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# SCHUR QUADRICS, CUBIC SURFACES AND RANK 2 VECTOR BUNDLES OVER THE PROJECTIVE PLANE

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Let $\Sigma \subset P^3$ be a smooth cubic surface. It is known that $S$ contains 27 lines. Out of these lines one can form 36 *Schläfli double - sixes* i.e., collections $\{l_1, ..., l_6\}, \{l'_1, ..., l'_6\}$ of 12 lines such that each $l_i$ meets only $l'_j$, $j \neq i$ and does not meet $l_j, j \neq i$, see n.0.1 below. In 1881 F. Schur proved **[S]** that any double - six gives rise to a certain quadric $Q$ , called *Schur quadric* which is characterized as follows: for any $i$ the lines $l_i$ and $l'_i$ are orthogonal with respect to (the quadratic form defining) $Q$.

The aim of the present paper is to relate Schur's construction to the theory of vector bundles on $P^2$ and to generalize this construction along the lines of the said theory.

Let us describe the vector bundle interpretation of the Schur quadric. Note that the first six lines $\{l_1, ..., l_6\}$ of a double - six on $\Sigma$ define a blow-down $\pi : \Sigma \to P^2$ which takes the lines $l_i$ into some points $p_i \in P^2$. These points are in general position i.e. no three of them lie on a line. Let $\check{P}^2$ be the dual projective plane and $H_i \subset \check{P}^2$ be the lines corresponding to $p_i$. The union $\mathcal{H}$ of these lines is a divisor with normal crossing in $\check{P}^2$. Let $E(\mathcal{H}) = \Omega^1_{\check{P}_2}(\log \mathcal{H})$ be the corresponding vector bundle (locally free sheaf) of logarithmic 1-forms on $\check{P}^2$. The twisted bundle $E = E(\mathcal{H})(-2)$ is a stable rank 2 bundle on $\check{P}^2$ with Chern classes $c_1 = -1, c_2 = 4$ (see **[DK]**). For such bundles K.Hulek **[Hu1]** has defined the notion of a *jumping line of the second kind* (shortly JLSK). This is a line $l \subset \check{P}^2$ such that the restriction of $E$ to the first infinitesimal neigborhood $l^{(1)}$ of $l$ is not isomorphic to $\mathcal{O}_{l^{(1)}} \oplus \mathcal{O}_{l^{(1)}}(-1)$. Hulek has shown that such lines form a

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curve $C(E)$ in the projective plane of lines in $\check{P}^2$ i.e. in $P^2$. Now the result is as follows.

**Theorem 1.** *The space $P^3$ containing the cubic surface $\Sigma$ is naturally identified with the projectivization of $H^1(\check{P}^2, E(-1))^*$. Under this identification the Schur quadric $Q$ becomes dual to the zero locus of the quadratic form given by the cup-product*

$$H^1(\check{P}^2, E(-1)) \otimes H^1(\check{P}^2, E(-1)) \to H^2(\check{P}^2, \bigwedge^2(E(-1))) = H^2(\check{P}^2, \mathcal{O}(-3)) = \mathbf{C}.$$

*The intersection $\Sigma \cap Q$ is mapped, under the projection $\pi : \Sigma \to P^2$, to the curve of JLSK $C(E)$.*

More generally, the whole theory of Hulek **[Hul]** of rank 2 vector bundles on $P^2$ with odd $c_1$ can be given a "geometric" interpretation involving some natural generalizations of cubic surfaces, double - sixes and Schur quadrics. This is done in §2 of the paper. This interpretation implies Theorem 1.

The outline of the paper is as follows. In §0 we recall some known (and less known) facts about cubic surfaces and Schur quadrics. In §1 we give a short overview of Hulek's theory of monads corresponding to vector bundles with $c_1 = -1$. In §2 we give an interpretation of Hulek's theory mentioned above. In §3 we consider bundles of logarithmic 1-forms corresponding to arrangements of $2d$ lines in $P^2$ in general position. The main result of this section is that all these bundles satisfy certain condition of $\Sigma$ - genericity in the sense defined in §2, which makes working with bundles satisfying this condition easier. Finally, in §4 we consider various examples of the previous constructions corresponding to some special types of vector bundles.

## §0. Cubic surfaces.

**0.1.** Here we recall some standard known facts about cubic surfaces. All the proofs can be found either in **[H]**, Ch.V, §4 or in **[M]** or can be easily reconstructed by the reader. Let $p_1, ..., p_6$ be six distinct points in the projective plane $P^2$. Assume that no three of these points lie on a line. Denote by $Z$ the union of the points $p_i$ and by $\mathcal{J}_Z \subset \mathcal{O}_{P(V)}$ the sheaf of ideals of $Z$. The linear system $P\big(H^0(\mathcal{J}_Z(3))\big)$ of cubic curves through $Z$ is of dimension 3 and defines a rational map

$$f : P^2 \to P(H^0(\mathcal{J}_Z(3)^*) = P^3$$

whose image is a cubic surface, denoted $\Sigma$. The rational map $f$ comes from a regular map $f' : \mathrm{Bl}_Z(P^2) \to P^3$ where $\mathrm{Bl}_Z(P^2)$ is the blow up of $Z$. Let $\pi : \mathrm{Bl}_Z(P^2) \to P^2$ be the projection. If we further assume that the points $p_i$ do not lie on a conic then $f'$ is an isomorphism and $\Sigma$ is nonsingular. If $p_i$ do lie on a conic then $\Sigma$ is singular and $f'$ blows down this conic to a singular point of $\Sigma$.

Suppose $\Sigma$ is nonsingular. Then $\Sigma$ has 27 lines on it. They can be grouped into three subsets:

$$\{l_1, ..., l_6\},\ \{l_1', ..., l_6'\},\ \{m_{ij},\ 1 \le i < j \le 6\}. \tag{0.1}$$

The lines $l_i$ are the images under $f'$ of the exceptional lines $\pi^{-1}(p_i)$. The lines $l_i'$ are images under $f'$ of proper transforms of the conics $C_i \subset P^2$ passing through $Z - \{p_i\}$. Finally the lines $m_{ij}$ are images of the proper transforms of the lines $< p_i, p_j >$ joining the points $p_i$ and $p_j$.

The first two groups of lines form a *double - six* which means that

$$l_i \cap l_j = \emptyset, \quad l_i' \cap l_j' = \emptyset, \quad l_i \cap l_j' \neq \emptyset \qquad \text{iff} \quad i \neq j. \tag{0.2}$$

Every set of 6 disjoint lines on $\Sigma$ can be included in a unique double - six from which $\Sigma$ can be reconstructed uniquely. There are 36 double - sixes of $\Sigma$. Every double - six defines two regular birational maps $\pi_1 : \Sigma \to P^2$, $\pi_2 : \Sigma \to P^2$, each blowing down one of the two sixes (sixtuples of disjoint lines) of the double - six.

The birational map $\pi_2 \circ \pi_1^{-1} : P^2 \to P^2$ is given by the linear system of quintics with double points at $p_i$. The two collections of 6 points in $P^2$ given by $\{\pi_1(l_i)\}$ and $\{\pi_2(l'_i)\}$ are associated to each other in the sense of Coble (cf.**[DO]**,**[DK]**).

**0.2.** Here we shall discuss somewhat less known facts about the determinantal representation of a cubic surface **[B]**. A modern treatment of this can be found in **[G]**,**[Gi]**. Consider the homogeneous ideal of the subscheme $Z$ i.e.

$$I_Z = \bigoplus_{n \geq 0} H^0(P^2, \mathcal{J}_Z(n)) \tag{0.3}$$

in the graded ring $R = \mathbf{C}\,[T_0, T_1, T_2]$. It is easy to see that the ring $R/I_Z$ is Cohen - Macaulay hence of homological dimension 1. Any four linearly independent cubic forms vanishing on $Z$ represent a minimal set of generators of $I_Z$. According to the Hilbert-Burch theorem ( see **[No]**,7.5) the ideal $I_Z$ is generated by the maximal minors of some $3 \times 4$ matrix of homogeneous linear forms. In other words, we have a resolution

$$0 \to R(-4)^3 \to R(-3)^4 \to I_Z \to 0.$$

This resolution gives the resolution of the sheaf $\mathcal{J}_Z(3)$:

$$0 \to \mathcal{O}_{P(V)}(-1)^3 \to \mathcal{O}^4_{P(V)} \to \mathcal{J}_Z(3) \to 0.$$

We can rewrite this resolution in the form

$$0 \to \mathcal{O}_{P^2}(-1) \otimes I^* \xrightarrow{\gamma} \mathcal{O}_{P^2} \otimes L^* \to \mathcal{J}_Z(3) \to 0 \tag{0.4}$$

where vector spaces $I^*$ and $L^*$ of respective dimensions 3 and 4 are defined intrinsically as follows:

$$L^* = H^0(P^2, \mathcal{J}_Z(3)); \tag{0.5}$$

$$I^* = \mathrm{Ker}\{H^0(P^2, \mathcal{O}(1) \otimes L^*) \to H^0(P^2, \mathcal{J}_Z(4))\}. \tag{0.6}$$

Note that one can also obtain (0.4) from the Beilinson spectral sequence applied to the sheaf $\mathcal{J}_Z(3)$. It gives also an isomorphism

$$I^* \cong H^1(P^2, \mathcal{J}_Z(1)).$$

It will be convenient for us to regard henceforth our projective plane $P^2$ as $P(V^*)$ where $V$ is a 3-dimensional vector space. With this choice of notation, the map $\gamma$ in (0.4) is given by a linear map $I^* \otimes V^* \to L^*$. We shall be more interested in the transpose of this map which we denote by

$$g : L \longrightarrow I \otimes V = \mathrm{Hom}(V^*, I). \tag{0.7}$$

Choosing bases in $V, I$ we can regard $g$ as a 3 by 3 matrix of linear forms on $L$. Here is the classical result on the determinantal representation.

**0.3. Proposition.** *The map $g$ is an embedding. The locus*

$$\Sigma = \{x \in P(L) : \ \mathrm{rank}\ g(x) \leq 2\} \tag{0.8}$$

*is a nonsingular cubic surface in $P(L) = P^3$ isomorphic to* $\mathrm{Bl}_Z(P(V^*)$. *An explicit blow-down $\pi_1 : \Sigma \to P(V^*)$ takes $x \in \Sigma$ into* Ker $g(x) \in P(V^*)$. *It is isomorphism outside the set $Z = \{p_1, ., p_6\} \subset P(V^*) = P^2$ (see n. 0.1). The dual blow-down $\pi_2 : \Sigma \to P(I^*)$ takes $x \in \Sigma$ into* $(\mathrm{Im} g(x))^\perp \in P(I^*)$. *It is an isomorphism outside a six - element set $Z^{as} = \{q_1, ..., q_6\} \subset P(I^*)$ (this is the set associated to $Z$).*

Note that a given cubic surface $\Sigma \subset P^3$ has many non-equivalent determinantal representations corresponding to different ways of blowing down $\Sigma$ onto a $P^2$ (i.e. to different choices of a double - six).

