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A. M. STEPIN

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by A.M. Stepin

In the last ten years a new method of investigating dynamical systems, called the (L,A)-pair method, has been developed. Essentially, it consists of the construction of prime integrals and the action-angle variables for hamiltonian systems as spectral invariants of some operator function L defined on the phase space of a system and satisfying the equation $\dot{L} = (L, A)$ for some operator function A . The (L, A)-pair method has made possible the integration of many important nonlinear partial differential equations and has been succesfully applied to the systems of analytical mechanics: it has been shown that the system of n particles on the real line with the Weierstrass $\wp$-function as the potential of their interaction and the problem of the solid body dynamic in R^3 are fully integrable. [7] and [3].

There is a distinguishing feature of prime integrals constructed by the (L, A)-pair method, namely the polynomiality of their variational derivatives for infinite-dimensional systems and the polynomiality of the integrels with respect to the impulses in the case of finite-dimensional systems. We note also that almost all of the integrated problems of analytical dynamic with the natural hamiltonian possess a full set of involutive integrals polynomial with respect to the impulses. Therefore, it is natural to approach the description of all integrable dynamical systems by determining Hamiltonians admitting prime integrals polynomial with respect to the impulses and independent of the energy and the kinetical moments of the given system. By kinetical moments we call the prime integrals associated, in virtue of the Noether's theorem, with the configurational symmetries of the Hamiltonian.

In the present paper this problem will be investigated in the case of n-particle linear system with the hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i<j} V(x_i - x_j) \tag{1}$$

where x_i and p_i are, for $i = 1,\dots,n$, the coordinates and impulses of the particles, respectively. The function V will be called in the sequel the potential. It will be said that a potential V admits a prime integral F if F is a prime integral for the system with the hamiltonian (1).

Let us choose now a suitable space of possible potentials. Namely, let $\mathcal{V}_\rho$, $\rho>0$ be the space of meromorphic functions holomorphic in a punctured neighbourhood of zero of the form $\{z : 0<|z|<\rho\}$, endowed with the norm

$$\|V\|_\rho = \sup_{|z|<\rho} |V_-(\tfrac{1}{z})| + \sup_{z\leq\rho} |V_+(z)|,$$

where V_- and V_+ are fundamental and regular part of the Laurent expansion of V in 0, respectively.

Theorem 1. The family of all potentials $V \in \mathcal{V}_\rho$ admitting a prime integral being a polynomial of a given degree with respect to the impulses and with non-constant uniform coefficients of the fundamental part V_- is finite-dimensional in $\mathcal{V}_\rho$.

Theorem 2. The family of all such potentials $V \in \mathcal{V}_\rho$ that

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + c_3 V(x_1 - x_2) + c_2 V(x_1 - x_3) + c_1 V(x_2 - x_3) \tag{2}$$

admits a prime integral involutive with respect to the full impulse P of the system, polynomial with respect to the impulses and functionally independent of H and P is finite-dimensional in $\mathcal{V}_p$ for almost all choices of the triple (c_1,c_2,c_3).

Remark: A finite-dimensionality theorem similar to the theorem 2 holds also for the hamiltonian

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2m_3} + V(x_1-x_2) + V(x_2-x_3) + V(x_1-x_3)$$

for almost all masses. If we assume, in addition, that the hamiltonian (1) admits a "typical integral", we are able to refine both theorems 1 and 2.

Definition: A prime integral F, polynomial with respect to the impulses of a given degree d is in general position if

(a) F is in involution with the full impulse P,

(b) the uniform fundamental part $\tilde{F}$ of F has constant coefficients,

(c) $\left(\sum_{i=1}^{n} \frac{\partial}{\partial p_i}\right)^{d-3} \tilde{F}$ is not a function of elementary symmetric polynomials δ_1 and δ_2.

A function $F(x_1,\dots,x_n,p_1,\dots,p_n)$ is called symmetric if for every permutation τ of the indexes $1,\dots,n$

$$F(x_{\tau(1)},\dots,x_{\tau(n)},p_{\tau(1)},\dots,p_{\tau(n)}) = F(x_1,\dots,x_n,p_1,\dots,p_n).$$

Theorem 3. If the potential V admits a symmetric integral in general position then V satisfies the differential equation

$$3V' V''' - 4V''^2 = aV'' + bV + c ,$$

where a,b,c are arbitrary parameters.

Remark. It is also possible to show a differential equation for potentials without the assumption b and the symmetry of the integral.

