{F}_{\text{inv}}$ relative to a given multirotation $\underset{\bullet}{R}$ but to different charts, are all conjugate according to (8.14) with **convergent** *connection factors Θ^{ij}_{inv}. The "convergent conjugacy class" of $\underset{\bullet}{F}_{\text{inv}}$ is therefore well-defined and dependent only on the analytic conjugacy class of Ob.*

Now, lėt us look for the mould expansion of the holonomy $\underset{\bullet}{F}_{\text{inv}}$ of a girator $2\pi i X$, assuming for simplicity that the condition for the validity of the star-marked formulae are met. Using definition (8.11) along with expansions (8.3)* and (8.3′) we find :

$$\left\{ \begin{aligned} \underset{\bullet}{F}_{\text{inv}} &= (\textstyle\sum \mathcal{S}^{\bullet}_{\text{co}}(z)\mathbf{B}_{\bullet})\, \underset{\bullet}{R}\, (\textstyle\sum S^{\bullet}_{\text{co}}(z)\mathbf{B}_{\bullet}) \\ &= \underset{\bullet}{R}(\textstyle\sum (\underset{\bullet}{R}^{-1}\, \mathcal{S}^{\bullet}_{\text{co}}(z))\mathbf{B}_{\bullet})(\textstyle\sum S^{\bullet}_{\text{co}}(z)\mathbf{B}_{\bullet}) \end{aligned} \right. \tag{8.16}$$

and finally :

$$\underset{\bullet}{F}_{\text{inv}} = \underset{\bullet}{R}\, (\sum D^{\bullet}(z)\mathbf{B}_{\bullet}) \tag{8.17}$$

with the customary order reversal :

$$D^\bullet(z) = (S^\bullet_{\mathrm{co}}(z)) \times (\underset{\bullet}{R}^{-1}\, \mathcal{S}^\bullet_{\mathrm{co}}(z)) \tag{8.18}$$

Due to (4.2) (4.3) or (6.13)(6.14) this becomes :

$$D^\bullet(z) = (\mathcal{S}^\bullet(z)) \times (\underset{\bullet}{R}^{-1} S^\bullet(z)) \tag{8.19}$$

For girators with semi-mixed spectrum (see §5) we have only one ramified variable :

$$z = z_1 = x_1^{-1/\lambda_1} \qquad (\lambda_1 < 0) \tag{8.20}$$

and, due to (4.2) (4.3), the mould $D^\bullet(z)$ simplifies to :

$$D^\bullet(z) = z^{\|\bullet\|}\, \mathcal{S}^\bullet_{\mathrm{co}}(e_1) \tag{8.21}$$

with $e_1 = \exp(2\pi i r_1/\lambda_1)$ and $z^{\|\omega\|} = x_1^{-\|\omega\|/\lambda_1}$. In other words, the ramified variable x_1 factors away and a mere *constant* e_1 becomes responsible for the compensation effect. For girators with truly mixed spectrum, on the other hand, we have (see §6) two ramified variables :

$$z = (z_1, z_2) = (x_1^{-1/\lambda_1}, x_2^{-1/\lambda_2}) \qquad (\text{with } \lambda_1 < 0 < \lambda_2) \tag{8.21}$$

which do not factor out and remain essentially involved in the compensation, which is now more directly apparent in (8.18) than (8.19).

To conclude, let us observe that the correspondence $\underset{\bullet}{f} \to \underset{\bullet}{g}$ studied in the preceding section (for girations) is essentially the same (up to ramification) as the correspondence $\underset{\bullet}{f} \to \underset{\bullet}{f}^{\mathrm{inv}}$ of the present section, since $\underset{\bullet}{g}$ is analytically conjugate to $P_{1/\lambda^*} \circ \underset{\bullet}{f}^{\mathrm{inv}} \circ P_{\lambda^*}$. Moreover, like in Remarks 5 and 6 of §7, the "holonomy" correspondence $\mathrm{Ob} \to \underset{\bullet}{F}_{\mathrm{inv}}$ extends, thanks to seriability, to non-analytic classes $\mathcal{C}(M)$ while retaining its basic invariance property.

9. LINK BETWEEN COMPENSATION AND RESURGENCE.

In the case of resonant objects, the seriable linearisators Θ_{ser} still exist, with unchanged expansions into series of compensators, but instead of their usual factorisation $\Theta_{\mathrm{ent}}\Theta_{\mathrm{inv}}^{-1}$ into two hopelessly divergent factors, they admit (see §10) an alternative factorisation $\Theta_{\mathrm{nor}}\Theta_{\mathrm{inv}}^{-1}$ into two *resurgent* (and usually resummable) factors. But since *effective resurgence* prevents regular summation in a *full* neighbourhood of $\underset{\bullet}{0}$ and since on the other hand Θ_{ser}, on account of its seriability, does have a regular sum in such a full neighbourhood, it follows that the resurgence present in both factors Θ_{nor} and Θ_{inv}^{-1} must somehow cancel out. This is indeed the case, as we shall see in the next section. But as a preparation, we must first study the interplay of resurgence, seriability and compensation in the simplest possible context, namely that of resurgent monomials. To that end we will have to study the following moulds :

$$(9.1)\qquad \left\{\begin{array}{ccc} \{S^\bullet, \mathcal{S}^\bullet\} & \xrightarrow{1} & \{S^\bullet_{co}(z), \mathcal{S}^\bullet_{co}(z)\} \\ \downarrow_3 & & \downarrow_4 \\ \{\mathcal{V}^\bullet(z), \mathcal{V}\!\!\!/^\bullet(z)\} & \xrightarrow{2} & \{\mathcal{V}^\bullet_{co}(z), \mathcal{V}\!\!\!/^\bullet_{co}(z)\} \end{array}\right.$$

and pay full attention to their intricate symmetries and interconnections.

All eight moulds listed in (9.1) are ***symmetral*** (see §12) and those bracketted together are ***mutually inverse*** (for mould multipliation, see §12). The first pair are scalar-valued moulds; all others are divergent series of decreasing powers of a complex variable z, but for simplicity we drop in this section the usual twiddle ($\sim$) indicative of formalness.

The moulds of the first line are indexed by sequences $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$ with real or complex ω_i. They were already defined in (1.22) (1.23) for the first pair and in (4.2) (4.3) for the second pair.

The moulds of the second line are indexed by sequences $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_r)$ with $\eta_i = \binom{\omega_i}{\sigma_i}$ and real or complex ω_i and σ_i.

The $\mathcal{V}^{\boldsymbol{\eta}}, \mathcal{V}\!\!\!/^{\boldsymbol{\eta}}$ are so-called ***resurgence monomials*** and their definition is given by the following induction :

$$(9.2)\qquad \mathcal{V}^{\emptyset}(z) \stackrel{\text{def}}{\equiv} \mathcal{V}\!\!\!/^{\emptyset}(z) \stackrel{\text{def}}{\equiv} 1$$

$$(9.3)\qquad (\omega_1 + \ldots \omega_r + \partial_z)\mathcal{V}^{\eta_1,\ldots,\eta_r}(z) \stackrel{\text{def}}{=} -\mathcal{V}^{\eta_1,\ldots,\eta_{r-1}}(z) z^{\sigma_r}$$

$$(9.4)\qquad (\omega_1 + \ldots \omega_r + \partial_z)\,\mathcal{V}\!\!\!/^{\eta_1,\ldots,\eta_r}(z) \stackrel{\text{def}}{=} +z^{\sigma_1}\,\mathcal{V}\!\!\!/^{\eta_2,\ldots,\eta_r}(z)$$

The $\mathcal{V}^{\boldsymbol{\eta}}_{co}, \mathcal{V}\!\!\!/^{\boldsymbol{\eta}}_{co}$ depend on a complex parameter ρ, which is left unwritten for the sake of brevity. They are called ***compensated resurgence monomials***, for reasons that will become plain in a moment (Proposition 9.1). Their shortest definition is by means of the mould factorisation :

$$\mathcal{V}^{\bullet}_{\mathrm{co}}(z) \stackrel{\mathrm{def}}{=} (z^{\|\bullet\|\rho}\, \mathcal{V\!\!\!/}^{\bullet}(z+\rho\log z)) \times (\mathcal{V}^{\bullet}(z)) \tag{9.5}$$

$$\mathcal{V\!\!\!/}^{\bullet}_{\mathrm{co}}(z) \stackrel{\mathrm{def}}{=} (\mathcal{V\!\!\!/}^{\bullet}(z)) \times (z^{\|\bullet\|\rho}\mathcal{V}^{\bullet}(z+\rho\log z)) \tag{9.6}$$

(with $\|\bullet\| = \omega_1 + \ldots \omega_r$ as usual) which mirrors the definition of $S^{\bullet}_{co}, \mathcal{S\!\!\!/}^{\bullet}_{co}$ in terms $S^{\bullet}, \mathcal{S\!\!\!/}^{\bullet}$.

