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Astérisque, tome 173-174 (1989), p. 257-269

<http://www.numdam.org/item?id=AST_1989__173-174__257_0>

Société Mathématique de France
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ON THE HOMOLOGY CLASSES FOR THE COMPONENTS OF SOME FIBRES OF SPRINGER'S RESOLUTION

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ABSTRACT: We compute the homology classes of the components of the fibres of Springer's resolution in terms of Schubert classes when the unipotent element is of "one hook" type.

0. Introduction

Let G be a connected reductive group over $\mathbf{C}$. Denote by $\mathcal{B}$ the variety of all Borel subgroups of G. If u is a unipotent element of G, the fibre of Springer's resolution $\mathcal{B}_u$ is the variety of Borel subgroups containing u. The inclusion $\mathcal{B}_u \hookrightarrow \mathcal{B}$ induces a homomorphism of homology groups $H_*(\mathcal{B}_u;\mathbf{Z}) \longrightarrow H_*(\mathcal{B};\mathbf{Z})$, which is injective if $G = GL_n(\mathbf{C})$ [8]. When w runs over the elements in the Weyl group W of G, the Schubert classes $[\bar{X}_w]$ form a basis of $H_*(\mathcal{B};Z)$ [2]. If C is a component of $\mathcal{B}_u$, it defines a homology class in $H_*(\mathcal{B}_u;\mathbf{Z})$, whose image in $H_*(\mathcal{B};\mathbf{Z})$ is denoted by [C]. We can then write

$$[C] = \sum_{w \in W} n_C(w)[\bar{X}_w] \qquad \text{with} \quad n_C(w) \in Z$$

In this paper we shall consider the case with $G = GL_n(\mathbf{C})$ and with u a unipotent element whose Jordan decomposition is of "one hook" type, i.e. such that there is at most one Jordan block of size greater than one. The result in that case is that $n_C(w)$ is the cardinal of a set of reduced expressions of w, depending on C. We believe that a similar result could be true in general, at least for $GL_n(\mathbf{C})$. For example, we have obtained such a result in the case that the Jordan decomposition of u has only two blocks.

We want to express our deep gratitude to Professor Springer. He proposed the problem [13], and inspired all our work. He also read the paper and implemented it considerably. The clarity the reader can find comes from him.

1. Combinatorial results about tableaux and permutations

1.1. A good reference for some terminology about tableaux is Macdonald's book [10], for links between tableaux and reduced decompositions the reader is referred to [5], [9], [14].

Consider "strict standard staircase tableaux" with entries in the set $\{1,\dots,n-1\}$, i.e. tableaux T for a partition $(m,m-1,\dots,1)$ such that the integers $a_{p,q}$ in the place (p,q) satisfy $a_{p,q} < a_{p+1,q}$, $a_{p+1,q} \leq a_{p,q+1}$ (the columns are strictly increasing and the diagonals are increasing, it follows that the rows are strictly increasing).

In the symmetric group S_n, let $s_{p,q}$ denote the transposition (p,q), let s_i be the fundamental transposition $s_{i,i+1}$, $1 \leq i \leq n-1$. Let $l(w)$ denote the length of an element $w \in S_n$ and $w < w'$ the Bruhat order relative to the set of generators (s_i) $1 \leq i \leq n-1$ [3].

Denote by $|X|$ the cardinality of a set X.

Associate to such a tableau T a permutation $w = w_T \in S_n$, namely $w = c_1 \dots c_m$ where $c_p = s_{a_{m-p+1,p}} \dots s_{a_{1,p}}$.

We say T is <u>reduced</u> if $l(w_T) = \frac{1}{2}m(m+1)$.

1.2. We list a number of properties.

1.2.1. Write $r_p = s_{a_{p,1}} \dots s_{a_{p,m-p+1}}$. Then $w = r_m \dots r_1$.

1.2.2. If T is reduced then $a_{p+1,q} = a_{p,q+1}$ implies $a_{p,q+1} = a_{p,q}+1$.

1.2.3. Let i defined by $wi = 1$. Then $a_{p,q} = p+q-1$ for $p+q \leq i$ and $a_{1,i} > i$.

Define the number $\tau = \tau_p = \tau(T,p)$ as follows: in the p^{th} column of T we have

$$a_{1,p} = a_{2,p}-1 = \dots = a_{\tau,p} - \tau + 1 < a_{\tau+1,p} - \tau .$$

1.2.4. For $p < q \leq i-1$ we have $\tau_p > \tau_q$. If $p < i$ then $w_p = \tau_p+1$. Hence if $p < q \leq i$ then $wp > wq$.

1.2.5. If T is reduced and $a_{1,i} = i+1$ then $\tau_i < \tau_{i-1}$. Moreover $\tau_{i-1} = \tau_i+1$ if and only if $\tau_i+2 = w(i-1) < w(i+1)$ and $\tau_{i-1} > \tau_i+1$ if and only if $w(i-1) > w(i+1) = \tau_i+2$.

