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§1. INTRODUCTION

In 1983 Haynes Miller [7] proved a conjecture of Sullivan and used it to show that if π is a locally finite group and X is a simply connected finite dimensional CW-complex then the space of pointed maps from the classifying space $B\pi$ to X is weakly contractible, ie. $\mathrm{Map}_*(B\pi, X) \simeq *$. This result had immediate applications. Alex Zabrodsky [11] used it to study maps between classifying spaces of compact Lie groups. McGibbon and Neisendorfer [6] applied Miller's theorem to answer a question of Serre; they proved that if X is a simply connected finite dimensional CW-complex with $\tilde{H}^*(X, \mathbf{F}_p) \neq 0$ then there are infinitely many dimensions in which $\pi_*(X)$ has p-torsion.

The goal of this note is to use the functor T^V of [2] to generalize Miller's theorem and some of its corollaries to a large class of infinite dimensional spaces (see [5] for closely related earlier work in this direction). This generalization comes at the expense of working with one component of the function complex $\mathrm{Map}_*(B\pi, X)$ at a time.

Fix a prime number p.

THEOREM 1.1. *Let π be a locally finite group and X a simply connected p-complete space. Assume that $H^*(X, \mathbf{F}_p)$ is finitely generated as an algebra. Then the component of* $\mathrm{Map}_*(B\pi, X)$ *which contains the constant map is weakly contractible.*

REMARK: There is a standard way [7, 1.5] to relax the assumption in 1.1 that X is p-complete.

Theorem 1.1 is actually a special case of a more general assertion. Recall that an unstable module M over the mod p Steenrod Algebra $\mathbf{A}_p$ is said to be *locally finite* [4] if each element $x \in M$ is contained in a finite $\mathbf{A}_p$ submodule. If R is a connected unstable algebra over $\mathbf{A}_p$ then the *augmentation ideal* $I(R)$ is by definition the ideal of positive-dimensional elements and the *module of indecomposables* $Q(R)$ is the unstable $\mathbf{A}_p$ module $I(R)/I(R)^2$. An unstable algebra R over $\mathbf{A}_p$ is of *finite type* if each R^k is finite-dimensional as an $\mathbf{F}_p$ vector space.

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THEOREM 1.2. *Let π be a locally finite group and X a simply connected p-complete space such that $H^*(X, \mathbf{F}_p)$ is of finite type. Assume that the module of indecomposables $Q(H^*(X, \mathbf{F}_p))$ is locally finite as a module over $\mathbf{A}_p$. Then the component of* $\mathrm{Map}_*(B\pi, X)$ *which contains the constant map is weakly contractible.*

REMARK: Theorem 1.1 does in fact follow from Theorem 1.2, since if $H^*(X, \mathbf{F}_p)$ is finitely generated as an algebra then $Q(H^*(X, \mathbf{F}_p))$ is a *finite* $\mathbf{A}_p$ module.

REMARK: Theorem 1.2 has a converse, at least if $p = 2$ (see Theorem 3.2). There is also a generalization of 1.2 that deals with other components of the mapping space $\mathrm{Map}_*(B\pi, X)$ (see Theorem 4.1) but for this generalization it is necessary to assume that π is an elementary abelian p-group.

Given 1.2, the arguments of [**6**] go over more or less directly and lead to the following result. A CW-complex is of *finite type* if it has a finite number of cells in each dimension.

THEOREM 1.3. *Suppose that X is a two-connected CW-complex of finite type. Assume that $\tilde{H}^*(X, \mathbf{F}_p) \neq 0$ and that $Q(H^*(X, \mathbf{F}_p))$ is locally finite as a module over $\mathbf{A}_p$. Then there exist infinitely many k such that $\pi_k(X)$ has p-torsion.*

REMARK: The example of CP^∞ shows that it would not be enough in Theorem 1.3 to assume that X is 1-connected.

Some instances of 1.3 were previously known; for instance, if $X = BG$ for G a suitable compact Lie group then the conclusion of 1.3 can be obtained by applying [**6**] to the loop space on X. However, Theorem 1.3 applies in many previously inaccessible cases; for example, it applies if X is the Borel construction $EG \times_G Y$ of the action of a compact Lie group G on a finite complex Y or if X is a quotient space obtained from such a Borel construction by collapsing out a skeleton.

