Astérisque

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Astérisque, tome 113-114 (1984), p. 312-336
<http://www.numdam.org/item?id=AST_1984__113-114__312_0>

AN ALGEBRAIC MODEL FOR G-HOMOTOPY TYPES

by

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(*) University of Minnesota Mathematics Report 82-1^3

This work was partially supported by a grant from the National Science Foundat ion.

1. INTRODUCTION

Let G be a finite group. Throughout this paper we consider simplical complexes X on which G acts simplicially. We assume that all spaces in sight are G-nilpotent in the sense that the fixed point spaces X^H are nonempty and nilpotent for all subgroups H of G . We also assume that the rational homology of each X^H is of finite type.

In [7] we constructed an algebraic invariant $\mathfrak{M}_X$ for a G-complex X , the equivariant minimal model of X , which generalizes D. Sullivan's [6] minimal model in the nonequivariant case (trivial G-action). The equivariant minimal model has properties analogous to the nonequivariant one and determines X up to rational G-homotopy type.

The object of this paper is to use $\mathfrak{M}_X$ together with certain additional structure to classify G-complexes up to (integral) G-homotopy type.

$\mathfrak{M}_X$ is a minimal system of differential graded algebras (Definition 2.3), a concept which plays the same role in our context as the concept of minimal algebra in the nonequivariant case. We specify the notion of lattices of a minimal system of DGA's and the notion of torsion bound of G-complexes in section 3, and we prove

Theorem 1.1: Let $\mathfrak{M}$ be a minimal system of DGA's with lattices Z and Z' and let M be a positive integer. Then there are only finitely many finite G-complexes, up to G-homotopy type, of torsion bound M with equivariant minimal model $\mathfrak{M}$ and lattices Z and Z' .

The proof is based on the study of the group $aut_G(X)$ of G-homotopy classes of G-self homotopy equivalences of a G-space X. Let $f: X \to X_o$ be an equivariant rationalization of X.

Theorem 1.2: (i) The group $aut_G(X_o)$ is an algebraic Q - matrix group.

(ii) $aut_G(X)$ is commensurable with an arithmetic subgroup of $aut_G(X_o)$. Hence $aut_G(X)$ is a finitely presented group.

Theorem 1.1 generalizes work by D. Sullivan [6] and Theorem 1.2 work by C. Wilkerson [9] and Sullivan [6] in the nonequivariant case. Because of the nature of our algebraic invariant we shall follow Sullivan's approach. The main arguments work equivariantly but they are technically much more delicate. Since the proofs in [6] are very sketchy we shall prove 1.1 and 1.2 (ii) in detail in section 3.

In section 4, we prove 1.2(i).

Whereas the algebraic models of rational homotopy theory did not produce any new information about the usual spaces like spheres, projective spaces, H-spaces etc., in the equivariant case there are already nontrivial questions to ask about the rational homotopy type of G-actions on spheres, H-spaces and so on. This indicates that in some sense these models could be more useful equivariantly than non-equivariantly. Applications of this type will appear elsewhere.

I would like to thank Peter May and Donald Kahn for helpful discussions and Dinakar Ramakrishnan for teaching me some elements of the theory of algebraic groups.

2. THE NILPOTENT CASE

In this section we generalize the construction of the equivariant minimal model for G-simple G-complexes [7] to the nilpotent case. The two constructions are basically the same. We give an outline of the results which will serve as a reference for later sections.

We recall that a group K is nilpotent if its lower central series $\Gamma^1(K) = K$, $\Gamma^{i+1}(K) = [K,\Gamma^i K]$, reaches $\{1\}$ in a finite number of steps. Let K act from the left on an abelian group A by automorphisms. Let $\Gamma^1(A) = A$ and let $\Gamma^{i+1}(A)$ be the subgroup generated by $\{xa-a,\ x \in K,\ a \in \Gamma^i(A)\}$. We say that K operates nilpotently on A if $\Gamma^i(A) = \{0\}$ for some i .

Definition 2.1: A space X is said to be nilpotent if $\pi_1(X)$ is nilpotent and acts nilpotently on the higher homotopy groups of X .

Definition 2.2: A G-space X is said to be G-nilpotent if all fixed point sets X^H are nilpotent spaces, H subgroup of G $(H < G)$.

We consider the category of canonical orbits $\mathcal{O}_G$ of G the objects of which are quotient spaces G/H, $H < G$, and the morphisms are G-maps, where G acts on G/H by left multiplication. A (coefficient) system of groups for G is a functor from $\mathcal{O}_G$ into the category of groups. Similarly, systems of abelian groups, systems of vector spaces, systems of chain complexes etc. are defined.

We denote by $\underline{\pi}_n(X)$ the system of groups associated to a G-space X defined by

$$\underline{\pi}_n(X)(G/H) \equiv \pi_n(X^H)$$

on objects of $\mathcal{O}_G$, where $X^H = \{x \in X \mid gx=x, g \in H\}$. Because of the functoriallity of the lower central series of a group we can define systems

$\Gamma^i(\underline{\pi}_n(X))$ by

$$\Gamma^i(\underline{\pi}_n(X))(G/H) = \Gamma^i(\pi_n(X^H)) ,$$

$n \geq 1$. Then $\Gamma^i\underline{\pi}_1(X))/\Gamma^{i+1}\underline{\Gamma}_1(X) \equiv \Gamma^{(i)}\underline{\pi}_1(X)$ is a system of abelian groups and $\Gamma^i\underline{\pi}_n(X)/\Gamma^{i+1}\underline{\pi}_n(X) \equiv \Gamma^{(i)}\underline{\pi}_n(X)$ is a system of abelian groups with trivial $\underline{\pi}_1(X)$-action (at every $G/H \in \mathcal{O}_G$).