**0.4.** All the other attributes of the cubic surface $\Sigma$ can be easily found from the map $g$. For example, the set $Z$ can be recovered in terms of $g$ as follows. Consider the partial transposes of (0.7):

$$g_V : V^* \to I \otimes L^* = \mathrm{Hom}(L, I).$$

$$g_I : I^* \to V \otimes L^* = \mathrm{Hom}(L, V).$$

Then

$$Z = \{z \in P(V^*) : \ \mathrm{rank}\ g_V(z) \leq 2\}. \tag{0.9}$$

The 12 lines of the double - six can be written in the form $A_z = P(\mathbf{A}_z), A'_z = P(\mathbf{A}'_z)$, $z \in Z$ where $\mathbf{A}_z$ and $\mathbf{A}'_z$ are 2-dimensional vector subspaces in $L$ defined for $z \in Z$ as follows:

$$\mathbf{A}_z = Ker(g_V(z)); \tag{0.10}$$

$$\mathbf{A}'_z = Ker(g_I(z)), \tag{0.11}$$

where $z^{\perp} \subset V$ is the 2-plane orthogonal to $z \in P(V^*)$. Thus if $Z = \{p_1, ..., p_6\}$ then the line $A_{p_i}$ is what was denoted in n.0.1 by $l_i$ and the line $A'_{p_i}$ is $l'_i$.

The classical theorem of F. Schur **[S]** can be stated as follows.

**0.5. Theorem.** *There exists a unique, up to a scalar factor, symmetric bilinear form $C(x,y)$ on $L$ with the following property: $C(x,y) = 0$ whenever $x \in A_z, y \in A'_z$ for some $z \in Z$ (i.e. the corresponding lines of the double - six are orthogonal with respect to $C$). This form is non-degenerate.*

*Proof.* a) Non-degeneracy: Suppose such a form $C$ exists and is degenerate. Let $K$ be the kernel of $C$. Suppose dim $K = 1$. Then for any 2-dimensional subspace $\Lambda \subset L$ not meeting $K$ its orthogonal (with respect to $C$) is a 2-subspace containing $K$. Since $P(A_z)$, $P(A'_z)$ form a double - six, $K$ can lie on no more than one among the $A_z$ and no more than one among the $A'_z$. Hence there is a 4-element subset $Z_0 \subset Z$ such that for $z \in Z_0$ both $A_z$ and $A'_z$ do not contain $K$. For such $z$ the space $A'_z$ should coincide with $A_z^{\perp}$ and hence contain $K$. Hence for $z_1 \neq z_2 \in Z_0$ we have $A'_{z_1} \cap A'_{z_2} \neq \{0\}$ which is a contradiction. The cases dim $K$ = 2,3 are similar and left to the reader.

b) Uniqueness: If there are two non-proportional forms $C_1, C_2$ with the required property then for any $\lambda, \mu$ the linear combination $\lambda C_1 + \mu C_2$ also satisfies this property. However, there will be always such $\lambda, \mu$ that the linear combination is non-zero but degenerate. This contradicts a).

**0.6.** It remains to prove the existence part of Theorem 0.5. To do this, let us take the second symmetric power of the map $g$ in (0.7) and use the natural decomposition

$$S^2(I \otimes V) = \left(\bigwedge^2 I \otimes \bigwedge^2 V\right) \quad \oplus \quad (S^2 I \otimes S^2 V). \tag{0.12}$$

By projecting $S^2 g$ to the first summand, we get a linear map

$$S^2 L \longrightarrow S^2(I \otimes V) \longrightarrow \left(\bigwedge^2 I \otimes \bigwedge^2 V\right). \tag{0.13}$$

Note that dim $S^2 L = 10$, and $\dim(\bigwedge^2 I \otimes \bigwedge^2 V) = 9$. Hence the map (0.13) has non-trivial kernel. (We shall see later that this kernel is in fact 1-dimensional).

**0.7. Proposition.** *If $B$ is a non-zero form from the kernel of (0.13) then $B : L^* \to L$ is invertible and $C = B^{-1} : L \to L^*$ is a bilinear form on $L$ satisfying the conditions of Theorem 0.5.*

We shall concentrate on the proof of this proposition.

**0.8.** A form $B \in S^2L$ lying in the kernel of (0.13) is classically called "apolar to all the quadratic forms given by $2 \times 2$ minors of $g$", cf. [**B**]. In general, if $E$ is a vector space then quadratic forms $G \in S^2E$, $H \in S^2E^*$ are called *apolar* if $(G, H)_2 = 0$ where $(\cdot,\cdot)_2$ is the natural pairing $S^2E \otimes S^2E^* \to \mathbf{C}$. Note the particular case when $G$ has rank 2 i.e. $G = e \cdot f$ is the symmetric product of two vectors $e, f \in E$. In this case the apolarity of $G$ and $H$ means that $H(e, f) = 0$.

We shall need a different description of the map dual to (0.13). Let us denote this map by

$$\delta : \bigwedge^2 I^* \otimes \bigwedge^2 V^* \longrightarrow S^2L^*. \qquad (0.14)$$

Let us chose volume forms on $V$ and $I$. Then we can write $\bigwedge^2 V^* = V$, $\bigwedge^2 I^* = I$. It is immediate to see that there are identifications

$$\bigwedge^2 V^* \cong V \quad = \quad H^0(P(V^*), \mathcal{O}_{P(V^*)}(1)); \qquad (0.15)$$

$$\bigwedge^2 I^* \cong I \quad = \quad H^0(P(V^*), \mathcal{J}_Z^2(5)); \qquad (0.16)$$

$$S^2L^* \quad = \quad H^0(P(L), \mathcal{O}(2)) \quad \cong \quad H^0(P(V^*), \mathcal{J}_Z^2(6)). \qquad (0.17)$$

Indeed, (0.15) follows by definition of $\mathcal{O}(1)$; the identification (0.16) expresses the fact that the Cremona transformation $\pi_2 \circ \pi_1^{-1} : P(V^*) \to P(I^*)$ is given by the linear system of quintics with singular points $p_i$, see n. 0.1. Finally, to see (0.17) we note that the embedding of the cubic surface $\Sigma$ into $P(L) = P^3$ is given by the linear system of cubics in $P(V^*)$ through $p_i$, so $L^*$ is the space of cubic polynomials on $V^*$ vanishing at $p_i$. The second symmetric power of this space maps therefore to the space of polynomials of degree 6 vanishing at $p_i$ together with their first derivatives i.e, to the RHS of (0.17); this map is easily seen to be an isomorphism.

**0.9. Lemma.** *Under identifications (0.15) - (0.17) the map $\delta$ corresponds to the multiplication map*

$$H^0(P(V^*),\mathcal{O}(1))\otimes H^0(P(V^*),\mathcal{J}_Z^2(5))\to H^0(P(V^*),\mathcal{J}_Z^2(6)).$$

In other words, quadrics in $P(L)$ are identified with sextics in $P(V^*)$ with double points at $p_i \in Z$ and quadrics from the image of $\delta$ correspond to sextics containing a line.

*Proof of the lemma:* We have the commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\mathrm{Sq}} & S^2L \\ g\downarrow & & \downarrow \\ \mathrm{Hom}(V^*,I) & \xrightarrow{\lambda} & \mathrm{Hom}(\bigwedge^2 V^*, \bigwedge^2 I) \end{array} \tag{0.18}$$

where the map Sq takes $x \mapsto x^2$, the map $\lambda$ takes $\phi \mapsto \bigwedge^2 \phi$ and the map on the right is the same as in (0.13). We keep the volume forms in $I$ and $V$ and identify correspondingly the spaces $\bigwedge^2 I$ and $\bigwedge^2 V$ with $V^*$ and $I^*$. For any $\phi \in \mathrm{Hom}(V^*, I)$ of rank 2 the second exterior power $\bigwedge^2 \phi \in I^* \otimes V^*$ is a tensor of rank 1. Hence it can be written in the form $i^* \otimes v^*$ for some $i^* \in I^*$, $v^* \in V^*$. This shows that the restriction of the map $\lambda \circ g$ to the cubic surface $\Sigma \subset P(L)$ coincides with the composition

$$\Sigma \xrightarrow{\pi_1\times\pi_2} P(I^*)\times P(V^*) \xrightarrow{\mathrm{Segre}} P(I^*\otimes V^*) \tag{0.19}$$

where $\pi_j$ are the blow-downs from n. 0.3.

The map $\lambda\circ g : P(L) \to P(I^*\otimes V^*)$ is given by the linear system $\mathcal{Q}$ of quadrics which is the projectivization of the image of the linear map $\delta$ from (0.14). The system $\mathcal{Q}$ is spanned by the $2\times 2$ minors of the matrix of linear forms on $L$ defining the determinantal representation of $\Sigma$. In other words, the preimage of the linear system of hyperplane sections of $P(I^*\otimes V^*)$ under $\lambda \circ g$ is the linear system of quadric sections on $\Sigma$ which is (the projectivization of) the image of the canonical pairing

$$H^0(P(I^*),\mathcal{O}(1))\otimes H^0(P(V^*),\mathcal{O}(1)) \longrightarrow H^0(\Sigma,\mathcal{O}(2)).$$

By Theorem 0.3, we can make an identification of the projective spaces $P(I^*)$ and

$P(H^0(P(V^*), \mathcal{J}_Z^2(5))^*)$. Under the rational map $P(V^*) \to P(I^*)$ given by the linear system of sections of $\mathcal{J}_Z^2(5)$, zeroes of these sections are preimages of the lines in $P(I^*)$ and the resulting map

$$H^0(P(V^*), \mathcal{O}(1)) \otimes H^0(P(V^*), \mathcal{J}_Z^2(5)) \to H^0(\Sigma, \mathcal{O}(2)) = H^0(P(V^*), \mathcal{J}_Z^2(6))$$

coincides with the natural multiplication map from the assertion of the lemma. So the lemma is proven.

