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ELLIPTIC OPERATORS, DISCRETE GROUPS AND VON NEUMANN ALGEBRAS *

by

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§ 1 - Introduction.

The global theory of elliptic equations on compact manifolds is very well established. In particular one has finite-dimensionality for the spaces of solutions and an explicit topological formula for the index [1]. For non-compact manifolds, on the other hand, the situation is much more difficult and there are few general results. The essential difficulties are :

(i) one has to decide which growth conditions to impose at infinity,

(ii) the spaces of solutions are usually infinite-dimensional.

In practice the most useful condition to impose under (i) is square integrability with respect to some natural inner product : the spaces of solutions are then Hilbert spaces. In view of (ii) one would not expect a meaningful index formula. Nevertheless, and this constitutes

* The results in this paper are essentially part of a larger investigation carried out in collaboration with I.M. Singer (see §(6.1)).

our main result, for a large class of problems it is possible to derive an index formula based on the dimension theory of von Neumann algebras - in which Hilbert spaces have a real-valued dimension.

The class of problems to which our theory applies are those in which there is a discrete group Γ acting freely on our non-compact manifold $\tilde{X}$, having a compact quotient $X = \tilde{X}/\Gamma$, and commuting with our elliptic differential operator $\tilde{D}$. In other words our operator $\tilde{D}$ is the lift to the covering space $\tilde{X}$ of an elliptic differential operator D on the compact manifold X. To form our Hilbert spaces on $\tilde{X}$ we use any Γ-invariant smooth positive measure $d\tilde{\mu}$, i.e. the lift of a smooth positive measure $d\mu$ on X. The bounded operators on† $L^2(\tilde{X})$ which commute with the action of Γ form a von Neumann algebra $\mathcal{A}$ and this has a natural trace function denoted by trace_Γ . In particular if $P \in \mathcal{A}$ is an orthogonal projection onto a subspace H, so that H is a Γ-module, one defines

$$\dim_\Gamma H = \mathrm{trace}_\Gamma P$$

which is a real number d with $0 \leqslant d \leqslant \infty$. Applying this to the spaces $\mathcal{H}(\tilde{D})$ and $\mathcal{H}(\tilde{D}^*)$ of L^2-solutions of $\tilde{D}\phi = 0$, $\tilde{D}^*\psi = 0$ we get a <u>finite</u> real-valued index

$$\mathrm{index}_\Gamma \tilde{D} = \dim_\Gamma \mathcal{H}(\tilde{D}) - \dim_\Gamma \mathcal{H}(\tilde{D}^*) .$$

Our main result is then

$$\mathrm{index}_\Gamma \tilde{D} = \mathrm{index}\, D . \tag{1.1}$$

In other words, for the Γ-periodic operator $\tilde{D}$, the Γ-index of L^2-solutions is the same as the ordinary index of Γ-periodic solutions. Combined with the explicit formula of [1] for index D, (1.1) gives a corresponding formula for $\mathrm{index}_\Gamma \tilde{D}$.

† For brevity we omit here any reference to vector bundles. These omissions will be rectified in the detailed text.

When Γ is finite of order $|\Gamma|$, so that $\tilde{X}$ is also compact, we have $\dim_\Gamma = \frac{1}{|\Gamma|} \dim$, so that (1.1) reduces to

$$\text{index } \tilde{D} = |\Gamma| \text{ index } D, \tag{1.2}$$

which is a well-known consequence of the index formula of [1]. In fact (1.2) can be proved quite easily, independently of the final formula of [1], and our proof of (1.1) will be a straightforward generalization of this direct proof.

Before proceeding further it is perhaps desirable to describe the Γ-dimensions, used in (1.1), more explicitly. If $\{\phi_n\}$ is an orthonormal base for the Hilbert space $\mathcal{H}(\tilde{D})$ we put

$$\tilde{f}(\tilde{x}) = \sum_n |\phi_n(\tilde{x})|^2$$

This series converges and the function $\tilde{f}$ is C^∞ and Γ-invariant, hence is the lift of a function f on X. Then

$$\dim_\Gamma \mathcal{H}(\tilde{D}) = \int_X f(x)\, d\mu \tag{1.3}$$

and similarly for $\tilde{D}^*$. Clearly if Γ is finite then

$$\dim \mathcal{H}(\tilde{D}) = \int_{\tilde{X}} \tilde{f}(\tilde{x}) d\tilde{\mu} = |\Gamma| \int_X f(x) d\mu = |\Gamma| \dim_\Gamma \mathcal{H}(\tilde{D})$$

as stated above. This shows that (1.3) gives a natural "normalized dimension".

Formula (1.1) embodies an existence theorem, namely if one knows that index $D > 0$ (a topological criterion in view of [1]) then index $\tilde{D} > 0$ and hence $\mathcal{H}(\tilde{D}) \neq 0$; in other words there exist non-zero L^2-solutions of the equation $\tilde{D}\phi = 0$. Notice that, at this stage, von Neumann algebras and normalized dimensions have disappeared from the scene : they appear only in the proof. This existence theorem is quite easy to apply. For example, if $\tilde{X}$ is the upper half plane and $\tilde{D}$ is the $\bar{\partial}$-operator on (0,1)-forms, we can choose X to be any compact

Riemann surface of genus $g \geq 2$. Then index $D = g-1 > 0$ and so we deduce the existence of non-zero holomorphic L^2 forms on $\tilde{X}$. Of course this fact is well-known and of great interest since this gives a Hilbert space representation of $G = SL(2,R)$, belonging to the discrete series. Note however that our proof applies directly to the universal covering of the Riemann surface X (of genus ≥ 2) without using its identification to the upper-half plane, i.e. without using the Riemann mapping theorem.

