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THE FOURIER TRANSFORM ON SYMMETRIC SPACES

BY

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1. Introduction

In his paper "Sur la détermination d'un système orthogonal complet dans un espace de Riemann symétrique clos" (*Rend. Circ. Mat. Palermo*, 53 (1929), 217–252), Élie CARTAN modified and extended the Peter-Weyl theorem [32] to compact symmetric spaces. This incorporated for example the classical theory of spherical harmonics and thereby this latter theory merged with the representation theory of the orthogonal group. CARTAN's paper is therefore a very important link in the chain which reaches from classical Fourier series to harmonic analysis on Lie groups and their homogeneous spaces.

In my lecture I am going to discuss some further developments in Fourier analysis on symmetric spaces. This will be in three parts :

(i) Noncompact Riemannian symmetric spaces.

(ii) Compact Riemannian symmetric spaces (refinements of CARTAN's theory, suggested by (i)).

(iii) Noncompact, non-Riemannian symmetric spaces (including noncompact semisimple Lie groups).

We shall use the standard notation $\mathcal{D}(X) = C_c^\infty(X)$, $\mathcal{E}(X) = C^\infty(X)$, X being any manifold.

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The Peter-Weyl theorem for a compact Lie group U can be stated

$$f = \sum_{\delta \in \widehat{U}} d(\delta) f * \chi_\delta , \qquad f \in \mathcal{D}(U). \tag{1.1}$$

Here $\widehat{U}$ is the set of equivalence classes of irreducible representations of U, $d(\delta)$ and χ_δ denote the degree and character of δ, respectively, and $*$ denotes convolution. Let V_δ be a representation space of δ with inner product $< \ , \ >$. Then (1.1) implies

$$L^2(U) = \bigoplus_{\delta \in \widehat{U}} \mathcal{H}_\delta(U), \tag{1.2}$$

where

$$\mathcal{H}_\delta(U) = \{f(u) = \mathrm{Tr}(\delta(u)C) : C \in \mathrm{Hom}(V_\delta, V_\delta)\}$$

and this space is irreducible under the action

$$f(u) \longrightarrow f(u_1^{-1} u u_2)$$

of $U \times U$. For a homogeneous space U/K (1.2) takes the form

$$L^2(U/K) = \bigoplus_{\delta \in \widehat{U}_K} \mathcal{H}_\delta(U/K), \tag{1.3}$$

where

$$\widehat{U}_K = \{\delta \in \widehat{U} \mid \delta(K) \quad \text{has a fixed vector} \neq 0\}$$

and

$$\mathcal{H}_\delta(U/K) = \{f(uK) = \mathrm{Tr}(\delta(u)C) : C \in \mathrm{Hom}(V_\delta, V_\delta^K)\}$$

V_δ^K denoting the space of vectors in V_δ fixed under $\delta(K)$. In his paper [4], Cartan investigated the case when U/K is symmetric. In this case, V_δ^K is spanned by one vector, say v_0 (of norm 1), and $\mathcal{H}_\delta(U/K)$ contains a unique function φ_δ such that

$$\varphi_\delta \in C(K \backslash U/K), \quad \varphi_\delta(e) = 1; \tag{1.4}$$

in fact the function is the spherical function

$$\varphi_\delta(uK) = < \delta(u)v_0, v_0 > . \tag{1.5}$$

Cartan's result can then be stated as follows.

Theorem 1.1. — *For the symmetric space U/K*

$$f = \sum_{\delta \in \widehat{U}_K} d(\delta) f * \varphi_\delta, \qquad f \in \mathcal{D}(U/K). \tag{1.6}$$