For any G-nilpotent space X there is an equivariant Postnikov decomposition ([7],[4]) which is a sequence of principal G-fibrations $p_{n,i} : X_{n,i} \to X_{n,i-1}$ with fibers $K(\Gamma^{(i)}\underline{\pi}_{n+1}(X) ; n+1)$. Here the fibers are Eilenberg-Mac Lane G-spaces in the sense of [3]. We have the familiar property for Bredon equivariant cohomology

$$H_G^{n+2} (X_{n,i-1} ; \Gamma^{(i)}\underline{\pi}_{n+1}(X)) \cong [X_{n,i-1}, K(\Gamma^{(i)}\underline{\pi}_{n+1}(X), n+2)]_G,$$

where $[\ ,\ ]_G$ means G-homotopy classes of G-maps, and the equivariant k-invariants lie in these cohomology groups. A rationalization in this context is a G-map $f: X \to X_0$ such that each $f^H: X^H \to X_0^H$ is an ordinary rationalization.

The algebraic analogue of the situation described so far is as follows:

We consider systems of differential graded commutative algebras over $\mathbb{Q}$ (systems of DGA's) $\mathfrak{G}$ which have the following property: $\mathfrak{G}$ is an injective object in the abelian category of systems of rational vector spaces by neglect of structure. For instance, let $\underline{\mathcal{E}}_X$ be the system of DGA's defined by

$$\underline{\mathcal{E}}_X(G/H) = \mathcal{E}_{X^H}$$

on the objects of $\mathcal{O}_G$, where X is a G-complex and $\mathcal{E}_X$ is the de Rham - Sullivan algebra of PL forms. $\underline{\mathcal{E}}_X$ is injective in the above sense ([7]). This property turns out to be crucial for the construction of the minimal model. We consider only injective systems of DGA's henceforward.

Definition 2.3: A system of DGA's $\mathfrak{M}$ is said to be minimal if it has the following properties:

(i) $\mathfrak{M}(G/H)$ is a free DGA, $H < G$,

(ii) $\mathfrak{M}(G/G)$ is a minimal DGA (initial condition) and

(iii) $d|\bigcap_{H' \gneq H} \ker \mu_{H,H'}, \subset \mathfrak{M}(G/H)$ is decomposable, where

$\mu_{H,H'}: \mathfrak{M}(G/H) \to \mathfrak{M}(G/H')$ is the map induced by the projection $G/H \to G/H'$.

We recall ([6]) that a minimal DGA $\mathcal{M}$ is said to be nilpotent if each subalgebra $\mathcal{M}(n)$ is constructed from $\mathcal{M}(n-1)$ by a finite number of elementary extensions; here $\mathcal{M}(n)$ is the subalgebra generated by elements of degree less or equal to n. An arbitrary connected $(H^0 = \mathbb{Q})$ DGA is said to be nilpotent if its minimal model is nilpotent.

Definition 2.4: A system of connected DGA's $\mathfrak{a}$ is said to be nilpotent if each $\mathfrak{a}(G/H)$ is a nilpotent DGA, $H < G$.

By methods entirely similar to those of [7], we can prove

Theorem 2.5: For any system of nilpotent DGA's $\mathfrak{a}$ there exists a minimal system of DGA's $\mathfrak{M}$ and a cohomology isomorphism $\rho : \mathfrak{M} \to \mathfrak{a}$ (for every $G/H \in \mathcal{O}_G$).

We omit the proof here and refer to [7]. There we specify the notion of equivariant elementary extension which is described as follows: Let $\mathfrak{a}$ be a system of DGA's and let W_n be a system of rational vector spaces (concentrated in degree n). Let

$d: W_n \to Z^{n+1}(\mathcal{a})$ be a map into the cocycles of $\mathcal{a}$. We consider a minimal injective resolution

$$0 \to W_n \to W_n^0 \to W_n^1 \to \cdots \to W_n^k \to 0$$

of W_n and form the system of DGA's

$$\mathcal{a}(W_n) \equiv \mathcal{a} \otimes \mathbb{Q}(W_n^0) \otimes \cdots \otimes \mathbb{Q}(W_n^k) \ , \tag{2.6}$$

where $\mathbb{Q}(W_n^i)$ is the system of free DGA's generated by W_n^i with $\deg(W_n^i) = n + i$. There is an appropriate differential on $\mathcal{a}(W_n)$ which extends the differential of $\mathcal{a} \subset \mathcal{a}(W_n)$ and the differential $d: W_n \to Z^{n+1}(\mathcal{a})$. $\mathcal{a}(W_n)$ is called the elementary extension of $\mathcal{a}$ with respect to $d: W_n \to Z^{n+1}(\mathcal{a})$.

Corollary 2.7: A minimal system of nilpotent DGA's $\mathfrak{M}$ can be written as the union of an expanding sequence of elementary extensions

$$t: \bigcup_{n,i} \mathfrak{M}(n)_i \xrightarrow{\cong} \mathfrak{M}$$

Proof: The proof of the injectivity of t is word for word the same as in [7].

For the surjectivity of t we give the following new argument. Let $\mathfrak{M}' \equiv \bigcup_{n,i} \mathfrak{M}(n)_i$. We will show that $t(G/H): \mathfrak{M}'(G/H) \to \mathfrak{M}(G/H)$ is a surjective map of DGA's for every $H < G$. Observe that $t(G/G)$ is an isomorphism. This follows from the non-equivariant case since $\mathfrak{M}'(G/G)$ and $\mathfrak{M}(G/G)$ are minimal DGA's. Assume inductively that $t(G/H')$ is an isomorphism for every $H' \gneq H$. The algebras $\mathfrak{M}'(G/H)$ and $\mathfrak{M}(G/H)$ are free and $t(G/H)$ is an injection. So

$$\mathfrak{M}(G/H) \cong \mathfrak{M}'(G/H) \otimes \mathcal{A} ,$$

where $\mathcal{A}$ is free and acyclic ([6]). Let y be a generator in $\mathcal{A}$ with dy indecomposable and let $y_{H'} = \mu_{H,H'}(y) \in \mathfrak{M}(G/H')$, $H' > H$, where $\mu_{H,H'}$ is the map induced by the projection $G/H \to G/H'$. Consider the preimages $\bar{y}_{H'} \in \mathfrak{M}'(G/H')$.