We first noticed Theorem 1.1 as part of our work [**1**] on calculating fragments of T^V with Smith theory techniques. The proof of 1.1 given here does not use the localization approach of [**1**]; it is partly for this reason that the proof generalizes to give 1.2.

Organization of the paper. Section 2 recalls some properties of the functor T^V. In §3 there is a proof of 1.2 and in §4 a generalization of 1.2 to other components of the mapping space. Section 5 uses the ideas of [**6**] to deduce 1.3 from 1.2.

Notation and terminology. The prime p is fixed for the rest of the paper; all unspecified cohomology is taken with $\mathbf{F}_p$ coefficients. The symbol $\mathcal{U}$ (resp. $\mathcal{K}$) will denote the category of unstable modules (resp. algebras) [**2**] over $\mathbf{A}_p$. If $R \in \mathcal{K}$ then $\mathcal{U}(R)$ (resp. $\mathcal{K}(R)$) will stand for the category of objects of $\mathcal{U}$ (resp. $\mathcal{K}$) which are also R-modules (resp. R-algebras) in a compatible way [**1**].

For a pointed map $f : K \to X$ of spaces we will let $\mathrm{Map}_*(K, X)_f$ denote the component of the pointed mapping space $\mathrm{Map}_*(K, X)$ containing f. The component of the *unpointed* mapping space containing f is $\mathrm{Map}(K, X)_f$.

§2 The functor T^V

Let V be an elementary abelian p-group, ie., a finite-dimensional vector space over $\mathbf{F}_p$, and H^V the classifying space cohomology H^*BV. Lannes [**2**] has constructed a functor $T^V : \mathcal{U} \to \mathcal{U}$ which is left adjoint to the functor given by tensor product (over $\mathbf{F}_p$) with H^V and has shown that T^V lifts to a functor $\mathcal{K} \to \mathcal{K}$ which is similarly left adjoint to tensoring with H^V.

Proposition 2.1 [**2**]. *For any object R of $\mathcal{K}$ the functor T^V induces functors $\mathcal{U}(R) \to \mathcal{U}(T^V(R))$ and $\mathcal{K}(R) \to \mathcal{K}(T^V(R))$. The functor T^V is exact, and preserves tensor products in the sense that if M and N are objects of $\mathcal{U}(R)$ there is a natural isomorphism*

$$T^V(M \otimes_R N) \cong T^V(M) \otimes_{T^V(R)} T^V(N)$$

Now suppose that $\gamma : R \to H^V$ is a $\mathcal{K}$-map. The adjoint of γ is a map $T^V(R) \to \mathbf{F}_p$ or in other words a ring homomorphism $\hat{\gamma} : T^V(R)^0 \to \mathbf{F}_p$. For $M \in \mathcal{U}(R)$, let $T^V_\gamma(M)$ be the tensor product $T^V(M) \otimes_{T^V(R)^0} \mathbf{F}_p$, where the action of $T^V(R)^0$ on $\mathbf{F}_p$ is given by $\hat{\gamma}$. Note that $T^V_\gamma(R) \in \mathcal{K}$.

Proposition 2.2 [**1**, 2.1]. *For any $\mathcal{K}$-map $\gamma : R \to H^V$ the functor $T^V_\gamma(-)$ induces functors $\mathcal{U}(R) \to \mathcal{U}(T^V_\gamma(R))$ and $\mathcal{K}(R) \to \mathcal{K}(T^V_\gamma(R))$. The functor T^V_γ is exact, and preserves tensor products in the sense that if M and N are objects of $\mathcal{U}(R)$ there is a natural isomorphism*

$$T^V_\gamma(M \otimes_R N) \cong T^V_\gamma(M) \otimes_{T^V_\gamma(R)} T^V_\gamma(N)\,.$$

The following proposition is a straightforward consequence of the above two.