2. Spherical Functions

The noncompact analog of (1.5) was investigated by GELFAND-NAIMARK [14] and in [12], GELFAND made the important observation that if G/K is a symmetric space, compact or not, the convolution algebra $C^\natural = C_c(K\backslash G/K)$ of bi-invariant functions under K is commutative. The continuous homomorphisms of $C^\natural$ into $\mathbf{C}$ are the maps

$$(2.1)\qquad F \longrightarrow \int_G F(g)\overline{\varphi(g)}\,dg = \widehat{F}(\varphi), \qquad F \in C_c(K\backslash G/K),$$

where φ is characterized by the functional equation

$$(2.2)\qquad \int_K \varphi(xky)\,dk = \varphi(x)\varphi(y).$$

These functions φ are called *spherical functions* because they generalize the function φ_δ in (1.5). The function $\widehat{F}$ (on the set of spherical functions) is called the *spherical transform* of F. By general Banach algebra theory, sharpened suitably for the case at hand (GODEMENT [16]) one has a decomposition

$$(2.3)\qquad F(g) = \int_\Phi \widehat{F}(\varphi)\varphi(g)\,d\mu(\varphi), \qquad F \in \mathcal{D}(K\backslash G/K),$$

where μ is fixed measure on the space Φ of positive definite spherical functions φ on G. If $f \in \mathcal{D}(G/K)$ we consider the K-biinvariant function

$$F(h) = \int_K f(gkh)\,dk$$

and deduce from (2.3)

$$(2.4)\qquad f(g) = \int_\Phi (f * \varphi)(g)\,d\mu(\varphi), \qquad f \in \mathcal{D}(G/K).$$

The space $\mathcal{D} * \varphi$ can be given a positive definite inner product by

$$\langle f_1 * \varphi, f_2 * \varphi\rangle_\varphi = \int_{G\times G} \varphi(g^{-1}h)f_1(g)\overline{f_2(h)}\,dg\,dh$$
$$\big(= (f_1 * \varphi, f_2)_{L^2(G)} = (f_1, f_2 * \varphi)_{L^2(G)}\big).$$

Denoting its completion by $\mathcal{H}_\varphi$ we have the direct integral decomposition

$$(2.5)\qquad L^2(G/K) = \int_\Phi \mathcal{H}_\varphi\, d_\mu(\varphi), \qquad \|f\|^2 = \int_\Phi \|f * \varphi\|^2 d\mu(\varphi),$$

and the natural representation of G on $\mathcal{H}_\varphi$ is irreducible.

Now assume

G : connected, noncompact semisimple Lie group with finite center,

K : maximal compact subgroup.

Here HARISH-CHANDRA ([17], [18]) put (2.3) and (2.4) into a more explicit form by relating it to the structure of G. Consider the Iwasawa decomposition

$$\begin{aligned} G &= NAK, & \underline{g} &= \underline{n} + \underline{a} + \underline{k} \\ g &= n \exp A(g) k, & \rho(H) &= (1/2)\operatorname{Tr}(\operatorname{ad} H|\underline{n}). \end{aligned} \tag{2.6}$$

Then the spherical functions and the decomposition (2.4) can be stated in a more explicit form.

THEOREM 2.1. — *The spherical functions are given by the integrals*

$$\varphi_\lambda(g) = \int_K e^{(i\lambda+\rho)(A(kg))}\, dk, \quad \lambda \in \underline{a}_c^*.$$

Moreover, the decomposition (2.4) *can be written*

$$f(g) = \int_{\underline{a}^*} (f * \varphi_\lambda)(g)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \tag{2.7}$$

where $d\lambda$ is a suitably normalized Euclidean measure on $\underline{a}^*$ and

$$\mathbf{c}(\lambda) = \lim_{H\to+\infty} e^{(-i\lambda+\rho)(H)} \varphi_\lambda(\exp H). \tag{2.8}$$

Formula (2.7) is the noncompact analog of the expansion in (1.6). Also the function $\mathbf{c}(\lambda)$ can be expressed explicitly in terms of the structure of G as shown by GINDIKIN-KARPELEVIÇ on the basis of work by HARISH-CHANDRA and BHANU-MURTHY.

3. The Fourier Transform on G/K

The direct integral decomposition (2.7) of $L^2(G/K)$ that was completed by HARISH-CHANDRA about 20 years ago is very explicit. Nevertheless, if we compare with $L^2(\mathbf{R}^n)$ we see that something seems missing because the theory above does not involve any *Fourier transform* concept for general functions $f \in \mathcal{D}(G/K)$. (The spherical transform F only applies for F bi-invariant under K.) The following definition of such a Fourier transform was proposed in [19].