Claim: There is an element $\bar{y} \in \mathfrak{M}(G/H)$ such that

$$\mu'_{H,H'}(\bar{y}) = \bar{y}_{H'} \ , \ H' > H \ .$$

This claim is proved in [7]. Now replace y by $y - t(G/H)(\bar{y})$, which has still indecomposable differential, and so assume that

$$\mu_{H,H'}(y) = 0 \ ,$$

for every $H' \gneq H$. But this contradicts the minimality of $\mathfrak{M}$.

Theorem 2.8: If $f\colon \mathfrak{M} \to \mathfrak{N}$ is a cohomology isomorphism between minimal systems of nilpotent DGA's then f is an isomorphism.

Proof: Consider $t\colon \bigcup_{n,i} \mathfrak{M}(n)_i \cong \mathfrak{M}$ as before. For the same reasons $f \circ t$ is an isomorphism. This proves the theorem.

Homotopy between two maps of systems of DGA's means the following: Let $\mathcal{C}(t,dt)$ be the system of DGA's defined by

$$\mathcal{C}(t,dt)(G/H) \equiv \mathcal{C}(G/H) \otimes \mathbb{Q}(t,dt) .$$

Two maps $f,g\colon \mathcal{C} \to \mathcal{B}$ are homotopic if there is a map $H\colon \mathcal{C} \to \mathcal{B}(t,dt)$ such that $H|_{t=0} = f$ and $H|_{t=1} = g$. The same universal lifting property

$$\begin{array}{ccc} & \overset{\tilde{f}}{\nearrow} & \mathcal{B} \\ & & \downarrow \pi \\ \mathfrak{M} & \xrightarrow[f]{} & \mathcal{C} \end{array} \tag{2.9}$$

holds for maps of nilpotent systems of DGA's, where $\mathfrak{M}$ is minimal and π is a cohomology isomorphism.

We establish some notation: Let $\mathfrak{M}$ be a minimal system of DGA's. Define $\pi_n(\mathfrak{M})$ by

$$\pi_n(\mathfrak{M})(G/H) \equiv \pi_n(\mathfrak{M}(G/H))$$

and $H^n(\mathfrak{M})$ by

$$H^n(\mathfrak{M})(G/H) \equiv H^n(\mathfrak{M}(G/H)) .$$

Also define $\Gamma^{(i)}\pi_n(\mathfrak{M}) \equiv H^n(\mathfrak{M}(n-1)_i, \mathfrak{M}(n-1)_{i-1})$, $n \geq 1$.

Let X be a G-space. In analogy to earlier notation, we denote by $\underline{H}^*(X;\mathbb{Q})$ the system of graded commutative algebras

$$\underline{H}^*(X;\mathbb{Q})(G/H) \equiv H^*(X^H;\mathbb{Q}) .$$

Now let X be a G-nilpotent G-complex and let $\rho: \mathfrak{M}_X \to \underline{\mathcal{E}}_X$ be a cohomology isomorphism from a minimal system $\mathfrak{M}_X$. (This exits by Theorem 2.5). $\mathfrak{M}_X$ is called the equivariant minimal model of X . The main result of this section is the following.

Theorem 2.10: The correspondence $X \mapsto \mathfrak{M}_X$ induces a bijection between equivariant rational homotopy types of nilpotent G-spaces on the one hand and isomorphism classes of minimal nilpotent systems of DGA's for G on the other. Moreover, the following relations hold:

(1) $H^*_G(X;N) \cong H^*(\underline{\mathcal{E}}_X, N^*) \cong H^*(\mathfrak{M}_X;N^*)$, where N is a (contravariant) system of rational vector spaces and N^* is its dual (covariant) system. For a definition of these cohomologies see [3], [7] (and 4.11 of this paper).

(2) $\underline{H}^n(X;\mathbb{Q}) \cong H^n(\mathfrak{M}_X)$,

(3) $(\pi_n(X) \otimes \mathbb{Q})^* \cong \pi_n(\mathfrak{M})$, $n > 1$

(4) $((\Gamma^{(i)}\underline{\pi}_n(X)) \otimes \mathbb{Q})^* \cong \Gamma^{(i)}\pi_n(\mathfrak{M})$, $n \geq 1$,

(5) (i) $\mathfrak{M}_X(n)_i$ is the equivariant minimal model of $X_{n,i}$ and (ii) the differential $d: W_{n+1,i} \to Z^{n+2}(\mathfrak{M}_X(n)_{i-1})$ of the

elementary extension $\mathfrak{M}_X(n)_i = \mathfrak{M}_X(n)_{i-1}(W_{n+1,i})$ determines the equivariant rational k-invariant

$$k_i^{n+2} \in H_G^{n+2}(X_{n,i-1}; \Gamma^{(i)}\underline{\pi}_{n+1}(X) \otimes \mathbb{Q}) .$$

Theorem 2.11: There exists a bijection

$$[X,Y]_G = [\mathfrak{M}_Y, \mathfrak{M}_X] ,$$

where Y is a rational G-space and [,] denotes homotopy classes of maps.

Again we refer to [7] for an inductive proof (over the equivariant Postnikov decomposition) of these results.

3. THE MAIN RESULTS

In this section we shall prove the main results of this paper granting 1.2(i) (that $\mathrm{aut}_G(X_0)$ is an algebraic matrix group over $\mathbb{Q}$) and certain facts about algebraic groups which we discuss in section 4.

Definition 3.1: Two groups A and B are commensurable if there exists a finite chain of maps $A \to C_1 \leftarrow C_2 \to \cdots \leftarrow B$ such that each map has finite kernel and its image has finite index.

Theorem 3.2: Let X be a G-nilpotent space which is a finite G-complex or has a finite equivariant Postnikov system. Then $\mathrm{aut}_G(X)$ is commensurable to an arithmetic subgroup of $\mathrm{aut}_G(X_0)$.