**0.10.** We continue to prove Theorem 0.5 and shall now use Lemma 0.9. Let us consider some particular sextics with double points at $Z = \{p_1, ..., p_6\}$. Let $C_i$ be the unique conic through $Z - \{p_i\}$. We can take a sextic curve which is the union of two lines $< p_i, p_j >$, $< p_k, p_s >$ and two conics $C_k, C_s$. By means of (0.17) this sextic corresponds to some quadric $Q_{ij,ks}$. Moreover, since the quintic $< p_k, p_s > \cup C_k \cup C_s$ belongs to the linear system of quintics singular at points of $Z$, the quadric $Q_{ij,ks}$ lies in the image of the map $\delta$ from (0.14). Now let us take $j = s$. Then our sextic can be represented as the union of two cubic curves through $Z$ namely

$$< p_i, p_j > \cup C_j \quad \text{and} \quad < p_k, p_j > \cup C_k.$$

Since such cubics correspond to hyperplanes in $P(L)$, we conclude that the quadric $Q_{ij,kj}$ is in fact the union of two planes, say $H_{ij}$ and $H_{kj}$. Moreover, $H_{ij}$ cuts out the cubic surface $\Sigma$ along 3 lines $l_i, l'_j, m_{ij}$ (see n. 0.1). The plane $H_{jk}$ cuts out the lines $l_j, l'_k, m_{kj}$ on $\Sigma$. Since the (quadratic form defining the) quadric $Q_{ij,kj} = H_{ij} \cup H_{kj}$ is apolar to our chosen $B \in S^2L$, we conclude that the equations of $H_{ij}$ and $H_{jk}$ (belonging to $L^*$) are $B$ - orthogonal.

**0.11.** Let us now prove Proposition 0.7 and hence Theorem 0.5. The form $B$ is a linear map $L^* \to L$. For any linear subspace $U \subset L$ we define its *polar subspace* (with respect to $B$) to be

$$U_B^\perp = B(U^\perp),$$

where $U^\perp$ denotes the orthogonal subspace of $U$ in $L^*$. If $B$ is non-degenerate then $U_B^\perp$ is the orthogonal complement of $U$ in $L$ with respect to the inverse form $B^{-1} \in S^2L^*$. If $B$ is degenerate and $K \subset L^*$ is its kernel then $U_B^\perp$ is contained in $K^\perp$ for any $U$.

We shall apply the previous notation for projective subspaces in $P(L)$. In particular, if $H \subset P(L)$ is a hyperplane whose equation does not lie in $K = \mathrm{Ker}\ B$ then $H_B^{\perp}$ is a point called the *pole* of $H$.

Let us prove that $B$ is non-degenerate. Let the double - six be $\{l_1, ..., l_6\}$, $\{l'_1, ..., l'_6\}$. Assume first that $B$ is of rank at least 3. Then at most one hyperplane $H_{ij}$ belongs to the kernel of $B$. Without loss of generality we may assume that all planes $H_{ij}$ are not in the kernel except maybe $H_{56}$. Consider the plane $H_{12}$ spanned by lines $l_1$ and $l'_2$ (which intersect). Its equation (in $V^*$) is orthogonal with respect to $B$ to equations of similar planes $H_{21}, H_{23}, H_{31}$ (see n. 0.10). Hence $(H_{12})_B^{\perp} = H_{21} \cap H_{23} \cap H_{31}$ and this intersection is easily seen to be the point $l'_1 \cap l_2$. In this way we show that each $l'_i \cap l_j$ is the pole of some plane $H_{ji}$, where $(i,j) \neq (5,6)$. Since these points obviously span $P(L)$, the form $B$ must be non-degenerate. Now assume that $B$ is of rank at most 2. Since the planes $H_{12}, H_{13}, H_{24}, H_{25}$ are linearly independent, at least one of them is not in the kernel of $B$. Let it be $H_{12}$. Similarly we find that $H_{34}$ and $H_{56}$ are not in the kernel. Their three poles $l'_1 \cap l_2, l'_3 \cap l_4, l'_5 \cap l_6$ are not on a line. This contradicts the assumption that $B$ is of rank at most 2.

It remains to show that $l_i^{\perp} = l'_i$. We have already seen that the point $l'_1 \cap l_2$ is the pole of the plane $H_{12}$ spanned by $l_1$ and $l'_2$. Similarly, $l'_1 \cap l_3$ is the pole of $H_{13} = \mathrm{Span}(l_1, l'_3)$. Hence $l'_1 = \mathrm{Span}(l'_1 \cap l_2, l'_1 \cap l_3)$ is the orthogonal complement of $H_{12} \cap H_{23} = l_1$. Similarly we prove that $l'_i = l_i^{\perp}$ for other $i$.

Theorem 0.5 is completely proven. The reader should compare this rather cumbersome proof with a more straightforward one based on the theory of vector bundles (Theorem 2.17 below).

**0.12. Definition.** *The quadric $Q \subset P(L)$ defined by $C(x,x) = 0$ where $C$ is the quadratic form given by Theorem 0.5, is called the Schur quadric (associated with the double - six $\{A_z, A'_z\}$).*

**0.13. Example.** Let us consider the following 4-dimensional space $L$:

$$L = \left\{ (x_1, ..., x_5) \in \mathbf{C}^5 : \quad \sum x_i = 0 \right\}$$

and define the cubic surface $\Sigma \subset P(L)$ by the equation $\sum x_i^3 = 0$ (the Clebsch diagonal surface). The symmetric group $S_5$ acts on $\mathbf{C}^5$ by permutations of coordinates and preserves $L$ and $\Sigma$. The line

$$l = \left\{ x \in P(L) : \quad x_1 + \frac{1+\sqrt{5}}{2} x_2 + x_3 = x_2 + \frac{1+\sqrt{5}}{2} x_3 + x_4 = 0 \right\}$$

lies on $\Sigma$ and so do all the lines obtained from $l$ by the action of $S_5$. It is known **[Bu]** that the $S_5$ - orbit of $l$ consists of 12 lines which form a double - six. Their equations can be found in **[B]**, p.168. The two sextuples of lines constituting this double - six are orbits of the alternating group $A_5 \subset S_5$. So one sextuple is the $A_5$ - orbit of $l$ and the other is the $A_5$ - orbit of the line

$$l' = \left\{ x : \quad x_1 + x_2 + \frac{1-\sqrt{5}}{2} x_4 = \frac{1-\sqrt{5}}{2} x_1 + x_3 + x_4 = 0 \right\}$$

So $l'$ is line of the second sextuple corresponding to $l$ (because $l \cap l' = \emptyset$). The lines $l$ and $l'$ are orthogonal with respect to the bilinear form $C(x,y) = \sum_{i=1}^{5} x_i y_i$ on $L$. By symmetry, all the other corresponding pairs of lines of our double - six are also orthogonal with respect to $C$. Thus the Schur quadric $Q$ associated to this double - six is given by the equation $\sum_{i=1}^{5} x_i^2 = 0$.

## §1. An overview of Hulek's theory.

**1.1.** Let $E$ be a stable rank 2 vector bundle on $P^2 = P(V)$ with $c_1(E) = -1$, $c_2(E) = n$. According to Le Potier **[L]** and Hulek **[Hu1]**, the bundle $E$ can be realized as the middle cohomology of a monad

$$H \otimes \mathcal{O}_{P(V)}(-1) \xrightarrow{\alpha} M \otimes \Omega^1_{P(V)}(1) \xrightarrow{\beta} H' \otimes \mathcal{O}_{P(V)}. \tag{1.1}$$

where

$$H = H^1(E(-2)) \cong \mathbf{C}^{n-1}, \quad M = H^1(E(-1)) \cong \mathbf{C}^n, \quad H' = H^1(E) \cong \mathbf{C}^{n-1} \tag{1.2}$$

and the maps $\alpha$ and $\beta$ are defined as follows. Let $\Omega^1(1)$ be identified with $\Theta(-2)$ where $\Theta$ is the tangent bundle of $P(V)$. Let $t : V \otimes \mathcal{O}_{P(V)}(-1) \to \Omega^1(1)$ be the Euler homomorphism twisted by $\mathcal{O}(-1)$ (see **[OSS]**). It allows one to identify

$$\mathrm{Hom}\big(H \otimes \mathcal{O}_{P(V)}(-1), M \otimes \Omega^1(1)\big) \quad \cong \quad \mathrm{Hom}_{\mathbf{C}}(H \otimes V^*, M). \tag{1.3}$$

The map $\alpha$ is induced by the cup - product

$$a : H^1(E(-2))\otimes V^* \;=\; H^1(E(-2))\otimes H^0(\mathcal{O}_{P(V)}(1))\longrightarrow H^1(E(-1)). \qquad (1.4)$$

Similarly, we have a map $t^* : \Omega^1(1) \to V^* \otimes \mathcal{O}_{P(V)}$ which allows us to identify

$$\mathrm{Hom}(M \otimes \Omega^1(1), H' \otimes \mathcal{O}_{P(V)}) \;\cong\; \mathrm{Hom}_{\mathbf{C}}(M \otimes V^*, H'). \qquad (1.5)$$

After this identification the map $\beta$ is induced by the cup - product

$$b : H^1(E(-1))\otimes V^* \;=\; H^1(E(-1))\otimes H^0(\mathcal{O}_{P(V)}(1))\longrightarrow H^1(E). \qquad (1.6)$$

The cup - product pairing

$$B : M \otimes M \;=\; H^1(E(-1))\otimes H^1(E(-1))\longrightarrow H^2\Big(\Big(\bigwedge^2 E\Big)(-2)\Big) \;=$$

$$= \; H^2(\mathcal{O}_{P(V)}(-3)) \;=\; \mathbf{C} \qquad (1.7)$$

is a symmetric non - degenerate bilinear form on $M$. We regard it as an isomorphism

$$B : M \to M^*. \qquad (1.8)$$

The spaces $H$ and $H'$ are dual to each other by means of the Serre duality and the isomorphism

$$E \;\cong\; E^* \otimes \bigwedge^2 E \;\cong\; E^*(-1).$$

With respect to the constructed pairings the monad (1.1) is self - dual in the sense that $\beta = \alpha^*(-1)$. Equivalently, if $\lambda \in V^*$ and $a(\lambda) : H \to M$ is the linear map defined by the pairing $a$ and similarly $b(\lambda) : M \to H'$ is the map defined by $b$ then

$$b(\lambda) = a(\lambda)^* \circ B.$$

This shows that the monad (1.1) is completely determined by the pairing (1.4) and the symmetric bilinear form $B$. The pairing must satisfy the following properties (cf. **[Hu1]**):

($\alpha$1) The map $a(\lambda)$ is injective for generic $\lambda \in V^*$.

($\alpha$2) For any $h \in H$ the map $a_H(h) : V^* \to M$ defined by the pairing $a$ is of rank $\geq 2$.

($\alpha$3) For any $\lambda, \lambda' \in V^*$ we have $b(\lambda') \circ a(\lambda) = b(\lambda) \circ a(\lambda')$ where $b(\lambda) = a(\lambda)^* \circ B$ and similarly for $b(\lambda')$.

Note that by a theorem of Grauert-Mülich, the last two properties imply the first one.