LEMMA 2.3. *Suppose that $\alpha : R_1 \to R_2$ and $\beta : R_2 \to H^V$ are morphisms of $\mathcal{K}$, and let $\gamma : R_1 \to H^V$ denote the composite $\beta \cdot \alpha$.*

(1) *If α is a surjection and $M \in \mathcal{U}(R_2)$ is treated via α as an object of $\mathcal{U}(R_1)$, then the natural map $T^V_\gamma(M) \to T^V_\beta(M)$ is an isomorphism.*
(2) *If $M \in \mathcal{U}(R_1)$ then the natural map $T^V_\beta(R_2) \otimes_{T^V_\gamma(R_1)} T^V_\gamma(M) \to T^V_\beta(R_2 \otimes_{R_1} M)$ is an isomorphism.*

There is a natural map $\lambda_X : T^V(H^*X) \to H^*\,\mathrm{Map}(BV, X)$ for any space X. If $g : BV \to X$ is a map which induces the cohomology homomorphism $\gamma : H^*X \to H^V$ then λ_X passes to a quotient map

$$\lambda_{X,g} : T^V_\gamma(H^*X) \to H^*\,\mathrm{Map}(BV, X)_g\,.$$

A lot of the geometric usefulness of T^V is explained by the following theorem.

THEOREM 2.4 [3]. *Let X be a 1-connected space, $g : BV \to X$ a map, and $\gamma : H^*X \to H^V$ the induced cohomology homomorphism. Assume that H^*X is of finite type, that $T^V_\gamma H^*X$ is of finite type, and that $T^V_\gamma H^*X$ vanishes in dimension 1. Then $\lambda_{X,g}$ is an isomorphism.*

For any object M of $\mathcal{U}$ the adjunction map $M \to H^V \otimes_{\mathbf{F}_p} T^V(M)$ can be combined with the unique algebra map $H^V \to \mathbf{F}_p$ to give a map $M \to T^V(M)$; call this map ϵ. (If $M = H^*X$ for some space X, then ϵ fits into a commutative diagram involving λ_X and the cohomology homomorphism induced by the basepoint evaluation map $\mathrm{Map}(BV, X) \to X$.)

THEOREM 2.5 [4, 6.3.2]. *The map $\epsilon : M \to T^V(M)$ is an isomorphism iff M is locally finite as a module over $\mathbf{A}_p$.*

If $R \in \mathcal{K}$, $M \in \mathcal{U}(R)$ and $\gamma : R \to H^V$ is a $\mathcal{K}$-map, we will denote the composite $M \xrightarrow{\epsilon} T^V(M) \to T^V_\gamma(M)$ by ϵ_γ. Theorem 2.5 leads to the following result, which we will need in §4.

PROPOSITION 2.6. *Let M be an object of $\mathcal{U}(H^V)$ and $\iota : H^V \to H^V$ the identity map. Then $\epsilon_\iota : M \to T^V_\iota(M)$ is an isomorphism iff M splits as a tensor product $H^V \otimes_{\mathbf{F}_p} N$ for some $N \in \mathcal{U}$ which is locally finite as a module over $\mathbf{A}_p$.*

PROOF: The fact that ϵ_ι is an isomorphism if M has the stated tensor product decompositon follows directly from 2.3(2), 2.5 and [**2**, 4.2]. Conversely, under the assumption that ϵ_ι is an isomorphism Proposition 2.4 of [**1**] guarantees that M splits as a tensor product $H^V \otimes_{\mathbf{F}_p} N$ for some $N \in \mathcal{U}$; the fact that N is locally finite is again a consequence of 2.3(2) and 2.5.

§3 The null component

In this section we will prove Theorem 1.2. Before doing this we will recast the conclusion of the theorem in a slightly different form.

Lemma 3.1. *Let K be a finite pointed CW-complex, X a 1-connected space, and $f : K \to X$ a pointed map. Then $\mathrm{Map}_*(K, X)_f$ is weakly contractible if and only if the inclusion of the basepoint in K induces a weak equivalence $\mathrm{Map}(K, X)_f \to X$.*

Proof: As in [**7**, 9.1] the inclusion $* \to K$ gives rise to a fibration sequence $\mathrm{Map}_*(K, X)_f \to \mathrm{Map}(K, X)_f \to X$.

The arguments of [**7**, §9] now show that Theorem 1.2 follows directly from the following result.