Proof: Let $\{X_{n,i}\}$ be an equivariant Postnikov system of X such that all spaces are G-complexes and all maps are simplicial equivariant inclusions. The first diagram below induces a diagram of systems of de Rham algebras and we can construct the equivariant minimal model of X as the union of the models of the $X_{n,i}$'s.

$$\begin{array}{ccc} & \vdots & \\ & \downarrow & \\ & X_{n,i+1} & \\ X \nearrow & \downarrow & \\ \searrow & X_{n,i} & \\ & \vdots & \end{array} \qquad \begin{array}{ccccc} & \vdots & & \vdots & \\ & & & \cup & \\ \swarrow & \underline{\mathcal{E}}_{X_{n,i+1}} & \longleftarrow & \mathfrak{M}(n)_{i+1} & \\ \underline{\mathcal{E}}_X & \uparrow & & \cup & \\ \nwarrow & \underline{\mathcal{E}}_{X_{n,i}} & \xleftarrow{\rho_{n,i}} & \mathfrak{M}(n)_i & \\ & \vdots & & \vdots & \end{array}$$

Let f_{i+1} be an equivariant self homotopy equivalence of $X_{n,i+1}$. Just as in the nonequivariant case, this map can be extended to an equivariant self homotopy equivalence $f_i : X_{n,i} \to X_{n,i}$, unique up to G-homotopy. Moreover it induces an automorphism g of $\Gamma^{i+1}\underline{\pi}_{n+1}(X)/\Gamma^{i+2}\underline{\pi}_{n+1}(X) \equiv \Gamma^{(i+1)}$.

Given the morphisms $\rho_{n,i}$, each map f_i induces an automorphism $\tilde{f}_i$ of the model $\mathfrak{M}(n)_i$, unique up to homotopy, such that the diagram

$$\begin{array}{ccc} \underline{\mathcal{E}}_{X_{n,i}} & \xrightarrow{f_i^*} & \underline{\mathcal{E}}_{X_{n,i}} \\ \rho_{n,i} \uparrow & & \uparrow \rho_{n,i} \\ \mathfrak{M}(n)_i & \xrightarrow{\tilde{f}_i} & \mathfrak{M}(n)_i \end{array}$$

commutes up to homotopy. Moreover, $\tilde{f}_{i+1}$ restricts to an automorphism of $\mathfrak{M}(n)_i$ which is homotopic to $\tilde{f}_i$. Therefore it induces an authomorphism of

$$\Gamma^{(i+1)}\pi_{n+1}(\mathfrak{M}) = \underline{H}^{n+1}(\mathfrak{M}(n)_{i+1},\mathfrak{M}(n)_i) \cong \Gamma^{(i+1)} \otimes \mathbb{Q}$$

as well.

So we have a commutative diagram

$$\begin{array}{ccc} \mathrm{aut}_G(X_{n,i+1}) & \xrightarrow{A} & \mathrm{aut}_G(X_{n,i}) \times \mathrm{aut}(\Gamma^{(i+1)}) \\ \downarrow & & \downarrow \\ \mathrm{aut}(\mathfrak{M}(n)_{i+1}) & \xrightarrow{A_{\mathbb{Q}}} & \mathrm{aut}(\mathfrak{M}(n)_i) \times \mathrm{aut}(\Gamma^{(i+1)} \otimes \mathbb{Q}) . \end{array}$$

By Theorem 4.7, $\mathrm{aut}(\mathfrak{M})$, the set of homotopy classes of automorphisms of a minimal system of DGA's is an algebraic matrix group over $\mathbb{Q}$ and by Lemma 4.5, $\mathrm{aut}(\Gamma^{(i+1)})$ is commensurable to an arithmetic subgroup of $\mathrm{aut}(\Gamma^{(i+1)} \otimes \mathbb{Q})$. Using the latter fact to start the induction, we assume that $\mathrm{aut}_G(X_{n,i})$ is commensurable to an arithmetic subgroup of $\mathrm{aut}(\mathfrak{M}(n)_i)$. We will prove the analogous statement for $\mathrm{aut}_G(X_{n,i+1})$. We study the kernel and the image of the maps **A** and A_0.

The image of A, say M, consists of pairs $([f_i],g)$ such that the diagram

$$\begin{array}{ccc} X_{n,i} & \xrightarrow{k} & K(\Gamma^{(i+1)},n+2) \\ f_i \uparrow & & \uparrow g \\ X_{n,i} & \xrightarrow{k} & K(\Gamma^{(i+1)},n+2) \end{array}$$

commutes up to homotopy, where k is a representative of the equivariant k-invariant $\bar{k} \in H_G^{n+2}(X_{n,i},\Gamma^{(i+1)})$. On the minimal model level this translates into the following: the element $([f_i],g) \in \operatorname{aut}(\mathfrak{M}(n)_i) \times \operatorname{aut}(\Gamma^{(i+1)} \otimes \mathbb{Q})$ is in the image M_0 of A_0 if it is in the isotropy group of the k-invariant $\bar{k}_0$ under the following action.

$$\operatorname{aut}(\mathfrak{M}(n)_i) \times \operatorname{aut}(\Gamma^{(i+1)} \otimes \mathbb{Q}) \times H^{n+2}(\mathfrak{M}(n)_i; (\Gamma^{(i+1)} \otimes \mathbb{Q})^*) \to H^{n+2}(\mathfrak{M}(n)_i;(\Gamma^{(i+1}\otimes\mathbb{Q})^*)$$

$$([f_i],g) \times \bar{k}_0 \longmapsto (f_i)_*(g^{-1})^*(\bar{k}_0)$$

It follows from Proposition 4.11 that this action is algebraic. Hence the isotropy group M_0 is an algebraic group ([1], p. 97) and M is commensurable to an arithmetic subgroup of M_0 .

Now consider the kernel N of A . We will prove the following statement: N is an abelian subgroup of $\operatorname{aut}_G(X_{n,i})$ and N_0 , the kernel of A_0 , is isomorphic to $N \otimes \mathbb{Q}$.