Let $X = G/K$, $B = K/M$ where M is the centralizer of A in K. For $x = gK$, $b = kM$ put

$$A(x,b) = A(gK, kM) = A(k^{-1}g). \tag{3.1}$$

Given a function f on X its *Fourier transform* is defined by

$$\widetilde{f}(\lambda, b) = \int_X f(x) e^{(-i\lambda+\rho)(A(x,b))}\, dx \tag{3.2}$$

for all $\lambda \in \underline{a}_c^*$, $b \in B$ for which the integral converges. Here dx is the volume element on X. If f is K-invariant, i.e., $f(k \cdot x) \equiv f(x)$ and we put $F(g) = f(gK)$, then $\widetilde{f}(\lambda, b) = \widehat{F}(\varphi_{-\lambda})$ for all b; thus the definition generalizes the spherical transform.

However, there are other ways of motivating the definition (3.2).

In order to invert the Fourier transform (3.2) we have to prove a functional equation for the spherical function.

LEMMA 3.1. — *Let $g, h \in G$. Then*

$$\varphi_\lambda(g^{-1}h) = \int_K e^{(-i\lambda+\rho)(A(kg))} e^{(i\lambda+\rho)(A(kh))}\, dk. \tag{3.3}$$

With this lemma formula (2.7) gives the inversion formula for the Fourier transform as follows.

THEOREM 3.2. — *If $f \in \mathcal{D}(G/K)$ then*

$$f(x) = \int_{\underline{a}^*} \int_B \widetilde{f}(\lambda, b) e^{(i\lambda+\rho)(A(x,b))}\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda\, db$$

db being a fixed suitably normalized invariant measure on B.

Formula (3.4) is analogous to the inversion formula for the Fourier transform on $\mathbf{R}^n$ when this is written in polar coordinate form :

$$F(x) = \int_{\mathbf{S}^{n-1}} \Big(\int_0^\infty \widetilde{F}(\lambda\omega) e^{i\lambda(x,\omega)} \lambda^{n-1} d\lambda \Big)\, d\omega.$$

It is therefore natural to try to find analogs for the transform $f(x) \to \widetilde{f}(\lambda, b)$ of classical theorems in Fourier analysis on $\mathbf{R}^n$ and then apply these to the study of differential equations on X. The analog of the classical Paley-Wiener theorem is such a result ([21]).

THEOREM 3.3. — *The Fourier transform $f(x) \to \widetilde{f}(\lambda, b)$ is a bijection of $\mathcal{D}(X)$ onto the space of functions $\varphi \in C^\infty(\underline{a}_c^* \times B)$ satisfying*

(i) *$\lambda \to \varphi(\lambda, b)$ is an entire function on $\underline{a}_c^*$ of exponential type uniform in $b \in B$.*

(ii) *$\int_B \varphi(\lambda, b) e^{(i\lambda+\rho)(A(x,b))} db$ is Weyl group invariant as a function of λ.*

The range $L^2(X)^\sim$ can also be described ([20]) and so can the range $\mathcal{S}(X)^\sim$ where $\mathcal{S}(X) \subset L^2(X)$ is the Schwartz subspace ([7], [8]).

THEOREM 3.3 can be used to prove the following result ([21]).

COROLLARY 3.4. — *Let $D \neq 0$ be a differential operator on X, invariant under G. Then $DC^\infty(X) = C^\infty(X)$.*

In other words, the differential equation $Du = f$ has a solution u for each $f \in C^\infty(X)$.