Theorem 3.2. *Let V be an elementary abelian p-group and X a 1-connected p-complete space such that H^*X is of finite type. Let $f : BV \to X$ be a constant map and $\phi : H^*X \to H^V$ the induced cohomology homomorphism. Consider the following three conditions:*

(1) *QH^*X is locally finite as an $\mathbf{A}_p$ module*
(2) *the map $\epsilon_\phi : H^*X \to T^V_\phi H^*X$ is an isomorphism*
(3) *the inclusion of the basepoint $* \to BV$ induces a weak equivalence $\mathrm{Map}(BV, X)_f \to X$.*

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). Moreover, if $p = 2$ then (3) $\Longrightarrow$ (1).

Remark 3.3: It is likely that the three conditions of Theorem 1.2 are equivalent for *any* prime p; the proof would depend on the odd primary version of the results in [**9**].

Proof of 3.2: First consider the implication (1) $\Longrightarrow$ (2). Let $R = H^*X$ and let $I \subset R$ be the augmentation ideal. Pick $s \geq 0$. The fact that the action of R on I^s/I^{s+1} factors through the augmentation $R \to \mathbf{F}_p$ implies that the action of $T^V(R)$ on $T^V(I^s/I^{s+1})$ factors through the map $T^V(R) \to T^V(\mathbf{F}_p) \cong \mathbf{F}_p$ induced by augmentation; since this last map is adjoint to $\phi : R \to H^*(BV)$ it follows from 2.3(1) that the quotient map $T^V(I^s/I^{s+1}) \to T^V_\phi(I^s/I^{s+1})$ is an isomorphism. Moreover, I^s/I^{s+1}, as a quotient of $(I/I^2)^{\otimes s}$, is the union of its finite $\mathbf{A}_p$ submodules so by 2.5 the map $\epsilon : I^s/I^{s+1} \to T^V(I^s/I^{s+1})$ is an isomorphism. Putting these two facts together shows that $\epsilon_\phi : I^s/I^{s+1} \to T^V_\phi(I^s/I^{s+1})$ is an isomorphism. By induction and exactness, then, the map $\epsilon_\phi : R/I^{s+1} \to T^V_\phi(R/I^{s+1})$ is an isomorphism. The map $T^V_\phi(R) \to T^V_\phi(\mathbf{F}_p) \cong \mathbf{F}_p$ induced by augmentation is an epimorphism, so by exactness $T^V_\phi(I)$ vanishes in dimension 0.

By Lemma 2.2 and exactness, $T_\phi^V(I^{s+1})$ vanishes up to and including dimension s, and hence again by exactness the map $T_\phi^V(R) \to T_\phi^V(R/I^{s+1})$ induced by the quotient projection $R \to R/I^{s+1}$ is an isomorphism up through dimension s. It follows immediately that $\epsilon_\phi : R \to T_\phi^V(R)$ is an isomorphism.

The implication (2) $\Longrightarrow$ (3) is an easy consequence of Theorem 2.4.

For (3) $\Longrightarrow$ (1), assume $p = 2$. According to [**9**, proof of 3.1] condition (3) implies that the loop space cohomology $H^*(\Omega X)$ is locally finite as an $\mathbf{A}_p$ module, ie., in the terminology of [**9**], that $H^*(\Omega X) \in \mathcal{N}\mathrm{il}_k$ for all k. According to [**9**, 2.1(iii)], this implies that $\Sigma^{-1}QH^*X \in \mathcal{N}\mathrm{il}_k$ for all k. This amounts to the assertion that $\Sigma^{-1}QH^*X$ (or equivalently QH^*X) is locally finite [**9**, proof of 3.1].

§4 Other mapping space components

In this section we will give a generalization of Theorem 1.2 to mapping space components other than the component containing the constant map; this generalization is limited, however, in that it deals with elementary abelian p-groups rather than with arbitrary locally finite groups.