Let

$$\Omega K \to X'_{n,i+1} \to X_{n,i} \xrightarrow[k]{} K(\Gamma^{(i+1)},n+2)$$

be the Barratt-Puppe sequence. Here all spaces are G-spaces and the maps are equivariant. Consider the orbit of $[\mathrm{id}] \in [X'_{n,i+1},X'_{n,i+1}]_G$ under the action from the right of $[X'_{n,i+1}, \Omega K]_G$. N is a subset of this orbit consisting of G-homotopy classes of maps which induce the identity on $\Gamma^{(i+1)}$. The proof of this is entirely anologous to the nonequivariant case ([8]). So, we consider classes $[\alpha] \in [X'_{n,i+1},\Omega K]_G$ such that $[\mathrm{id}\alpha|_{\Omega K}] = [\mathrm{id}_{\Omega K}]$, i.e. $[\alpha|_{\Omega K}] = 0$. The set of such classes is the kernel of $i^*: H_G^{n+1}(X'_{n,i+1};\Gamma^{(i+1)}) \to H_G^{n+1}(\Omega K;\Gamma^{(i+1)})$ which is equal to $H_G^{n+1}(X_{n,i};\Gamma^{(i+1)})$ as follows from the long exact sequence. Hence N is a quotient of $H_G^{n+1}(X_{n,i};\Gamma^{(i+1)})$.

It remains to show that the group structure of N , namely composition of maps $[\mathrm{id}\alpha \circ \mathrm{id}\alpha']$, coincides with the addition $[\mathrm{id}\,\alpha\alpha']$ in H_G^{n+1}. Since $[\alpha]$ and $[\alpha']$ belong to the kernel of i^* , the maps α and α' can be replaced by G-homotopic maps which factor through $X_{n,i}$, i.e. do not depend on the path variable of $X'_{n,i+1}$.

Similarly N_0 is isomorphic to a quotient of

$$H^{n+1}(\mathfrak{M}(n)_i;(\Gamma^{(i+1)} \otimes \mathbb{Q})^*) \cong H_G^{n+1}(X_{n,i};\Gamma^{(i+1)} \otimes \mathbb{Q}) \cong H_G^{n+1}(X_{n,i};\Gamma^{(i+1)}) \otimes \mathbb{Q}.$$

In a trivial way, a vector space is an algebraic group and a finitely generated subgroup of maximal rank is an arithmetic subgroup.

Now we have the diagram

$$\begin{array}{ccccccccc} 0 & \to & N & \to & \mathrm{aut}_G(X_{n,i+1}) & \to & M & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & N_0 & \to & \mathrm{aut}(\mathfrak{M}(n)_{i+1}) & \to & M_0 & \to & 0 \end{array} ,$$

where the lower row consists of algebraic groups and N and M are commensurable to arithmetic subgroups of N_0 and M_0 respectively. By a result in [2], $\mathrm{aut}_G(X_{n,i+1})$ must be also commensurable to an arithmetic subgroup of $\mathrm{aut}(\mathfrak{M}(n)_{i+1})$. This completes the proof of the theorem for G-spaces with finite equivariant Postnikov systems. For finite G-complexes we only need to observe that $\mathrm{aut}_G(X) \cong \mathrm{aut}_G(X_n)$ for n sufficiently large.

Now we will use the above result to prove the classification theorem 1.1. Before this we establish some terminology.

Let X be a G-complex and let $\rho: \mathfrak{M} \to \underline{\mathcal{E}}_X$ be an equivariant minimal model of X . By the universal lifting property of $\mathfrak{M}$ (2.9) the map ρ determines maps $\rho_{n,i} : \mathfrak{M}(n)_i \to \underline{\mathcal{E}}_{X_{n,i}}$ uniquely up to homotopy, where the spaces $X_{n,i}$ are as in the previous theorem. The composite map

$$\underline{\text{Hom}}\ (\Gamma^{(i)}\underline{\pi}_n(X),\mathbb{Z}) \to \underline{\text{Hom}}\ (\Gamma^{(i)}\underline{\pi}_n(X),\mathbb{Q}) \cong \Gamma^{(i)}\pi_n(\mathfrak{M})\ ,\ n \geq 1\ ,$$

defines a <u>system of lattices</u> $Z_{n,i}$ of $\Gamma^{(i)}\pi_n(\mathfrak{M})$, i.e. a system of finitely generated abelian groups $Z_{n,i}(G/H)$ such that $Z_{n,i} \hookrightarrow Z_{n,i} \otimes \mathbb{Q} \cong \Gamma^{(i)}\pi_n(\mathfrak{M})$ for every $n \geq 1$ and $i \geq 1$. Here $\underline{\text{Hom}}(A,\mathbb{Z})$ is defined by $\underline{\text{Hom}}(A,\mathbb{Z})(G/H) \equiv \text{Hom}\ (A(G/H),\mathbb{Z})$ for a system of abelian groups A. Also the composite

$$H^{n+2}_G(X_{n,i};\ \Gamma^{(i+1)}\underline{\pi}_{n+1}(X)) \to H^{n+2}_G(X_{n,i};\ \Gamma^{(i+1)}\underline{\pi}_{n+1}(X) \otimes \mathbb{Q}) \to H^{n+2}(\mathfrak{M}(n)_i;\Gamma^{(i+1)}\pi_{n+1}(\mathfrak{M}))$$

defines a lattice $Z'_{n+2,i+1}$ of the vector space over $\mathbb{Q}$ $H^{n+2}(\mathfrak{M}(n)_i;\ \Gamma^{(i+1)}\pi_{n+1}(\mathfrak{M}))$ for every $i \geq 1$ and $n \geq 0$.

Again we restrict attention to G-spaces X which are finite G-complexes or whose equivariant Postnikov system is finite and $\pi_n(X^H)$ is finitely generated for every $H < G$ and every n. Such a space is said to have a <u>torsion bound</u> M if

$$|\text{torsion}(\pi_n(X^H))| < M\ ,\ n \leq N,$$

for every subgroup $H < G$, where N is the dimension of X (if X is a finite CW-complex) or the length of the Postnikov tower. Now we can express the main result of this paper precisely.