Proposition 9.1. (Compensation of resurgence in $\mathcal{V}^{\boldsymbol{\eta}}_{\mathrm{co}}(z)$ and $\mathcal{V\!\!\!/}^{\boldsymbol{\eta}}_{\mathrm{co}}(z)$)

Each monomial $\mathcal{V}^{\eta}_{\mathrm{co}}(z)$ and $\mathcal{V\!\!\!/}^{\eta}_{\mathrm{co}}(z)$ is a resurgence constant, meaning that all their alien derivatives vanish :

$$\Delta_{\omega_0}\mathcal{V}^{\boldsymbol{\eta}}_{\mathrm{co}}(z) \equiv \Delta_{\omega_0}\mathcal{V\!\!\!/}^{\boldsymbol{\eta}}_{\mathrm{co}}(z) \equiv 0 \qquad (\forall \omega_0 \in \dot{\mathbb{C}};\ \forall \boldsymbol{\eta} = (\eta_1,\ldots,\eta_r)) \tag{9.7}$$

Proof : We start from the alien derivation rules for resurgent monomials (see e.g. [E.1] or [E.3]) :

$$\Delta_{\omega_0}\mathcal{V}^{\boldsymbol{\eta}}(z) = \sum_{\boldsymbol{\eta}=\boldsymbol{\eta}^1\boldsymbol{\eta}^2} V^{\boldsymbol{\eta}^1}_{\omega_0}\mathcal{V}^{\boldsymbol{\eta}^2}(z) \tag{9.8}$$

$$\Delta_{\omega_0}\mathcal{V\!\!\!/}^{\boldsymbol{\eta}}(z) = -\sum_{\boldsymbol{\eta}=\boldsymbol{\eta}^1\boldsymbol{\eta}^2} \mathcal{V\!\!\!/}^{\boldsymbol{\eta}^1}(z)V^{\boldsymbol{\eta}^2}_{\omega_0} \tag{9.9}$$

Applying this to the factorisations (9.5) and (9.6) in conjunction with the rule (12.39), which in the present instance yields :

$$\Delta_{\omega}(z^{\|\bullet\|\rho}\mathcal{A}^{\boldsymbol{\eta}}(z+\rho\log z)) \equiv (\Delta_{\omega}\mathcal{A}^{\boldsymbol{\eta}})(z+\rho\log z) \tag{9.10}$$

(with $\mathcal{A}^{\bullet} = \mathcal{V}^{\bullet}$ or $\mathcal{V\!\!\!/}^{\bullet}$) we immediately arrive at (9.7).

Proposition 9.2. (Seriability of $\mathcal{V}^{\boldsymbol{\eta}}_{\mathrm{co}}(z)$ and $\mathcal{V\!\!\!/}^{\boldsymbol{\eta}}_{\mathrm{co}}(z)$)

*Each monomial $\mathcal{V}^{\boldsymbol{\eta}}_{\mathrm{co}}(z)$ and $\mathcal{V\!\!\!/}^{\boldsymbol{\eta}}_{\mathrm{co}}(z)$, belongs to $z^{\|\boldsymbol{\sigma}\|}\mathbb{C}[[z^{-1}, z^{-1}\log z]]$ (with $\|\boldsymbol{\sigma}\| = \sigma_1 + \ldots \sigma_r$) as well as to the algebra $\mathrm{Ser}(z^{-1})$ of **seriable series** ($z \sim \infty$) and can therefore be resummed by the procedure of §3 in a full neighbourhood of $\dot{0} \in \dot{\mathbb{C}}$. Moreover, each $\mathcal{V}^{\boldsymbol{\eta}}_{\mathrm{co}}(z)$ and $\mathcal{V\!\!\!/}^{\boldsymbol{\eta}}_{\mathrm{co}}(z)$ is also **compensable**, with the explicit expansions (9.29) and (9.30) into convergent series of compensators.*

Proof : The factorisations (4.2) (4.3) and (9.5) (9.6) make explicit the horizontal arrows 1 and 2 in diagram (9.1). The way to prove Proposition 9.2 is by studying the vertical arrows 3 and 4. The following two lemmas will do this for us :

Lemma 9.1. *($\{\mathcal{V}^{\bullet}(z), \not{\mathcal{V}}^{\bullet}(z)\}$ in terms of $\{S^{\bullet}, \not{S}^{\bullet}\}$).* .

(9.11) $$\mathcal{V}^{\boldsymbol{\eta}}(z) = (z + \partial_{\omega_1})^{\sigma_1} \dots (z + \partial_{\omega_r})^{\sigma_r} S^{\boldsymbol{\omega}}$$

(9.12) $$\not{\mathcal{V}}^{\boldsymbol{\eta}}(z) = (z + \partial_{\omega_1})^{\sigma_1} \dots (z + \partial_{\omega_r})^{\sigma_r} \not{S}^{\boldsymbol{\omega}}$$

Lemma 9.2. *($\{\mathcal{V}^{\bullet}_{\mathrm{co}}(z), \not{\mathcal{V}}^{\bullet}_{\mathrm{co}}(z)\}$ in terms of $\{S^{\bullet}_{\mathrm{co}}(z), \not{S}^{\bullet}_{\mathrm{co}}(z)\}$)* .

(9.13) $$\mathcal{V}^{\boldsymbol{\eta}}_{\mathrm{co}}(z) = (z + \partial_{\omega_1})^{\sigma_1} \dots (z + \partial_{\omega_r})^{\sigma_r} S^{\boldsymbol{\omega}}_{\mathrm{co}}(z)$$

(9.14) $$\not{\mathcal{V}}^{\boldsymbol{\eta}}_{\mathrm{co}}(z) = (z + \partial_{\omega_1})^{\sigma_1} \dots (z + \partial_{\omega_r})^{\sigma_r} \not{S}^{\boldsymbol{\omega}}_{\mathrm{co}}(z)$$

Here $(z + \partial_\omega)^\sigma$ is short-hand for :

(9.15) $$(z + \partial_\omega)^\sigma = z^\sigma(1 + z^{-1}\partial_\omega)^\sigma = z^\sigma + \sum_{n \geq 1} \frac{\sigma(\sigma-1)\dots(\sigma-n+1)}{n!} z^{\sigma-n}(\partial_\omega)^n$$

with $\partial_\omega = \partial/\partial\omega$. To prove the above lemmas, we now subdot all $\boldsymbol{\omega}$-indexed or $\boldsymbol{\eta}$-indexed moulds to denote multiplication by the same factor $e^{\|\boldsymbol{\omega}\| z}$ in both cases :

(9.16) $$A^{\boldsymbol{\omega}} \mapsto \underset{\bullet}{A}^{\boldsymbol{\omega}} \overset{\text{def}}{=} e^{\|\boldsymbol{\omega}\| z} A^{\boldsymbol{\omega}}$$

(9.17) $$A^{\boldsymbol{\eta}} \mapsto \underset{\bullet}{A}^{\boldsymbol{\eta}} \overset{\text{def}}{=} e^{\|\boldsymbol{\omega}\| z} A^{\eta}$$

and we observe that the identites (9.11), (9.12), (9.13) (9.14) simplify to :

(9.18) $$\underset{\bullet}{\mathcal{V}}^{\boldsymbol{\eta}}(z) = (\partial_{\omega_1})^{\sigma_1} \dots (\partial_{\omega_r})^{\sigma_r} \underset{\bullet}{S}^{\boldsymbol{\omega}}$$