PROOFS

1.2.1.* If $p>u$, $q>v$ then $a_{p,q}-a_{u,v}\geq 2$. It follows that $s_{a_{p,q}}$ and $s_{a_{u,v}}$ commute. The proof follows from this observation.

1.2.2.* Let T be not necesarilly reduced. We show by induction on m-i that $l(r_m\ldots r_i)>l(r_m\ldots r_i s_{a_{i,j}})$ $(1\leq i\leq m,\ 1\leq j\leq m-i+1)$. This is clear if $j=m-i+1$ or $a_{i,j+1}>a_{i,j}+1$, otherwise $s_{a_{i,j}}s_{a_{i,j+1}}s_{a_{i,j}}=s_{a_{i,j+1}}s_{a_{i,j}}s_{a_{i,j+1}}$, $a_{i,j+1}=$ $=a_{i+1,j}$ (diagonals increase) so that $r_i s_{a_{i,j}}=s_{a_{i+1,j}}r_i$. By the induction hypothesis we have $l(r_m\ldots r_{i+1}s_{a_{i+1,j}})<l(r_m\ldots r_{i+1})$, whence the asserted inequality.

It follows that if $a_{p+1,q}=a_{p,q+1}$ and $a_{p,q}+1<a_{p,q+1}$, we have $l(r_m\ldots r_{p+1}s_{a_{p+1,q}})<l(r_m\ldots r_{p+1})$, showing that T is not reduced.

1.2.3.* We have $w^{-1}=r_1^{-1}\ldots r_m^{-1}$ and r_p fixes 1 if $p\geq 2$, so $i=w^{-1}(1)=r_1^{-1}(1)=$ $=s_{a_{1,m}}s_{a_{1,m-1}}\ldots s_{a_{1,1}}(1)$. Since $a_{1,p}\geq p$ it follows that $a_{1,p}=p$ for $p\leq i-1$, $a_{1,p}>p$ for $p\geq i$.

1.2.4.* That $\tau_p>\tau_q$ if $p<q<i$ follows from the definitions (T is a strict tableau). Now c_j fixes p if $j>p$, $c_p p=p+\tau_p$ for $p<i$ and $c_j t=t-1$ if $j<t\leq j+\tau_j$; thus because $p<i$ we have $wp=c_1\ldots c_p(p)=c_1\ldots c_{p-1}(p+\tau_p)=\tau_p+1$.

1.2.5.* That $\tau_i<\tau_{i-1}$ follows from (1.2.2). From (1.2.4) we have $w(i-1)=\tau_{i-1}+1$. Also $w(i+1)=r_m\ldots r_1(i+1)=r_m\ldots r_2(i+2)=\ldots=r_m\ldots r_{\tau_i+1}(i+\tau_i+1)$; now r_{τ_i+1} fixes $i+\tau_i+1$, $r_{\tau_i+2}(i+\tau_i+1)=\tau_i+2$ if $\tau_{i-1}>\tau_i+1$ and $r_{\tau_i+2}(i+\tau_i+1)>\tau_i+2$ if $\tau_{i-1}=\tau_i+1$. The result now follows from the observation that r_j fixes $\{1,\ldots,\tau_i+2\}$ if $j>\tau_i+2$.

1.3. The following known result (see [6, pg. 156] as reference) is useful.

LEMMA 1.- Let $w\in S_n$, assume $1\leq p\leq q\leq n$. We have $l(ws_{p,q})<l(w)$ if and only if $wp>wq$, moreover in this case $l(w)-l(ws_{p,q})=1+2|\{k \text{ s.t. } p<k<q \text{ and } wp>wk>wq\}|$. As a consequence if $ws>w$ for some fundamental transposition s and $l(ws\,s_{p,q})=l(w)$ then $ws(p)>ws(q)$ and there is no k with $p<k<q$ and $ws(p)>ws(k)>ws(q)$.

LEMMA 2.- Let $w=w_T$, $w'=w_{T'}$ be permutations corresponding to tableaux $T=(a_{p,q})$, $T'=(a'_{p,q})$ as in the beginning of the section. Suppose there are positive integers t,j,k, $j<k$, with $a_{p,q}=a'_{p,q}$ if $q\neq t$ or $p>k$, $a'_{p,t}=a_{p,t}=t+p-1$ if $1\leq p<j$ and $a'_{p,t}=a_{p,t}+1=t+p$ if $j\leq p\leq k$. Then $w^{-1}w'$ is the cyclic permutation (t,b,c), where b is defined by $wb=j$ and c by $w'c=k+1$.