The essence of the above example is that the operator $\tilde{D}$ not only commutes with the discrete group Γ (the fundamental group of X) but with the transitive Lie group G. The space $\mathcal{H}(\tilde{D})$ is then a G-module and not only a Γ-module. Much more generally one can consider G-invariant elliptic operators $\tilde{D}$ on a homogeneous space $\tilde{X} = G/K$. When G is semi-simple and K is a maximal compact subgroup, so that $\tilde{X}$ is the symmetric space, there are suitable operators $\tilde{D}$ to which (1.1) applies and shows that $\mathcal{H}(\tilde{D}) \neq 0$. Thus our theory can be used as an analytical starting point for the investigation of the discrete series representations of G. This will be taken up in a subsequent paper where it will be shown in particular that our Γ-dimensions are closely related to the "formal degrees" of the discrete series. An interesting feature in all these cases is that, because Γ is highly non-commutative, the von Neumann algebra $\mathcal{A}$ is actually a factor (of type II_∞) so that the dimension function is unique up to a scalar.

We proceed now to explain the method of proof of (1.1) and, as a preliminary, we shall show how to compute index D (on the compact manifold X) in terms of any parametrix Q. By definition of a parametrix we have

$$(1.4) \qquad QD = 1 - S_0 \qquad DQ = 1 - S_1$$

where S_0, S_1 are operators with C^∞ kernel. A special case is that of the Green's operator G which satisfies

(1.5) $$GD = 1 - H_0 , \qquad DG = 1 - H_1$$

where H_0, H_1 are the projections onto $\mathcal{H}(D)$ and $\mathcal{H}(D^*)$ respectively. From (1.4) we deduce

$$H_0 = S_0 H_0 , \qquad H_1 = H_1 S_1 , \qquad DS_0 = S_1 D .$$

Using these formulae we now compute traces using the fact that S_0, S_1 and DS_0 have C^∞ kernel, hence are of trace class, and that G, DG and GD are bounded :

$$\text{trace } DS_0G = \text{trace } GDS_0 = \text{trace } S_0GD = \text{trace } S_0 - \text{trace } H_0$$

$$\text{trace } S_1DG = \text{trace } DGS_1 = \text{trace } S_1 - \text{trace } H_1 .$$

Since $DS_0G = S_1DG$ we deduce

(1.6) $$\text{index } D = \text{trace } H_0 - \text{trace } H_1 = \text{trace } S_0 - \text{trace } S_1 .$$

The advantage of (1.6) is that the parametrix Q, and hence the S_i, can be constructed locally out of the operator D, whereas the H_i depend globally on D. In particular we can always construct Q so that it is <u>almost local</u>, i.e. so that its Schwartz kernel has support close to the diagonal. The same will then be true of the S_i.

Suppose now that $\tilde{X} \to X$ is a <u>finite</u> covering, then the almost local property of Q, S_0, S_1 means that they have natural liftings to almost local operators on $\tilde{X}$ and equation (1.4) implies

(1.7) $$\tilde{Q}\tilde{D} = 1 - \tilde{S}_0 \qquad \tilde{D}\tilde{Q} = 1 - \tilde{S}_1 .$$

Hence, applying (1.6) to $\tilde{D}$ we have

(1.8) $$\text{index } \tilde{D} = \text{trace } \tilde{S}_0 - \text{trace } \tilde{S}_1 .$$

But $\tilde{S}_i(\tilde{x},\tilde{x})$ is by construction the lift to $\tilde{X}$ of $S_i(x,x)$, hence

$$(1.9) \qquad \operatorname{trace} \tilde{S}_i = \int_{\tilde{X}} S_i(\tilde{x},\tilde{x})\, d\tilde{\mu} = k \int_X S_i(x,x)\, d\mu = k \operatorname{trace} S_i$$

where k is the degree of the covering $\tilde{X} \to X$. From (1.6) and (1.8) we deduce

$$\operatorname{index} \tilde{D} = k \operatorname{index} D$$

which proves (1.2).

For an infinite covering with group Γ we proceed in exactly the same way as far as (1.7). The main difference is that (1.8) has to be replaced by

$$(1.10) \qquad \operatorname{index}_\Gamma \tilde{D} = \operatorname{trace}_\Gamma \tilde{S}_0 - \operatorname{trace}_\Gamma \tilde{S}_1$$

and (1.9) by

$$(1.11) \qquad \operatorname{trace}_\Gamma \tilde{S}_i = \int_X S_i(x,x)\, d\mu = \operatorname{trace} S_i \; .$$

The proof of (1.10) is formally similar to that of (1.6) with $\operatorname{trace}_\Gamma$ replacing trace throughout. A technical difficulty however is that the Green's operator $\tilde{G}$ need not be bounded (e.g. $\tilde{X} = R$, $D = \frac{d}{dx}$, $\Gamma = Z$), so that more care is needed in the use of the commutation formula $\operatorname{trace}_\Gamma AB = \operatorname{trace}_\Gamma BA$.

The detailed contents of the paper are as follows. In § 2 we review the basic properties of the kernel function associated to a general elliptic operator. This generalizes the classical theory of the Bergman kernel. In § 3 we introduce the discrete group Γ into the picture and we prove that, for an elliptic operator $\tilde{D}$ commuting with Γ, the minimal and maximal domains coincide : this means there is no ambiguity about its adjoint $\tilde{D}^*$. We then give the precise formulation of our main theorem. In § 4 we introduce the von Neumann algebra $\mathcal{A}$ of Γ-invariant operators and the trace function $\operatorname{trace}_\Gamma$. We establish the basic properties of trace_Γ and then use these in §5 to prove (1.1) on the lines indicated above. Note that our treatment in § 4 and § 5 is elementary and self-contained : knowledge of von Neumann algebras is not assumed. We conclude in § 6 with some further observations and open problems.

I am indebted to L. Hörmander and J. Duistermaat for help with a number of the analytical questions.

§ 2 - The kernel function.

In this section we shall review some essentially well-known material which extends the classical results of S. Bergman on the kernel function. The proofs rely heavily on the kernel theorems of L. Schwartz.

Let X be a smooth paracompact manifold with a smooth measure $d\mu$ and let E,F be two complex vector bundles on X with hermitian inner products (all structures being smooth). We can then form the Hilbert spaces $L^2(X,E)$ and $L^2(X,F)$ of square-integrable sections, the inner product on sections being given by

$$\langle u,v\rangle = \int_X (u,v)\, d\mu$$

where (u,v) is the function obtained by taking the inner product in each fibre.