Theorem 4. If the hamiltonian

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + V(x_1 - x_2) + V(x_1 - x_3) + V(x_2 - x_3)$$

admits a symmetric prime integral in general position then the potential V is either the Weierstrass $\wp$-function or one of its degenerated forms : $\frac{a}{x^2}$, $\frac{a}{\sin^2 kx}$, $\frac{a}{\mathrm{sh}^2 kx}$.

Systems with the hamiltonian (1) and the potentials enumerated in the theorem 4 have been investigated by Moser [4], Calogero [1], Olshanetski and Perelomov [5]. The paper [5] is devoted to the investigation of systems with the hamiltonian of the form

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + a\sum_{k<l}\left[V(x_k - x_l) + V(x_k + x_l)\right] + b\sum_{k=1}^{n} V(x_k) + c\sum_{k=1}^{n} V(2x_k)$$

Facts, analogous to the theorems 1 - 4 hold also for such systems. We note moreover that Marcus and Meyer [2] have shown that the set of hamiltonians admitting no prime integrals independent of the energy is massive (infinite-dimensinal) for all systems with two-dimensional compact phase space. The proof involves the Robinson's theorem about periodic trajectories with multipliers +1 of hamiltonian systems. It is not known whether or not a theorem of this form holds for hamiltonians of the form (1).

The results formulated above can be obtained in the following way: the first step is the proof of polynomiality with respect to the x_i-s of the leading coefficients of a primeintegral regarded as a polynomial in the impulses and the description of the remaining coefficients; this inductive procedure of computing the coefficients ends with the integral-differential equation for the potential similar to an addition formula for elliptic functions; then we check that the resulting formula is non-trivial (this can be done either by computation of the order of the pole in origin of a potential satisfying the addition formula or by the check of the fact, that the leading coefficients of the integral are constant for the systems of n particles on a line interacting with the Weierstrass $\wp$-function as the potential because of their complete integrability [7]) after all that, the addition formula can be reduced to an algebraic differential equation.

The following two lemmas contain the description of the coefficients of prime integrals polynomial with respect to the impulses

Lemma 1. Let F be a polynomial with respect to the impulses prime integral of a system with the Hamiltonian of the form (1) or (2) and let

$$\tilde{F} = \sum_{k_1+\dots+k_n=d} E^{k_1,\dots,k_n}(x_1,\dots,x_n)\, p_1^{k_1}\dots p_n^{k_n}$$

be its leading uniform component. Then $E^{k_1,\dots,k_n}$ is a polynomial of $x_1,\dots,x_n$ and $\deg_{x_i} E^{k_1,\dots,k_n} \leq d-k_i$.

We denote now by A_n the differential ring of polynomials of n variables $x_1,\dots,x_n$ with the distinguished differentiations $\frac{\partial}{\partial x_i}$ and by $A_n\{V\}$ the smallest differential ring obtained by attaching to A_n the functions $V\, x_i - x_j$. More precisely $A_n\{V\}$ is the ring of polynomials of infinitely many variables:

$$A_n\{V\} = A_n[V_{ij}^{(s)},\ i,j = 1,\dots,n;\ s = 0,1,\dots]$$

and $\frac{\partial}{\partial x_k} V_{ij}^{(s)} = 0$, $\frac{\partial}{\partial x_i} V_{ij}^{(s)} = V_{ij}^{(s+1)}$, $\frac{\partial}{\partial x_j} V_{ij}^{(s)} = -V_{ij}^{(s+1)}$.

The smallest differential ring containing the primitive functions of all elements of a given differential ring $\mathcal{R}$ will be called the 1-extension of $\mathcal{R}$. We define higher extensions inductively, letting the n+1-extension to be the 1-extension of the n-extension of $\mathcal{R}$.

Lemma 2. For every natural d there exists $N = N(d)$ such that the coefficients of a prime integral of a system with the Hamiltonian (1) or (2), being a polynomial of degree d of the impulses belong to the N-extension of the ring $A_n\{V\}$.