PROOF.- Write $w = c_1 \dots c_m$, $w' = c'_1 \dots c'_m$ then $c_p = c'_p$ if $p \neq t$ and $c_t(h) \neq c'_t(h)$ exactly for three values of h, namely h=t, t+j, t+k. Therefore $w^{-1}w'$ is a cyclic permutation (a,b,c). Moreover wt = k+1 and $w't = c_1 \dots c_{t-1}(t+j-1) = j$, so we can take a = t and b,c defined by wb = w't = j, w'c = wt = k+1.

2. Combinatorial correspondences

2.1. Let $\mathcal{L}$ be the set of tableaux $T = (a_{p,q})$ as in section 1, with w(i+1) = 1.

Let $\mathcal{M}$ be the set of tableaux with wi = 1 and $a_{1,i} = i+1$.

Let $\mathcal{N}$ be the set of tableaux with w(i-1) = 1, $a_{1,i-1} = i$ and $a_{1,i} = i+1$.

2.2. Define a map $\psi : \mathcal{L} \longrightarrow \mathcal{M}$ as follows: ψT is obtained by replacing the numbers $i, i+1, \dots, i+\tau_i-1$ in the i^{th} column of T by $i+1, i+2, \dots, i+\tau_i$. Define $\overline{\psi} : \mathcal{M} \longrightarrow \mathcal{N}$ similarly (change i for i-1). Define $e = e_T$ by $w_{\psi T}e = \tau_i+1$ ($\tau_i = \tau(T,i)$) if $T \in \mathcal{L}$ (similarly if $T \in \mathcal{N}$).

If $T \in \mathcal{N}$ then $\tau_{i-1} > \tau_i$ by definitions. Define a map $\chi : \mathcal{N} \longrightarrow \mathcal{M}$ as follows: χT is obtained by replacing the numbers $i, i+1, \dots, i+\tau_i$ in the $(i-1)^{th}$ column of T by $i-1, i, \dots, i+\tau_i-1$.

We list a number of results.

2.2.1. We have $\overline{\psi} \circ \chi =$ identity, in particular χ is injective.

2.2.2. Suppose $T \in \mathcal{L}$, then $e > i+1$, $w_T = w_{\psi T} s_i s_{i,1}$ and ψT is reduced if T is reduced.

2.2.3. Suppose $T \in \mathcal{N}$ is reduced, then $w_T = w_{\chi T} s_i s_{i-1,i+1}$ and χT is reduced.

2.2.4. We have that $\psi T = \psi T'$ and $e_T = e_{T'}$, implies T = T'.

2.2.5. If $T \in \mathcal{M}$ is reduced and $l(ws_i s_{t,i+1}) = l(w)$ for some t < i, then t = i-1 and $\tau_{i-1} = \tau_i+1$, in particular $T = \chi\overline{\psi}T$, $\overline{\psi}T \in \mathcal{N}$.

2.2.6. Given $T' \in \mathcal{M}$ reduced, $e > i+1$, $l(w_T s_i s_{i,1}) = l(w_{T'})$ then there exists a reduced $T \in \mathcal{L}$ with $\psi T = T'$ and $e = e_T$, if and only if $w_{T'}e \leq \tau(T',i)+1$.

PROOFS

2.2.1.* This follows from the constructions of the maps.

2.2.2.* Write $w_{\psi T} = c_1 \dots c_m$ as in section 1. We have:

$$e = w_{\psi T}^{-1}(\tau_i+1) = c_m^{-1} \dots c_1^{-1}(\tau_i+1) = c_m^{-1} \dots c_2^{-1}(\tau_i+2) = \dots = c_m^{-1} \dots c_{i+1}^{-1}(\tau_i+i+1).$$

Now $c_{i+1}, \dots, c_m$ fix $\{1, \dots, i+1\}$ therefore $e > i+1$.

Applying lemma 2 to T, ψT with $t=i$, $j=1$ and $k=\tau_i$ we obtain $w_T = w_{\psi T}(i,e,i+1)$ because $b = i+1$ and $c=e$, thus $w_T = w_{\psi T}s_i s_{i,e}$.

We shall prove $w_{\psi T}(i+1) > \tau_i+1 = w_{\psi T}e_T$. Then we shall have $l(w_T) < l(w_T s_{i,e}) = l(w_{\psi T}s_i)$ (c.f. lemma 1), thus ψT is reduced if T is. Write $w_{\psi T} = r_m \dots r_i$ then $w_{\psi T}(i+1) = r_m \dots r_1(i+1) = r_m \dots r_{\tau_i+1}(i+\tau_i+1)$, now $r_m,\dots,r_{\tau_i+1}$ fix $\{1,\dots,\tau_i\}$ and $w_{\psi T}(i+1) \neq \tau_i+1$ because $e > i+1$.