Suppose now that $D : C^\infty(X,E) \to C^\infty(X,F)$ is a C^∞ elliptic differential operator of order m. We denote by $\mathcal{H}(D)$ the space of all L^2-solutions of the equation $Du = 0$. Since D is elliptic every weak solution is actually C^∞ so that

$$\mathcal{H}(D) \subset C^\infty(X,E) \cap L^2(X,E).$$

If u_j is a sequence in $\mathcal{H}(D)$ converging to u in $L^2(X,E)$ then

$Du_j \to Du$ weakly (i.e. as distributions) hence $Du = 0$ and so $u \in \mathcal{H}(D)$: thus $\mathcal{H}(D)$ is a closed subspace of $L^2(X,E)$.

Let P denote the orthogonal projection onto $\mathcal{H}(D)$. Since P is a continuous operator on L^2 it is certainly continuous as an operator $\mathcal{D} \to \mathcal{D}'$ and so has a distributional kernel $p(x,y)$ in the sense of Schwartz. Since P acts on sections of E, $p(x,y) \in \mathrm{Hom}(E_y,E_x) \cong E_x \otimes E'_y$ so that p is a distributional section of the bundle $\mathrm{Hom}(E_1,E_2)$ on $X \times X$, where $E_i = \pi_i^* E$, π_i being the projection $X \times X \to X$ onto the appropriate factor. In fact p is a C^∞ section because it satisfies an elliptic differential equation. To see this we note first that $DP = 0$ (by definition) and so $D_x p(x,y) = 0$. Taking adjoints and using the fact that $P^* = P$ we get another equation

$$\bar{D}_x p(x,y) = 0 ,$$

where $\bar{D} = hDh^{-1}$ and $h : C^\infty(X,E) \to C^\infty(X,E')$ is the antilinear isomorphism defined by the metric on E. Combining the two equations we see that $p(x,y)$ satisfies the differential equation

$$(2.1) \qquad (D_x^* D_x + \overline{D_y}^* \bar{D}_y)\, p(x,y) = 0$$

which is clearly elliptic.

If $\{\phi_n(x)\}$ is an orthonormal base of $\mathcal{H}(D)$ we consider the series

$$(2.2) \qquad \sum_{n=1}^{\infty} \phi_n(x)\, \bar{\phi}_n(y)$$

where $\bar{\phi}_n = h\, \phi_n$ gives the corresponding orthonormal base of $\mathcal{H}(\bar{D}) \subset C^\infty(X,E')$. If we put

$$P_N\,(x,y) = \sum_{n=1}^{N} \phi_n(x)\, \bar{\phi}_n(y)$$

then P_N is a C^∞ kernel and it defines a corresponding projection operator P_N (of finite rank) on L^2. The sequence P_N converges to P in the strong operator topology, that is $P_N(\phi) \to P(\phi)$ in L^2 for any $\phi \in L^2$. In particular $P_N(\phi) \to P(\phi)$ in $\mathcal{D}'$ for any $\phi \in \mathcal{D}$ which means that $P_N \to P$ weakly in $\mathcal{L}(\mathcal{D}_y, \mathcal{D}_x')$. But on bounded sets (and hence for convergent sequences) the weak and strong topologies on $\mathcal{L}(\mathcal{D}_y, \mathcal{D}_x')$ coincide (cf. [2 ; (4.3)]), so that $P_N \to P$ strongly in $\mathcal{L}(\mathcal{D}_y, \mathcal{D}_x')$. But the Schwartz kernel theorem [8 ; Prop.25] asserts that we have a topological isomorphism

$$\mathcal{L}(\mathcal{D}_y, \mathcal{D}_x') \cong \mathcal{D}'_{x,y}$$

the former having the strong topology. Thus $p_N \to p$ as distributions on $X \times X$.

Now it is clear that p_N also satisfies the elliptic equation (2.1) and we have the following general lemma :

LEMMA (2.3) <u>If a sequence</u> f_j <u>of solutions of an elliptic equation converges to</u> f <u>in</u> $\mathcal{D}'$ <u>then</u> $f_j \to f$ <u>in</u> C^∞.

<u>Proof</u> : In any relatively compact set we have [7 ; Theorem 23] $f_j = D^p g_j$, $f = D^p g$ with $g_j \to g$ a uniformly convergent sequence of continuous functions, hence $f_j \to f$ in the local Sobolev space H^{loc}_{-p} [6, (2.6)]. But the space of solutions of the given elliptic equation is a closed subspace of the Frechet space H^{loc}_s for every s. It is therefore a Frechet space and, by the closed-graph theorem, the induced topology is independent of s. Thus $f_j \to f$ in all H^{loc}_s and hence, by the Sobolev lemma, in C^∞.

Summarizing these results we have :

PROPOSITION (2.4) <u>The kernel</u> $p(x,y)$ <u>of the projection</u> P <u>onto the space</u> $\mathcal{H}(D)$ <u>of</u> L^2 <u>solutions of the elliptic equation</u> $Du = 0$ <u>is</u> C^∞. <u>Moreover, if</u> $\phi_n(x)$ <u>is an orthonormal base of</u> $\mathcal{H}(D)$ <u>and</u> $\bar{\phi}_n = h\phi_n$

<u>the corresponding base of</u> $\mathcal{H}(\bar{D})$, <u>we have</u>

$$p(x,y) = \sum_{n=1}^{\infty} \phi_n(x)\ \bar{\phi}_n(y)$$

<u>the sequence convergine uniformly on compact sets of</u> $X \times X$, <u>together with all its derivatives</u>.

If we put $x = y$, then $p(x,x) \in \mathrm{Hom}(E_x,E_x)$ and we can therefore take its trace to obtain a function. Proposition (2.4) implies that

$$\text{(2.5)} \qquad \mathrm{tr}\, p(x,x) = \sum_{n=1}^{\infty} |\phi_n(x)|^2 \quad .$$

§ 3 - <u>The index theorem</u>.