In THEOREM 3.3 the range $\mathcal{D}(X)^\sim$ is not topologized intrinsically. In order to get a topological statement it is better to consider the subspace $\mathcal{D}_\delta(X) \subset \mathcal{D}(X)$ of K-finite functions of a fixed type $\delta \in \widehat{K}$. It is then possible to characterize and topologize the range $\mathcal{D}_\delta(X)^\sim$ explicitly and to prove that the Fourier transform connecting $\mathcal{D}_\delta(X)$ and $\mathcal{D}_\delta(X)^\sim$ is a homeomorphism. One can then draw the following consequence ([23]).

COROLLARY 3.5. — *The K-finite joint eigenfunctions of the G-invariant differential operators on X are precisely the integrals*

$$f(x) = \int_B e^{(i\lambda+\rho)(A(x,b))} F(b)\, db,$$

where F is a K-finite function on B.

A more general result, dropping the K-finiteness condition on f and replacing $F(b)\, db$ by a hyperfunction on B, was given in [20], [22] for $X = \mathbf{H}^2$ (the hyperbolic plane) and in [24] for general X, by powerful new methods. A different, representation-theoretic approach, involving COROLLARY 3.5, and yielding other results as well, has been outlined by SCHMID (see these proceedings).

4. Compact Symmetric Spaces

Since the inversion formula (3.4) refines the decomposition (2.7) by introducing a genuine Fourier transform one can ask whether a similar refinement of the compact case (1.6) is possible. This was done by SHERMAN [38]. Paradoxically however, the compact case leads to some convergence difficulties, which have not been fully resolved except for the rank one case.

Consider again the compact symmetric space $S = U/K$ and now we assume U simply connected and K connected. We can assume U/K and G/K dual symmetric spaces so that we have the orthogonal decompositions with respect to the Killing form $\langle\ ,\ \rangle$ of $\underline{g}^{\mathbf{C}} = \underline{u}^{\mathbf{C}}$,

$$\underline{g} = \underline{k} + \underline{p} = \underline{k} + \underline{a} + \underline{q}\,, \tag{4.1}$$

$$\underline{u} = \underline{k} + i\underline{p} = \underline{k} + i\underline{a} + i\underline{q}\,. \tag{4.2}$$

Let Σ be the set of roots of $\underline{g}$ with respect to $\underline{a}$ and Σ^+ the set of positive roots corresponding to the subalgebra $\underline{n}$ in (2.6). Let

$$\Lambda = \{\mu \in \underline{a}^* = \frac{\langle \mu, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbf{Z}^+ \quad \text{for all } \alpha \in \Sigma^+\}. \tag{4.3}$$

Given $\mu \in \Lambda$ there exists a unique irreducible finite-dimensional representaion π_μ of U which has a K-fixed vector and whose highest weight has restriction to $\underline{a}$ given by μ. The representation space $V_\mu = V_{\pi_\mu}$ can be taken as the subspace $L^2(S) * \varphi_\delta \subset L^2(S)$ if δ and π_μ are contragredient. We put $d(\mu) = d(\pi_\mu)$. Let us now determine the highest weight vector e_μ of π_μ as a function of S. We have

$$G = NAK, \qquad \underline{g} = \underline{n} + \underline{a} + \underline{k},$$
$$g = n \exp A(g) k, \qquad \underline{u}^{\mathbf{C}} = \underline{g}^{\mathbf{C}} = \underline{n}^{\mathbf{C}} + \underline{a}^{\mathbf{C}} + \underline{k}^{\mathbf{C}},$$

and it is convenient to assume G contained in $G^{\mathbf{C}}$, the simply connected Lie group with Lie algebra $\underline{g}^{\mathbf{C}}$. The mapping

$$(X, H, T) \longrightarrow \exp X \exp H \exp T \tag{4.4}$$

is a holomorphic diffeormorphism of a neighborhhod of O in $\underline{n}^{\mathbf{C}} + \underline{a}^{\mathbf{C}} + \underline{k}^{\mathbf{C}} = \underline{u}^{\mathbf{C}}$ onto a neighborhood $U_o^{\mathbf{C}}$ of e in $U^{\mathbf{C}} = G^{\mathbf{C}}$. We can then as in [6], [40] consider the map