**1.2. Theorem.** *Let $V, H, M$ be linear spaces of respective dimensions $3, n-1$ and $n$ and $n \geq 2$. Let us fix a non-degenerate symmetric bilinear form $B$ on $M$. By assigning to each $a \in \mathrm{Hom}(H \otimes V^*, M)$ satisfying $(\alpha 1)$ – $(\alpha 3)$ the map*

$$\alpha = (\mathrm{Id} \otimes t) \circ (a \otimes \mathrm{Id})\colon \; H \otimes \mathcal{O}_{P(V)}(-1) \longrightarrow M \otimes \Omega^1(1),$$

*we get a bijective correspondence between equivalence classes of self - dual monads (1.1) modulo action of the group $O(M, B) \times GL(H)$ and isomorphism classes of stable rank 2 vector bundles $E$ on $P(V)$ with $c_1(E) = -1$ and $c_2(E) = n$.*

**1.3.** Let $l$ be a line in $P^2$ and $E$ be a stable bundle as in Theorem 1.2. Let $\lambda \in V^*$ be a linear form defining $l$. We have a canonical exact sequence

$$0 \longrightarrow E(-1) \xrightarrow{\lambda} E \longrightarrow E|_l \longrightarrow 0,$$

which together with the fact $H^0(E) = 0$ which follows from the stability of $E$, gives an isomorphism

$$H^0(E|_l) = \mathrm{Ker}\{H^1(E(-1)) \to H^1(E)\} = \mathrm{Ker}\{a(\lambda) : M \to H\}. \tag{1.9}$$

Since $E|_l \cong \mathcal{O}(p) \oplus \mathcal{O}(q)$ with $p + q = -1$, we obtain that

$$E|_l \cong \mathcal{O} \oplus \mathcal{O}(-1) \qquad \Longleftrightarrow \qquad \mathrm{rank}\, a(\lambda) = n - 1.$$

A line $l$ is called a *jumping line* if $E|_l \neq \mathcal{O} \oplus \mathcal{O}(-1)$. It follows from the Grauert - Mülich theorem **[OSS]** that the set of jumping lines is a proper Zariski closed subset of the dual plane $P(V^*)$. This set is known to be 0 -dimensional for a generic $E$.

**1.4.** In **[Hu1]** the notion of a *jumping line of the second kind* (shortly JLSK) was introduced. Let $l^{(1)}$ be the first infinitesimal neighborhood of $l$ in $P(V)$. We use the exact sequence

$$0 \to \mathcal{O}_{P(V)}(-2) \xrightarrow{\lambda^2} \mathcal{O}_{P(V)} \to \mathcal{O}_{l^{(1)}} \to 0 \tag{1.10}$$

to obtain

$$H^0(E|_{l^{(2)}}) = \mathrm{Ker}\{s(\lambda) : H^1(E(-2)) \to H^1(E)\}. \tag{1.11}$$

Here the map $s(\lambda)$ corresponds to the canonical pairing

$$S^2(V^*) \otimes H^1(E(-2)) \longrightarrow H^1(E))$$

evaluated at $\lambda^2$. In the notation of the previous subsections, $s(\lambda)$ is the composition

$$a(\lambda)^* \circ B \circ a(\lambda): \ H \to M \to M^* \to H^*.$$

We say that $l$ is a JLSK if $s(\lambda)$ is not bijective. Since the source and target of $s(\lambda)$ have the same dimension, $l$ is a JLSK if and only if $H^0(E|_{l(1)}) \neq 0$.

Let us introduce a rational quadratic map

$$\gamma : P(V^*) \to P(S^2 H^*), \quad \lambda \mapsto s(\lambda)$$

By property $(\alpha 1)$, for a generic line $l \in P(V^*)$ the value $\gamma(\lambda)$ is well defined and is an non-degenerate quadric in $P(H)$. We denote by $C(E)$ the set of all JLSK of $E$. Thus outside a finite set of points in $P(V^*)$ the set $C(E)$ is equal to the preimage, under $\gamma$, of the locus of degenerate quadrics in $P(H)$. So we get that $C(E)$ is a closed subscheme in $P(V^*)$ defined by the equation $\det \gamma(l) = 0$. We shall consider $C(E)$ as a closed subscheme of $P(V^*)$ defined by this equation. So $C(E)$ is a (possibly reducible) curve of degree $2n-2$ containing the set of jumping lines of $E$ in the usual sense.

**1.5.** One can give another interpretation of the curve $C(E)$. Consider the rational map

$$\sigma : P(V^*) \to P(M^*), \quad \lambda \mapsto \mathrm{Im}(a(\lambda))^{\perp} \subset M^*.$$

It is defined on the complement of the set of jumping lines of $E$. A non-jumping line $l$ is a JLSK of and only if the hyperplane $\sigma(l) \subset P(M)$ is tangent to the quadric defined by $B(m,m) = 0$. Let us denote by $Q$ the dual quadric in $P(M^*)$ (which parametrizes the hyperplane tangent to $\{B(m,m) = 0\}$; so it is given by the inverse quadratic form $C = B^{-1}$). Then

$$l \text{ is a JLSK} \quad \text{if and only if} \quad \sigma(l) \in Q.$$

## §2. Generalized Schur quadrics and cubic surfaces.

**2.1.** Let $E$ be a stable rank 2 vector bundle on $P^2 = P(V)$ with $c_1 = -1$, $c_2 = n$. As we mentioned in the previous section, its monad (1.1) defines (and is uniquely defined by) the following linear algebra data: a linear map (tensor)

$$a : H \otimes V^* \to M \tag{2.1}$$

and a quadratic form (the cup - product)

$$B : M \otimes M \to \mathbf{C}. \tag{2.2}$$

Our aim in this section is the study of the geometry of some algebraic varieties naturally associated to $a$ and $B$ (and hence to $E$).

**2.2.** We denote by $Q \subset P(M^*)$ the quadric defined by the equation $C(m,m) = 0$ where $C$ is the quadratic form on $M^*$ inverse to $B$, see n.1.5. We shall call $Q$ the *Schur quadric* of $E$. We shall see later in this section how the classical Schur quadric of a double - six is a particular case of this construction.

**2.3.** By taking various partial transposes of the tensor $a$, we construct the following linear operators:

$$a_M : M^* \to H^* \otimes V = \mathrm{Hom}(H, V); \tag{2.3}$$

$$a_V : V^* \to H^* \otimes M = \mathrm{Hom}(H, M); \tag{2.4}$$

$$a_H : H \to M \otimes V = \mathrm{Hom}(M^*, V). \tag{2.5}$$

These operators define determinantal varieties in $P(M^*), P(V^*), P(H)$ consisting of points whose images (under the corresponding $a$) are operators not of maximal rank. Before going into details, let us recall some well known facts about varieties of matrices of given rank.

Let $L_1, L_2$ be vector spaces of respective dimensions $n_1, n_2$. We denote by $\mathrm{Hom}(L_1, L_2)_r \subset \mathrm{Hom}(L_1, L_2)$ the variety of linear maps of rank $\leq r$. We assume that $r \leq \min(n_1, n_2)$. Then the following is true **[ACGH]**,**[R]**.

**2.4. Proposition.**

a) *The codimension of* $\mathrm{Hom}(L_1, L_2)_r$ *in* $\mathrm{Hom}(L_1, L_2)$ *is equal to* $(n_1 - r)(n_2 - r)$.

b) $\mathrm{Hom}(L_1, L_2)_r$ *is irreducible and Cohen - Macaulay;*

c) *The degree of (the projectivization of)* $\mathrm{Hom}(L_1, L_2)_r$ *is equal to*

$$\prod_{i=0}^{n_1-r-1} \frac{(n_2+i)!\ i!}{(r+i)!\ (n_2-r-i)!}.$$

d) *Let* $\phi \in \mathrm{Hom}(L_1, L_2)_r$ *be a linear map of rank* $k \leq r$. *Then the multiplicity of* $\mathrm{Hom}(L_1, L_2)_r$ *at* $\phi$ *is given by*

$$\mathrm{mult}_\phi(\mathrm{Hom}(L_1, L_2)_r) = \prod_{i=0}^{n_1-r-1} \frac{(n_2-k-i)!\ i!}{(r-k-1)!(n_2-r-i)!}.$$

**2.5.** Let us return to the situation of n. 2.3. We define the variety $\Sigma \subset P(M^*)$ as follows

$$\Sigma = \{\mu \in P(M^*) : \mathrm{rank}\ a_M(\mu) \leq 2\}. \tag{2.6}$$

This is an analog of a cubic surface in $P^3$, cf. Proposition 0.3.

Note that $\dim M = n$, $\dim H = n-1$, $\dim V = 3$. Therefore, by Proposition 2.4, the variety $\mathrm{Hom}(H, V)_2$ has codimension $n-3$ in $\mathrm{Hom}(H, V)$ and so $\dim \Sigma \geq 2$. Generically, one would expect that $\dim \Sigma = 2$.

We shall call the tensor $a$ (and the bundle $E$) $\Sigma$ - *generic* if for any $\mu \in \Sigma$ the rank of $a_M(\mu)$ is exactly 2. We shall see in section 3 that if $n$ is a square then $\Sigma$ - generic bundles exist. Since being $\Sigma$ - generic is an open condition, this will imply that such bundles form an open dense subset in the moduli space. We shall also see that for (some other) open dense subset in the moduli space the variety $\Sigma$ is indeed a surface. However, there are important particular cases when $\Sigma$ is reducible and contains components of higher dimension, see n. 3.5 below.

**2.6.** Consider now the partial transpose $a_V$ of the tensor $a$ given in (2.4). We define the determinantal variety $Z \subset P(V^*)$ by

$$Z = \{\lambda \in P(V^*) : \quad \mathrm{rank}\ a_V(\lambda) \leq n-2\}. \tag{2.7}$$

It will be important for us to consider $Z$ as a scheme with the scheme structure given naturally by (2.7). This means that we choose bases in $H$ and $M$ and regard $a_V$ as a $(n-1)\times n$ -matrix whose entries are linear forms in $\lambda$. The $n$ maximal minors of this matrix are taken to be the equations of the subscheme $Z$.

Since $\mathrm{Hom}(H,M)_{n-2}$ has codimension 2 in $\mathrm{Hom}(H,M)$, generically one expects $Z$ to be 0-dimensional and reduced. If this is indeed the case, we shall call the tensor $a$ (and the bundle $E$) $Z$ - *generic.* It follows from **[Hul]** that $Z$ -generic bundles exist for any values of $n$. Namely, the so-called Hulsbergen bundles will be $Z$ -generic (see also §4 for discussion of these bundles). Thus $Z$ -generic bundles form an open dense subset in the moduli space.

If $a$ is $Z$-generic then, by Proposition 2.4. c), the degree of the 0- dimensional scheme $Z$ equals deg $\mathrm{Hom}(H,M)_{n-2} = \binom{n}{2}$. Moreover, the multiplicity of any point $\lambda \in Z$ in $Z$ is at least $\binom{n-r(\lambda)}{2}$ where $r(\lambda) = \mathrm{rank}\ a_V(\lambda)$

The meaning of $Z$ is as follows.