2.2.3.* We have $w_{\chi T} = w_{\overline{\psi}\chi T}s_{i-1}s_{i-1,e} = w_T s_{i-1}s_{i-1,e}$, $e = e_{\chi T}$ (c.f. 2.2.2., 2.2.1.). Now $w_T e = \tau(\chi T,i-1)+1$ (definition) and $\tau(\chi T,i-1)=\tau(T,i)+1$ by construction. We shall prove $w_T(i+1) = \tau(T,i)+2$ then $e = i+1$ and $w_T = w_{\chi T}s_i s_{i-1,i+1}$. We have: $w_T(i+1) = c_1 \dots c_m(i+1) = c_1 \dots c_i(i+1) = c_1 \dots c_{i-1}(i+\tau_i+1) = c_1 \dots c_{i-2}(i+\tau_i)$, T is reduced and $\tau_{i-2} > \tau_{i-1}$ (c.f. 1.2.5.), hence

$$c_1 \dots c_{i-2}(i+\tau_i) = c_1 \dots c_{i-3}(i+\tau_i-1) = \dots = c_1(\tau_i+3) = \tau_i+2$$

Also $l(w_T) < l(w_T s_{i-1,i+1}) = l(w_{\chi T}s_i)$ (use that $w_T(i-1) = 1$ c.f. lemma 1), thus χT is reduced.

2.2.4.* If $T,T' \in \mathcal{L}$ and $\psi T = \psi T'$, $e_T = e_{T'}$ implies $\tau(T,i) = \tau(T',i)$ (definition of e). Then the construction of $\psi T = \psi T'$ shows $T = T'$.

2.2.5.* We have $ws_i(i+1) = wi = 1$. We deduce $t = i-1$ (c.f. 1.2.4., and lemma 1). We must have also (c.f. lemma 1) $ws_i(i) = w(i+1) > ws_i(i-1) = w(i-1)$, then (1.2.5) implies $\tau_{i-1} = \tau_i+1$.

2.2.6.* If $T' = \psi T$, $e = e_T$ then by definition $w_{T'}e = \tau_i+1 \leq \tau(T',i)+1$ (construction). Conversely if $w_{T'}e \leq \tau(T',i)+1$ put $k+1 = w_{T'}e$ and obtain T from T' by replacing the numbers $i+1,\dots,i+k$ in the i^{th} column of T' by the numbers $i,\dots,i+k-1$ (T is a strict tableau because $\tau(T',i-1) > \tau(T',i)$ (c.f. 1.2.5.)). Then $\psi T = T'$, $w_{\psi T}e = w_{T'}e = k+1 = \tau(T,i)+1$, i.e. $e = e_{T'}$ and T is reduced because $w_T = w_{T'}s_i\, s_{i,e}$ (c.f. 2.2.2).

2.3. Let $T \in \mathcal{M}$ be reduced and let $t < i$ be such that $l(ws_i s_{t,i}) = l(w)$, then t is uniquely determined by T (c.f. 1.2.4. and lemma 1). We have also (lemma 1 and 1.2.4) $w(i+1) < wt = \tau_t+1$.

In this situation construct $\lambda T \in \mathcal{M}$ in the following way:
Replace the numbers $t+w(i+1)-1,\dots,t+\tau_t-1$ in the t^{th} column of T by the numbers $t+w(i+1),\dots,t+\tau_t$. (We obtain a strict tableau because $w(t+1) < w(i+1)$ (c.f. lemma 1).

Define $k = k_T$ by $w_{\lambda T}k = \tau(T,t)+1$.

We list a number of results.

2.3.1. We have $k > i+1$ and $w_T s_i s_{t,i} = w_{\lambda T} s_i s_{i,k}$.

2.3.2. λT is reduced and there is no reduced $T' \in \mathcal{L}$ with $\psi T' = \lambda T$ and $k = e_{T'}$.

2.3.3. Suppose we have $\lambda T = \lambda T'$, $k_T = k_{T'}$ for two reduced $T, T' \in \mathcal{M}$ with $l(w_T s_i s_{t,i}) = l(w_T)$, $l(w_T s_i s_{t',i}) = l(w_{T'})$, then $t = t'$ and $T = T'$.

2.3.4. Let $T \in \mathcal{M}$ be reduced, let $e > i+1$ be such that $l(ws_i s_{i,e}) = l(w)$, and suppose that there is no reduced $T' \in \mathcal{L}$ with $T = \psi T'$ and $e = e_{T'}$. Then there is a reduced $T' \in \mathcal{M}$ and $t < i$ with $l(w_T s_i s_{t,i}) = l(w_{T'})$, $T = \lambda T'$ and $e = k_{T'}$.