(9.19) $$\underset{\bullet}{\not{\mathcal{V}}}^{\boldsymbol{\eta}}(z) = (\partial_{\omega_1})^{\sigma_1} \dots (\partial_{\omega_r})^{\sigma_r} \underset{\bullet}{\not{S}}^{\boldsymbol{\omega}}$$

(9.20) $$\underset{\bullet\,\mathrm{co}}{\mathcal{V}}^{\boldsymbol{\eta}}(z) = (\partial_{\omega_1})^{\sigma_1} \dots (\partial_{\omega_r})^{\sigma_r} \underset{\bullet\,\mathrm{co}}{S}^{\boldsymbol{\omega}}(z)$$

$$\underset{\bullet\,co}{\mathcal{V}}^{\boldsymbol{\eta}}(z) = (\partial_{\omega_1})^{\sigma_1} \dots (\partial_{\omega_r})^{\sigma_r} \underset{\bullet\,co}{\mathcal{S}}^{\boldsymbol{\omega}}(z) \tag{9.21}$$

although their interpretation is still according to (9.15) after elimination of the exponential factors $e^{\|\boldsymbol{\omega}\| z}$. We readily check that the series $\underset{\bullet}{\mathcal{V}}^{\boldsymbol{\eta}}(z)$ and $\underset{\bullet}{\mathcal{V}}^{\boldsymbol{\eta}}(z)$ *as defined* by (9.18) (9.19) satisfy the induction :

$$\partial_z \underset{\bullet}{\mathcal{V}}^{\eta_1,\dots,\eta_r}(z) = -\underset{\bullet}{\mathcal{V}}^{\eta_1,\dots,\eta_{r-1}}(z).(e^{\omega_r z} z^{\sigma_r}) \tag{9.22}$$

$$\partial_z \underset{\bullet}{\mathcal{V}}^{\eta_1,\dots,\eta_r}(z) = (e^{\omega_1 z} z^{\sigma_1}).\underset{\bullet}{\mathcal{V}}^{\eta_2,\dots,\eta_r}(z) \tag{9.23}$$

which is equivalent to the relations (9.3) (9.4) which characterize $\mathcal{V}^{\eta}(z)$ and $\mathcal{V}^{\eta}(z)$. This proves lemma 9.1.

The derivation of (9.13) (9.14) is not so straightforward. Putting $z_* = z + \rho log z$ we observe that (9.5) (9.6) translate into the relations :

$$\partial_z \underset{\bullet\,co}{\mathcal{V}}^{\eta_1,\dots,\eta_r}(z) = \begin{cases} +\left(\frac{dz_*}{dz}\right)(e^{\omega_1 z_*} z_*^{\sigma_1})\left(\underset{\bullet\,co}{\mathcal{V}}^{\eta_2,\dots,\eta_r}(z)\right) \\ -\left(\underset{\bullet\,co}{\mathcal{V}}^{\eta_1,\dots,\eta_{r-1}}(z)\right)(e^{\omega_r z} z^{\sigma_r}) \end{cases} \tag{9.24}$$

$$\partial_z \underset{\bullet\,co}{\mathcal{V}}^{\eta_1,\dots,\eta_r}(z) = \begin{cases} +\left(e^{\omega_r 1 z} z^{\sigma_1}\right)\left(\underset{\bullet\,co}{\mathcal{V}}^{\eta_2,\dots,\eta_r}(z)\right) \\ -\left(\underset{\bullet\,co}{\mathcal{V}}^{\eta_1,\dots,\eta_{r-1}}(z)\right)(e^{\omega_r z_*} z_*^{\sigma_r})\left(\frac{dz_*}{dz}\right) \end{cases} \tag{9.25}$$

which provides an inductive definition of $\underset{\bullet\,co}{\mathcal{V}}^{\boldsymbol{\eta}}(z)$ and $\underset{\bullet\,co}{\mathcal{V}}^{\boldsymbol{\eta}}(z)$. Next, using the differentiation rules (4.4) (4.5) for the compensators $\underset{\bullet\,co}{\mathcal{S}}^{\boldsymbol{\omega}}(z)$ and $\underset{\bullet\,co}{\mathcal{S}}^{\boldsymbol{\omega}}(z)$, we check that, *if we now regard* $\underset{\bullet\,co}{\mathcal{V}}^{\boldsymbol{\eta}}(z)$ *and* $\underset{\bullet\,co}{\mathcal{V}}^{\boldsymbol{\eta}}(z)$ *as being defined by* (9.20) (9.21), they still satisfy the characteristic induction (9.24) (9.25). This shows the equivalence of both definitions and proves lemma 9.2.

Now, we require complex coefficient $c^{\mathbf{n}}_{\boldsymbol{\sigma}}$ and $d^{\mathbf{n}}_{\boldsymbol{\sigma}}$ defined by the generating function (9.27) :

$$d^{n_1,\dots,n_r}_{\sigma_1,\dots,\sigma_r} \stackrel{\text{def}}{=} (-1)^{n_1+\dots n_r}\, c^{n_r,\dots,n_1}_{\sigma_r,\dots,\sigma_1} \tag{9.26}$$

$$\prod_{i=1}^{r}(z+\partial_{\omega_i})^{\sigma_i} \stackrel{\text{def}}{=} \sum c^{n_1,\dots,n_r}_{\sigma_1,\dots,\sigma_r}\, z^{(\sigma_1+\dots\sigma_r)-(n_1+\dots n_r)}\, \frac{(\partial_{\tau_1})^{n_1}}{n_1!} \dots \frac{(\partial_{\tau_r})^{n_r}}{n_r!} \tag{9.27}$$

$$\partial_{\omega_i} \stackrel{\text{def}}{=} \partial_{\tau_i} + \partial_{\tau_{i+1}} + \ldots \partial_{\tau_r} \qquad (i = 1, \ldots, r) \tag{9.28}$$

Using these coefficients, we find that equations (9.20) (9.21) immediately translate into the following expansions of $\mathcal{V}_{\mathrm{co}}^{\boldsymbol{\eta}}(z)$ and $\mathcal{V}_{\mathrm{co}}^{\boldsymbol{\eta}}(z)$ into ***convergent series of compensators*** :

$$\mathcal{V}_{\mathrm{co}}^{\eta_1,\ldots,\eta_r}(z) = z^{\sigma_1+\ldots\sigma_r} \sum_{n_i} c_{\sigma_1,\ldots,\sigma_r}^{n_1,\ldots,n_r} \, (\rho/z)^{n_1+\ldots n_r} \, \mathcal{S}^{\omega_1,0^{(n_1)},\ldots,\omega_r,0^{(n_r)}}(z^\rho) \tag{9.28bis}$$

$$\mathcal{V}_{\mathrm{co}}^{\eta_1,\ldots,\eta_r}(z) = z^{\sigma_1+\ldots\sigma_r} \sum_{n_i} d_{\sigma_1,\ldots,\sigma_r}^{n_1,\ldots,n_r} \, (\rho/z)^{n_1+\ldots n_r} \, \mathcal{S}^{0^{(n_1)},\omega_1,\ldots,0^{(n_r)},\omega_r}(z^\rho) \tag{9.28ter}$$

where $0^{(n_i)}$ denotes a subsequence of n_i consecutive zeros. This vindicates the claim about the ***compensability*** (in the sense of §3) of $\mathcal{V}_{\mathrm{co}}^{\boldsymbol{\eta}}(z)$ and $\mathcal{V}_{\mathrm{co}}^{\boldsymbol{\eta}}(z)$ and terminates the proof of Proposition 9.2.

Remark : For the simplest conceivable instance of *compensated resurgence*, corresponding to $r = 1$ and $\eta_1 = \binom{\omega_1}{\sigma_1} = \binom{+1}{-1}$, see (3.28) (3.29).

10. RAMIFIED LINEARISATION OF RESONANT OBJECTS.

Resonant local objects present us with a rich situation, as they fall within the purview of two of the formal statements F_i and two of the analytic statements A_i of §1. Indeed, on the one hand, resonant local objects can be brought to a *normal form* (containing only resonant monomials with formally invariant coefficients) by a change of coordinates that is not merely *formal* (statement F_2) but also *resurgent* (statement A_2) with regular summability in *sectorial neighbourhoods* of $\underset{\bullet}{0}$. On the other hand, resonant, non-degenerate local objects admit *ramified linearisations* which are not merely *formal* (statement F_3) but may be chosen to be *seriable* (statement A_3) with regular summability in full spiral-like neighbourhoods of $\underset{\bullet}{0}$.