**2.7. Lemma.** *The support of the scheme $Z$ is precisely the set of jumping lines of $E$.*

*Proof:* This immediately follows from considerations of n.1.3.

**2.8.** Let $\mathcal{J}_Z \subset \mathcal{O}_{P(V^*)}$ be the sheaf of ideals of the subscheme $Z$. By construction of $Z$ (see n. 2.6), maximal minors of the $(n-1)\times n$ - matrix $a_V$ are global sections of $\mathcal{J}_Z(n-1)$. In invariant terms, we consider the linear map

$$a_V^* : H \otimes M^* \to V \tag{2.9}$$

and, by taking its $(n-1)$ -st symmetric power, we get a linear map

$$\bigwedge^{n-1} H \otimes \bigwedge^{n-1} M^* \hookrightarrow S^{n-1}(H\otimes M^*) \longrightarrow S^{n-1}V = H^0(P(V^*),\mathcal{O}(n-1)) \tag{2.10}$$

whose image is contained in $H^0(P(V^*),\mathcal{J}_Z(n-1))$. It will be convenient for us to rewrite (2.10) as

$$A : M \otimes \left(\bigwedge^{n-1} H \otimes \bigwedge^{n} M^*\right) \quad\longrightarrow\quad H^0(P(V^*),\mathcal{J}_Z(n-1)). \tag{2.11}$$

The 1-dimensional vector space $\bigwedge^{n-1} H \otimes \bigwedge^n M^*$ can be chased away by choosing bases in $H$ and $M$.

**2.9. Proposition.** *If* $\dim Z = 0$ *then the operator* $A$ *in (2.11) is an isomorphism. In other words, the linear system of curves of degree* $n-1$ *through* $Z$ *is generated by maximal minors of* $a_V$.

*Proof:* We associate to $a_V$, in a standard way, a morphism $\tilde{a}$ of sheaves on $P(V^*)$ and denote its cokernel by $\mathcal{F}$:

$$0 \to H \otimes \mathcal{O}_{P(V^*)}(-1) \xrightarrow{\tilde{a}} M \otimes \mathcal{O}_{P(V^*)} \to \mathcal{F} \to 0 \qquad (2.12)$$

(the fact that $\tilde{a}$ is injective, follows from $\dim Z = 0$). We claim the following:

**2.10. Lemma.** $\mathcal{F}$ *is isomorphic to* $\mathcal{J}_Z(n-1)$. *Under this isomorphism the natural map* $M \to H^0(\mathcal{J}_Z(n-1))$ *corresponds, up to a scalar multiple, to the map* $A$ *from (2.11).*

Clearly, Lemma 2.10 implies our proposition in virtue of the exact cohomological sequence of (2.12).

*Proof of the lemma:* The assertion follows from the well-known resolution of Eagon-Northcott of the ideal of a determinant variety defined by maximal minors (see **[No]**, Appendix C.1). However we prefer to give an elementary proof here. We choose a bases $h_1, ..., h_{n-1} \in H$ and $m_1, ..., m_n \in M$. This makes it possible to speak about the determinant $\det[v_1, ..., v_n]$ of a system of $n$ vectors in $M$ (this is just $|b_{ij}|$ where $v_i = \sum b_{ij} m_j$). We define a morphism of sheaves $\psi : \mathcal{F} \to \mathcal{J}_Z(n-1)$ i.e. a morphism $\Psi : M \otimes \mathcal{O}_{P(V^*)} \to \mathcal{J}_Z(n-1)$ vanishing on $\operatorname{Im}(\tilde{a})$, as follows. Let $m = m(\lambda)$ be a local section of $M \otimes \mathcal{O}_{P(V^*)}$ i.e. an $M$ - valued function in $\lambda$ homogeneous of degree 0. We put $\Psi(m)$ to be the homogeneous (of degree $n-1$) function

$$\lambda \mapsto \det\big[m(\lambda), a_V(\lambda)(h_1), ..., a_V(\lambda)(h_{n-1})\big].$$

This defines $\psi$. It is clear that $\psi$ is injective. The fact that $\psi$ is surjective follows by comparing Chern classes of $\mathcal{F}$ and $\mathcal{J}_Z(n-1)$. The rest of the lemma is obvious.

**2.11.** We continue to assume that $\dim Z = 0$. Let $S$ be the blow up of $P(V^*)$ along $Z$ and $\pi_S : S \to P(V^*)$ be the canonical projection. In virtue of Proposition 2.9 the linear system of curves of degree $n-1$ through $Z$ defines a regular map $p : S \to P(M^*)$. A generic point $s = \pi_S^{-1}(\lambda) \in S$, $\lambda \in P(V^*)$ goes under $p$ into the hyperplane in $P(M)$ consisting of $m$ such that $A(m) \in H^0(P(V^*), \mathcal{J}_Z(n-1))$ vanishes at $\lambda$ as well. Here $A$ is as in (2.11). The interpretation of $A$ in Lemma 2.10 shows that $p(S)$ is contained in the determinantal variety $\Sigma \subset P(M^*)$ as an irreducible component. We shall denote variety $p(S)$ (typically a surface) by $\Sigma' \subset \Sigma$.

**2.12.** Suppose that our bundle $E$ is $\Sigma$ - generic. Then we have a regular map

$$\pi_\Sigma : \Sigma \to P(V^*)$$

which takes $\mu \in \Sigma \subset P(M^*)$ to the linear subspace $\mathrm{Im}(a_M(\mu)) \subset V$ (this subspace has dimension 2 by the assumption of $\Sigma$ -genericity). The map $\pi_\Sigma$ is the analog of the blow-down of a cubic surface onto a plane.

If the bundle $E$ is not $\Sigma$ -generic, the map $\pi_\Sigma$ will be defined on the open part $\Sigma_0 \subset \Sigma$ consisting of $\mu$ such that $a_M(\mu)$ has rank exactly 2.

For $\lambda \in P(V^*)$ the fiber $\pi_\Sigma^{-1}(\lambda)$ is the projective space $P(\mathrm{Ker}\ a_V(\lambda)^*)$. The dimension of this fiber is equal to $n-\mathrm{rank}\ a_V(\lambda)-1$. Hence $\pi_\Sigma$ is an isomorphism over the complement of $\mathrm{Supp}(Z)$. On the other hand, if the rank of $a_V(\lambda)$ is small the fiber $\pi_\Sigma^{-1}(\lambda)$ will have dimension $\geq 2$ and the variety $\Sigma$ will be reducible. We shall see in §3 that such situations do occur for stable bundles.

**2.13. Proposition.** *Assume that $E$ is $Z$ - generic and no $n-1$ points of $Z$ lie on a line. Then:*
*(a) The map $p : S \to \Sigma$ is an isomorphism (so, in particular, $\Sigma' = \Sigma$);*
*(b) $\Sigma$ is a projectively Cohen - Macaulay surface in $P(M^*)$ of degree $(n-1)^2 - \binom{n}{2}$.*

*Proof:* Introducing the Hilbert function $H(Z,t) = h^0(\mathcal{O}_{P(V^*)}(t)) - h^0(\mathcal{J}_Z(t))$, and applying exact sequence (2.12), we have

$$H(Z, n-1) = (1/2)n(n+1) - n = (1/2)n(n-1) =$$

$$H(Z, n-2) > H(Z, n-3) = (1/2)(n-1)(n-2).$$

This gives

$$n - 1 = \min\{t : H(Z,t) = H(Z, t-1)\}.$$

By **[DG]**, this implies that the linear system of curves of degree $n-1$ through $Z$ maps $S = \mathrm{Bl}_Z(P^2)$ isomorphically into $P(H^0(\mathcal{J}_Z(n))^*) = P(M^*)$. By **[Gi]** the image of this map i.e., the variety $\Sigma'$, is projectively Cohen - Macaulay. Recall that this means that the projective coordinate ring of $\Sigma'$ is Cohen - Macaulay. In particular, we get that $\Sigma'$ is projectively normal i.e., for any $k \geq 0$ the restriction map

$$H^0(P(M^*), \mathcal{O}(k)) \longrightarrow H^0(\Sigma', \mathcal{O}(k))$$

is surjective. Since the rational map $P(V^*) \to \Sigma$ is given by the linear system of curves of degree $n-1$ through $Z$, we obtain the assertion about the degree of $\Sigma'$. Since $E$ is $Z$ - generic, the fiber of the map $\pi_\Sigma : \Sigma \to P(V^*)$ over each point $z \in Z$ is isomorphic to $P^1$. Since $\Sigma'$ and $\Sigma$ coincide outside the union of the fibers $\pi_\Sigma^{-1}(z)$, $z \in Z$, this implies that $\Sigma' = \Sigma$. Q.E.D.

**2.14.** Let $z \in Z \subset P(V^*)$. We denote the fiber

$$\pi_\Sigma^{-1}(z) = P(\mathrm{Ker}(a_V(z)^*) \subset P(M^*) \quad \text{by} \quad A_z.$$

The corresponding linear subspace $\mathrm{Ker}(a_V(z)^*) \subset M^*$ of which $A_z$ is the projectivization, will be denoted by $\mathbf{A}_z$.

Consider the space

$$H_z = \mathrm{Ker}\ a_V(z) \subset H.$$

We also consider the linear subspace

$$\mathbf{A}'_z = \bigcap_{h \in H_z} \mathrm{Ker}\ a_H(h) \quad \subset \quad M^*$$

and denote its projectivization by $A'_z \subset P(M^*)$.

The collection of projective subspaces $A_z, A'_z$, $z \in Z$, forms the analog of a Schläfli double - six on a cubic surface in $P^3$.

In our case $A'_z$ lies on $\Sigma$ but $A_z$ does not, in general, do so. Indeed, the typical situation (see Proposition 2.13) is that $\Sigma$ is a surface, that for any $z \in Z$ we have $\mathrm{rk}(a(z)) = n-2$ and so $\dim A_z = 1$, $\dim A'_z = n-3$. So for $n \geq 5$ $A'_z$ cannot lie on $\Sigma$. The relation of $A'_z$ with the component $\Sigma' = p(S) \subset \Sigma$ is as follows.

**2.15. Proposition.** *Assume that $E$ is $Z$ - generic. Then $A'_z$ is a subspace of codimension 2 in $P(M^*)$ which intersects the surface $\Sigma'$ along a curve. The image* *of this curve under the projection $\pi_\Sigma : \Sigma \to P(V^*)$ is the unique curve of degree $n-2$ which passes through the points $z' \in Z \setminus \{z\}$.*

For the case $n = 4$ we get the standard description of the second sextuple of lines of the double - six on the cubic surface as the inverse image of quadrics containing some 5 of the 6 points of $Z$. For $n \geq 5$ instead of the property that $A'_z$ lies on $\Sigma'$ we have that $A'_z \cap \Sigma'$ is a curve (instead of a set of isolated points, as one would expect by dimension count).

