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ON THE PROPERTY $(f(n),g(n)) = 1$ FOR CERTAIN MULTIPLICATIVE FUNCTIONS

by

Eira J. SCOURFIELD

The problem of investigating the sum

$$\Sigma_h(x) = \sum_{\substack{n \leq x \\ (n,h(n))=1}} 1 \qquad (1)$$

for certain integer-valued arithmetic functions h has been considered by several authors in cases when the arithmetic properties of n and $h(n)$ are not too closely related, and the expected result

$$\Sigma_h(x) \sim 6x/\pi^2$$

established ; for references, see this author's paper [2] . Multiplicative functions, however, present a rather different problem, and in 1948 [1], Erdös obtained the result

$$\Sigma_\varphi(x) \sim e^{-\gamma}x/\log\log\log x \quad \text{as} \quad x \to \infty$$

for Euler's function φ . In [2], we considered the sum (1) for a class of integer-valued multiplicative functions, called polynomial-like, that includes φ and the divisor functions σ_ν $(\nu \geq 0)$; f is <u>polynomial-like</u> if there exists a polynomial $W \in \mathbb{Z}[x]$ such that

$$f(p) = W(p) \qquad \text{for all primes } p. \quad (2)$$

For these functions, we proved in [2] :

THEOREM 1. *If the polynomial* W *of* (2) *satisfies* $\deg W > 0$, $W(0) \neq 0$, *then there exist constants* $C > 0$, λ $(0 < \lambda \leq 1$, λ *rational*), *depending on* f, *such that*

$$\Sigma_f(x) \sim C\,x(\log\log\log x)^{-\lambda} \qquad \text{as } x \to \infty .$$

If W *is a non-zero constant, then there exists a constant* C $(0 < C \leq 1)$ *such that*

$$\Sigma_f(x) \sim C\,x \qquad \text{as } x \to \infty .$$

If $W(0) = 0$,

$$\Sigma_f(x) = O(x^{\frac{1}{2}}) .$$

Example. $f = \sigma_\nu$ $(\nu > 0)$. For ν odd, $\lambda = 1$, $C = e^{-\gamma}$, whilst for ν even, $\lambda = 2^{-\beta}$, where $2^\beta \| \nu$.

We obtain a generalization of the sum in theorem 1 by noting that n itself is a polynomial-like multiplicative function. Let f, g be multiplicative polynomial-like functions, and let W_1, $W_2 \in \mathbb{Z}[x]$ be the polynomials such that

$$f(p) = W_1(p),\ g(p) = W_2(p) \qquad \text{for all primes } p .$$

Suppose that the following conditions hold :

(i) $\deg W_i > 0$ $(i = 1,2)$;

(ii) $W_1(x) = x^\alpha W_1^*(x)$ where $\alpha \geq 0$, $W_1^*(0) \neq 0$, $\deg W_1^* > 0$, and $W_2(0) \neq 0$;

(iii) W_1, W_2 are coprime polynomials.

It follows from (iii) that the set

$$S_o = \{p : p|(f(q),g(q)) \text{ for all primes } q \neq p\}$$

of primes is finite (possibly empty). If $p \in S_o$, $p|(f(n),g(n))$ whenever there exists a prime $q \neq p$ with $q \| n$, and hence for "most" n. This suggests that for our generalization of the sum $\Sigma_f(x)$, we consider

$$\Sigma_{f,g}(x) = \sum_{\substack{n \leq x \\ p\nmid(f(n),g(n)) \ \forall p \notin S_o}} 1 .$$

Using results from sieve theory, we can prove

THEOREM 2. _If conditions_ (i), (ii), (iii) _above hold, there exist constants_ $C > 0$, λ $(0 < \lambda \leq 1$, λ _rational)_ _such that_

$$\frac{x}{\log x} \log \log x \ll \Sigma_{f,g}(x) \ll \frac{x}{\log x} \exp\left(\frac{C \log \log x}{(\log \log \log x)^{\lambda}}\right) .$$

Conditions (i), (ii), (iii) ensure that the sum $\Sigma_{f,g}(x)$ is not too small and does not reduce to the sum considered in theorem 1 or in other published papers.

Examples. (i) $f = \varphi$, $g = \sigma_\nu$ $(\nu > 0)$, when $S_o = \{2\}$, $\lambda = 2^{-\beta}$ where $2^\beta \| \nu$.
(ii) $f = \sigma_\nu$, $g = \sigma_\varkappa$ $(\nu, \varkappa > 0$, $\beta > \gamma$, where $2^\beta \| \nu$, $2^\gamma \| \varkappa)$, when $S_o = \{2\}$, $\lambda = 2^{-\beta}$.

The method used to prove the upper bound in theorem 2 also establishes

THEOREM 3. _The number of positive integers_ $n \leq x$ _with the property that_ n _does not have a prime divisor in every residue class_ (mod p) _coprime to_ p _for any odd prime_ p _is_

$$\ll \frac{x}{\log x} \exp\left(\frac{B \log \log x}{\log \log \log x}\right) ,$$

where $B > 0$ _is constant._

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