Given an elementary abelian p-group V, call an object M of $\mathcal{U}(H^V)$ *f-split* if M is isomorphic to $H^V \otimes_{\mathbf{F}_p} N$ for some $N \in \mathcal{U}$ which is locally finite as a module over $\mathbf{A}_p$. Suppose that $\gamma : R \to H^V$ is a map in $\mathcal{K}$ with image $S \subset H^V$ and kernel $I \subset R$. Say that γ is *almost f-split* if

(*i*) S is a Hopf subalgebra of H^V, and

(*ii*) for each $s \geq 0$ the tensor product $H^V \otimes_S (I^s/I^{s+1})$ is f-split as an object of $\mathcal{U}(H^V)$.

Recall from 3.1 that $\mathrm{Map}_*(K, X)_f$ is weakly contractible iff evaluation at the basepoint gives an equivalence $\mathrm{Map}(K, X)_f \cong X$.

THEOREM 4.1. *Let V be an elementary abelian p-group and X a 1-connected p-complete space such that H^*X is of finite type. Let $g : BV \to X$ be a map and $\gamma : H^*X \to H^V$ the induced cohomology homomorphism. Consider the following three conditions:*

(1) *γ is almost f-split*

(2) *the map $\epsilon_\gamma : H^*X \to T_\gamma^V H^*X$ is an isomorphism*

(3) *the inclusion of the basepoint $* \to BV$ induces a weak equivalence $\mathrm{Map}(BV, X)_g \to X$.*

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). *Morever, if $p = 2$ then* (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1).

REMARK 4.2: As in the case of Theorem 3.2, it is likely that the three conditions of Theorem 4.1 are equivalent for any prime p.

LEMMA 4.3. *Let K be a pointed CW-complex, X a pointed 0-connected space, $g : K \to X$ a map, and $f : K \to X$ a constant map. Assume that there exists a map $m : K \times X \to X$ which is 1_X on the axis $* \times X$ and $g : K \to X$ on the axis $K \times *$. Then the basepoint evaluation map $e_f : \mathrm{Map}(K, X)_f \to X$ is a weak equivalence if and only if the corresponding map $e_g : \mathrm{Map}(K, X)_g \to X$ is a weak equivalence.*

PROOF: Construct a commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{=} & K \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ K \times X & \xrightarrow[(pr_1, m)]{} & K \times X \end{array}$$

in which $a(k) = (k, *)$, $b(k) = (k, g(k))$ and pr_1 is projection on the first factor. Since the lower horizontal map is a weak equivalence, it follows that the induced map $c : \mathrm{Map}(K, K \times X)_a \to \mathrm{Map}(K, K \times X)_b$ is a weak equivalence. It is clear that c commutes with the natural projections from its domain and range to $\mathrm{Map}(K, K)_i$, where i is the identity map of K. The lemma follows from the fact that the domain of c is $\mathrm{Map}(K, K)_i \times \mathrm{Map}(K, X)_f$ while the range is $\mathrm{Map}(K, K)_i \times \mathrm{Map}(K, X)_g$.

LEMMA 4.4. *Let K be a pointed CW-complex, X a pointed 0-connected space, $g : K \to X$ a map, and $f : K \to X$ a constant map. Assume that the basepoint evaluation map $e_g : \mathrm{Map}(K, X)_g \to X$ is a weak equivalence. Then the basepoint evaluation map $e_f : \mathrm{Map}(K, X)_f \to X$ is also a weak equivalence.*

PROOF: The map m required in 4.3 is provided up to weak equivalence by the evaluation map $K \times \mathrm{Map}(K, X)_g \to X$.

LEMMA 4.5. *Let V be an elementary abelian p-group, R a connected object of $\mathcal{K}$, $\gamma : R \to H^V$ a map, and $\phi : R \to H^V$ the trivial map (ie. the map which factors through the augmentation $R \to \mathbf{F}_p$). Assume there exists a map $\mu : R \to H^V \otimes_{\mathbf{F}_p} R$ which gives 1_R when combined with the augmentation map of H^V and $\gamma : R \to H^V$ when combined with the augmentation map of R. Then $\epsilon_\phi : R \to T_\phi^V(R)$ is an isomorphism if and only if $\epsilon_\gamma : R \to T_\gamma^V(R)$ is an isomorphism.*

PROOF: This is essentially the proof of 4.3 with the arrows reversed.