We continue with the situation of § 2 but it will be convenient to modify the notation, replacing X by $\tilde{X}$ etc. Suppose now that Γ is a discrete group of automorphisms of the whole structure, that is Γ acts smoothly on $\tilde{X},\tilde{E},\tilde{F}$ preserving the measure $d\tilde{\mu}$ on $\tilde{X}$, the inner products on $\tilde{E},\tilde{F}$ and commuting with $\tilde{D}$. We assume further that

(i) Γ acts <u>freely</u> so that $\tilde{X}/\Gamma = X$ is again a smooth manifold and $\tilde{E}/\Gamma = E$, $\tilde{F}/\Gamma = F$ are vector bundles on X,

(ii) X is <u>compact</u>.

We denote by D the operator $C^\infty(X,E) \to C^\infty(X,F)$ induced by $\tilde{D}$.

Conversely if D is an elliptic operator on a compact manifold X and if $\tilde{X} \to X$ is a Galois covering with group Γ we can lift everything to $\tilde{X}$ and we shall recover the above situation.

It will sometimes be convenient to introduce a fundamental domain U of Γ. We recall that this means an open set of $\tilde{X}$, disjoint from all its translates by Γ and such that $\tilde{X} - \bigcup_{\gamma\in\Gamma} \gamma(U)$ has measure zero. To construct such a U is a simple matter. Let V_i be a finite open covering of X by small balls, so that we have a continuous section s_i of $\tilde{X} \to X$ over V_i, and put $W_i = V_i - \bigcup_{j<i} \bar{V}_j \cap V_i$. Then $U = \bigcup s_i(W_i)$

is a fundamental domain. Note that the characteristic function χ of U satisfies $\sum_{\gamma\epsilon\Gamma} \gamma(\chi) = 1$ almost everywhere. When dealing with differential operators it is preferable to have a smooth partition of unity relative to Γ, namely a non-negative C^∞ function σ on $\tilde{X}$ with compact support and such that $\sum_{\gamma\epsilon\Gamma} \gamma(\sigma) = 1$: note that at any $\tilde{x} \epsilon \tilde{X}$ only finitely many of the functions $\gamma(\sigma)$ are non-zero, so that the summation is essentially finite. To construct such a σ we take a C^∞ partition of unity $\{\phi_i\}$ on X with supp $\phi_i \subset V_i$, lift ϕ_i to a function $\tilde{\phi}_i$ on $\tilde{X}$ using the section s_i and put $\sigma = \Sigma\tilde{\phi}_i$.

The fact that the operator $\tilde{D}$ commutes with Γ has strong implications for its domain of definition as an operator on L^2. For any differential operator A, defined in the first instance on C^∞_{comp}, we can consider its closure $\bar{A}$ as an operator on L^2. The domain of $\bar{A}$ consists of $u \epsilon L^2$ for which there exists a convergent sequence $u_j \to u$ and $Au_j \to Au$ in L^2 : this domain is usually called the <u>minimal</u> domain of A. The <u>maximal</u> domain of A is the space of all $u \epsilon L^2$ such that $Au \epsilon L^2$ (as a distribution). If B denotes the formal adjoint of A, i.e. the differential operator with domain C^∞_{comp} such that $\langle Au,v\rangle = \langle u,Bv\rangle$, we see that the maximal domain of B is just the domain of the Hilbert space adjoint $A^* = \bar{A}^*$. The minimal domain is of course contained in the maximal domain. For the operator $\tilde{D}$ we have the converse :

PROPOSITION (3.1) <u>The minimal and maximal domains of the operator $\tilde{D}$ coincide</u>.

<u>Proof</u> : Given $u \epsilon L^2(\tilde{X},\tilde{E})$ with $\tilde{D}u \epsilon L^2(\tilde{X},\tilde{F})$ we must produce a sequence $u_j \epsilon C^\infty_{comp}(\tilde{X},\tilde{E})$ such that $u_j \to u$ in $L^2(\tilde{X},\tilde{E})$ and $\tilde{D}u_j \to \tilde{D}u$ in $L^2(\tilde{X},\tilde{F})$. We shall carry this out in two stages, first by regularization and then by cutting down the support. For the first stage we shall use a parametrix $\tilde{Q}$ for $\tilde{D}$ obtained by lifting an

almost local pseudo-differential parametrix Q for D as explained in §1. Then

$$(3.2) \qquad \tilde{Q}\tilde{D} = 1 - \tilde{S}_0 , \qquad \tilde{D}\tilde{Q} = 1 - \tilde{S}_1$$

with $\tilde{S}_i$ having a C^∞ kernel and $\tilde{Q}$, $\tilde{S}_0$ and $\tilde{S}_1$ all bounded operators on L^2. Now choose a sequence $v_j \in C^\infty_{comp}(\tilde{X},\tilde{F})$ converging in L^2 to $v = \tilde{D}u$, then $\omega_j = \tilde{Q}v_j \in C^\infty_{comp}(\tilde{X},\tilde{E})$ since $\tilde{Q}$ is pseudo-differential and almost local. Applying (3.2) we find

$$\omega_j \to \tilde{Q}\tilde{D}u = u - \tilde{S}_0 u$$

$$\tilde{D}\omega_j = v_j - \tilde{S}_1 v_j \to v - \tilde{S}_1 v$$

the convergence being in L^2. Thus $u - \tilde{S}_0 u$ is in the minimal domain of $\tilde{D}$, and so we are reduced to showing that $\omega = \tilde{S}_0 u$ is in this minimal domain. But $\tilde{S}_0$ has a C^∞ kernel supported near the diagonal, hence $\omega \in C^\infty \cap L^2$. This completes the first stage in the proof and we come now to the second stage. Here we shall use a C^∞ function σ on $\tilde{X}$ with $\sum_{\gamma\in\Gamma} \gamma(\sigma) = 1$ as explained above. Since $\tilde{X}$ is paracompact, Γ is countable so let $\Gamma_N \subset \Gamma$ be the first N elements and put $\sigma_N = \sum_{\gamma\in\Gamma_N} \gamma(\sigma)$. Then $\sigma_N \in C^\infty_{comp}$ and $\sigma_N\omega \to \omega$ in L^2. We will show that $\phi_N = \tilde{D}(\sigma_N\omega)$ converges to $\tilde{D}\omega$ in L^2, which will show that ω is in the minimal domain of $\tilde{D}$ and will complete the proof. Since $\tilde{D}\circ\sigma$ is an operator of order m with compact support and since $\tilde{D}$ is elliptic of order m, we have an inequality for L^2 norms