<u>Theorem 3.3</u>: Let $\mathfrak{M}$ be a minimal system of finitely generated nilpotent DGA's with systems of lattices $Z_{n,i} \subset \Gamma^{(i)}\pi_n(\mathfrak{M}), n, i \geq 1$,

and lattices $Z'_{n+2,i+1} \subset H^{n+2}(\mathfrak{M}(n)_i, \Gamma^{(i+1)}\pi_{n+1}(\mathfrak{M}))$, $n \geq 0$, $i \geq 1$, and let M be a positive integer. Then there are only finitely many G-homotopy types [X] of torsion bound M such that there exist cohomology isomorphisms $\rho_X: \mathfrak{M} \to \underline{\mathcal{E}}_X$ which induce isomorphisms of the lattices.

<u>Proof</u>: We proceed by induction on a filtration $\underline{\mathbb{Q}} \subset \cdots \subset \mathfrak{M}(n)_i \subset \mathfrak{M}(n)_{i+1} \subset \cdots$ of $\mathfrak{M}$. We start the induction with the observation that there are only finitely many G-homotopy types which are rationally equivalent to a point because of the given torsion constraint.

We fix a space $X_{n,i}$ with a morphism $\rho_{n,i}: \mathfrak{M}(n)_i \to \underline{\mathcal{E}}_{X_{n,i}}$ which induces an isomorphism of the lattices on homotopy and cohomology. Let $\Gamma^{(i+1)}$ be a system of finitely generated abelian groups such that

$$\underline{\text{Hom}}\ (\Gamma^{(i+1)}, \mathbb{Z}) \overset{\sigma}{\cong} Z_{n+1,i+1} .$$

There are only finitely many such systems since we restrict the possible torsion by the given bound M . We have a map

$$H^{n+2}_G(X_{n,i}; \Gamma^{(i+1)}) \xrightarrow{A} H^{n+2}_G(X_{n,i}, \Gamma^{(i+1)} \otimes \mathbb{Q}) \xleftarrow[\cong]{B} H^{n+2}(\mathfrak{M}(n)_i; \Gamma^{(i+1)}\pi_{n+1}(\mathfrak{M})) ,$$

where B is induced by $\rho_{n,i}$ and σ . Let $d: \Gamma^{(i+1)}\pi_{n+1}\mathfrak{M} \to \mathfrak{M}(n)_i$ be the differential of the elementary extension $\mathfrak{M}(n)_{i+1} \supset \mathfrak{M}(n)_i$ and let $\bar{d}$ be its cohomology class in $H^{n+2}(\mathfrak{M}(n)_i; \Gamma^{(i+1)}\pi_{n+1}(\mathfrak{M}))$. The preimage of $\bar{d}$ under $B^{-1} \circ A$ consists of all possible k-invariants for the next stage of the Postnikov system (for fixed $\rho_{n,i}$ and σ). This preimage is a finite set because A is an isomorphism after tensoring over $\mathbb{Q}$.

Now suppose that $\rho'_{n,i}: \mathfrak{M}(n)_i \to \underline{\mathcal{E}}_{X_{n,i}}$ is another cohomology isomorphism preserving the lattices. Instead of B above we have an isomorphism B' and an element $k' = B'(\bar{d})$.

By (2.9) there is an isomorphism $a\colon \mathfrak{M}(n)_i \to \mathfrak{M}(n)_i$ such that the diagram

$$\begin{array}{ccc} \mathfrak{M}(n)_i & \xrightarrow{\rho_{n,i}} & \\ a \uparrow & & \mathcal{E}_{X_{n,i}} \\ \mathfrak{M}(n) & \xrightarrow[\rho'_{n,i}]{} & \end{array}$$

commutes up to homotopy. The isomorphism a preserves the lattices isomorphically since $\rho_{n,i}$ and $\rho'_{n,i}$ have this property.

We consider the following equivalence relation in $\operatorname{aut}(\mathfrak{M}(n)_i)$: $[a] \sim [b]$ if a and b induce the same map on $H^{n+2}(\mathfrak{M}(n)_i;\, \Gamma^{(i+1)}\pi_{n+1}\mathfrak{M})$ and we call $\operatorname{aut}(\mathfrak{M}(n)_i)/H$ the group of such equivalence classes. By Proposition 4.11, $\operatorname{aut}(\mathfrak{M}(n)_i)/H$ is an algebraic group and the projection $p\colon \operatorname{aut}(\mathfrak{M}(n)_i) \to \operatorname{aut}(\mathfrak{M}(n)_i)/H$ is algebraic. Let Γ be the subgroup of $\operatorname{aut}(\mathfrak{M}(n)_i)/H$ consisting of those elements which preserve the lattice in $H^{n+2}(\mathfrak{M}(n)_i;\, \Gamma^{(i+1)}\pi_{n+1}(\mathfrak{M}))$ isomorphically. Then Γ is an arithmetic subgroup. We observe that the set of all images $k' = B'(\bar{d})$ of $\bar{d}$ in $H^{n+2}_G(X_{n,i};\, \Gamma^{(i+1)} \otimes \mathbb{Q})$, as we vary $\rho_{n,i}$, is the orbit of $k = B(\bar{d})$ under the action of Γ.

We have a commutative diagram

$$\begin{array}{ccc} \operatorname{aut}(\mathfrak{M}(n)_i) & \xrightarrow{p} & \operatorname{aut}(\mathfrak{M}(n)_i)/H \\ \uparrow & & \cup \\ \operatorname{aut}_G(X_{n,i}) & \longrightarrow & \Gamma \end{array} .$$

Now we use a theorem in the theory of arithmetic groups which states that if $f\colon C \twoheadrightarrow C'$ is a surjective homomorphism of

$\mathbb{Q}$-algebraic matrix groups then the image of an arithmetic subgroup of C is commensurable with an arithmetic subgroup of C' ([2]). Hence the image of $\mathrm{aut}_G(X_{n,i})$ is a subgroup of finite index in Γ.

In order to complete the proof of the theorem, it suffices to observe that the orbit of k under the action of $\mathrm{aut}_G(X_{n,i})$ (and therefore the orbit of k under the action of any coset of $\mathrm{aut}_G(X_{n,i})$ in Γ) contains equivalent k-invariants, i.e. elements which induce the same G-homotopy types $[X_{n,i+1}]$.