This dual nature, and in particular the passage from sectorial to full neighbourhoods of $\underset{\bullet}{0}$, calls for a close study of the interplay between, first F_2 and F_3, then A_2 and A_3.

Interplay of F_2 and F_3 : from formal-entire normalisation to formal-ramified linearisation.

Statement F_2 is classic and statement F_3 follows from statement A_3 with its explicit expansions of the linearisation map into convergent sums of compensators. Nonetheless, it is interesting to study directly the passage from *normal* to *linear* forms, if only because it helps understand the compensation of resurgence. (See (10.25) and (10.26)).

We will restrict ourselves to vector fields, which we will write as dynamical systems (with respect to a complex time z) first in a ***normal, entire*** (but generally ***non-analytic)*** ***chart*** $x = (x_i)$, then in a ***linear, ramified chart*** $y = (y_i)$:

$$\partial_z \; x_i = x_i . \{\lambda_i + \sum c_{i,m} x^m\} \qquad (1 \leq i \leq \nu; \; < \lambda, m >= 0) \tag{10.1}$$

$$\partial_z \; y_i = y_i \lambda_i \qquad (1 \leq i \leq \nu) \tag{10.2}$$

and we will look for ramified changes of coordinates :

$$x_i = k_i^{\mathrm{lin}}(y) \quad \textit{and} \quad y_i = h_i^{\mathrm{lin}}(x) \qquad (1 \leq i \leq \nu) \tag{10.3}$$

of the form (1.35) or the still better form (1.36)

Example 1. Vector fields with one degree of resonance : Simplest type.

This is the type $(p, \rho) = (1, 0)$. See e.g. [E.3]. In this case, the normal form is as follows :

$$\partial_z \; x_i = x_i \{\lambda_i + \tau_i x^m\} \qquad (1 \leq i \leq \nu) \tag{10.4}$$

with $x^m = x_1^{m_1} \ldots x_\nu^{m_\nu}$ $\quad (m_i \in \mathbb{N})$ and :

$$< m, \lambda >= 0 \; ; \quad < m, \tau >= -1 \tag{10.5}$$

For any real vector $\alpha = (\alpha_1, \ldots, \alpha_\nu)$ normalised to λ :

$$< \alpha, \lambda > \stackrel{\mathrm{def}}{=} \sum \alpha_i \lambda_i = 1 \qquad (\alpha_i \in \mathbb{R}) \tag{10.6}$$

we have the ramified linearisation :

$$x_i = k_i^{\mathrm{lin}}(y) = y_i \{1 + y^m \log y^\alpha\}^{\tau_i} \qquad (1 \leq i \leq \nu) \tag{10.7}$$

with an explicit reciprocal linearisation :

$$y_i = h_i^{\mathrm{lin}}(x) = x_i \{1 + x^m \log x^\alpha\}^{\tau_i} \qquad (1 \leq i \leq \nu) \tag{10.8}$$

which, unlike (10.7), is valid only if α is also orthogonal to τ (i.e. $< \alpha, \tau >= 0$). Both linearisations have the required form (1.35) and when :

$$\{m_i = 0\} \Rightarrow \{\alpha_i = 0\} \qquad (\forall i) \tag{10.9}$$

they even have the better form (1.36).

The quickest way to check (10.7) (10.8) is to compare the formal integrals of (10.4) and (10.2), which read :

(10.10) $$x_i = u_i e^{\lambda_i z} z^{\tau_i} \qquad \text{with} \quad u^m \equiv 1$$

(10.11) $$y_i = v_i e^{\lambda_i z} \qquad \text{with} \quad v^m \log v^\alpha \equiv -1$$

with the identities :

(10.12) $$z \equiv x^{-m} \equiv y^{-m} + \log y^\alpha$$

(10.13) $$u_i \equiv v_i (v^m)^{\tau_i} \qquad (\forall i)$$

Example 2. Vector fields with one degree of resonance : General type.

This is the type with general "level" $p \in N^*$ and general "residue" $\rho \in \mathbb{C}$. The normal form is now as follows :

(10.14) $$\partial_z x_i = x_i (1 + \rho x^{mp})^{-1} . (\lambda_i + \sum_{q=1}^{p-1} \lambda_i^q x^{mq} + \tau_i x^{mp})$$

with (10.5) still in force and

(10.15) $$< m, \lambda^q > \stackrel{\text{def}}{=} \sum m_i \lambda_i^q = 0 \qquad (1 \leq q \leq p-1)$$

From (10.14) we calculate the formal integral :

(10.16) $$x_i = u_i z_*^{\tau_i} \exp \left\{ \lambda_i z_* + \sum_{q=1}^{p-1} \Lambda_i^q z_*^{1-q/p} \right\}$$

with

(10.17) $$z \equiv z_* + \rho \log z_*; \;\; u^m \equiv 1; \;\; (1 - q/p) \Lambda_i^q = \lambda_i^q \;\; (\forall i)$$

This can be rewritten as :

(10.18) $$x_i = u_i \;\; \exp \Lambda_i(z) \qquad (\text{with} \;\; \Lambda_i(z) = \lambda_i z + o(z))$$

and yields an explicit ramified linearisation :

(10.19) $$x_i = k_i^{\text{lin}}(y) = y_i \exp\{\Lambda_i(y^{-pm} + \log y^\alpha) - \Lambda_i(y^{-pm}) - \lambda_i \log y^\alpha\}$$

which is indeed of the form :

$$k_i^{\text{lin}}(y) = y_i\{1 + \varphi_i(y^m, y^m \log y^\alpha)\} \tag{10.20}$$

$$\varphi_i(a,b) \in \mathbb{C}\{a,b\} \tag{10.20bis}$$

and hence of the form (1.35) or even (1.36) if α is chosen so as to meet condition (10.9) (on top of (10.6)).

Example 3. Vector fields with one degree of resonance : Nihilent type.

This is the case $p = +\infty$, but although fields with finite level p have $(\nu - 1)(p+1)+1$ formal invariants in their normal form, the fields of infinite level still have finitely many invariants. We may however start from any normal or prenormal (see statement F_2 in §1) form :

$$\partial_z x_i = x_i\{\lambda_i + \sum_{1\leq q} \lambda_i^q x^{mq}\} \tag{10.21}$$

and arrive at the ramified linearisation :

$$x_i = k_i^{\text{lin}}(y) = y_i \exp\left\{(\log y^\alpha)(\sum_{1\leq q} \lambda_i^q y^{mq})\right\} \tag{10.22}$$

with (10.6) and (10.20) as usual.

Example 4. Vector fields with several degrees of resonance.

Whenever the resonance degree μ is ≥ 2, there exist infinitely many formal invariants and the normal form (10.1) is generally divergent. One may still construct formal integrals (1.49) and deduce from them ramified linearisations, as was done in the previous examples, but one may also take recourse to explicit expansions into sums of compensators, such as (5.5) or (6.21). Note that these expansions, when applied to normal (or prenormal) forms, undergo a drastic simplification, because the compensators in them carry only vanishing indexes ($\omega_i \equiv 0$), so that they become polynomials of $\log x_i$, for one or several variables x_i. In the case of vector fields with a general complex spectrum (irreducible to girators), it is advisable to introduce *one ramified variable per degree of resonance* and to make repeated use of *mould mixing* (see §6) in order to construct compensators of several variables.

Example 5. One-dimensional, resonant diffeos.

They are local diffeos $x \mapsto f(x)$ with multipliers $\ell = f'(0)$ equal to roots of unity. If nilpotent (i.e. if some of their iterates reduce to the identity map) such diffeos possess analytic linearisations. Otherwise, they have a definite level $p \in \mathbb{N}^*$ and can be linearised by a formal ramified change of coordinates $y = \tilde{h}(x)$ with $\tilde{h}(x)$ in $\mathbb{C}[[x, x^p \log x]]$.

But let us leave these formal aspects for the real thing, namely the analytic study, with the phenomenon of resurgence compensation leading to seriability.