PROOFS

2.3.1.* Write $w_{\lambda T} = c_1 \ldots c_m$ then

$$k = w_{\lambda T}^{-1}(\tau_t+1) = c_m^{-1}\ldots c_1^{-1}(\tau_t+1) = c_m^{-1}\ldots c_t^{-1}(\tau_t+t) = c_m^{-1}\ldots c_{t+1}^{-1}(\tau_t+t+1).$$

Observe that $c_j^{-1}(s) > j+\tau_j$ if $j < i$ and $s > j+\tau_j$. Also $c_m, \ldots, c_i$ fix $\{1,\ldots,i\}$, we conclude $k > i$.

Applying lemma 2 to T, λT, t, $w(i+1)$, τ_t we obtain $w_T = w_{\lambda T}(t,c,b)$ where $b = i+1$ and $w_{\lambda T}c = \tau_t+1$. Therefore $k \neq i+1$ and $w_T s_i s_{t,i} = w_{\lambda T} s_i s_{i,k}$.

2.3.2.* We have $l(w_{\lambda T} s_i s_{i,k}) = l(w_T s_i s_{t,i}) = l(w_T)$ (c.f. 2.3.1). We shall prove $w_{\lambda T}(i+1) > \tau_t+1 = w_{\lambda T}k$. Then $l(w_{\lambda T} s_i) > l(w_{\lambda T} s_i s_{i,k}) = l(w_T)$ (lemma 1) and λT is reduced.

Write $w_T = c_1 \ldots c_m$, $w_{\lambda T} = c'_1 \ldots c'_m$. Also $w(i+1) < \tau_t+1$ by definition of t. Then $w(i+1) = c_1 \ldots c_t(t+w(i+1))$ and $w_{\lambda T}(i+1) = c_1 \ldots c_{t-1} c'_t(t+w(i+1)) = c_1 \ldots c_{t-1}(u)$ where $u \geq t+\tau_t+1$, hence $w_{\lambda T}(i+1) \geq \tau_t+2$.

We have $w_{\lambda T}k = \tau_t+1 > \tau_i+1$ (c.f. 1.2.4, 1.2.5). Hence the second part of the statement follows from (2.2.6).

2.3.3.* In this situation $w_T s_i s_{t,i} = w_{T'} s_i s_{t',i}$ (c.f. 2.3.1). If $t \neq t'$ suppose for instance $t' < t$, then $w_T t' > w_T t$ (c.f. 1.2.4) and $w_T t' = w_{T'}(i+1) > w_{T'} t' = w_T t$ against the definition of t'.

Therefore $t = t'$ and $w_T = w_{T'}$. We conclude $T = T'$ because they can only differ in one column.

2.3.4.* In this situation $we > \tau_i+1$ (c.f. 2.2.6). We deduce $\tau_{i-1} = \tau_i+1$ because if $\tau_{i-1} > \tau_i+1$ then $w(i+1) = \tau_i+2$ (c.f. 1.2.5) against $we < w(i+1)$ (c.f. lemma 1). We have in fact $we > \tau_{i-1}+1$ ($we \neq \tau_{i-1}+1 = w(i-1)$ (c.f. 1.2.4) because $e \neq i-1$).

Define t to be the smallest number with $we > \tau_t+1$.

We claim: the t^{th} column of T is of the form, $t, t+1,\dots,t+\tau_t-1, t+\tau_t+1, t+\tau_t+2,\dots,t+we-1,\dots$. If $w = c_1\dots c_m$ this is equivalent to $c_t(t+\tau_t+1) \geq t+we$. Suppose $c_t(t+\tau_t+1) < t+we$, because $we \leq \tau_{t-1}+1$ (by definition and 1.2.4), we deduce $c_1\dots c_t(t+\tau_t+1) \leq we$. If we show $c_{t+1}\dots c_m(i+1) = t+\tau_t+1$, we arrive to the contradiction $w(i+1) \leq we$ and the claim follows.

For $j = i-1$ we have $c_{j+1}\dots c_m(i+1) = c_i(i+1) = i+\tau_i+1 = i+\tau_{i-1} = j+\tau_j+1$. Now we show $c_j(j+\tau_j+1) = j-1+\tau_{j-1}+1$ if $t < j < i$ by decreasing induction.

If the j^{th} column of T has the form $j, j+1,\dots,j+\tau_j-1, j+\tau_j+1, j+\tau_j+2,\dots\dots,j+\tau_j+s,\dots$ then $\tau_{j-1} \geq \tau_j+s+1$ (c.f. 1.2.2). If $\tau_{j-1} \geq \tau_j+s+1$ then $w(i+1) < w(j-1) = \tau_{j-1}+1$, this is against $w(i+1) > \tau_t+1$ (c.f. 1.2.4). We have $\tau_{j-1} = \tau_j+s+1$ and the result follows.

Now construct T' by replacing the numbers $t+\tau_t+1,\dots,t+we-1$ in the t^{th} column of T by the numbers $t+\tau_t,\dots,t+e-2$.