Construct a commutative diagram

$$\begin{array}{ccc} H^V & \xleftarrow{=} & H^V \\ \alpha\uparrow & & \beta\uparrow \\ H^V \otimes_{\mathbf{F}_p} R & \xleftarrow{in_1 \cdot \mu} & H^V \otimes_{\mathbf{F}_p} R \end{array}$$

in which α is the product of 1_{H^V} with the augmentation of R, β is $(1_{H^V}) \cdot \gamma$, and in_1 is the map from H^V to the tensor product obtained using the unit of R. Since the lower horizontal map is an isomorphism, it follows that the induced map $\chi : T^V_\beta(H^V \otimes_{\mathbf{F}_p} R) \to T^V_\alpha(H^V \otimes_{\mathbf{F}_p} R)$ is an isomorphism. It is clear that χ respects the natural structures of its domain and range as modules over $T^V_\iota(H^V)$, where ι the identity map of H^V. The lemma follows from the fact [**1**, 2.2] that the domain of χ is $T^V_\iota(H^V) \otimes_{\mathbf{F}_p} T^V_\gamma(R)$ while the range is $T^V_\iota(H^V) \otimes_{\mathbf{F}_p} T^V_\phi(R)$.

LEMMA 4.6. *Let V be an elementary abelian p-group, R a connected object of $\mathcal{K}$, $\gamma : R \to H^V$ a map and $\phi : R \to H^V$ the trivial map. Assume that $\epsilon_\gamma : R \to T^V_\gamma(R)$ is an isomorphism. Then $\epsilon_\phi : R \to T^V_\phi(R)$ is also an isomorphism.*

PROOF: The map μ required in 4.5 is provided by the map $R \to H^V \otimes_{\mathbf{F}_p} T^V_\gamma(R)$ which is adjoint to the identity map of $T^V_\gamma(R)$.

REMARK 4.7: It follows from 4.5, 4.6 and 3.2 that at least if $p = 2$ the three conditions of 4.1 are equivalent to a fourth, namely, that QH^*X is locally finite as an $\mathbf{A}_p$ module and there exists a $\mathcal{K}$ map $H^*X \to H^V \otimes_{\mathbf{F}_p} H^*X$ which satisfies the conditions of 4.5.

LEMMA 4.8. *Let V be an elementary abelian p-group and $\nu : S \to H^V$ the inclusion of a subalgebra over $\mathbf{A}_p$. Then $\epsilon_\nu : S \to T^V_\nu(S)$ is an isomorphism if and only if ν includes S as a Hopf subalgebra of H^V.*

PROOF: Suppose that ϵ_ν is an isomorphism. In this case the adjunction homomorphism $S \to H^V \otimes_{\mathbf{F}_p} T^V_\nu(S)$ provides a map $\Delta_S : S \to H^V \otimes_{\mathbf{F}_p} S$ which fits into a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\Delta_S} & H^V \otimes_{\mathbf{F}_p} S \\ \nu\downarrow & & \downarrow\iota\otimes\nu \\ H^V & \xrightarrow[\Delta_{H^V}]{} & H^V \otimes_{\mathbf{F}_p} H^V \end{array}$$

$T_\gamma^V(H^V)$ is injective, and it follows from naturality and the fact that $H^V \to T_\gamma^V(H^V)$ is injective [**2**, 4.2] that $S \to T_\gamma^V(S)$ is injective. By 2.3(1) the map $\epsilon_\nu : S \to T_\nu^V(S)$ is an isomorphism and hence (4.8) S is a Hopf subalgebra of H^V.

By exactness the map $I^s \to T_\gamma^V(I^s)$ is seen to be an isomorphism if $s = 1$ and a monomorphism if $s > 1$; this first fact, though, combines with the tensor product formula (2.2) and exactness to show that $I^s \to T_\gamma^V(I^s)$ is an epimorphism for $s \geq 1$. Thus by exactness and 2.3(1) the maps $\epsilon_\nu : I^s/I^{s+1} \to T_\nu^V(I^s/I^{s+1})$ are isomorphisms. The proof is finished by running in reverse the argument used above at the end of the proof of (1) $\Longrightarrow$ (2).