*Proof:* Since the rank of $a_V(z)$ equals $n-2$, we have $\dim(H_z) = 1$. Thus $\mathbf{A}'_z$ is the kernel of the map $a_V(h) : M^* \to V$ where $h$ is any non-zero vector from $H_z$. Note that the rank of this map equals 2. In fact, otherwise $Z$ would contain a line as an irreducible component. This shows that $\dim(\mathbf{A}_z) = n-2$. Now let us observe that $A'_z = P(\mathbf{A}'_z)$ intersects each $A_{z'}$ for $z' \neq z$. Indeed, the sum of linear subspaces $\mathbf{A}'_z + \mathbf{A}_{z'}$ is contained in the hyperplane of zeroes of the linear form $a(h, z') \in M = (M^*)^*$, where $a$ is as in (2.1).

Let $\{H(\lambda)\}_{\lambda \in P^1}$ be the pencil of hyperplanes in $P(M^*)$ which contain the subspace $A'_z$. It cuts out a pencil $\mathcal{P}$ of curves on $\Sigma$ with the base locus $A'_z \cap \Sigma$. For each $z' \neq z$ one of the hyperplanes $H(\lambda)$ contains the line $A_{z'}$. Thus each $A_{z'}$ contains one of the base points of the pencil $\mathcal{P}$. Under the rational map $P(V^*) \to P(M^*)$ (given by curves of degree $n-1$ through $Z$) the preimage of the pencil $\{H(\lambda)\}$ is some pencil of curves of degree $n-1$ passing through $Z$. Let $\mathcal{C}$ be its moving part and $F$ be its fixed curve. Let $d$ be the degree of $F$ (zero if $F = \emptyset$). Curves of the pencil $\mathcal{C}$ have degree $n-1-d$. Suppose that they pass through some $m$ points say, $z_1, ..., z_m$ of $Z$. Then, since $z_i$ remain basic for $\mathcal{C}$ after the blow - up, curves from $\mathcal{C}$ have the same tangent direction at each $z_i \neq z$. The curve $F$ passes through the remaining $(1/2)n(n-1) - m$ points of $Z$. Consider a typical curve $C \in \mathcal{C}$. Let $\tilde{C}$ be its proper transform in $S = \mathrm{Bl}_Z(P(V^*))$. Since $\tilde{C}$ moves, its self - intersection index is non - negative so we get

$$0 \leq \tilde{C}^2 \leq (n-d-1)^2 - 2(m-1) - 1 = (n-d)(n-d-1) - 2m - (n-d-2).$$

If $n-d-2 \geq 0$, we obtain that $(n-d)(n-d-1) - 2m \geq 0$ thus there exists a plane curve of degree $n-d-2$ passing through $z_1, ..., z_m$. Together with the curve $F$, it defines a curve of degree $n-2$ passing through all the points of $Z$. But Lemma 2.10 and the exact sequence (2.12) show that this is impossible. So we must have $d = n-2$ and hence $m = 1$, so $F$ is a curve of degree $n-2$ which passes through all the points of $Z$ except $z$. If there is another curve, say, $F'$, with this property then we would have a pencil of curves of degree $n-2$ through $Z - \{z\}$. This pencil must then contain a curve passing also through $z$. This, as we have just seen, is impossible.

**2.17.** Up until now we worked exclusively with the tensor $a$ from (2.1). Now we take into account the non-degenerate quadratic form $B \in S^2M^*$ from (2.2). Let $C = B^{-1}$ be the inverse quadratic form on $M^*$. The following result justifies the name "Schur quadric" for the quadric defined by $C$.

**2.17. Theorem.** *Let $z \in \mathrm{Supp}(Z)$. Then $A_z$ is contained in the orthogonal complement $(A'_z)^{\perp}_{C}$ of $A'_z$ with respect to $C$ . If, moreover, rk $a_V(z) = n-2$ then we have equality $A'_z = (A_z)^{\perp}_{C}$.*

*Proof:* For any $\lambda \in V^*$ let

$$b(\lambda) = a(\lambda)^* \circ B : \quad M \longrightarrow H^*$$

where $a(\lambda)$ is the map induced by the $a$ from (2.1). Then

$$B^{-1}(\mathbf{A}'_z) = \{m \in M : (b(\lambda)(m), h) = 0, \quad \forall \lambda \in V^*, h \in H_z = \mathrm{Ker}\ (a_V(z))\}.$$

For any $m \in \mathbf{A}_z^{\perp} = a_V(z)(H)$ we write $m = a(z)(h')$ for some $h' \in H$ and obtain

$$(b(\lambda)(a_V(z)(h')), h) \quad = \quad (b(\lambda)(a_V(z)(h), h') = (0, h') = 0.$$

Here we use the property ($\alpha 3$) from n.1.1. Thus we obtain

$$\mathbf{A}_z^{\perp} \subset B^{-1}(\mathbf{A}'_z).$$

If rank $(a(z)) = n-2$ then dim $\mathbf{A}_z = 2$, dim $H_z = 1$ and dim $\mathbf{A}'_z = n-2$. Thus the dimensions of the spaces $\mathbf{A}_z^{\perp}$ and $B^{-1}(\mathbf{A}'_z)$ are the same so these spaces are equal. Theorem is proven.

**2.18. Remark.** Let $Z$ be any set of $\binom{n}{2}$ points in $P^2$ such that no curve of degree $n-2$ contains $Z$ and no lines pass through $n-1$ points of $Z$. The linear system of curves of degree $n-1$ through $Z$ defines a rational map of $P^2$ into $P^{n-1}$

whose image is a nonsingular surface $X$ classically known as a *White surface* **[R]**. If $n = 4$, this is a cubic surface. The surface $X$ is given by vanishing of maximal minors of a $3 \times (n-1)$ matrix of linear forms. A modern proof of these results can be found in **[DG]** and **[Gi]**.

Every White surface comes equipped with a set of $\binom{n}{2}$ lines $E_z, z \in Z$ corresponding to exceptional curves of the blow - up $\mathrm{Bl}_Z(P^2)$ and a set of $\binom{n}{2}$ curves $C_z$ of degree $(n-2)(n-4)/2+1$. The curve $C_z$ is the image of the (unique) plane curve of degree $n-2$ passing through $Z - \{z\}$. Each curve $C_z$ spans a subspace $E'_z$ of codimension 2 in $P^{n-1}$. We have $E_z \cap E'_z = \emptyset$ but $E_z \cap E'_{z'} \neq \emptyset$ for $z' \neq z$. This situation is analogous to a configuration of a double - six on a cubic surface.

Propositions 2.13 and 2.15 imply that for a $\Sigma$ - generic stable bundle $E$ the variety $\Sigma$ is a White surface. However, by counting constants it follows that not every White surface comes in this way, as soon as $n \geq 5$. Although one can reconstruct a linear map

$$a : H \otimes V^* \cong \mathbf{C}^{n-1} \otimes \mathbf{C}^3 \longrightarrow M = \mathbf{C}^n$$

from a determinantal representation of $X$, there does not exist, in general, a quadratic form $B$ on $M$ such that $a$ satisfies the property $(\alpha 3)$ from n.1.1. By Theorem 1.2 the existence of such a $B$ is necessary and sufficient in order that $X = \Sigma$ for some $Z$ - generic bundle $E$. It seems likely that these conditions are equivalent to the existence of a "Schur quadric" for the "double - six" $\{E_z, E'_z\}$ i.e., a quadric $Q$ in $P^{n-1}$ such that $E_z$ and $E'_z$ are orthogonal with respect to the (quadratic form defining) $Q$.

**2.20.** The role of the Schur quadric $Q$ (see n.2.2) in the description of jumping lines of the second kind is given by the following remark.

**2.21. Proposition.** *Let $\Sigma_0 \subset \Sigma$ be the open set of $\mu$ such that the rank of $a_V(\mu)$ equals 2 (so $\Sigma_0 = \Sigma$ if the bundle is $\Sigma$ - generic). Let $\pi_\Sigma : \Sigma_0 \to P(V^*)$ be the projection defined in n. 2.12. Then the curve $C(E)$ of jumping lines of second kind coincides with the closure of $\pi_\Sigma(Q \cap \Sigma_0)$.*

In particular, when the bundle $E$ is $\Sigma$ - generic, we have $C(E) = \pi_\Sigma(Q \cap \Sigma)$

*Proof:* This is a reformulation of what has been done in n.1.5.

As an application of our formalism of Schur quadrics let us prove a statement about the singular tangent lines of the curves of JLSK which strengthens, under assumptions of genericity, a theorem of Hulek. More precisely, Hulek **[Hu1]** has proven the following fact.

**2.22. Theorem.** *Let $l \in C(E)$ be a JLSK of $E$. Suppose that $E|_l \cong \mathcal{O}(-1-k) \oplus \mathcal{O}(k)$ with $k \geq 1$. Then $l$ is a singular point of the curve $C(E)$ of multiplicity $2k$ and for any line $T$ in the tangent cone of $C(E)$ at $l$ the intersection index of $C(E)$ and $T$ at $l$ is at least $2k+2$.*

Assume that $E$ is $Z$ - generic. Then every singular point $l$ of $C(E)$ is a double point (a node or, possibly, a cusp of type $y^2 = x^r$). Theorem 2.22 gives that in this case there exist at least one line $T$ through the point $l$ with intersection index $\geq 4$.

We claim that the case of the cusp does not occur for $Z$-generic $E$. Call an ordinary double point $p$ of a plane curve $C$ a *biflexnode* if each of the two branches has a flex at this point i.e. each of the two tangents has the intersection index $\geq 4$ with $C$ at $p$.

**2.23. Theorem.** *Assume that the bundle $E$ is $Z$ - generic. Then every singular point of $C(E)$ corresponding to a jumping line is a biflexnode.*

*Proof:* Let $z \in Z$ be a singular point of $C(E)$ corresponding to a jumping line. Then $z \in Z$. The branches of $C(E)$ at $z$ correspond to the points of intersection of the line $A_z$ and the Schur quadric $Q$. Note that $Q$ cannot be tangent to $A_z$ since otherwise we would have $A_z \cap A'_z \neq \emptyset$ which contradicts Proposition 2.15. This proves that the point $z$ is an ordinary node.

Although Theorem 2.22 allows us to finish the proof, we prefer to give an independent proof based on the properties of the Schur quadric.