$$\|\tilde{D}(\sigma g)\| \leq C \{ \|\chi g\| + \|\chi \tilde{D} g\| \}$$

where C is a constant depending on $\tilde{D}$ and σ, the section g is in $C^\infty(\tilde{X},\tilde{E})$ and $\chi \in C^\infty_{comp}$ is equal to one in a neighbourhood of supp σ. Taking $g = \gamma^{-1}(\omega)$, and using the fact that every $\gamma \in \Gamma$ commutes with $\tilde{D}$ and is unitary, we get

$$\begin{aligned}\|\tilde{D}(\gamma(\sigma)\omega)\|^2 &= \|\tilde{D}(\sigma\gamma^{-1}(\omega))\|^2 \\ &\leqslant C\{\|\chi\gamma^{-1}(\omega)\|^2 + \|\chi\tilde{D}\gamma^{-1}(\omega)\|^2\} \\ &\leqslant C\{\|\gamma(\chi)\omega\|^2 + \|\gamma(\chi)\tilde{D}\omega\|^2\}.\end{aligned}$$

Hence

$$\begin{aligned}\|\tilde{D}\omega - \phi_N\|^2 &\leqslant C \sum_{\gamma\notin\Gamma_N} \{\|\gamma(\chi)\omega\|^2 + \|\gamma(\chi)\tilde{D}\omega\|^2\} \\ &\leqslant CM \int_{\tilde{X}_N} (|\omega|^2 + |\tilde{D}\omega|^2)\, d\tilde{\mu}\end{aligned}$$

where $M = \sup \sum_{\gamma\varepsilon\Gamma} |\gamma(\chi)|^2$ (which is finite) and $\tilde{X}_N = \bigcup_{\gamma\notin\Gamma_N} \operatorname{supp} \gamma(\chi)$. But given any compact $K \subset \tilde{X}$ there are only a finite number of $\gamma \varepsilon \Gamma$ such that $K \cap \gamma(\operatorname{supp} \chi) \neq \emptyset$, so that for sufficiently large N, $K \cap \tilde{X}_N = \emptyset$. Since ω and $\tilde{D}\omega$ belong to L^2 this implies that

$$\int_{\tilde{X}_N} (|\omega|^2 + |\tilde{D}\omega|^2) < \varepsilon \qquad \text{for } N > N_0$$

and hence $\|\tilde{D}\omega - \phi_N\|^2 < CM\varepsilon$ for large N. Thus $\phi_N \to \tilde{D}\omega$ in L^2 as required.

<u>Remark</u> If $\tilde{X} = R^n$, $D = 1 + \Delta$ where Δ is the Laplacian $-\Sigma\frac{\partial^2}{\partial x_i^2}$, Proposition (3.1) amounts to the fact that C^∞_{comp} is dense in the Sobolev space H^2. Thus if we introduce generalized Sobolev spaces on $\tilde{X}$ they will have the usual properties with respect to Γ-invariant elliptic operators.

With these technicalities out of the way we return to the kernel function studied in §2. Thus we consider the space $\mathcal{H}(\tilde{D})$ of L^2-solutions of the equation $\tilde{D}u = 0$, the orthogonal projection $\tilde{P}$ onto this subspace of $L^2(\tilde{X},\tilde{E})$ and its Schwartz kernel $\tilde{p}(\tilde{x},\tilde{y})$. As proved in §2 $\tilde{p}$ is C^∞. Since Γ commutes with $\tilde{D}$ and preserves inner products it acts on $\mathcal{H}(\tilde{D})$ and commutes with P. Hence

$$\tilde{p}(\gamma\tilde{x}, \gamma\tilde{y}) = \tilde{p}(\tilde{x},\tilde{y}) \qquad \text{for all } \gamma \varepsilon \Gamma.$$

In particular, putting $\tilde{x} = \tilde{y}$, we see that $\tilde{p}(\tilde{x},\tilde{x})$ is a Γ-invariant section of $\mathrm{Hom}(\tilde{E},\tilde{E})$ and so is the lift of a section $p(x)$ of $\mathrm{Hom}(E,E)$ on X.

Taking its trace pointwise we get a C^∞ function and we shall define the Γ-dimension of $\mathcal{H}(\tilde{D})$ by

$$(3.3) \qquad \dim \mathcal{H}(\tilde{D}) = \int_X \mathrm{tr}\, p(x)\, d\mu\ .$$

Note that, by (2.4), $\mathrm{tr}\, p(x)$ can be computed in terms of an orthonormal base $\{\phi_n(\tilde{x})\}$ of $\mathcal{H}(\tilde{D})$ by

$$(3.4) \qquad \mathrm{tr}\, p(x) = \sum_n |\tilde{\phi}_n(\tilde{x})|^2$$

where $\tilde{x} \in \tilde{X}$ is any point lying over $x \in X$. Thus (3.3) can be reformulated as

$$(3.5) \qquad \dim_\Gamma \mathcal{H}(\tilde{D}) = \int_U \sum_n |\tilde{\phi}_n(\tilde{x})|^2\, d\tilde{\mu}$$

where U is a fundamental domain for Γ. Alternatively, using a C^∞ function σ with $\sum_{\gamma\varepsilon\Gamma} \gamma(\sigma) = 1$ as before, we have

$$(3.6) \qquad \dim_\Gamma \mathcal{H}(\tilde{D}) = \int_{\tilde{X}} \sigma(\tilde{x})\tilde{p}(\tilde{x},\tilde{x})\, d\tilde{\mu}\ .$$

For the moment (3.3), and its equivalent forms (3.5) or (3.6), should be regarded as an ad hoc definition. In §4 we shall reinterpret this in terms of von Neumann algebras as explained briefly in §1.