We have $wt = \tau_t+1$ (c.f. 1.2.4), $w_{T'}t = we$ (by construction of T') and $w_{T'}(i+1) = \tau_t+1$ (we had $c_{t+1}\dots c_m(i+1) = t+\tau_t+1$). Then (by lemma 2) $w_{T'} = w(i+1,t,e)$ and $ws_is_{i,e} = w_{T'}s_is_{t,i}$.

We have $l(w_{T'}s_is_{t,i}) = l(ws_is_{i,e}) = l(w)$ and $we = w_{T'}t > w_{T'}(i+1) = \tau_t+1$, then $l(w_{T'}s_i) > l(w)$ (c.f. lemma 1) and T' is reduced.

Also $l(w_Ts_is_{t,i}) = l(w_{T'})$. One see easily $\lambda T' = T$ (we have $w_{T'}(i+1) = \tau_t+1$) and $e = k_{T'}$.

3. The main result

3.1. We start with a unipotent u in the general linear group $Gl(n,\mathbf{C})$, which in Jordan normal form has a block of size n-m and m blocks of size one.

Let us recall that the variety $\mathcal{B}_u$ can be identified with the variety of flags fixed by u.

It is known [11], that the standard tableaux of shape $(n-m,1,\dots,1)$ parametrize the components of $\mathcal{B}_u$. Here we shall follow the convention that a standard tableau has strictly decreasing rows and columns.

3.1.1. THEOREM.- <u>The expression of the homology class of a component</u> C <u>of</u> $\mathcal{B}_u$ <u>corresponding to the tableau</u> T_C <u>in terms of Schubert classes is</u> $[C] = \sum [\bar{X}_{w_T}]$, <u>where T runs over the set of reduced tableaux</u> $T = (a_{p,q})$ <u>with</u> $a_{1,q} = a_q$ $1 \le q \le m$.

n	...	
a_m		
:		T_C
a_1		

3.1.2 Example.- The component C corresponding to the tableau Tb is non singular and is not a Schubert cycle. There is only one reduced tableau corresponding to the component, namely

6	4	2
5		
3	Tb	
1		

1	3	5
2	4	
3		

Thus $[C] = [\bar{X}_w]$ with $w = s_3 s_2 s_1 s_4 s_3 s_5$ ($w = 415263$) and $[C]$ is a single Schubert class. The corresponding Schubert cycle is singular and has different Poincaré polynomial and intersection homology Poincaré polynomial ($P_C = (q^2+q+1)(q+1)^4$, $P_w = (q^3+3q^2+2q+1)(q+1)^3$, $IHP_w = (q+1)^6$).

3.2. Let us recall some results on the components and on the action of the Weyl group.

Let Δ be the root system of a reductive group; Π is a system of simple roots; Δ_+ is the set of positive roots; $< , >$ is the duality pairing between roots and coroots; $\beta^\vee$ is the coroot associated to the root β; $s_\beta \in W$ is the reflection defined by β [3].

3.2.1. According to Bernstein-Gelfand-Gelfand [1 th. 3.12] and Demazure [4 pg.80] the action of a simple reflection $s = s_\alpha, \alpha \in \Pi$, on Schubert classes is given by:

$$(1) \qquad s[\bar{X}_w] = \begin{cases} -[\bar{X}_w] & \text{if } ws < w \\ [\bar{X}_w] + \displaystyle\sum_{\substack{\beta \in \Delta_+ - \{\alpha\} \\ l(wss_\beta) = l(w)}} <\alpha, \beta^\vee> [\bar{X}_{wss_\beta}] & \text{if } ws > w \end{cases}$$

Here $[\bar{X}_w]$ is the Schubert class corresponding to $w \in W$.

3.2.2 Let $\mathcal{P}_s$ be the variety of parabolic lines of type s [15], and $\Pi : \mathcal{B} \longrightarrow \mathcal{P}_s$ the natural projection. Following Hotta [7] we say that a pair of components (C,C') form an <u>s-pair</u> if $\Pi(C') \subset \Pi(C)$ but $\Pi(C') \neq \Pi(C)$; in particular C and C' intersect in codimension one.

The action of s on the homology classes of the components is given by:

$$s[C] = \begin{cases} -[C] & \text{if } \dim \Pi(C) < \dim C \\ [C] + \sum n_{C,C'} [C'] & \text{if } \dim \Pi(C) = \dim C \end{cases}$$

Here the summation is over all the components C' with (C,C') and s-pair and the numbers $n_{C,C'}$ are strictly positive integers [8].

REMARK.- In the formulas of Demazure and Hotta we observe

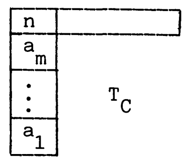
i) $ws < w$ if and only if the Schubert cycle corresponding to w contains lines of type s.

ii) $\dim \Pi(C) < \dim C$ if and only if the component C contains lines of type s.

3.3. In our case the Weyl group W is S_n and the set of positive roots Δ_+ is in one-to-one correspondece with set of transpositions in S_n.