$$\exp X \exp H \exp T \longrightarrow H \tag{4.5}$$

as a well-defined holomorphic map of $U_o^{\mathbf{C}}$ into $\underline{a}^{\mathbf{C}}$, also denoted A. We take $U_o^{\mathbf{C}}$ as the diffeomorphic image (under exp) of a ball $B(0) \subset \underline{n}^{\mathbf{C}}$. Then $U_o^{\mathbf{C}}$

is invariant under the maps $u \longrightarrow kuk^{-1}$ and so is the set $U_0 = U_o^{\mathbf{C}} \cap U$. Viewing π_μ as a representation of $G^{\mathbf{C}}$ we find

$$e_\mu(uK) = e^{-\mu(A(u))}, \qquad u \in U_o. \tag{4.6}$$

We put $S_0 = \{uK : u \in U_o\}$ and define the function f_μ by

$$f_\mu(uK) = e^{(\mu+2\rho)(A(u))}, \qquad u \in U_o. \tag{4.7}$$

LEMMA 4.1. — *For uK, $s \in S_o$ and $\mu \in \Lambda$ we have*

$$\int_K e_\mu(k \cdot s) f_\mu(kuK)\, dk = \varphi_\delta(u^{-1} \cdot s). \tag{4.8}$$

Proof. — Put $\lambda = -i(\mu + \rho)$ so $\mu = i\lambda - \rho$. Then the lemma follows from (3.3) by analytic continuation [38].

THEOREM 4.2. — *For $f \in \mathcal{D}(S_o)$ define the Fourier transform $\widetilde{f}$ on $\Lambda \times K/M$ by*

$$\widetilde{f}(\mu, kM) = \int_S f(s) f_\mu(k^{-1}s)\, ds. \tag{4.9}$$

Then

$$f(s) = \sum_{\mu \in \Lambda} d(\mu) \int_{K/M} \widetilde{f}(\mu, kM) e_\mu(k^{-1}s)\, dk_M, \qquad s \in S_o. \tag{4.10}$$

In fact, the latter integral is by LEMMA 4.1

$$\int_{K/M} \left[\int_U f(uK) f_\mu(k^{-1}uK)\, du \right] e_\mu(k^{-1}s)\, dk_M$$
$$= \int_U f(uK) \varphi_\delta(u^{-1}s)\, du = f * \varphi_\delta(s),$$

so we are reduced to the expansion (1.6).

Example. — We shall now put this in a more explicit form in the case of the unit sphere $\mathbf{S}^d \subset \mathbf{R}^{d+1}$. Let $a \in \mathbf{S}^d$ be the North pole, B the equator and $(\ ,\ )$ the inner product. For $n \in \mathbf{Z}^+$, $b \in B$ consider

$$e_{b,n}(s) = (a + ib, s)^n, \qquad s \in \mathbf{S}^d,$$
$$f_{b,n}(s) = \{\operatorname{sgn}(a, s)\}^{d-1} (a + ib, s)^{-n-d-1}, \qquad s \in \mathbf{S}^d - B.$$

The Fourier transform of $f \in \mathcal{D}(\mathbf{S}^d)$ is defined as the regularized integral ([37])

$$\widetilde{f}(b,n) = \int_{\mathbf{S}^d} f(s) f_{b,n}(s)\, ds = \lim_{\alpha \to 0^+} \int_{|(a,s)|>\alpha} f(s) f_{b,n}(s)\, ds$$

and the following result holds.

THEOREM 4.3. — *For $f \in \mathcal{D}(\mathbf{S}^d)$ we have*

$$f(s) = \int_{B \times \mathbf{Z}^+} \widetilde{f}(b,n) e_{b,n}(s)\, d\mu(b,n) \tag{4.11}$$

where $d\mu(b,n) = \dim E_n(\mathbf{S}^d) \times db$.

Here

$$\dim E_n(\mathbf{S}^d) = \binom{d+n-1}{n} + \binom{d+n-2}{n-1} = d(n), \tag{4.12}$$

the dimension of the space of spherical harmonics of degree n.