If D^* is the adjoint of D, $\tilde{D}^*$ its lift to $\tilde{X}$, we now define

$$(3.7) \qquad \mathrm{index}_\Gamma\, \tilde{D} = \dim_\Gamma \mathcal{H}(\tilde{D}) - \dim_\Gamma \mathcal{H}(\tilde{D}^*)\ .$$

Our main result can now be formulated :

THEOREM (3.8) $\mathrm{index}_\Gamma\, \tilde{D} = \mathrm{index}\, D$, <u>or more explicitly</u>

$$\int_U \sum |\tilde{\phi}_n(\tilde{x})|^2\, d\tilde{\mu} - \int_U \sum_n |\tilde{\psi}_n(\tilde{x})|^2\, d\tilde{\mu} = \mathrm{index}\, D$$

where $\{\tilde{\phi}_n\}$ and $\{\tilde{\psi}_n\}$ are orthonormal bases for the L^2 solutions of $\tilde{D}u = 0$, $\tilde{D}^*v = 0$ respectively, and U is a fundamental domain for the action of Γ on $\tilde{X}$.

§ 4 - The von Neumann algebra.

We continue with the situation of §3 in which Γ is a discrete group acting freely on the manifold $\tilde{X}$ with $X = \tilde{X}/\Gamma$ compact. Then Γ acts unitarily on $L^2(\tilde{X})$ and we consider the algebra $\mathcal{A}$ of all bounded operators on $L^2(\tilde{X})$ which commute with Γ. It is weakly closed and self-adjoint which makes it a von Neumann algebra. Its structure becomes clear if we use a fundamental domain U to make the identification

$$(4.1) \qquad L^2(\tilde{X}) \cong L^2(\Gamma) \otimes L^2(U) \cong L^2(\Gamma) \otimes L^2(X).$$

The action of Γ on $L^2(\tilde{X})$ corresponds by (4.1) to the left regular representation of Γ on $L^2(\Gamma)$ extended by the identity on $L^2(X)$. The commutant of the left regular representation on $L^2(\Gamma)$ is well-known to be (the von Neumann algebra generated by) the right regular representation [5; p. 282]. Hence (cf. [5 ; p. 24]) the commutant $\mathcal{A}$ of Γ acting on $L^2(\tilde{X})$ is given by

$$(4.2) \qquad \mathcal{A} \cong \mathcal{R} \otimes \mathcal{B}$$

where $\mathcal{R}$ is the algebra generated by right translations $\mathcal{R}_\gamma$ on $L^2(\Gamma)$ and $\mathcal{B}$ denotes all bounded operators on $L^2(X)$.

For the algebra $\mathcal{R}$ there is a well-known trace defined on the generators by

$$\begin{aligned} \operatorname{trace} \mathcal{R}_\gamma &= 0 \qquad \gamma \neq 1 \\ &= 1 \qquad \gamma = 1 . \end{aligned}$$

For the algebra $\mathcal{B}$ we have the usual trace of Hilbert space theory (defined on operators of trace class). These two traces then define a trace on the tensor product algebra $\mathcal{A}$.

Instead of proceeding as above we shall give a direct self-contained treatment which does not appeal to the general theory of von Neumann algebras. This will have the added advantage of clarifying the "trace-class" operators of the algebra $\mathcal{A}$, and will be better adapted to the C^∞ framework. We assume as known the usual theory of Hilbert-Schmidt and trace-class operators in Hilbert space.

Any bounded operator A on $L^2(\tilde{X})$ has a Schwartz kernel $A(\tilde{x},\tilde{y})$ which is a distribution on the product $\tilde{X} \times \tilde{X}$. Clearly $A \in \mathcal{A}$ if and only if its kernel is Γ-invariant, i.e.

$$A(\gamma\tilde{x}, \gamma\tilde{y}) = A(\tilde{x},\tilde{y}) \qquad \text{for all } \gamma \in \Gamma.$$

When this is satisfied we may view this kernel as a distribution on the quotient $\frac{\tilde{X} \times \tilde{X}}{\Gamma}$.

We recall that, in the algebra of all bounded operators, an operator is said to be Hilbert-Schmidt if it has an L^2-kernel. For the algebra $\mathcal{A}$ we therefore make the following definition :

Definition (4.3) $A \in \mathcal{A}$ <u>is Γ-Hilbert Schmidt if its kernel is in $L^2(\frac{\tilde{X} \times \tilde{X}}{\Gamma})$. The Γ-HS norm of A is then taken to be the L^2-norm of its kernel</u>.

An alternative definition, easily seen to be equivalent to (4.3), is

Definition (4.3)' $A \in \mathcal{A}$ <u>is Γ-Hilbert-Schmidt if ϕA is Hilbert-Schmidt for all bounded measurable functions ϕ on $\tilde{X}$ with compact support</u>.

The symmetry of (4.3) in the two factors of $\tilde{X}$ shows that

(4.4) A is Γ-Hilbert-Schmidt $\Leftrightarrow A^*$ is Γ-Hilbert-Schmidt.

On the other hand, using (4.3)' and the fact that the usual Hilbert-Schmidt operators form an ideal in the algebra of all bounded operators, we see that

(4.5) A is Γ-Hilbert- Schmidt and $B \in \mathcal{A} \Rightarrow AB$ is Γ-Hilbert-Schmidt.

Taking adjoints and using (4.4) it follows that, in (4.5) we also have BA is Γ-Hilbert-Schmidt, so that the Γ-Hilbert-Schmidt operators form a 2-sided *-ideal of $\mathcal{A}$.

In analogy with the usual theory we now make the following definition :

Definition (4.6) $A \in \mathcal{A}$ *is of* Γ-*trace class if* $A = T_1 T_2$ *with* $T_i \in \mathcal{A}$ *being* Γ-*Hilbert-Schmidt*.