Interplay of F_2 and F_3 : from resurgent normalisation to seriable linearisation.

As we saw in §8, for non-resonant local objects, seriable linearisators admit the canonical factorisation

$$\Theta_{\mathrm{ser}} = \Theta_{\mathrm{ent}}\ \Theta_{\mathrm{inv}}^{-1} \tag{10.23}$$

For resonant local objects this becomes :

$$\Theta_{\mathrm{ser}} = \Theta_{\mathrm{nor}}\ \Theta_{\mathrm{inv}}^{-1} \tag{10.24}$$

or alternatively :

$$\Theta_{\mathrm{ser}} = (\Theta_{\mathrm{nor}}\ \Theta_{\mathrm{lin}}) \cdot (\Theta_{\mathrm{inv}}\ \Theta_{\mathrm{lin}})^{-1} \tag{10.24bis}$$

with an *entire change of coordinates* Θ_{nor} taking the object into a normal form, and a simple *ramified change of coordinates* Θ_{lin} taking that normal form into the linear form (such as in the above examples). Like (10.23), the factorisation (10.24) is essentially unique, at least for a given normal form, but the factorisation (10.24bis) has a larger element of arbitrariness in it.

Now, *for any seriable linearisation* Θ_{ser}, *both factors in* (10.24) *are resurgent and satisfy the same bridge equation* :

$$[\ \dot{\Delta}_\omega, \Theta_{\mathrm{nor}}] = -\Theta_{\mathrm{nor}}\mathbb{A}_\omega \qquad (\forall \omega \in \Omega) \tag{10.25}$$

$$[\ \dot{\Delta}_\omega, \Theta_{\mathrm{inv}}] = -\Theta_{\mathrm{inv}}\mathbb{A}_\omega \qquad (\forall \omega \in \Omega) \tag{10.26}$$

with respect to the same "resurgence lattice" Ω *and the same holomorphic invariants* $\mathbb{A}_\omega$. Here, the $\dot{\Delta}_\omega$ are the usual *pointed alien derivations* (see §12) acting on $z = x^{-m}$ (for level one) or $z = x^{-mp}$ (for level p) or some suitable resonance monomial in the multiresonant case (see [E.3] or [E.7]) and the $\mathbb{A}_\omega$ are ordinary differential operators intrinsically attached to the object (see §1).

Equation (10.25) is of course nothing but equation (1.48) in operatorial form (and with a minus sign), but the remarkable thing is that Θ_{inv} should satisfy the same equation. The immediate consequence is that :

$$[\ \dot{\Delta}_\omega, \Theta_{\mathrm{inv}}^{-1}] = +\mathbb{A}_\omega\Theta_{\mathrm{inv}}^{-1} \tag{10.26bis}$$

so that for each ω :

$$[\ \overset{\bullet}{\Delta}_\omega, \Theta_{\text{ser}}] = [\ \overset{\bullet}{\Delta}_\omega, \Theta_{\text{nor}}]\Theta_{\text{inv}}^{-1} + \Theta_{\text{nor}}[\ \overset{\bullet}{\Delta}_\omega, \Theta_{\text{inv}}^{-1}] = 0 \tag{10.27}$$

which means that Θ_{ser} is a ***resurgence constant*** : the resurgence in both factors of (10.24) cancels out. This in turn explains how Θ_{ser} can have a regular sum in a full neighbourhood of $\underset{\bullet}{0}$, while the resurgent factors Θ_{nor} and Θ_{inv}^{-1} have regular sums in sectorial neighbourhoods only.

We may also observe (reasoning on a girator for definiteness) that while the factors $\Theta_{\text{nor}}^{\pm 1}$ and $\Theta_{\text{inv}}^{\pm 1}$ admit convergent mould expansions involving the moulds $\mathcal{V}^\bullet(z)$ and $\mathcal{V}\!\!\!/^\bullet(z)$ of §8, the product $\Theta_{\text{ser}}^{\pm 1}$ admits simultaneously two types of expansions : one (already encountered in §5 and §6) involving the ***plain compensators*** $\mathcal{S}_{\text{co}}^\bullet(z)$ and $\mathcal{S}\!\!\!/_{\text{co}}^\bullet(z)$; and another involving the ***compensated resurgence monomials*** $\mathcal{V}_{\text{co}}^\bullet(z)$ and $\mathcal{V}\!\!\!/_{\text{co}}^\bullet(z)$ studied in §9.

We cannot enter into details for lack of space, but the following two examples will clarify the preceding points.

Example 6. Vector fields with one degree of resonance. Compensation of resurgence.

This is example 1 studied from the analytic angle. Here, the general bridge equation (1.48) involves holomorphic invariants of the form (see [E.3] or [E.7]) :

$$\mathbb{A}_\omega = u^{n(\omega)}.\left\{A_\omega^0 \partial_z + \sum_{i=1}^{\nu-1} A_\omega^i\ u_i \partial_{u_i}\right\} \tag{10.28}$$

with constant coefficients A_ω^i. In the normal chart this becomes :

$$\mathbb{A}_\omega = I^\omega\ {}^*\mathbb{A}_\omega \tag{10.29}$$

with :

$${}^*\mathbb{A}_\omega = \left\{{}^*A_\omega^0 X^{\text{nor}} + \sum_{i=1}^{\nu}\ {}^*A_\omega^i x_i \partial_{x_i}\right\} \tag{10.30}$$

$$\sum_{i=1}^{\nu} m_i\ {}^*A_\omega^i = 0 \tag{10.31}$$

$$I^\omega = x^{\sigma(\omega)} e^{-\omega x^{-m}} \tag{10.32}$$

$$\sigma_i(\omega) = n_i(\omega) + \left(\sum_{i=1}^{\nu} \tau_i n_i(\omega)\right)\ m_i \quad \text{if} \quad \omega = \sum_{i=1}^{\nu} n_i(\omega)\lambda_i \tag{10.33}$$

Due to the resonance, the $n_i(\omega)$ are not determined by ω, but the $\sigma_i(\omega)$ are. If we now revert to the notations of example 1 and choose :

$$\alpha_1 = 1/\lambda_1;\ \alpha_2 = \alpha_3 = \dots \alpha_\nu = 0 \tag{10.34}$$

the normal variable x_i and the linear variables y_i are connected by :

$$x^{-m} \equiv y^{-m} + (1/\lambda_1)\log y_1 \tag{10.35}$$

Using the *unpointed* alien derivations Δ_ω and Δ'_ω relative to $z = x^{-m}$ and $z' = y^{-m}$, we obtain for the factors (10.24) of Θ_{ser} two parallel bridge equations :

$$[\Delta_\omega, \Theta_{\mathrm{nor}}] = -\Theta_{\mathrm{nor}}.(x^{\sigma(\omega)}\ {}^*\mathbf{A}_\omega) \tag{10.36}$$

$$[\Delta'_\omega, \Theta_{\mathrm{inv}}] = -\Theta_{\mathrm{inv}}.(y^{\sigma(\omega)}\ y_1^{-\omega/\lambda_1}{}^*\mathbf{A}_\omega) \tag{10.37}$$

which make perfect sense if we interpret them component-wise, isolating on both sides the power series in x^m and y^m and regarding the other variables (minus one) as mere parameters, inert under alien differentiation. In this identification process, the factor $y_1^{-\omega/\lambda_1}$ must of course be interpreted as an entire series of $\log y_1$, via the obvious formula.

Example 7. Resonant local diffeos of C. Compensation of resurgence.

This is the *analytic counterpart* of example 5, as well as the *resonant counterpart* of the example studied under remark 5 of section 7.