Formula (4.11) has an advantage over the customary spherical harmonics expansion

$$f(s) = \sum_{n=0}^{\infty} \sum_{1 \le m \le d(n)} a_{m,n} S_{n,m}\,, \tag{4.13}$$

($S_{n,m}$ $(1 \le m \le d(n))$ orthonormal basis of $E_n(\mathbf{S}^d)$) in that it is canonical whereas the basis $S_{n,m}$ is not.

A similar, but a bit more complicated, regularization works for $S = U/K$ of rank one ([38]) so THEOREM 4.2 holds for all $f \in \mathcal{D}(S)$. For U/K of higher rank this remains however to be carried out.

Analogies. — Let us now return to th general situation and compare the inversion formulas discussed above for G/K and U/K, respectively :

$$f(x) = \int_{\underline{a}^*} \int_B \widetilde{f}(\lambda, b) e^{(i\lambda+\rho)(A(x,b))} |\mathbf{c}(\lambda)|^{-2}\, db\, d\lambda,$$

$$f(s) = \sum_{\mu \in \Lambda} d(\mu) \int_{K/M} \widetilde{f}(\mu, kM) e_\mu(k^{-1}s)\, ds_M.$$

To what extent are they analytic continuations of each other as the kernels

$$e^{(i\lambda+\rho)(A(x,b))} \quad \text{and} \quad e_\mu(k^{-1}s) \tag{4.14}$$

certainly are (because of (4.6))? The analogy woulde be complete if the density $|\mathbf{c}(\lambda)|^{-2}$ became $d(\mu)$ upon substituting $\lambda = -i(\mu+\rho)$, μ being the highest restricted weight above. Then the Gindikin-Karpelevič product formula for $\mathbf{c}(\lambda)$ [15] would correspond to Weyl's product formula [43] for the degree $d(\mu)$. Such a formula was in fact obtained by VRETARE [42 a,b]; an independent proof was kindly communicated to me by OSHIMA. We sketch the idea. The spherical function $\varphi_\lambda(g)$ can for $\lambda = -i(\mu+\rho)$ be identified with the coefficient $\langle v_o, \delta(g)v_o\rangle$ where δ is the irreducible representation of G (and G^C) with K-fixed vector $v_o \neq 0$ and highest restricted weight μ. By the Schur orthogonality relations, $\varphi_\lambda\overline{\varphi}_\lambda$ has integral $d(\mu)^{-1}$ over U. On the other hand, $\varphi_{\lambda_1}\overline{\varphi}_{\lambda_2}$ can be written as a sum of spherical functions and the coefficients of the leading terms can be related to the c-function. This leads to the desired relationship

$$d(\mu) = \left\{\frac{c(\lambda+i\mu)c(-\lambda-i\mu)}{c(\lambda)c(-\lambda)}\right\}_{\lambda=-i(\mu+\rho)}.$$

5. Semisimple Symmetric Spaces

By a semisimple symmetric space is usually meant a coset space G/H where H is the fixed point group (not necessarily compact) of an involutive automorphism of a semisimple group G. CARTAN remarks in [5], p. 84 : "L'étude géométrique de ces espaces ne manquerait pas d'intérêt." The irreducible spaces of this type were classified by BERGER [1], [2] (see also FEDENKO [10] for G classical).

Harmonic analysis for such spaces, with particular emphasis on the discrete series, has been developing vigorously in recent years (for a sample see [11], [29], [31], [35], [36], [42]). Fourier transform theory in the spirit of § 3 is less advanced; in the present context it seems most illuminating to describe the results for the quadric

$$\text{(5.1)}\qquad \begin{aligned} &X = G/H = 0(p,q)/0(p,q-1), \quad pq>1.\\ &(x,x) = -x_1^2 - \cdots - x_p^2 + x_{p+1}^2 + \cdots + x_{p+q}^2 = 1.\end{aligned}$$