§5 Torsion in homotopy groups

In this section we will use a slight variation on the ideas of [**6**] to prove Theorem 1.3.

Let $\mathbf{Z}$ denote the ring of integers, $\mathbf{Z}_p^\wedge$ the additive group of p-adic integers, and $\mathbf{Z}/p^n$ the cyclic group of order p^n. The group $\mathbf{Z}/p^\infty$ is by definition the locally finite group obtained by taking the direct limit of the groups $\mathbf{Z}/p^n$ under the standard inclusion maps.

LEMMA 5.1. *For any finitely-generated abelian group A the cohomology group $H^k(B\mathbf{Z}/p^\infty, A)$ is isomorphic to $\mathbf{Z}_p^\wedge \otimes A$ if $k > 0$ is even and is zero if k is odd. The natural map $A \to \mathbf{Z}_p^\wedge \otimes A$ induces isomorphisms $H^k(B\mathbf{Z}/p^\infty, A) \cong H^k(B\mathbf{Z}/p^\infty, \mathbf{Z}_p^\wedge \otimes A)$ for all $k > 0$.*

SKETCH OF PROOF: One way to do this is to calculate the homology $H_*(B\mathbf{Z}/p^\infty, \mathbf{Z})$ as a direct limit $\varinjlim_n H_*(B\mathbf{Z}/p^n, \mathbf{Z})$ and then pass to cohomology by using the universal coefficient theorem. The key algebraic ingredient is the fact that

$$\operatorname{Ext}_{\mathbf{Z}}(\mathbf{Z}/p^\infty, \mathbf{Z}) \cong \operatorname{Ext}_{\mathbf{Z}}(\mathbf{Z}/p^\infty, \mathbf{Z}_p^\wedge) \cong \mathbf{Z}_p^\wedge .$$

Let P_nX stand for the n'th Postnikov stage of the space X and $k^{n+1}(X)$ for the Postnikov invariant of X which lies in $H^{n+1}(P_{n-1}X, \pi_nX)$(see [**10**, IX]).

LEMMA 5.2. *If Y is a loop space ΩX and Y has finitely-generated homotopy groups, then the Postnikov invariants of Y are torsion cohomology classes.*

PROOF: This follow from [**8**, p. 263]. In effect, Milnor and Moore show that the rationalized Postnikov invariants

$$k^{n+1}(Y) \otimes \mathbf{Q} \in H^{n+1}(P_{n-1}Y, \pi_n(Y) \otimes \mathbf{Q})$$

where ι is the identity map of H^V and we have used the fact [**2**, 4.2] that $\epsilon_\iota : H^V \to H^V$ is an isomorphism. It is easy to see that Δ_{H^V} is the Hopf algebra comultiplication map on H^V. It now follows from the fact that the comultiplication on H^V is cocommutative that $\Delta_S(S) \subset S \otimes_{\mathbf{F}_p} S$ and thus that S is a Hopf subalgebra of H^V.

Suppose conversely that S is a Hopf subalgebra of H^V, and let $\phi : S \to H^V$ be the trivial map which factors through the augmentation $S \to \mathbf{F}_p$. The Hopf algebra H^V is primitively generated, and the associated restricted Lie algebra of primitives [**8**, 6.7] is a free abelian restricted Lie algebra on a finite collection of generators (in dimensions 1 and 2). It follows from [**8**, 6.13–6.16] that S is primitively generated and is isomorphic as an algebra to a finite tensor product of exterior and polynomial algebras; in particular, $Q(S)$ is a finite unstable $\mathbf{A}_p$ module. By the proof of (1) $\Longrightarrow$ (2) in Theorem 3.2 the map $\epsilon_\phi : S \to T_\phi^V(S)$ is an isomorphism. Since the comultiplication of S produces the map μ required for Lemma 4.5, an application of this lemma finishes the proof.

Proof of 4.1: Let R denote H^*X, I the kernel of $\gamma : R \to H^V$, S the image of γ and $\nu : S \to H^V$ the inclusion map. We will use f to stand for a constant map $BV \to X$ and ϕ for the cohomology homomorphism induced by f.