Let us consider two local diffeos f_j with multipliers $\ell_j = 1$ and levels $p_j = 1$ (to simplify) but with general residues ρ_j :

$$f_j(x) = x + a_j x^2 + (a_j)^2(1+\rho_j)x^3 + \dots \qquad (j = 1,2;\ a_j \neq 0) \tag{10.38}$$

Let us also consider the usual formal iterators $\tilde{f}_j^*$ of the form :

$$\tilde{f}_j^*(x) = x^{-1} - \rho_j \log x + \sum_{n\geq 1} c_{j,n}x^n \tag{10.39}$$

and characterised by :

$$\tilde{f}_j^* \circ f_j(x) \equiv 1 + \tilde{f}_j^*(x) \tag{10.40}$$

so that for any $r \in \mathbf{Z}$:

$$\tilde{f}_j^*(e^{2\pi i r}x) \equiv (e^{-2\pi i r}\tilde{f}_j^*(x) - 2\pi i r\rho_j \tag{10.41}$$

Next, consider the formal series $\tilde{f}_{ij}$ of the form :

$$\tilde{f}_{ij}(x) = x + x\tilde{\varphi}_{ij}(x) \text{ with } \tilde{\varphi}_{ij}(x) \in \mathbb{C}[[x, x\log x]] \tag{10.42}$$

and characterised by :

$$\tilde{f}_i^* = \tilde{f}_{ij} \circ \tilde{f}_j^* \qquad \text{with } (i,j) = (1,2) \text{ or } (2,1) \tag{10.43}$$

If we now denote by $\underset{\bullet}{r}$ any rotation of r turns :

$$\underset{\bullet}{r}\,(x) \overset{\text{def}}{=} e^{2\pi i r}x \qquad (x \in \underset{\bullet}{\mathbb{C}},\ r \in \mathbb{Z}^*) \tag{10.44}$$

and if we assume :

$$a_1 = a_2 = 1; \qquad 2\pi i r(\rho_1 - \rho_2) = 1 \tag{10.45}$$

then, using (10.41), we readily check the following formal linearisation equations :

$$\underset{\bullet}{f_1} = \tilde{f}_{12} \circ \underset{\bullet}{r} \circ \tilde{f}_{21} \text{ with } \underset{\bullet}{f_1} \overset{\text{def}}{=} \underset{\bullet}{r} \circ f_1 \tag{10.46}$$

$$\underset{\bullet}{f_2}^{-1} = \tilde{f}_{21} \circ \underset{\bullet}{r} \circ \tilde{f}_{12} \text{ with } \underset{\bullet}{f_2} \overset{\text{def}}{=} \underset{\bullet}{r}^{-1} \circ f_2 \tag{10.47}$$

But iterators $\tilde{f}_j^*(x)$ are resurgent in $z = x^{-1}$ and satisfy the bridge equation :

$$\overset{\bullet}{\Delta}_\omega \tilde{f}_j^* = -A_\omega \exp(-\omega \tilde{f}_j^*) \qquad (\forall \omega \in 2\pi i \mathbb{Z}^*) \tag{10.48}$$

Therefore if we assume that f_1 and f_2 have the same holomorphic invariants, that is to say, if the coefficient A_ω in (10.48) do not depend on j, we immediately check that the series $\tilde{f}_{ij}$ are *resurgence constants* :

$$\overset{\bullet}{\Delta}_\omega \tilde{f}_{ij} \equiv 0 \qquad (\forall \omega) \tag{10.49}$$

with *seriable sums* f_{ij} which, unlike the sectorial sums f_i^* of the iterators, are defined and regular in a full, spiral-like neighbourhood of $\underset{\bullet}{0}$, so that the linearisation equations (10.46) (10.47), from formal, become effective. This is possibly (after the examples of §9) one of the simplest instances of resurgence compensation.

11. CONCLUSION AND SUMMARY.

We have shown in this paper that ***girators and girations admit seriable linearisations valid in ramified neighbourhoods*** of $\underset{\bullet}{0} \in \underset{\bullet}{\mathbb{C}}^{\nu}$. Actually, as we propose to show in a follow-up investigation, all local analytic objects (whether vector fields or diffeos) under the sole assumption of non-degeneracy (no vanishing multipliers) also admit seriable linearisations, although for them the relevant constructions are somewhat less explicit than for girators and girations, and the geometric-dynamical interpretation somewhat less direct than (1.63) (1.64).

To conclude, let us review the main merits of seriable linearisation.

(M1) Like analytic-entire linearisation, seriable-ramified linearisation $x_i = k_i^{\mathrm{ser}}(y)$ is "quasianalytic" in the sense that the formal series $\tilde{k}_i^{\mathrm{ser}}$ constructively determine k_i^{ser}.

(M2) But unlike analytic linearisation, whose existence is guaranteed only in the absence of the three complications C_1, C_2, C_3 (resonance, quasiresonance, nihilence), seriable linearisation holds ***in all cases***, subject only to non-degeneracy.

(M3) Even when analytic linearisation is available (according to statement A_2 of section 1), the size of the Siegel linearisation domain is no continuous function of the multipliers of the object. It is highly sensitive to their arithmetical properties : when the sum S in (1.19) is large, the Siegel domain tends to be very small. The spiral-like domain of seriable linearisation, on the other hand, depends continuously on the multipliers and is always fairly large.

(M4) The optimal spiralling speed of the domain of seriable linearisation can be explicitly specified in function of the resonance or quasiresonance of the multipliers.

(M5) For resonant objects, there is a remarkable interplay between ***seriability*** and ***compensated resurgence***, which gets reflected in the symmetry between ***plain compensators*** and ***compensated resurgence monomials*** (see §9 ans §10).

(M6) Seriable linearisation leads to a generalisation of the notion of ***holonomy*** (see §8) and provides a systematic, unified method for constructing intrinsic correspondences between analytic conjugacy classes of local objects (see §8).

(M7) Seriable linearisation, suitably reinterpreted, ***extends to non-analytic local objects*** (defined for instance by power series in Gevrey or more general Carleman classes), which are ***ex hypothesi*** beyond the pale of geometry, and yet give rise to non-trivial conjugacy problems (see for ex. §7, remark 6, for the extension of the Douady-Ghys lemma to non-analytic classes).

12. REMINDER ABOUT MOULDS; ARBORIFICATION; RESURGENT FUNCTIONS; ALIEN DERIVATIONS.

The following ultraconcise reminders are no substitute for full definitions, but references are appended to each subsection.

A. Moulds and comoulds.

A *mould* $A^\bullet$ is a family $\{A^{\boldsymbol{\omega}} = A^{\omega_1,\dots,\omega_r}\}$ of elements of a commutative algebra $\mathcal{A}$, indexed by sequences $\boldsymbol{\omega}$ of ω_i ranging through a commutative semigroup Ω. ***Mould multiplication*** is defined by :

$$(12.1) \qquad A_3^\bullet = A_1^\bullet \times A_2^\bullet \Leftrightarrow A_3^{\omega_1,\dots,\omega_r} = \sum_{i=0}^{r} A_1^{\omega_1,\dots,\omega_i} \; A_2^{\omega_{i+1},\dots,\omega_r}$$

For any three sequences $\boldsymbol{\omega}, \boldsymbol{\omega}', \boldsymbol{\omega}''$ we put :

$$(12.2) \qquad \mathrm{sh}\begin{pmatrix} \boldsymbol{\omega}', \boldsymbol{\omega}'' \\ \boldsymbol{\omega} \end{pmatrix} = \text{number of order} - \text{preserving bijections of } \boldsymbol{\omega}' \oplus \boldsymbol{\omega}'' \text{ into } \boldsymbol{\omega}$$

$$(12.3) \qquad \mathrm{ctsh}\begin{pmatrix} \boldsymbol{\omega}', \boldsymbol{\omega}'' \\ \boldsymbol{\omega} \end{pmatrix} = \text{number of order} - \text{preserving surjections of } \boldsymbol{\omega}' \oplus \boldsymbol{\omega}'' \text{ into } \boldsymbol{\omega}$$

A mould $A^\bullet$ is said to be *symmetral* or *symmetrel* if $A^\emptyset = 1$ and if for any pair $\boldsymbol{\omega}', \boldsymbol{\omega}''$:

$$(12.4) \qquad A^{\boldsymbol{\omega}'} A^{\boldsymbol{\omega}''} = \sum \mathrm{sh}\begin{pmatrix} \boldsymbol{\omega}', \boldsymbol{\omega}'' \\ \boldsymbol{\omega} \end{pmatrix} A^{\boldsymbol{\omega}} \qquad (symmetral)$$

$$(12.5) \qquad A^{\boldsymbol{\omega}'} A^{\boldsymbol{\omega}''} = \sum \mathrm{ctsh}\begin{pmatrix} \boldsymbol{\omega}', \boldsymbol{\omega}'' \\ \boldsymbol{\omega} \end{pmatrix} A^{\boldsymbol{\omega}} \qquad (symmetrel)$$