Now let $x$ be one of the two points of $Q \cap A_z$ and let $\Pi$ be a hyperplane in $P(M^*)$ which is spanned by the point $x$ and the codimension 2 subspace $A'_z$. Let $Q(z)$ denote the quadric in $A'_z$ cut out by $Q$. For any point $y \in Q(z)$ the line $< x, y >$ is contained in $Q$. This implies that $\Pi$ is tangent to $Q$ at $x$. Let $\tilde{C}(E) = \Sigma \cap Q$ be the proper inverse transform of the curve $C(E)$ in $\Sigma$, under the blow-down $\pi_\Sigma : \Sigma \rightarrow P(V^*)$. Let $\tau$ be the tangent line to $C(E)$ at $z$ at the branch corresponding to $x$ and let $\tilde{\tau}$ be its proper inverse transform on $\Sigma$.

Under the correspondence between hyperplanes in $P(M^*)$ and curves of degree $n-1$ in $P(V^*)$ through $Z$, the hyperplane $\Pi$ corresponds to the reducible curve $\tau + C_z$ where $C_z$ is the plane curve of degree $n-2$ passing through $Z - \{z\}$. This implies that $\tilde{\tau} \subset \Pi$ and so

$$T_x(\tilde{\tau}) = \Pi \cap T_x(\Sigma) = T_x(Q) \cap T_x(\Sigma) = T_x(\tilde{C}(E)).$$

This shows that $\tilde{\tau}$ is tangent to $\tilde{C}(E)$ at the point $x$. Obviously this implies that $\tau$ is a flex tangent at the branch of $C(E)$ at $z$ corresponding to $x$. Theorem is proven.

## §3. Logarithmic bundles.

**3.1.** Consider a projective plane $P^2 = P(V)$, $\dim V = 3$. Let $\mathcal{H} = (H_1, ..., H_m)$ be an arrangement of $m$ lines in $P(V)$ in general position (i.e., no three of these lines have a common point). Let $E(\mathcal{H}) = \Omega^1_{P(V)}(\log \mathcal{H})$ be the sheaf of 1-forms on $P(V)$ with logarithmic poles along $H_i$. Since $\mathcal{H}$ is a divisor with normal crossings, $E(\mathcal{H})$ is locally free i.e. we can and will regard it as a rank 2 vector bundle. It was proven in **[DK]** that this bundle is stable.

We further suppose that the number of lines is even: $m = 2d$. In this case $c_1 E(\mathcal{H}) = 2d - 3$. The normalized bundle $E_{\text{norm}}(\mathcal{H}) = E(\mathcal{H})(-d+1)$ is a stable bundle with $c_1 = -1, c_2 = (d-1)^2$. In this section we apply considerations of §§1,2 to bundles $E_{\text{norm}}(\mathcal{H})$.

**3.2.** It was shown in **[DK]** that the bundle $E(\mathcal{H})$ has a resolution of the form

$$0 \to I \otimes \mathcal{O}_{P(V)}(-1) \xrightarrow{\tau} W \otimes \mathcal{O}_{P(V)} \to E(\mathcal{H}) \to 0. \tag{3.1}$$

In (3.1) the space $W$ is defined as

$$W = \left\{ (a_1, ..., a_{2d}) \in \mathbf{C}^{2d} : \sum a_i = 0 \right\}. \tag{3.2}$$

The space $I$ is defined as follows. Let $f_i \in V^*$ be a linear equation of the line $H_i$. Then $I$ is the space of relations among $(f_1, ..., f_{2d})$ i.e.,

$$I = \left\{ (a_1, ..., a_{2d}) \in \mathbf{C}^{2d} : \sum a_i f_i = 0 \right\}. \tag{3.3}$$

The map $\tau$ is induced by the canonical map

$$t : I \otimes V \to W, \quad (a_1, \ldots, a_{2d}) \otimes v \mapsto \big(a_1 f_1(v), \ldots, a_{2d} f_{2d}(v)\big) \tag{3.4}$$

called the fundamental tensor of $\mathcal{H}$.

**3.3.** By twisting the resolution (3.1) with $\mathcal{O}(-d+1)$ we get a resolution for $E_{\mathrm{norm}}(\mathcal{H}) = E(\mathcal{H})(-d+1)$. From this it is immediate to find the data defining the Hulek's monad for $E_{\mathrm{norm}}(\mathcal{H})$ (see §1). To formulate the answer neatly, let again $f_j \in V^*$ be the equation of $H_j$. For any $m \geq 1$ denote by $\partial/\partial f_j : S^m V \to S^{m-1} V$ the derivation corresponding to $f_j$ regarded as a constant vector field on $V^*$. We define the following map

$$t_{(m)} : S^m V \otimes I \to S^{m-1} V \otimes W, \tag{3.5}$$

$$p \otimes (a_1, \ldots, a_{2d}) \mapsto \left(a_1 \frac{\partial p}{\partial f_1}, ., a_{2d} \frac{\partial p}{\partial f_{2d}}\right) \tag{3.6}$$

where we regard $S^{m-1} V \otimes W$ as the space of collections $(q_1, \ldots, q_{2d})$ of polynomials $q_j \in S^{m-1} V$ summing up to 0.

Now the vector spaces in the monad for $E_{\mathrm{norm}}(\mathcal{H})$ have the form

$$H = H^1\big(E_{\mathrm{norm}}(\mathcal{H})(-2)\big) = H^1\big(E(\mathcal{H})(-d-1)\big) = \mathrm{Ker}\ (t_{(d-1)}); \tag{3.7}$$

$$M = H^1\big(E(\mathcal{H})(-d)\big) = \mathrm{Ker}\ (t_{(d-2)}); \tag{3.8}$$

$$H' = H^1\big(E(\mathcal{H})(-d+1)\big) = \mathrm{Ker}\ (t_{(d-3)}), \tag{3.9}$$

as it follows immediately from the resolution (3.1). For example, the map $t_{(d-1)} : S^{d-1} V \otimes I \to S^{d-2} V \otimes W$ in (3.7) appears as the map

$$H^2(P(V), \mathcal{O}(-d-2) \otimes I) \to H^2(P(V), \mathcal{O}(-d-1) \otimes W)$$

in the long exact sequence of cohomology of the resolution (3.1) tensored with $\mathcal{O}(-d-1)$.

As regards maps in the monad (1.1), we shall only need the explicit form of the operator

$$b_M : M \to V \otimes H' \tag{3.10}$$

defined by the map $b$ in (1.1). Namely, $b_M$ is induced by

$$\psi \otimes \mathrm{Id}_I : S^{d-2} V \otimes I \to V \otimes S^{d-3} V \otimes I \tag{3.11}$$

where $\psi : S^{d-2} V \to V \otimes S^{d-3} V$ is the canonical $GL(V)$ - equivariant embedding. The map $a$ in (1.1) is dual to $b$ by means of the form $B$.

The following is the main result of this section.

**3.4. Theorem.** *Any bundle $E_{\mathrm{norm}}(\mathcal{H})$ is $\Sigma$ -generic (see n. 2.5).*

*Proof:* In the notation of §2 we have to prove that

$$a_M(M^*) \quad \cap \quad \mathrm{Hom}(V^*, H^*)_1 \qquad = \qquad \{0\}. \tag{3.12}$$

We have a commutative diagram

$$\begin{array}{ccc} M^* & \xrightarrow{a_M} & \mathrm{Hom}(V^*, H^*) \\ \downarrow & & \downarrow \\ M & \xrightarrow{b_M} & \mathrm{Hom}(V^*, H') \end{array}$$

where the left vertical arrow is induced by the form $B$ and the right vertical arrow — by the isomorphism $H^* = H'$ (see n. 1.2). It is enough therefore to prove that

$$b_M(M) \cap \mathrm{Hom}(V^*, H')_1 = \{0\}. \tag{3.13}$$

Let $m = \sum p_i \otimes x_i$ be an element of $M \subset S^{d-2}V \otimes I$, so $p_i \in S^{d-2}V, x_i \in I$. The element $m$ is mapped by $b_M$ into an element of $\mathrm{Hom}(V^*, H')_1$ if and only if there is $v \in V$ such that each $p_i$ equals $vq_i$ for some $q_i \in S^{d-3}V$ and also $\sum q_i \otimes x_i \in H'$. Each $x_i \in I$ is in fact a vector $x_i = (x_i^{(1)}, ..., x_i^{(2d)})$ such that $\sum_{j=1}^{2d} x_i^{(j)} f_j = 0$. Since $m$ belongs to $M = \mathrm{Ker}\,(t_{(d-1)})$, we have, by (3.5) and (3.6):

$$\sum_i x_i^{(j)} \frac{\partial(vq_i)}{\partial f_j} = 0, \quad j = 1, ..., 2d. \tag{3.14}$$

By applying Leibnitz' rule for $\partial/\partial f_j$ and taking into account the fact that $\sum_i q_i \otimes x_i \in H' = \mathrm{Ker}(t_{(d-3)})$, we get the equalities

$$f_j(v) \sum_i x_i^{(j)} q_i = 0, \quad j = 1, ..., 2d. \tag{3.15}$$

We claim that these equalities imply that $q_i = 0$ for all $i$. Indeed, let $\lambda : S^{d-3}V \to \mathbf{C}$ be any linear functional. Consider the vector

$$y = \sum_i \lambda(q_i) x_i \in I.$$

If we write $y$ in terms of its components: $y = (y^{(1)}, ..., y^{(2d)})$ then (3.15) implies that

$$f_j(v) y^{(j)} = 0, \quad j = 1, ..., 2d.$$

Let $J = \{j : f_j(v) = 0\}$. Since the lines $\{f_j = 0\}$ are in general position, $|J| \leq 2$. For $j \notin J$ we have therefore $y^{(j)} = 0$. Since $y \in I$, we have

$$0 = \sum_{j=1}^{2d} y^{(j)} f_j = \sum_{j \in J} y^{(j)} f_j$$

which means that we have a nontrivial linear relation among $|J| \leq 2$ elements of $\{f_1, ..., f_{2d}\}$. This contradicts the general position of $\{f_i = 0\}$ so the vector $y \in I$ is zero. In other words, for any linear functional $\lambda : S^{d-3}V \to \mathbf{C}$ we have $\sum \lambda(q_i)x_i = 0$ in $I$. This means that $\sum q_i \otimes x_i = 0$ in $S^{d-3}V \otimes I$ and Theorem 3.4 is proven.

**3.5.** Let $Z$ be the subscheme of jumping lines of $E_{\text{norm}}(\mathcal{H})$. As was shown in **[DK]** (Proposition 7.4), the lines $H_i$ belong to $Z$. Moreover,

$$E_{\text{norm}}(\mathcal{H})|_{H_i} = \mathcal{O}_{H_i}(1-d) \oplus \mathcal{O}_{H_i}(d-2). \tag{3.16}$$

Denote, as usual, by $f_i \in V^*$ the equation of $H_i$. The equality (3.16) means that the matrix $a_V(f_i)$ (see formula (2.4)) has rank $n-d-1$. By Proposition 2.4 d) this implies that the multiplicity of each $H_i$ as a point of $Z$ is at least $(d-1)(d-2)/2$. The total degree of $Z$, however, equals to $\binom{n}{2}$ where $n = c_2(E_{\text{norm}}(\mathcal{H})) = (d-1)^2$. Thus one expects that for $d \geq 4$ there will be many other jumping lines apart from $H_1, ...., H_{2d}$.

