Astérisque

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Astérisque, tome 147-148 (1987), p. 315-317

<http://www.numdam.org/item?id=AST_1987__147-148__315_0>

Société Mathématique de France
Astérisque 147-148 (1987), p.315-317.

THIN BASES IN ADDITIVE NUMBER THEORY

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Let B be a set of nonnegative integers, and let hB denote the set of all sums of h elements of B. The set B is a basis (resp. asymptotic basis) of order h if hB contains all (resp. all sufficiently large) natural numbers. The squares, the k-th powers, and the primes are the classical examples of asymptotic bases in additive number theory.

Let B(x) denote the number of positive integers in the set B that do not exceed x. If B is an asymptotic basis of order h, then it is easy to show that $B(x)>c_1x^{1/h}$ for some constant $c_1>0$ and all $x>x_1$. An asymptotic basis B of order h is called thin if $B(x)<c_2x^{1/h}$ for some constant $c_2>0$ and all $x>x_2$. Thin bases exist. Indeed, for each $h\geq 2$, Cassels [1] constructed a family of bases B of order h such that $B(x)\sim cx^{1/h}$ as $x\to\infty$. It is not known if the classical sequences in additive number theory contain subsequences that are thin bases.

Let A be a finite set of nonnegative integers, and let $|A|$ denote the cardinality of A. If $\{0,1,\dots,n\}\subseteq hA$, then A is called a basis of order h for n. Clearly, if A is a basis of order h for n, then $|A|>n^{1/h}$.

In this report I state some recent results on the additive basis properties of subsets of the squares, k-th powers, and primes.

THEOREM (Choi-Erdős-Nathanson[2]. For every n>1 there exists a finite set A of squares such that A is a basis of order 4 for n and

$$|A|<cn^{1/3}\log n$$

where c = 4/log 2.

This is proved by means of an explicit construction. Note that the set of all squares up to n contains $[n^{\frac{1}{2}}]+1$ elements.

THEOREM (Erdős-Nathanson[3]). For every $\mathcal{E} > 0$ there exists a set B of squares such that

(i) B is a basis of order 4,

(ii) If $n \neq 4^r(8k+7)$, then $n \in 3B$,

(iii) $B(x) \sim cx^{(1/3)+\varepsilon}$ for some $c>0$ as $x \longrightarrow \infty$

The proof uses the probability method of Erdős and Rényi. The Theorem is best possible except for the $\mathcal{E}$ in the exponent in (iii).

Zöllner combined the two results above to obtain the following.

THEOREM (Zollner[7]. For every $\varepsilon > 0$ there exists n_0 such that if $n > n_0$ there is a finite set A of squares such that A is a basis of order 4 for n and

$$|A| < n^{(\frac{1}{4})+\varepsilon}$$

This result is best possible except for the ε in the exponent.

THEOREM (Zöllner[8]). Let $h \geq 4$. For every $\varepsilon > 0$ there exists a set B of squares such that B is a basis of order h and

$$B(x) < x^{(1/h)+\mathcal{E}}$$

for all $x > x_0$.

THEOREM (Wirsing[6]). Let $h \geqslant 4$. There exists a set B of squares such that B is a basis of order h and

$$B(x) < c(x\log x)^{1/h}$$

for some constant $c = c(h) > 0$ and all $x > x_0$.

Both Wirsing and Zöllner use probability methods to obtain their results, and, consequently, it is not yet possible to describe explicitly a sparse sequence of squares that is a basis of order 4.

There are some results on thin versions of Waring's problem.

THEOREM (Nathanson[4]). Let $k \geqslant 3$ and $s > s_0(k)$. Let $0 < \varepsilon < 1/s$. There exists a set B of nonnegative k-th powers such that B is a basis of order s and

$$B(x) \sim cx^{1-(1/s)+\varepsilon}$$

for some constant $c > 0$ as $x \longrightarrow \infty$.

The proof requires the Hardy-Littlewood asymptotic formula for the number of representations of an integer as the sum of s k-th powers, as well as the Erdős-Rényi probability method.

There is a finite version of the preceding theorem. Let $f(n,k,s)$ denote the cardinality of the smallest finite set A of k-th powers such that A is a basis of order s for n. Clearly, $f(n,k,s) > n^{1/s}$.

Define

$$\beta(k,s)=\limsup_{n\to\infty}\frac{\log f(n,k,s)}{\log n}$$

Let $g(k)$ denote the smallest integer h such that the set of all non-negative k-th powers is a basis of order h.

THEOREM (Nathanson[5]). For $k \geq 3$ and $s \geq g(k)$,

$$f(n,k,s) < 2(s-g(k)+1)\, n^{1/(s-g(k)+k)}$$

In particular, $\beta(k,s) \sim 1/s$ as $s \to \infty$.

Finally, there is the following beautiful result on sums of primes.

THEOREM (Wirsing[6]). For $h \geq 3$, there is a set P of primes such that

(i) $n \in hP$ for all $n>n_0$ such that $n \equiv h \pmod 2$,

(ii) $P(x) < c\,(x\log x)^{1/h}$ for some constant $c>0$ and all $x>x_0$.

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