For instance :

$$(12.6) \qquad A^{\omega_1} A^{\omega_2,\omega_3} = A^{\omega_1,\omega_2,\omega_3} + A^{\omega_2,\omega_1,\omega_3} + A^{\omega_2,\omega_3,\omega_1} \qquad (symmetral)$$

$$(12.7) \qquad A^{\omega_1} A^{\omega_2,\omega_3} = \text{idem } + A^{\omega_1+\omega_2,\omega_3} + A^{\omega_2,\omega_1+\omega_3} \qquad (symmetrel)$$

The simplest moulds are $1^\bullet$ and $I^\bullet$:

$$(12.8) \qquad 1^\emptyset = 1; \; 1^{\omega_1,\dots,\omega_r} = 0 \qquad \text{if } r \geq 1$$

$$(12.9) \qquad I^{\omega_1} \equiv 1 \; (\forall \omega_1); \; I^{\omega_1,\dots,\omega_r} = 0 \qquad \text{if } r = 0 \text{ or } r \geq 2.$$

$1^\bullet$ is neutral respective to mould multiplication X and $I^\bullet$ is neutral respective to mould composition 0 (which we don't require here).

Comoulds $\mathbf{B}_\bullet$ have sequences $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r)$ as lower indexes. They usually assume values in bialgebras $\mathcal{B}$ of differential operators, endowed with a non-commutative, associative product and a commutative co-product σ :

$$(12.10) \qquad \sigma : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$$

Most comoulds are either cosymmetral or cosymmetrel :

$$\sigma(\mathbf{B}_{\boldsymbol{\omega}}) = \sum \mathrm{sh}\binom{\boldsymbol{\omega}', \boldsymbol{\omega}''}{\boldsymbol{\omega}} \mathbf{B}_{\boldsymbol{\omega}'} \otimes \mathbf{B}_{\boldsymbol{\omega}''} \ (cosymmetral) \tag{12.11}$$

$$\sigma(\mathbf{B}_{\boldsymbol{\omega}}) = \sum \mathrm{ctsh}\binom{\boldsymbol{\omega}', \boldsymbol{\omega}''}{\boldsymbol{\omega}} \mathbf{B}_{\boldsymbol{\omega}'} \otimes \mathbf{B}_{\boldsymbol{\omega}''} \ (cosymmetrel) \tag{12.12}$$

Thus if the $\mathbf{B}_{\omega_i}$ are ordinary derivations, we have :

$$\sigma(\mathbf{B}_{\omega_i}) = \mathbf{B}_{\omega_i} \otimes 1 + 1 \otimes \mathbf{B}_{\omega_i} \tag{12.13}$$

and the comould $\mathbf{B}_\bullet$ defined by:

$$\mathbf{B}_{\omega_1,\ldots,\omega_r} \overset{\text{def}}{=} \mathbf{B}_{\omega_r} \ldots \mathbf{B}_{\omega_1} \tag{12.14}$$

is cosymmetral.

The *contraction* of a mould $A^\bullet$ with a comould $\mathbf{B}_\bullet$ is a sum of the form :

$$\Theta = \sum A^\bullet B_\bullet = \sum A^{\boldsymbol{\omega}} \mathbf{B}_{\boldsymbol{\omega}} \tag{12.15}$$

extending to *all* sequences $\boldsymbol{\omega}$, including $\boldsymbol{\omega} = \emptyset$. If $A^\bullet$ and $\mathbf{B}_\bullet$ are well-matched (i.e. symmetral and cosymmetral, or symmetrel and cosymmetrel), their contraction Θ is a formal automorphism :

(12.16). $$\sigma(\Theta) = \Theta \otimes \Theta$$

For details and examples, see [E.1],[E.3],[E.8].

B. Arborification and coarborification.

Arborification means replacing fully ordered sequences $\boldsymbol{\omega}$ by sequences $\overset{<}{\boldsymbol{\omega}}$ with an *arborescent* partial order on them (i.e. each element ω_i has at most one immediate antecedent ω_{i_-}).

For any pair $(\boldsymbol{\omega}, \overset{<}{\boldsymbol{\omega}})$ we define sh $\binom{\overset{<}{\boldsymbol{\omega}}}{\boldsymbol{\omega}}$ and ctsh $\binom{\overset{<}{\boldsymbol{\omega}}}{\boldsymbol{\omega}}$ as the number of order-preserving bijections (resp. surjections) of $\overset{<}{\boldsymbol{\omega}}$ into $\boldsymbol{\omega}$.

Symmetral arborification-coarborification obeys the formulae :

$$A^{\overset{<}{\boldsymbol{\omega}}} = \sum \mathrm{sh}\binom{\overset{<}{\boldsymbol{\omega}}}{\boldsymbol{\omega}} A^{\boldsymbol{\omega}} \tag{12.17}$$

$$\mathbf{B}_{\omega} = \sum \mathrm{sh}\begin{pmatrix} \overset{<}{\omega} \\ \omega \end{pmatrix} \mathbf{B}_{\overset{<}{\omega}} \tag{12.18}$$

Symmetrel arborification-coarborification obeys the formulae :

$$A^{\overset{<}{\omega}} = \sum \mathrm{ctsh}\begin{pmatrix} \overset{<}{\omega} \\ \omega \end{pmatrix} A^{\omega} \tag{12.19}$$

$$\mathbf{B}_{\omega} = \sum \mathrm{ctsh}\begin{pmatrix} \overset{<}{\omega} \\ \omega \end{pmatrix} \mathbf{B}_{\overset{<}{\omega}} \tag{12.20}$$

Whereas (12.17) or (12.19) define $A^{\overset{<}{\omega}}$, neither (12.18) nor (12.20) suffice to determine $\mathbf{B}_{\overset{<}{\omega}}$. So we add other natural conditions. For instance, for differential operators $\mathbf{B}_{\omega_i}$ as in (12.13) and a comould $\mathbf{B}_{\bullet}$ as in (12.14), we define $\mathbf{B}_{\overset{<}{\bullet}}$ by :

$$\mathbf{B}_{\overset{<}{\omega}} . \varphi \overset{\text{def}}{=} \{\mathbf{B}_{\omega_r}, \ldots, \mathbf{B}_{\omega_1}\}\, \varphi \tag{12.21}$$

by letting each $\mathbf{B}_{\omega_i}$ in $\{\ldots\}$ act on φ alone if ω_i has no antecedent in $\overset{<}{\omega}$, or on the coefficients of $\mathbf{B}_{\omega_{i\_}}$ if $\omega_{i\_}$ is the (unique) antecedent of ω_i in $\overset{<}{\omega}$.

Since arborification and coarborification (whether symmetral or symmetrel) are dual operations, we have :

$$\Theta = \sum A^{\bullet}\mathbf{B}_{\bullet} = \sum A^{\overset{<}{\bullet}}\mathbf{B}_{\overset{<}{\bullet}} \tag{12.22}$$

but in very numerous instances the seemingly harmless passage from $\bullet$ to $\overset{<}{\bullet}$ restores *normal convergence.*

For details and examples, see [E.8].

C. The algebras RES of resurgent functions.

There are three models : *formal, convolutive, geometric.* The *formal model* $\widetilde{\mathrm{R}}\mathrm{ES}$ consists of formal series like :

$$\tilde{\varphi}(z) = \sum a_n z^{-n} \qquad (n \in \mathbb{N} \text{ or } \mathbb{R}^+) \tag{12.23}$$

or of a more general type :

$$\tilde{\varphi}(z) = \sum a_n \epsilon_n(z) \qquad (\epsilon_n(z) \gg \epsilon_{n+1}(z)) \tag{12.24}$$

with each "monomial" $\epsilon_n(z)$ decreasing subexponentially as $z \to \underset{\bullet}{\infty} \in \underset{\bullet}{\mathbb{C}}$.

The product here is the (formal) multiplication of (formal) series.