(1) $\Longrightarrow$ (2). The assumption that S is a Hopf subalgeba of H^V implies by 4.8 that $\epsilon_\nu : S \to T_\nu^V(S)$ and hence (2.3(1)) $\epsilon_\gamma : S \to T_\gamma^V(S)$ are isomorphisms. Pick $s \geq 1$ and let $M = I^s/I^{s+1}$. If we can show that $\epsilon_\gamma : M \cong T_\gamma^V(M)$ we will be able to finish up by imitating the proof of (1) $\Longrightarrow$ (2) in Theorem 3.2. By 2.3(1) it is enough to show that $\epsilon_\nu : M \cong T_\nu^V(M)$. Proposition 2.6 ensures that $\epsilon_\iota : H^V \otimes_S M \to T_\iota^V(H^V \otimes_S M)$ is an isomorphism, where ι is the identity map of H^V. By 2.3(2) and [**2**, 4.2] , however, the map ϵ_ι is $\iota \otimes_S \epsilon_\nu$, so the desired result follows from the fact that H^V is free [**8**, 4.4] and therefore faithfully flat as a module over S.

(2) $\Longrightarrow$ (3). This is an immediate consequence of 2.4.

(3) $\Longrightarrow$ (2). By Lemma 4.4 and Theorem 3.2 the map $\epsilon_\phi : R \to T_\phi^V(R)$ is an isomorphism. The evaluation map $m : BV \times \mathrm{Map}(BV, X)_g \to X$ induces a cohomology homomorphism $\mu : R \to H^V \otimes_{\mathbf{F}_p} R$ which satisfies the conditions of 4.5, so the implication follows from the conclusion of 4.5.

(2) $\Longrightarrow$ (1). This implication does not in fact require the assumption that $p = 2$. The map $T_\gamma^V(R) \to T_\gamma^V(S)$ is surjective and it follows immediately from naturality that $\epsilon_\gamma : S \to T_\gamma^V(S)$ is surjective. The map $T_\gamma^V(S) \to$

are zero. Under the stated finite generation assumption this implies that the Postnikov invariants themselves are torsion.

Proof of 1.3: Let S_1 be the set of all k such that $\pi_k(X) \otimes \mathbf{Z}_p^{\hat{}} \neq 0$ and S_2 the set of all k such that $\pi_k X$ contains p-torsion. The set S_1 is non-empty (because $H^*(X, \mathbf{F}_p) \neq 0$) and clearly contains S_2. Suppose that S_2 is finite. In that case we can find an integer k in S_1 such that no integer j greater than k belongs to S_2. Let $Y = \Omega^{k-2}X$. (Note that because X is 2-connected the integer k is greater than 2 and Y is a loop space.) By Lemma 5.1 the space $\mathrm{Map}_*(B\mathbf{Z}/p^\infty, P_1Y)$ is contractible and hence $\mathrm{Map}_*(B\mathbf{Z}/p^\infty, P_2Y) \cong \mathrm{Map}_*(B\mathbf{Z}/p^\infty, K(\pi_2 Y, 2))$. Because of the way in which k was chosen we can thus, by Lemma 5.1 again, find an essential map $f : B\mathbf{Z}/p^\infty \to P_2Y$ which remains essential in the p-completion $(P_2Y)_p^{\hat{}}$. The obstructions to lifting f to a map $g : B\mathbf{Z}/p^\infty \to Y$ are the pullbacks to $B\mathbf{Z}/p^\infty$ of the Postnikov invariants of Y [**10**, p. 450]; by Lemma 5.2 these obstructions are torsion, but by Lemma 5.1 and the choice of k they lie in torsion-free abelian groups. Therefore the obstructions vanish, and the lift g exists. The composite h of g with the completion map $Y \to Y_p^{\hat{}}$ is non-trivial because the composite of h with the projection map $Y_p^{\hat{}} \to P_2(Y_p^{\hat{}}) \cong (P_2Y)_p^{\hat{}}$ is essential. The adjoint of h is then non-zero element of $\pi_{k-2}\,\mathrm{Map}_*(B\mathbf{Z}/p^\infty, X)$, an element which by Theorem 1.2 cannot exist. This contradiction shows that S_2 is infinite and proves the theorem.

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