4. THE AUTOMORPHISM GROUP OF THE EQUIVARIANT MINIMAL MODEL

In this section we develop the algebraic material needed in the proof of Theorems 3.2 and 3.3.

We recall some definitions and elementary facts from [1] and [2].

Definition 4.1: A group K is called an algebraic $\mathbb{Q}$-matrix group if K is a subgroup of $GL(n,\mathbb{Q})$ for some n and its elements have the following property: the coefficients of the matrices in K annihilate some set of polynomials in $M(n,\mathbb{Q})$ with rational coefficients.

Definition 4.2: Let $K_{\mathbb{Z}}$ be the subgroup of elements of K which have integral coefficients and determinants ± 1. Then $K_{\mathbb{Z}}$ is said to be an arithmetic subgroup of K

Typical examples are $SL(n,\mathbb{Z}) \subset SL(n,\mathbb{Q})$.

Let V be an n-dimensional rational vector space and let Z be a lattice in V i.e. a finitely generated subgroup of maximal rank. By choosing a basis of Z we can identify $(K=)\ GL(V) = GL(n,\mathbb{Q})$, and $K_{\mathbb{Z}}$ with the subgroup of those isomorphisms of V which yield isomorphisms of the lattice Z (not only preserve it) .

The concept algebraic group is more general than algebraic matrix group, namely

Definition 4.3: A group M is called an algebraic group over $\mathbb{Q}$ if M is an algebraic variety over $\mathbb{Q}$ and the multiplication and inverse $(x \mapsto x^{-1})$ maps are maps of varieties.

(4.4) Let $f: K \to K'$ be a group homomorphism between two algebraic $\mathbb{Q}$-matrix groups and let $f_{ij}(A)$ be the (i,j) entry of the matrix $f(A) \in K'$, where A is a matrix in K . If $f_{ij}(A)$ is a polynomial with rational coefficients of the entries of A

for every (i,j) then f is a morphism of algebraic $\mathbb{Q}$-matrix groups. The kernel and the image of such a map are algebraic $\mathbb{Q}$-matrix groups.

Let $\mathrm{Aut}(\mathfrak{M})$ denote the group of automorphisms of the minimal system of finitely generated DGA's $\mathfrak{M}$ and let $\mathrm{aut}(\mathfrak{M})$ be the group of homotopy equivalence classes of automorphisms of $\mathfrak{M}$.

The main result of this section is that $\mathrm{aut}(\mathfrak{M})$ is an algebraic $\mathbb{Q}$-matrix group. Since

$$\mathrm{aut}(\mathfrak{M}_X) \cong \mathrm{aut}_G(X_0)$$

(see Theorem 2.11), this will prove Theorem 1.2(i).

Lemma 4.5: Let M be a covariant system of finite-dimensional rational vector spaces and let $i\colon Z \longrightarrow M$ be a system of lattices in M. Then

(i) $\mathrm{Aut}(M)$, the group of isomorphisms of M, is an algebraic $\mathbb{Q}$-matrix group and

(ii) $\mathrm{Aut}(Z)$ is an arithmetic subgroup of $\mathrm{Aut}(M)$.

Proof: Let $f\colon M \to M$ be an isomorphism. The diagram

$$\begin{array}{ccc} M(G/H) & \xrightarrow{f(G/H)} & M(G/H) \\ {\scriptstyle M(a)}\downarrow & & \downarrow{\scriptstyle M(a)} \\ M(G/H') & \xrightarrow{f(G/H')} & M(G/H') \end{array}$$

must commute for every morphism $a\colon G/H \to G/H'$ in $\mathfrak{O}_G$ and $f(G/H)$ is an isomorphism of vector spaces for every $G/H \in \mathfrak{O}_G$. Observe $\mathrm{Aut}(M) \subset \underset{G/H \in \mathfrak{O}_G}{\times} GL(M(G/H)) \subset GL(\underset{G/H \in \mathfrak{O}_G}{\oplus} M(G/H))$, where the product is finite and the second inclusion is obviously an inclusion of algebraic $\mathbb{Q}$-matrix groups. Similarly,

$$\mathrm{Aut}(Z) \subset \underset{G/H \in \mathfrak{O}_G}{\times} \mathrm{Aut}(Z(G/H)) \subset \mathrm{Aut}(\underset{G/H \in \mathfrak{O}_G}{\oplus} Z(G/H)),$$

where the latter two groups are arithmetic subgroups of the algebraic groups above respectively. Here we consider a basis for $\underset{G/H}{\oplus} M(G/H)$ consisting of bases for each $Z(G/H)$. The elements of $\underset{G/H}{\times} GL(M(G/H))$ are matrices of the form

$$\begin{pmatrix} A & 0 & 0 & \cdots & 0 \\ 0 & B & 0 & & \\ 0 & 0 & C & \ddots & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix}$$

where $A, B, \cdots$ are invertible matrices. The commutative diagrams above impose a finite number of polynomial (in fact linear) equations on such matrices to give elements in $\mathrm{Aut}(M)$. This means that $\mathrm{Aut}(M)$ is an algebraic $\mathbb{Q}$-matrix group. Moreover, the elements of $\mathrm{Aut}(Z)$ are exactly those matrices in $\mathrm{Aut}(M)$ which have integral coefficients and determinant ± 1. So $\mathrm{Aut}(Z)$ is an arithmetic subgroup of $\mathrm{Aut}(M)$. This completes the proof of the lemma.

Proposition 4.6: If $\mathfrak{M}$ is a minimal system of finitely generated DGA's over $\mathbb{Q}$ then $\mathrm{Aut}(\mathfrak{M})$ is an algebraic $\mathbb{Q}$-matrix group.