The *convolutive model* $\overset{\triangledown}{\mathrm{RES}}$ consists of pairs :

$$\overset{\triangledown}{\varphi} = (\overset{\wedge}{\varphi}, \overset{\vee}{\varphi}) = \text{(minor, major)} \tag{12.25}$$

of germs $\overset{\wedge}{\varphi}(\zeta)$ and $\overset{\vee}{\varphi}(\zeta)$ at $\underset{\bullet}{0} \in \underset{\bullet}{\mathbb{C}}$ in the ζ-plane conjugate to the z-plane.

The *minors* $\overset{\wedge}{\varphi}(\zeta)$ are assumed to be *endlessly continuable* in the ζ-plane, along any (discretely punctured) broken line. The *majors* $\overset{\vee}{\varphi}(\zeta)$ are defined up to regular germs at $\underset{\bullet}{0}$ and relate to the minors as follows (for ζ close to $\underset{\bullet}{0}$) :

$$\overset{\wedge}{\varphi}(\zeta) = \overset{\vee}{\varphi}(\zeta) - \overset{\vee}{\varphi}(e^{-2\pi i}\zeta) \tag{12.26}$$

The product here is *convolution* $(*)$, which for integrable functions (at $\underset{\bullet}{0}$) reduces to *minor convolution* :

$$\overset{\wedge}{\varphi}_1 * \overset{\wedge}{\varphi}_2(\zeta) = \int_0^\zeta \overset{\wedge}{\varphi}_1(\zeta_1)\ \overset{\wedge}{\varphi}_2(\zeta - \zeta_1) d\zeta_1 \qquad (\text{for } \zeta \sim \underset{\bullet}{0}) \tag{12.27}$$

The *geometric model* ${}^{\theta}\mathrm{RES}$ of direction θ consists of holomorphic germs $\varphi(z)$ which are defined and have (at most) subexponential growth in half-planes $\mathrm{Re}(ze^{i\theta}) \geq x(\varphi) > 0$. The product here is ordinary, point-wise multiplication.

The passage from the convolutive model to the geometric model of direction θ is via the *Laplace transform* $\mathcal{L}$, which reduces to :

$$\overset{\triangledown}{\varphi}(\zeta) \to \varphi(z) = \int_{\arg \zeta = \theta} \overset{\wedge}{\varphi}(\zeta) e^{-\zeta z} d\zeta \tag{12.28}$$

when $\overset{\triangledown}{\varphi}(\zeta)$ is integrable at $\underset{\bullet}{0}$.

The passage from the geometric (resp. formal) models to the convolutive model is via the effective (resp. formal or term-wise) *Borel transform* $\mathcal{B}$:

$$\varphi(z) \mapsto \overset{\wedge}{\varphi}(\zeta) = \frac{1}{2\pi i} \int_{\infty_1}^{\infty_2} \varphi(z) e^{+\zeta z} dz \tag{12.29}$$

$$\varphi(z) \mapsto \overset{\vee}{\varphi}(\zeta) = \frac{1}{2\pi i}\int_{u}^{\infty_3} \varphi(z)e^{+\zeta z}dz \tag{12.30}$$

for $\infty_1, \infty_2, \infty_3$ suitably located at infinity.

For details and examples, see [E.1], [E.3], [E.5], [E.8].

D. Alien derivations.

These are linear operators Δ_ω, with indexes ω in $\underset{\bullet}{\mathbb{C}}$, which are useful on three accounts.

First, they are easy to handle, being derivations :

$$\Delta_\omega(\overset{\triangledown}{\varphi}_1 * \overset{\triangledown}{\varphi}_2) = (\Delta_\omega \overset{\triangledown}{\varphi}_1) * \overset{\triangledown}{\varphi}_2 + \overset{\triangledown}{\varphi}_1 * (\Delta_\omega \overset{\triangledown}{\varphi}_2) \tag{12.31}$$

Second, they measure the singularities of minors $\overset{\wedge}{\varphi}(\zeta)$ over the point ω, which is essential, since those singularities are responsible for the divergence of the corresponding series $\tilde{\varphi}(z)$ in the formal model.

Third, they enable us to describe, by means of so-called *resurgence equations* :

$$E_\omega((\overset{\triangledown}{\varphi}, \Delta_\omega \overset{\triangledown}{\varphi}) \equiv 0 \tag{12.32}$$

the close connection which usually exists between the behaviour of $\overset{\wedge}{\varphi}(\zeta)$ near $\underset{\bullet}{0}$ and near its other singularities ω. This self-reproduction property is an outstanding feature of all *resurgent functions* of natural origin.

Alien derivations act as follows on the *convolutive model* :

$$\Delta_\omega : \qquad \overset{\triangledown}{\varphi} = (\overset{\wedge}{\varphi}, \overset{\vee}{\varphi}) \mapsto \overset{\triangledown}{\varphi}_\omega = (\overset{\wedge}{\varphi}_\omega, \overset{\vee}{\varphi}_\omega) \tag{12.33}$$

with :

$$\overset{\wedge}{\varphi}_\omega(\zeta) \overset{\mathrm{def}}{=} \sum_{\epsilon_i=\pm} \epsilon_r \delta^{\epsilon_1,\dots,\epsilon_{r-1}} \overset{\wedge}{\varphi}{}^{\epsilon_1,\dots,\epsilon_r}_{\omega_1,\dots,\omega_r}(\zeta+\omega) \tag{12.34}$$

$$\overset{\vee}{\varphi}_\omega(\zeta) \overset{\mathrm{def}}{=} \sum_{\epsilon_i=\pm} \delta^{\epsilon_1,\dots,\epsilon_{r-1}} \overset{\wedge}{\varphi}{}^{\epsilon_1,\dots,\epsilon_{r-1},+}_{\omega_1,\dots,\omega_{r-1},\omega_r}(\zeta+\omega) \tag{12.35}$$

first for ζ close to $\overset{\bullet}{0}$ on the radius $\arg\zeta = \arg\omega$, and then in the large by analytic continuation. Here, the ω_i (with $\omega_r = \omega$) denote the successive singularities of $\overset{\wedge}{\varphi}(\zeta)$ on the segment $[\underset{\bullet}{0}, \omega]$ and $\overset{\wedge}{\varphi}{}^{\epsilon_1,\dots,\epsilon_r}_{\omega_1,\dots,\omega_r}(\zeta+\omega)$ denotes the determination of $\overset{\wedge}{\varphi}(\zeta+\omega)$ that corresponds to the right (resp. left) circumvention of ω_i if $\epsilon_i = +$ (resp. $\epsilon_i = -$). Lastly, the *weights* $\delta^{\bullet}$ are given by :

$$\delta^{\epsilon_1,\dots,\epsilon_{r-1}} = \delta_{p,q} = \frac{p!q!}{(p+q+1)!} = \frac{p!q!}{r!} \tag{12.36}$$

where p and q are the number of $+$ and $-$ signs in $(\epsilon_1, \dots, \epsilon_{r-1})$.

The alien derivations Δ_ω *generate a free Lie algebra.* There being no danger of confusion, we retain the same symbols Δ_ω to denote their (pulled-back) action in the *multiplicative models* (formal or geometric). We also introduce the *pointed alien derivations* :

$$\overset{\bullet}{\Delta}_\omega \overset{\text{def}}{=} e^{-\overset{\bullet}{\omega} z} \Delta_\omega \tag{12.37}$$

which act in the *multiplicative models tensored by exponentials* $\exp(-\overset{\bullet}{\omega} z)$, and obey the rules :

$$[\overset{\bullet}{\Delta}_\omega, \partial] \equiv 0 \qquad (\text{with } \partial = \partial/\partial z) \tag{12.38}$$

$$\overset{\bullet}{\Delta}_\omega (\varphi \circ f) \equiv (\overset{\bullet}{\Delta}_\omega \varphi) \circ f + (\partial\varphi) \circ f \,.\, \overset{\bullet}{\Delta}_\omega f \tag{12.39}$$

if $f(z) = z + o(z)$.

For details and examples, see [E.1], [E.3], [E.7], [E.8].

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The notion of compensation, as used in this article, along with its applications to vector fields and diffeomorphisms, was introduced by us in 1986 and expounded in an unpublished manuscript. We mentioned special applications in [E.4] and [E.5]. It was also brought to our attention that Eliason had already made use of the word "compensation" (see [Eℓ]) but with another meaning and in a different context (with entire series and without ramified variables).

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