Here the basic harmonic analysis has been worked out by a number of people with varying rigor, generality and methods ([13], [34], [39], [27], [28], [26], [41], [33], [9]). Here I describe ROSSMANN's formulation. Let

$$B = \{b \in \mathbf{R}^{p+q} : b_1^2 + \cdots + b_p^2 = 1, b_{p+1}^2 + \cdots + b_{p+q}^2 = 1\}$$

i.e., $B = \mathbf{S}^{p-1} \times \mathbf{S}^{q-1}$. Put $\rho = (1/2)(p+q-2)$. For $\epsilon = 0,1$ we define for $f \in \mathcal{D}(X)$ the *Fourier transform*

$$\text{(5.2)}\qquad \widetilde{f}_\epsilon(\lambda,b) = \int_X |(x,b)|^{i\lambda-\rho}\,\mathrm{sgn}^\epsilon(x,b)f(x)\,dx, \quad \lambda\in\mathbf{R},\ b\in B.$$

Thus the Fourier transform of f is a pair of functions $\widetilde{f_0}$, $\widetilde{f_1}$ on $\mathbf{R} \times B$. The integral (5.2) is in fact defined for all $\lambda \in \mathbf{R}$ by analytic continuation : the integral is convergent for $\Re(i\lambda) > \rho - 1$ and extends to a meromorphic function of λ with poles at most for $i\lambda - \rho + 1 \in -2\mathbf{Z}^+$. The Fourier transform (5.2) is now inverted by the following result.

THEOREM 5.1. — *For a certain dense subspace of $\mathcal{D}(X)$ we have*

$$f(x) = \sum_\epsilon \frac{1}{2\pi} \int_0^\infty |(x,b)|^{-i\lambda-\rho} \operatorname{sgn}^\epsilon(x,b) \widetilde{f_\epsilon}(\lambda,b) 2^{-\rho} |\mathbf{C}_\epsilon(\lambda)|^{-2} \, d\lambda \, db$$

$$+ \sum_{\lambda_0} \operatorname{Res}_{\lambda=\lambda_0} \left[\mathbf{C}_\gamma(\lambda)^{-1} \int_B |(x,b)|^{-i\lambda-\rho} \operatorname{sgn}^\epsilon(x,b) \widetilde{f_\epsilon}(\lambda,b) \, db \right].$$

Here λ_0 ranges over

$$i\lambda_0 > 0, \quad \rho + i\lambda_0 \in \mathbf{Z}^+, \quad \epsilon \equiv \rho + i\lambda_0 - q \pmod 2$$

and the $\mathbf{C}_\epsilon$-function is given by

$$\mathbf{C}_\epsilon(\lambda) = 2^{2\rho+1} \pi^\rho \frac{\Gamma(i\lambda)}{\Gamma(\rho + i\lambda)}, \qquad q \text{ odd};$$

$$\mathbf{C}_\epsilon(\lambda) = (-1)^\epsilon 2^{2\rho+1} \pi^\rho \frac{\Gamma(i\lambda)}{\Gamma(\rho + i\lambda)} \tan\left[\frac{\pi}{2}(\rho + i\lambda + \epsilon)\right], \qquad q \text{ even}.$$

The principal contrast with the Riemannian case is the appearance of the discrete series. The theorem generalizes ([9], [25]) with similar features to the quadrics over the complexes, or the quaternions and to the indefinite Cayley plane. These spaces are all symmetric and according to WOLF [44] they exhaust the non-Riemannian isotropic pseudo-Riemaniann manifolds up to local isometry. The proofs proceed via delicate spectral theory of the radial part of the Laplace-Beltrami operator on X — which is a singular ordinary 2^{nd} order differential operator.

In recent years OSHIMA [30] has attacked the case of a general non-Riemannian symmetric space. He has arrived at a reasonable general definition of the Fourier transform as well as at a working hypothesis for the inversion formula. He has announced the proof of such a formula for the case $X = G_c/K_c$ where G_c is a complex semisimple Lie group and K_c a complexification of a maximal compact subgroup K of a real form G of G_c (Example : $\mathbf{SL}(n,\mathbf{C})/\mathbf{SO}(n,\mathbf{C})$), and for some spaces X of rank one.

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