Denote by $\beta_{i,j}$ the positive root corresponding to the transposition $s_{i,j} = (i,j)$. The roots $e_i = \beta_{i,i+1}$ $1 \leq i \leq n-1$ corresponding to the fundamental transpositions $s_i = (i,i+1)$ form a system of simple roots. Say that a component contains lines of type i if it contains lines of type s_i. Say that two components form an i-pair if they form an s_i-pair.

3.3.1. PROPOSITION [12 pg. 87].- <u>The component corresponding to the tableau</u> T_C <u>contains exactly lines of type</u> $\{a_1, a_2, \ldots, a_m\}$.



The intersection pattern of these components is known [16], in particular we have:

3.3.2. PROPOSITION.- <u>Two components intersect in codimension one if and only if the corresponding tableaux differ by a transposition of consecutive integers not lying in the same row or column.</u>

As a consequence there are for given i and C at most two components C' with (C,C') an i-pair.

We see that the homology classes of the components are a very special basis for the action of the Weyl group (Springer representations), i.e. the matrices of the fundamental reflections have 1's and -1's in the diagonal and 0's or positive integers outside. For "hook" components we have:

3.3.3. PROPOSITION.- <u>All the integers</u> $n_{C,C'}$ <u>in Hotta's formula (2) are 1</u>.

PROOF.- Take a component A whose tableau has i in the first column and $i+1 \leq n-1$ in the first row. Let B be the component obtained by interchanging i and i+1. Assume:

a) $i \geq 2$ and $i-1$ is in the first column. Let C be the component whose tableau is obtained by interchanging i and i-1 in B.

b) $i+1 < n-1$ and $i+2$ is in the first column. Let D be the component whose tableau is obtained by interchanging $i+1$ and $i+2$ in A.

Then

$$s_i[B] = [B] + n^i_{BA}[A] + n^i_{BC}[C],$$

$$s_{i+1}s_i[B] = -[B]+n^i_{BA}([A]+n^{i+1}_{AB}[B]+n^{i+1}_{AD}[D]) - n^i_{BC}[C],$$

$$s_is_{i+1}s_i[B] = -[B]-n^i_{BA}[A]-n^i_{BC}[C]-n^i_{BA}[A]+n^i_{BA}n^{i+1}_{AB}([B]+n^i_{BA}[A]+n^i_{BC}[C])+n^i_{BC}[C]-n^i_{BA}n^{i+1}_{AD}[D],$$

$$s_{i+1}[B] = -[B],$$

$$s_is_{i+1}[B] = -[B] - n^i_{BA}[A] - n^i_{BC}[C],$$

$$s_{i+1}s_is_{i+1}[B] = [B] - n^i_{BA}([A] + n^{i+1}_{AB}[B] + n^{i+1}_{AD}[D]) + n^i_{BC}[C]$$

But $s_is_{i+1}s_i[B] = s_{i+1}s_is_{i+1}[B]$ and the homology classes of the components form a basis, so comparing coefficients:

$$-1 + n^i_{BA}n^{i+1}_{AB} = 1 - n^i_{BA}n^{i+1}_{AB} \quad \text{and} \quad n^i_{BA}n^{i+1}_{AB} = 1 \quad \text{therefore} \quad n^i_{BA} = n^{i+1}_{AB} = 1$$

If the assumtions a) and b) are not satisfied the components C or D do not appear but the proof is the same.

3.4. The proof of theorem 3.1.1. is by double induction on the length of the first row and on the integer in the upper right-hand corner of the tableau.

Read the tableau
$$\begin{matrix} n\ldots b_1 \\ a_m \\ \vdots \\ a_1 \end{matrix}$$
as the "word" $a_1\ldots n\ldots b_1$.

i) = case 1 2...n. If the lenght of the first row of the tableau is one, the unipotent is the indentity and $\mathcal{B}_u = \mathcal{B}$. On the other hand $s_{n-1}\cdots s_1 s_{n-1}\cdots s_2\cdots s_{n-1}$ is w_o the longest element in S_n which corresponds to $[\mathcal{B}]$.

ii) case $a_1\ldots a_m n\ldots b_3 b_2 1$. If the number in the upper right-hand corner of the tableau is 1, all the flags in the component have for one-dimensional subspace a fixed line [12].