From the properties of Γ-Hilbert-Schmidt operators it follows that the operators of Γ-trace class also form a 2-sided *-ideal of $\mathcal{A}$. Moreover if A is of Γ-trace class then, for any pair ϕ,ψ of bounded measurable functions on $\tilde{X}$ with compact support, we have

$$\phi A \psi = (\phi T_1)(T_2 \psi)$$

the product of two Hilbert-Schmidt operators. Thus

(4.7) A of Γ-trace class and ϕ,ψ bounded measurable functions with compact support $\Rightarrow \phi A \psi$ is of trace class.

For positive operators we also have the converse

LEMMA (4.8) *Let* $A \in \mathcal{A}$ *be a positive self-adjoint operator*, *then the following are equivalent*

i) A *is of* Γ-*trace class*

ii) $\phi A \psi$ *is of trace class for all* $\phi,\psi \in C^\infty_{comp}(\tilde{X})$

iii) $A^{1/2}$ *is* Γ-*Hilbert-Schmidt*.

Proof : i) $\Rightarrow$ ii) is just (4.7). The implication iii) $\Rightarrow$ i) is trivial since $A = A^{1/2} A^{1/2}$. The only point to note is that $A \in \mathcal{A} \Rightarrow A^{1/2} \in \mathcal{A}$ (i.e. $A^{1/2}$ commutes with all $\gamma \in \Gamma$), and this follows for example from the Cauchy integral representation for $A^{1/2}$. It remains to prove ii) $\Rightarrow$ iii). Take $\psi = \bar{\phi}$, then $\phi A \bar{\phi} = (\phi A^{1/2})(\phi A^{1/2})^*$ being of trace class implies that $\phi A^{1/2}$ is Hilbert-Schmidt and so, by (4.3)', $A^{1/2}$ is Γ-Hilbert-Schmidt.

We shall now use (4.7) to introduce the Γ-trace in $\mathcal{A}$. This depends on the following lemma :

LEMMA (4.9) *If* A *is of* Γ-*trace class and* ϕ,ϕ',ψ,ψ' *are bounded measurable functions with compact support on* $\tilde{X}$ *such that*

$$\sum_{\gamma\in\Gamma} \gamma(\phi\psi) = \sum_{\gamma\in\Gamma} \gamma(\phi'\psi') = 1$$

then trace $\phi A\psi$ = trace $\phi' A\psi'$.

Proof : Since ϕ,ϕ' have compact support there is a finite subset S of Γ such that

$$\operatorname{Supp} \gamma(\phi') \cap \operatorname{Supp} \phi \neq \emptyset \Rightarrow \gamma \text{ and } \gamma^{-1} \in S .$$

Hence
$$\sum_{\gamma\in S} \gamma(\phi'\psi')\phi = \sum_{\gamma\in\Gamma} \gamma(\phi'\psi')\phi = \phi$$

and
$$\sum_{\gamma\in S} \gamma(\phi\psi)\phi' = \sum_{\gamma\in\Gamma} \gamma(\phi\psi)\phi' = \phi' .$$

Therefore

$$\operatorname{trace} \phi A\psi = \operatorname{trace} \sum_{\gamma\in S} \gamma(\phi'\psi')\phi A\psi = \sum_{\gamma\in S} \operatorname{trace} \gamma(\phi'\psi')\phi A\psi$$

$$= \sum_{\gamma\in S} \operatorname{trace} \gamma(\phi')\phi\psi A\gamma(\psi') \quad \text{(using trace } ST = \text{trace } TS \text{ for } S \text{ of trace class and } T \text{ bounded)}$$

$$= \sum_{\gamma\in S} \operatorname{trace} \gamma^{-1}(\phi\psi)\phi' A\psi' \quad , \quad \text{since } \gamma(A) = A$$

$$= \text{trace} \sum_{\gamma \varepsilon S} \gamma^{-1}(\phi\psi)\phi' A\psi'$$

$$= \text{trace} \sum_{\gamma \varepsilon \Gamma} \gamma^{-1}(\phi\psi)\phi' A\psi' = \text{trace}\ \phi' A\psi' .$$

In view of this lemma we can now make the following definition :

Definition (4.10) <u>If</u> $A \varepsilon \mathcal{A}$ <u>is of</u> Γ-<u>trace class we put</u>

$$\text{trace}_\Gamma A = \text{trace}\ \phi A \psi$$

<u>for any pair</u> ϕ, ψ <u>of bounded measurable functions with compact support such that</u> $\sum_{\gamma \varepsilon \Gamma} \gamma(\phi\psi) = 1$.

Admissible pairs ϕ, ψ can be obtained either by

i) $\phi = \psi$ = characteristic function of a fundamental domain U for Γ acting on $\tilde{X}$

or ii) ϕ, ψ C^∞ functions on $\tilde{X}$ with compact support such that $\sum_{\gamma \varepsilon \Gamma} \gamma(\phi) = 1$, and $\psi = 1$ on supp ϕ .

Using (i) we shall derive a formula for trace_Γ involving an orthonormal base $\{e_i\}$ for $L^2(U)$. Since the elements $\{\gamma e_i\}$ form an orthonormal base of $L^2(\tilde{X})$, and since

$$\psi\gamma(e_i) = 0 \quad \text{for} \quad \gamma \neq 1$$

$$\psi e_i = e_i$$

(ψ being the characteristic function of U), we have

$$\text{trace}_\Gamma A = \text{trace}\ \psi A \psi = \Sigma \langle \psi A \psi \gamma e_i, \gamma e_i \rangle$$

(4.11) $$= \sum_i \langle A e_i, e_i \rangle \quad .$$

Applying this in particular to $A = T^*T$, where T is Γ-Hilbert-Schmidt, we get

$$||T||_{\Gamma HS} = \mathrm{Trace}_\Gamma\ T^*T = \sum_i \langle T^*Te_i, e_i\rangle = \sum_i ||Te_i||^2$$

$$= \sum_{i,j,\gamma} |\langle Te_i, \gamma e_j\rangle|^2 = \sum_{i,j,\gamma} |\langle e_i, T^*\gamma e_j\rangle|^2$$

$$= \sum_{i,j,\gamma} |\langle \gamma^{-1} e_i, T^* e_j\rangle|^2 \quad (\text{since } \gamma T = T \text{ and } \gamma\gamma^* = 1)$$

$$= \sum ||T^*e_j||^2 = \mathrm{Trace}_\Gamma\ TT^* = ||T^*||_{\Gamma HS}\ . \tag{4.12}$$