Definition. Let $P(V)$ be an element of the N-extension of $A_n\{V\}$. A potential V satisfies the addition formula $P(V) = 0$ if substituting V in P we obtain a functional equivalence. If we restore the coefficients of the k-th uniform component of the integral F by the coefficients of its k+2-component, according to the equation $\{F, H\} = 0$, we obtain at the end some addition formula. If there are no even-degree monomials in F then this formula becomes

$$E^{1,0,\dots,0}\dot{p}_1 + \dots + E^{0,\dots,0,1}\dot{p}_n = 0 \text{ , where } \dot{p}_i = -\frac{\partial H}{\partial x_i} \text{ ;}$$

if there are no odd-degree monomials in F then the addition formule is of the form:

$$\frac{\partial}{\partial x_1}\left(2E^{2,0,\dots,0}\dot{p}_1 + E^{1,1,0,\dots,0}\dot{p}_2 + \dots + E^{1,0,\dots,1}\dot{p}_n\right) =$$

$$= \frac{\partial}{\partial x_2}\left(2E^{0,2,0,\dots,0}\dot{p}_2 + E^{1,1,0,\dots,0}\dot{p}_1 + \dots + E^{0,1,0,\dots,0}\dot{p}_n\right)$$

Trying to reduce these equations to ordinary differential equations we will investigate their non-triviality. An addition formula is trivial if it is satisfied by any potential V . Such formula leads us to the integral depending polynomially on the full impulse and energy. An addition formula turns out to be non-trivial in the following cases:

(a) the integral F is in general position;

(b) the leading coefficients of F are not constant;

(c) the integral F of a system with the Hamiltonian 2 is not a function of the full impulse and energy.

Lemma 3. If the potential V admits an integral F in general position then either V is regular in zero or it has there a pole of the order 2.

Lemma 4. Let G be a polynomial of x_i, p_i $i=1,\dots,n$. Suppose that for every potential V there exists an integral G_V of a system with the Hamiltonian (1) such that G_V is a polynomial of p_i and its leading uniform component is G . Then G is a polynomial of p_i with constant coefficients.

The proof of the lemma 4 is based on the investigation of the system with the potential εP and its billiard limit for $\varepsilon \to 0$ and the fact that inveriant tori of the first degree are close to the polyhedral invariant tori of the billiard system. It is also to the point to recall the problem of the realization of integrable billiard systems in polyhedra as the billiard limits of classical integrable billiard systems.

Corollary. If F is a symmetric integral for the Hamiltonian (1) with the potential $\wp$, being in involution with the full impulse then F is a polynomial of the Moser-Calogero integrals $I_1, \dots, I_n$.

It has been proved in the paper [7] that the assumption of the symmetry of an integral F can be omitted if F is multilinear with respect to the impulses. It follows then, as it was established in [7], that the Moser-Calogero integrals are involutive. Therefore, ~~almost~~ all solutions of the system with the Hamiltonian $H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \sum_{i<j} \wp(x_i - x_j)$ are almost periodical. On the other hand, we can investigate the dispersion for the system with the decreasing potential $\frac{1}{x^2}$ or $\frac{1}{sh^2 x}$. From the mathematical point of view, the dispersion in these systems is connected with the Stokes pkenomenon for the functions describing changes of the phase variables with the time. The dispersion mapping is explicitly computable for the systems with the potential $\frac{1}{x^2}$, with the help of the polynomial integrals.

We note also in the connection with the lemma 4, that the potentials admitting polynomial with respect to the impulses integrals with non-constant leading coefficients have been also found in an explicit form (the leading coefficients are always polynomial with respect to the x_i-s - see lemma 1). For example, the general form of a symmetric, quadratical with respect to the impulses prime integral for a system with the Hamiltonian (1) and $n=3$ is, up to a summand depending on H, of the form:

$$F = -\frac{1}{2}(x_2-x_3)^2 p_1^2 + \ldots + (x_1-x_2)(x_2-x_3)p_1p_2 + c(x_1,x_2,x_3) .$$

If we write down the corresponding addition formula and reduce it to a differential equation for the potential we obtain:

$$x^2V'' - xV' + 4V = 8ax^2 + b ,$$

with the general solution $V = c_{-2}x^{-2} + c_0 + c_2x^2 + c_4x^4$. The integral F does not depend on the mass center u, hence it is involutive with the full impulse. It follows now that the system of three particles with the interaction potential described above is fully integrable.