Let us also note that the fibers of the map $\pi_\Sigma : \Sigma \to P(V^*)$ introduced in n. 2.12 over points $H_i \in P(V^*)$ are projective spaces of dimension $d-2$. This means that for $d \geq 4$ the determinantal variety $\Sigma$ ("cubic surface") will be always reducible.

## §4. Examples.

**4.1.** In this section we shall illustrate geometric constructions of §2 on some particular classes of bundles. The example with cubic surfaces and Schur quadrics (which motivated the present paper) will be considered in n.4.4.

In each of the examples below we shall indicate the value of $n = c_2$ (we assume $c_1 = -1$) and describe the following geometric objects (all introduced in §2):

a) The subscheme $Z \subset P(V^*)$ of jumping lines. If dim $Z = 0$ then deg $Z = \binom{n}{2}$.

b) The determinantal variety $\Sigma \subset P(M^*)$ (the analog of the cubic surface). It comes with a natural map $p : \mathrm{Bl}_Z P(V^*) \to \Sigma$ whose image is a component of $\Sigma$. The map $p$ is given by the linear system of curves of degree $n-1$ in $P(V^*)$ through $Z$.

c) The Schur quadric $Q \subset P(M^*)$.

d) The curve $C(E) \subset P(V^*)$ of JLSK. Its degree is $2n-2$. It can be described as $\overline{\pi_\Sigma(\Sigma_0 \cap Q)}$ where $\pi_\Sigma : \Sigma_0 \to P(V^*)$ is the projection of the generic part of $\Sigma$ introduced in n. 2.12.

e) The projective subspaces $A_z, A'_z, z \in Z$ (the analog of the double - six).

By $M(-1, n)$ we shall denote the moduli space of stable rank 2 vector bundles on $P^2$ with $c_1 = -1$, $c_2 = n$. It is an irreducible variety of dimension $4n-4$, see **[Hu1]**,**[OSS]**.

**4.2. The case** $n = 2$. This case was considered in **[Hu1]**. The features are as follows:

a) $Z$ consists of just one point $z_0 \in P(V^*)$. This point corresponds to the 1-dimensional kernel of

$$a_V : V^* \to \mathrm{Hom}(H, M) = \mathbf{C}^2.$$

b) The determinantal variety $\Sigma \subset P(M^*) = P^1$ coincides with $P(M^*)$. The regular map $p : \mathrm{Bl}_Z P(V^*) \to \Sigma$ is the natural projection $\mathrm{Bl}_{z_0} P^2 \to P^1$.

c) The Schur quadric $Q \subset P(M^*) = P^1$ consists of two distinct points.

d) The curve $C(E)$ is $\pi(p^{-1}(Q))$ where $\pi : \mathrm{Bl}_Z P(V^*) \to P(V^*)$ is the projection. In other words, $C(E)$ is the union of two distinct lines through $z_0$.

e) The "double - six" is as follows: $A_{z_0} = P(M^*), A'_{z_0} = \emptyset$.

**4.3. The case** $n = 3$. There may be several possibilities for $Z$ which were also listed by Hulek **[Hu1]**. We shall consider only the most generic case when $Z$ consists of three distinct non-collinear points. In this case the features are as follows:

b)The variety $\Sigma \subset P(M^*)$ again coincides with $P(M^*) = P^2$. The regular map $\mathrm{Bl}_Z P(V^*) \to P(M^*) = \Sigma$ resolves the standard Cremona transformation $c : P(V) = P^2 \to P^2 = P(M^*)$ defined by quadrics through $Z$ (three points). If we choose homogeneous coordinates $x_i$ in $P(V^*)$ in which $Z$ consists of points $(1,0,0),(0,1,0),(0,0,1)$ then $c$ is given by the formula $t_0 = x_1x_2, t_1 = x_0x_2, t_2 = x_0x_1$ where $t_i$ are appropriate coordinates in $P(M^*)$.

c) The Schur quadric $Q \in P(M^*)$ is the conic $t_0^2 + t_1^2 + t_2^2 = 0$.

d) The curve $C(E)$ is the inverse image of this conic under the Cremona transformation defined in b). In other words, the equation of $C(E)$ is $x_0^2x_1^2 + x_0^2x_2^2 + x_1^2x_2^2 = 0$.

e) The subspaces $A_z$ are coordinate lines $\{t_i = 0\}$ in $P(M^*)$, the subspaces $A'_z$ are the opposite points of the coordinate triangle i.e., points $\{t_i = t_j = 0\}$.

**4.4. The case** $n = 4$. The moduli space $M(-1,4)$ has dimension 12. As shown in **[DK]**, an open dense subset in $M(-1,4)$ is provided by normalized logarithmic bundles

$$E_{\mathrm{norm}}(\mathcal{H}) = \Omega^1_{P(V)}(\log \mathcal{H}) \otimes \mathcal{O}(-2)$$

where $\mathcal{H} = (H_1, ..., H_6)$ is an arrangement of 6 lines in $P(V) = P^2$ in general position. We consider only such bundles $E$. Let $p_i \in P(V^*)$ be points corresponding to lines $H_i \subset P(V)$. We first assume that $p_i$ do not lie on a conic (i.e., $H_i$ are not all tangent to a conic). In this case:

a) $Z = \{p_1, ..., p_6\}$.

b) The variety $\Sigma \subset P(M^*) = P^3$ is the cubic surface obtained by blowing up $Z$.

c) The quadric $Q$ is the classical Schur quadric associated with the double - six $\{l_i = A_{p_i},\ l'_i = A'_{p_i}\}$ (see §0). This follows from Theorem 2.17.

d) The curve $C(E)$ is the image under $\pi_\Sigma : \Sigma \to P(V^*)$ of the intersection $\Sigma \cap Q$. The intersection is non singular of degree 6 and genus 4; the projection will have nodes at $p_i$ since each the six lines $l_i \subset \Sigma$ blown down to $p_i$ by $\pi_\Sigma$ meets $Q$ twice.

e) The subspaces $A_{p_i}, A'_{p_i}$ form the standard double - six associated to the blow-down $\pi_\Sigma$.

If all $p_i$ do lie on a conic $\Gamma \subset P(V^*)$, the situation changes. In this case $E(\mathcal{H})$ is the Schwarzenberger bundle associated to $\Gamma$ (see **[Sch1,Sch2]**, **[DK]**) and the features are as follows:

a) $Z$ equals the conic $\Gamma$ (so dim $Z = 1$).

b) The variety $\Sigma$ is the union of a smooth quadric surface $Q$ and a plane $\Pi$. The projection $\pi_\Sigma : \Sigma \to P(V^*)$ maps $\Pi$ bijectively to $P(V^*)$ and projects $Q = P^1 \times P^1$ to one of its $P^1$ - factors which is then being embedded into $P(V^*)$ as the conic $\Gamma$.

c) The "Schur quadric" is the surface $Q$ from n. b).

d) The curve $C(E)$ coincides with $\Gamma$ (taken three times).

e) For any $z \in Z = \Gamma$ the lines $A_z$ and $A'_z$ both coincide with the generator of $Q = P^1 \times P^1$ mapped into $z$ by $\pi_\Sigma$, see n.b).

**4.5. Bring's curve as** $C(E)$. Consider the situation of Example 0.13: the cubic surface $\Sigma$ is given by equation $x_1^3 + \ldots + x_5^3 = 0$ where $x_i$ are linear functions on $M^*$ constrained by $\sum x_i = 0$. The Schur quadric corresponding to double - six described in n. 0.13 is given by $\sum x_i^2 = 0$. The intersection $C = \Sigma \cap Q$ i.e. the curve given in $P^4$ by equations

$$\sum x_i = \sum x_i^2 = \sum x_i^3 = 0$$

is known as *Bring's curve* **[K]** **[Hu2]**. The blow-down of the first six lines of the double - six described in n. 0.13 gives 6 points $p_1, \ldots, p_6$ in $P^2$ forming an orbit of the alternating group $A_5$ **[Hu2]**. These points will be the nodes of the sextic curve $\pi_\Sigma(C) \subset P^2$ i.e., of the projection of $C$ to $P^2$, which is also called Bring's curve. The equation of $\pi_\Sigma(C)$ can be found in **[Hu2]**, p. 82.

Thus Bring's curve can be represented as the curve of JLSK of a certain bundle on $\check{P}^2$: the (normalized) logarithmic bundle of the configuration of lines dual to $p_i$.

**4.6. Hulsbergen bundles.** Let $q_1, ..., q_n$ be $n$ points in general position in $P(V)$. There exists an $n-1$ -dimensional family of stable rank 2 bundles $E$ on $P(V)$ with $c_1 = -1, c_2 = n$ such that $\{q_1, ..., q_n\}$ is the set of zeros of a section of $E(1)$ (see **[Hul]**). They are called Hulsbergen bundles. For such $E$ the subscheme $Z$ of jumping lines of $E$ is reduced and consists of $\binom{n}{2}$ lines $< q_i, q_j >$. We denote by $f_j$ linear functions on $V^*$ corresponding to $q_i \in P(V)$. The linear system of curves of degree $n-1$ through $Z$ has a basis formed by the curves

$$F_j = \prod_{i \neq j} f_i = 0.$$

This system maps the surface $S = \mathrm{Bl}_Z(P(V^*))$ to the surface $\Sigma \subset P(M^*) = P^{n-1}$ given, in natural homogeneous coordinates $(t_1, ..., t_n)$, by equations

$$\left(\prod_{i=1}^{n} t_i\right)\left(\sum_{i=1}^{n} \frac{a_{ji}}{t_i}\right) = 0, j = 1, ..., n-3$$

where $(a_{j1}, ..., a_{jn})$, $j = 1, ..., n-3$ is a basis of the space of linear relations among the vectors $f_i$. In the coordinates $t_i$ the "Schur quadric" $Q$ is given by the equation $\sum c_i t_i^2 = 0$ so the curve of JLSK in $P(V^*)$ has the equation

$$\sum_{i=1}^{n} c_i F_i^2 = 0.$$

(cf. **[Hul]** n. 10.5). Note that $p : S \to \Sigma$ blows down the proper transforms of the lines $l_i$ to singular points of $\Sigma$ which have the coordinates $(1, 0, ..., 0), ...,$ $(0, ..., 1)$. These points belong to $a_M^{-1}(\mathrm{Hom}\ (V^*, H^*)_1)$. So Hulsbergen bundles are not $\Sigma$ - generic in the sense of n. 2.5, although they are $Z$ - generic.

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