The component is isomorphic to the component given by the tableau $a_1-1, a_2-1, \ldots, n-1, \ldots, b_3-1, b_2-1$; the isomorphism is given by the natural isomorphism between Flags (n-1) and the Schubert variety in Flags (n) of flags that contain the fixed line, and through this isomorphism the Schubert cycle corresponding to the permutation $w' \in S_{n-1}$ goes to the one corresponding to $w \in S_n$ ($w(1)=1$, $w(i) = w'(i-1)+1$ if $i>1$). (See for instance [6, Chapter III §4]). If one writes $w' = s_{\alpha_1} \ldots s_{\alpha_k}$ as product of fundamental transpositions then $w = s_{\alpha_1+1} \ldots s_{\alpha_k+1}$. The result now follows inmediately.

iii) Case $1\,2 \ldots i\ \widehat{i+1} \ldots n \ldots b_2\ i+1$. By induction we have now a fixed component A corresponding to a tableau of this form (if the tableau is $1\,2 \ldots m\ n\ n-1 \ldots\ m+1$ the proof is the same as in the first step of the induction).

Let B the component corresponding to the tableau which we obtain by interchanging i and i+1 in the tableau of A.

We will assume $i \geq 2$ and we shall not treat the case i=1 separately, (because the proof in that situation is a particular case of the proof for $i \geq 2$).

Let C be the component corresponding to the tableau which we obtain by interchanging i-1 and i in the tableau of B.

Put $s = s_i$, and let $e = e_i$ be the corresponding fundamental root (i is fixed).

Define Γ_A as the set of reduced tableaux $T=(a_{p,q})$ with $a_{1,q}=a_q$ $1 \leq q \leq m$. Define Γ_B, Γ_C similarly.

We have $\Gamma_A \subset \mathcal{L}$, $\Gamma_B \subset \mathcal{M}$, $\Gamma_C \subset \mathcal{N}$

Hotta formula gives $s[B] = [B] + [A] + [C]$

By the induction hypothesis the main result is true for B and C, so we have:

$$[B] = \sum_{T \in \Gamma_B} [\overline{X}_{w_T}] \qquad , \qquad [C] = \sum_{T \in \Gamma_C} [\overline{X}_{w_T}]$$

Now $\Gamma_B \subset \mathcal{M}$ then $w_T i=1$ if $T \in \Gamma_B$ (c.f. 1.2.3) and $w_T s > w_T$. Thus by Demazure formula (1) $[A] = \sum_{T \in \Gamma_A} [\overline{X}_{w_T}]$ is equivalent to:

$$(*) \qquad \sum_{T \in \Gamma_A} [\overline{X}_{w_T}] + \sum_{T \in \Gamma_C} [\overline{X}_{w_T}] = \sum_{(T,\beta) \in I} \langle e, \beta^\vee \rangle\ [\overline{X}_{w_T s s_\beta}]$$

where I is the set of pairs (T,β) with $T \in \Gamma_B$, $\beta \subset \Delta_+ - \{e\}$ and $l(w_T s s_\beta) = l(w_T)$.

We prove (*) by "counting" terms in both sides of the equality. Put $I = I_+ \cup I_-$ where $(T,\beta) \in I_+$ if $\langle e, \beta^\vee \rangle = 1$ and $(T,\beta) \in I_-$ if $\langle e, \beta^\vee \rangle = -1$.

We shall construct two maps:

ϕ: $\Gamma_A \cup \Gamma_C \longrightarrow I_+$ injective and satisfying $w_{T'} = w_T s s_\beta$ if $(T,v) = \phi(T')$

Ψ: $I_- \longrightarrow I_+ - \mathrm{im}\,\phi$ bijective and satisfying $w_{T'} s s_{\beta'} = w_T s s_\beta$ if $(T,\beta) = \Psi(T',\beta')$

$$\text{Define } \phi \text{ by: } \phi(T) = \begin{cases} (\psi T, \beta_{i,e}) \quad e = e_T & \text{if } T \in \Gamma_A \\ (\chi T, \beta_{i-1,i+1}) & \text{if } T \in \Gamma_C \end{cases}$$

ϕ is well defined and satisfies the previous requirement (c.f. 2.2.2, 2.2.3). ϕ is injective (c.f. 2.2.4, 2.2.1).

Suppose (T,v) is in I_-. Then β has the form $\beta = \beta_{t,i}$ with $t < i$ (if β has the form $\beta = \beta_{i+1,e}$ with $e > i+1$, then because $w_T i=1$ $l(w_T) < l(w_T s_i) < l(w_T s_i s_\beta)$ (c.f. lemma 1)).

Define Ψ by:

$$\Psi(T,\beta) = (\lambda T, k_T)$$

Ψ is well defined and satisfies the requirement (c.f. 2.3.2, 2.3.1). Ψ is injective (c.f. 2.3.3).

If $(t,\beta) \in I_+$ and β has the form $\beta = \beta_{t,i+1}$ then (T,β) is in im ϕ (c.f. 2.2.5), it follows from (2.3.4) that Ψ is surjective.

Example.- We have a non trivial example for the tableau

```
9 7 6 4 3 .
8
5
2
1
```

Here the cardinalities of the sets Γ_A, Γ_B, Γ_C are 10, 4, 3 respectively. Moreover I_- is a non empty set.

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