The general form of symmetric cubical integrals with non-constant leading coefficients has been computed by S. Pidkuiko:

$$F = A_1p_1^3 + \ldots + B_{12}p_1^2p_2 + \ldots + \sum_{i<j<k} C_{ijk}p_ip_jp_k + \text{linear terms},$$

where

$$A_1 = a(x_2^2 + \ldots + x_n^2) - \frac{2a}{n-2}\sum_{2\leq i<j\leq n} x_ix_j ,$$

$$B_{12} = a(x_2-2x_1x_2) + \frac{2a}{n-2}(x_3+\ldots+x_n)x_1 + b(x_3^2+\ldots+x_n^2) - \frac{3b}{n-3}\sum_{3\leq i<j\leq n} x_ix_j$$

$$C_{123} = -\frac{2a}{n-2}(x_1^2+x_2^2+x_3^2) - 2b(x_1x_2+x_2x_3+x_3x_1) + \frac{3b}{n-3}(x_1+x_2+x_3)(x_4+\ldots+x_n) - \frac{3b}{2\,n-3}(x_4^2+\ldots+x_n^2) ,$$

and $b = \frac{4a}{n-2}$.

Lemma 5. For an everywhere dense set of triples (c_1,c_2,c_3) there exists a potential $V\in\mathcal{V}_p$ such that the system with the Hamiltonian (2) admits no polynomial integral involutive with P and being no function of H and P.

The proof of this lemma adopts the Siegel s method of "blowing up" the convergence of Birkhoff transformation.

We shall now outline the reduction of a non-trivial addition formula to an algebraic differential equation. Let the addition formula be of the form $P(V)=0$, where $P(V)$ is an element of the N-extension of the differential ring $A_n\{V\}$. Fix $x_1,\dots,x_{n-1}$ and let $x_n \to x_{n-1}$. Denoting $x_n - x_{n-1}$ by y we develop every function $V, V', \dots$ and every polynomial in $P\,V$ with respect to the powers of y. After integration by y we can write $P(V)$ in the form

$$\sum_k \sum_{l \in L} P_{kl}(V)\, y^k \ln^l y\ ,$$

where L is a finite set of natural numbers since there is only finite number of integrations in $P(V)$. Integrating the formula $Q(y)\sum_m c_m y^m$, where Q is a polynomial, we obtain $\ln y$, then integrating $\sum_m c_m\, y^m \ln y$ we obtain $\ln^2 y$ and so on.

Therefore, our addition formula takes form

$$P(V) - \sum_k \sum_l P_{kl}(V) y^k \ln^l y = 0\ ,$$

where all factors $P_{kl}(V)$ are elements of the N-extension of the differential ring $A_{n-1}\{V\}$ of functions of n-1 variables $x_1,\dots,x_{n-1}$. The above given equation implies that any potential satisfying $P(V) = 0$ satisfies also $P_{kl}(V) = 0$ for every k,l.

Since our addition formula was non-trivial, there exists a pair k,l such that $P_{kl}(V) = 0$ is also non-trivial. We repeat now our procedure fixing $x_1,\dots,x_{n-2}$ and letting x_{n-1} converge to x_{n-2}. At the very end, we obtain an algebraic differential equation $Q(V) = 0$. "Algebraic" means here that Q is a polynomial of V, derivatives of V and the variable x.

Remark. Some information about the potential V can be lost sn the process of the reduction of addition formula to the resulting differential equation. This means that the final equation can be satisfied by a larger class of functions then these satisfying the initial formula. This loss of information can be however controlled. Namely, writing down all equations $Q_i(V) = 0$, $i=1,2,\dots$ resulting from the addition formula we obtain a set of equations equivalent to the initial one. Applying now the Ritt-Rodenbush basis theorem to the differential ideal generated by $Q_i(V)$ we obtain a finite set of differential equations $Q_{i_1}(V)=0$, ... $Q_{i_s}(V)=0$, equivalent to the initial addition formula. Potentials satisfying this set of equations admit integrals independent of H and P . For example, in the case of therem 4 the addition formula is equivalent to the differential equation for the Weierstrass $\wp$-function.

We turn now our attention back to the finite-dimensionality theorems 1 and 2 which can be obtained from the above given results in the following way. A function V , satisfying an algebraic differential equation cannot possess in origin a pole of arbitrarily high order.

Therefore, all solutions of the equation $Q(V)=0$ are of the form $\frac{U}{x^k}$, where U is holomorphic and satisfies another algebraic differential equation $\tilde{Q}(U)=0$. A solution U of the equation $\tilde{Q}(U)=0$ will be called regular if it has the only multiple roots in finite number of points in the disk $|z|\leq\rho$, when regarded as a polynomial of its highest-order derivative. Otherwise, such solution U of $\tilde{Q}(U)=0$ will be called singular. It satisfies the algebraic-differential equation $\frac{\partial\tilde{Q}}{\partial U^{(m)}}=0$, where $U^{(m)}$ is the highest order derivative in Q .