Proof: If $\mathcal{A}$ is a DGA which is finitely generated by elements of degree at most n then $\mathrm{Aut}(\mathcal{A})$ is an algebraic matrix group. In fact, $\mathrm{Aut}(\mathcal{A}) \subset \underset{i \leq 2n}{\times} GL(\mathcal{A}^i)$, where upper i means degree and where the matrices $\begin{pmatrix} A^1 & & 0 \\ & \ddots & \\ 0 & & A^{2n} \end{pmatrix}$ in $\mathrm{Aut}(\mathcal{A})$ satisfy the equations

$$\begin{array}{ccc} \mathcal{A}^i \otimes \mathcal{A}^j & \xrightarrow{m} & \mathcal{A}^{i+j} \\ {\scriptstyle A^i \otimes A^j}\downarrow & & \downarrow{\scriptstyle A^{i+j}} \\ \mathcal{A}^i \otimes \mathcal{A}^i & \xrightarrow{m} & \mathcal{A}^{i+j} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{A}^i & \xrightarrow{A^i} & \mathcal{A}^i \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ \mathcal{A}^{i+1} & \xrightarrow[A^{i+1}]{} & \mathcal{A}^{i+1} \end{array},$$

$i,j \leq n$.

This fact together with Lemma 4.5(i) proves the proposition.

Theorem 4.7: The group $\operatorname{aut}(\mathfrak{M})$ is an algebraic $\mathbb{Q}$-matrix group, where $\mathfrak{M}$ is a minimal system of finitely generated DGA's.

Proof: In order to prove that $\operatorname{aut}(\mathfrak{M})$ is an algebraic group, we will prove that the kernel of $\operatorname{Aut}(\mathfrak{M}) \to \operatorname{aut}(\mathfrak{M})$ is a unipotent subgroup of $\operatorname{Aut}(\mathfrak{M})$ which is constructed from a nilpotent Lie algebra $\mathcal{L}$ by the Campbell-Hausdorff formula. So the proof consists of the following two propositions.

Proposition 4.8: An automorphism $\sigma : \mathfrak{M} \to \mathfrak{M}$ is homotopic to the identity iff there is a derivation $i : \mathfrak{M} \to \mathfrak{M}$ of degree -1 (at each G/H) such that $\sigma = \exp(di+id)$.

Proposition 4.9: The set of "inner derivations"

$$\mathcal{L} = \{di+id,\ i: \mathfrak{M} \to \mathfrak{M} \text{ derivation of degree } -1\}$$

is a nilpotent Lie algebra under bracket $[X,Y] = X \circ Y - Y \circ X$.

The proofs of these propositions are essentially the same as in the non-equivariant case [5], [6] . We can apply word for word the methods given (in detail) in [5] using, in addition, the following technical arguments.

In order to prove that any inner derivation $di + id$ is nilpotent we use induction on the subgroups of G . We know that $(di+id)(G/G)$ is nilpotent because $\mathfrak{M}(G/G)$ is a minimal DGA (non-equivariant case). Let H be a subgroup of G and assume inductively that $(di+id)(G/H')$ is nilpotent for every subgroup H' of G which contains H as a proper subgroup. Then

$$(di+id)^m(G/H) \; : \mathfrak{M}(G/H) \to \mathfrak{M}(G/H) ,$$

for some m , takes values in $\bigcap_{H' \supsetneq H} \ker \mu_{H,H'}$. By definition of a minimal system of DGA's, the differential restricted on $\bigcap_{H' \supsetneq H} \ker \mu_{H,H'}$

is decomposable. Therefore d increases the weight of the monomials in this subset and i preserves it. So $(di+id)(G/H)$ is nilpotent.

As in [5], we construct $\mathfrak{M}^I$ and use the circle construction $(\Lambda u \otimes \mathfrak{M}, D_\varphi)$ for a given unipotent automorphism $\varphi: \mathfrak{M} \to \mathfrak{M}$. It follows from the characterization of injectives in [7] that these constructions preserve the injectivity of systems of DGA's. In particular, $\Lambda u \otimes \mathfrak{M}$ is a direct summand of $\mathfrak{M}(t,dt)$ which is injective.

Proposition 4.10: The map $p: \mathrm{Aut}(\mathfrak{M}) \to \mathrm{Aut}(H^i(\mathfrak{M}))$, and therefore the map $q: \mathrm{aut}(\mathfrak{M}) \to \mathrm{Aut}(H^i(\mathfrak{M}))$, is a map of algebraic groups.

Proof: We know that the map

$$\mathrm{Aut}(\mathcal{A}) \to GL(H^i(M))$$

is algebraic, where $\mathcal{A}$ is a finitely generated DGA ([6]). This and lemma 4.5(i) imply that the map p is algebraic. Then the map q is also algebraic by a result in [1] p. 174.

Proposition 4.11: The maps

$$\mathrm{Aut}(\mathfrak{M}) \to \mathrm{aut}(\mathfrak{M}) \to GL(H^i(\mathfrak{M};M))$$

are algebraic, where M is any covariant system of finitely dimensional rational vector spaces.

Proof: As before, it suffices to prove that the composite map p is algebraic. We recall the definition of $H^i(\mathfrak{M};M)$. We consider the set of natural transformations $\mathrm{Nat}(M,\mathfrak{M})$.

This set is a differential graded rational vector space. Its ith cohomology is denoted by $H^i(\mathfrak{M};M)$. We have the following diagram of algebraic maps

$$\begin{array}{ccc} \mathrm{Aut}(\mathfrak{M}) & \xrightarrow{\ r\ } & GL(\mathrm{Nat}(M,\mathfrak{M}^i)) \\ & {\scriptstyle s}\searrow & \cup \\ & & GL(\mathrm{Nat}(M,\mathfrak{M}^i))_{Z,B} \\ & & \downarrow{\scriptstyle t} \\ & & GL(H^i(M,\mathfrak{M})) \end{array} ;$$

here r is given by composing an automorphism of $\mathfrak{M}$ with any natural transformation (multiplication of matrices) and $GL(\mathrm{Nat}(M,\mathfrak{M}^i))_{Z,B}$ contains only those isomorphisms of $\mathrm{Nat}(M,\mathfrak{M}^i)$ which preserve the sub-vector space of cocycles and the subspace of coboundaries. All maps in the diagram are obviously algebraic and therefore $p = t \circ s$ is algebraic.

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