The following observations make up the proofs of the theorems 2 and 1:

(a) the theorem about smooth dependence of solutions on initial data tells us that the set of regular solutions is a smooth image of a finite-dimensional Euclidean space;

(b) singular solutions satisfy an algebraic differential equation of lower order, which can be obtained by elimination of the highest derivative in $\tilde{Q}(U)=0$ and $\frac{\partial\tilde{Q}}{\partial U^{(m)}}=0$. Then we regard regular solutions of the resulting equation and so on.

Remark. The set of solutions of a differential equation can be infinite-dimensional in the space of smooth functions. It is a by-effect of multiple switching from one singular solution to an another one.

It is very important that the proposed here approach to the description of potentials admitting supplementary prime integrals polynomial with respect to the impulses can be successfully applied to the multidimensional multiple-particles systems with the interaction potential of the form $V(r)=f(r^2)$, where f is a meromorphic function.

If the interaction is given then our method gives us, in theory, the opportunity of an explicit computation of the prime integrals polynomial with respect to the impulses. Computing Poisson brackets of these integrals we obtain some information about the group of all symmetries of the system, e.g. the well-known results of Foc and Moser can be obtained in this way in the case of Kepler systems. However, the computation of the Poisson brackets may be very complicated and technically difficult. To this point, we state a theorem which can help us in the check of the involutiveness of integrals found:

If F_1, F_2 are analytic integrals of a Hamiltonian system close to a nondegenerate, completely integrable system then $\{F_1, F_2\} = 0$. Therefore, if a system with degree of freenes n, close to a non-degenerate integrable system possesses n independent integrals then it is integrable.

Now we show an application of our method to Euler systems. Let $\mathfrak{g}$ be a semi-simple Lie algebra and let A be an operator self-adjoint with respect to the Killing form in $\mathfrak{g}$. The system $\dot{x} = [Ax, x]$ is called an Euler system in $\mathfrak{g}$. Mishtshenko and Fomenko

[3] have determined a class of operators A for which the Euler system is integrable on the orbits of the adjoint representation being in general position. The class contains operators of the form:

$$A = ad_a^{-1} ad_b \oplus A_0 , \qquad (3)$$

where a, b are in general position in the Cartan subalgebra $\mathfrak{g}_0$ of and A_0 is self-adjoint in $\mathfrak{g}_0$. Differentiation in the direction of the vector a maps the integrals of the Euler system for such A into themselves. Under the hypotheses of the theorems 3 and 4 the differentiation $\sum_{i=1}^{n} \frac{\partial}{\partial p_i}$ has analogous properties and this allowed us to describe potentials admitting integrals in general position. Similarly, the following opposite theorem holds also for Euler systems.

Operators A admitting a differentiation in general position with constant coefficients leaving the set of prime integrals invariant for the equation $\dot{x} = [Ax,x]$ are of the form (3).

An another interesting connection between Euler systems and systems with the hamiltonian (1) should be noted. The right-hand side of the equation $\dot{x} = [Ax,x]$ is uniform and therefore, as it was noticed by Mishtshenko, the problem of description of all analytic integrals of this system can be reduced to the description of the polynomial and even uniform integrals. It turns out also to be true that the system with the hamiltonian $\frac{1}{2} \sum_{i=1}^{n} p_i^2 + \sum_{i<j} \wp(x_i - x_j)$ admits a reformulation as a system of differential equations with rational right-hand sides.

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References

1. F. Calogero: Exactly solvable one-dimensional many-body problems. Lett. Nuovo Cimento 13:1975
2. L. Markus, K. Meyer: Generic Hamiltonian systems are not ergodic. Trudy V Mezhd. Konferencii po nelineinym kolebaniam. Kiev 1970.
3. A.S. Mishtshenko, A.T. Fomenko: Integrability of Euler equations on semi-simple Lie algebras. (russ.) Trudy seminara po vektornomu i kompleksnomu analizu 1977.
4. J. Moser: Three integrable Hamiltonian systems sonnected with isospectral deformations. Adv. in Math. 16: 1975
5. M.A. Olshanetskyi, A.M. Perelomov: Completely integrable Hamiltonian systems connected with semi-simple Lie algebras. (russ.) Invent. Math. 37:1976.
6. S.I. Pidkuiko, A.M. Stepin: On the solution of a differential-functional equation. Functional Anal. and its Appl. 10:1976.
7. S.I. Pidkuiko: On an integrable n-body problem. UMN 1977

A.M.STEPIN
М Г У
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117 234 МОСКВА
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