From this it is now a standard elementary argument to deduce

PROPOSITION (4.13) _If_ $A \in \mathcal{A}$ _and_ $S \in \mathcal{A}$ _is of_ Γ-_trace class then_ $\mathrm{trace}_\Gamma\ AS = \mathrm{trace}_\Gamma\ SA$

For completeness we recall the details. First let S be positive self-adjoint of Γ-trace class and $A \in \mathcal{A}$ unitary. Then $T = AS^{1/2}$ is Γ-Hilbert-Schmidt and so (4.12) gives

$$\mathrm{trace}_\Gamma\ ASA^{-1} = \mathrm{trace}_\Gamma (AS^{1/2})(AS^{1/2})^* = \mathrm{trace}_\Gamma (AS^{1/2})^*(AS^{1/2}) = \mathrm{trace}_\Gamma\ S\ . \tag{4.14}$$

Now the positive elements span (over $\mathbb{C}$) the whole ideal of Γ-trace class. To see this let $A = A^*$ be of Γ-trace class. Then P_+ the spectral projection corresponding to $\lambda \geqslant 0$ is in $\mathcal{A}$ (use the Cauchy integral formula), hence $A_+ = AP_+$ is of Γ-trace class and is positive. Hence $A = A_+ - A_-$, with $A_- = A(1 - P_+)$, is in the span over R of the positive elements of Γ-trace class. The result over C follows by using the decomposition : $B = 1/2(B + B^*) + \frac{i}{2}\frac{(B - B^*)}{i}$. Hence (4.14) holds for all S of Γ-trace class. Replacing S by SA then gives (4.13) for unitary A. It remains to note that the unitary elements span $\mathcal{A}$. To check this it is enough to consider a self-adjoint element A in $\mathcal{A}$ with $||A|| < 1$. Then $U = A + i(1 - A^2)^{1/2}$ is unitary in $\mathcal{A}$ and $A = 1/2(U + U^*)$.

Formula (4.11) also leads to the following important continuity property of trace_Γ :

PROPOSITION (4.15) Let $S \in \mathcal{A}$ be of Γ-trace class and let $A_j \in \mathcal{A}$ be a sequence of operators converging strongly to $A \in \mathcal{A}$ (i.e. $A_j f \to Af$ for every $f \in L^2(\tilde{X})$). Then

$$\mathrm{trace}_\Gamma\, SA_j \to \mathrm{trace}_\Gamma\, SA .$$

Proof : Since S is of Γ-trace class we have $S = T_1T_2$ where T_i is Γ-Hilbert-Schmidt. Hence, by (4.11),

$$\mathrm{Trace}_\Gamma\, SA_j = \sum_i \langle SA_j e_i, e_i\rangle = \sum_i \langle T_2A_j e_i, T_1^* e_i\rangle .$$

For a fixed i and $j \to \infty$ each term in this series converges to $\langle T_2Ae_i, T_1^* e_i\rangle = \langle SAe_i, e_i\rangle$, so it remains to show uniformity in j. Now the fact that A_j converges strongly to A implies in particular that we have a uniform bound $\|A_j\| \leqslant C$, and so (using (4.12))

$$\|T_2A_j\|_{\Gamma HS} = \|A_j^* T_2^*\|_{\Gamma HS} \leqslant C\|T_2^*\|_{\Gamma HS} .$$

Hence

$$\left\{\sum_N^\infty |\langle T_2A_j e_i, T_1^* e_i\rangle|\right\}^2 \leqslant \left\{\sum_N^\infty \|T_2A_je_i\|^2\right\}\left\{\sum_N^\infty \|T_1^* e_i\|^2\right\}$$

$$\leqslant C\,\|T_2^*\|_{\Gamma HS}\,\varepsilon_N$$

where $\varepsilon_N \to 0$ as $N \to \infty$ (independently of j), giving the desired uniform convergence.

We come now to operators in $\mathcal{A}$ with C^∞ kernel. For these we have :

PROPOSITION (4.16) Let $A \in \mathcal{A}$ have a C^∞ kernel $A(\tilde{x},\tilde{y})$ and assume either

i) A <u>is positive self-adjoint</u>

<u>or</u> ii) A <u>has a kernel with compact support on</u> $\frac{\tilde{X} \times \tilde{X}}{\Gamma}$.

<u>Then</u> A <u>is of</u> Γ-<u>trace class and</u>

$$\operatorname{trace}_\Gamma A = \int_X A(x)\, d\mu$$

<u>where</u> $A(x)$ <u>is the</u> C^∞ <u>function on</u> X <u>defined by the</u> Γ-<u>invariant function</u> $A(\tilde{x},\tilde{x})$ <u>on</u> $\tilde{X}$.

<u>Proof</u> : (i) For any $\phi,\psi \in C^\infty_{\text{comp}}(\tilde{X})$ the operator $\phi A \psi$ has a kernel in $C^\infty_{\text{comp}}(\tilde{X} \times \tilde{X})$, hence is of trace class, and so by (4.9), A is of Γ-trace class. Taking ϕ,ψ such $\Sigma\gamma(\phi) = 1$ and $\psi = 1$ on supp ϕ we have

$$\operatorname{trace}_\Gamma A = \int_{\tilde{X}} \phi(\tilde{x}) A(\tilde{x},\tilde{x})\, d\tilde{\mu} = \int_X A(x)\, d\mu .$$

ii) Let B be a Γ-invariant elliptic differential operator of order $k > \frac{n}{2}$ (e.g. a suitable power of a Laplace-type operator). As in §1 we then construct a pseudo-differential parametrix Q which is almost local (hence has a kernel which is compactly supported on $\frac{\tilde{X} \times \tilde{X}}{\Gamma}$) and of order $-k$. Thus

$$QB = 1 - T$$

with T having a C^∞ kernel compactly supported on $\frac{\tilde{X} \times \tilde{X}}{\Gamma}$. Multiplying by A we get

$$A = TA + QBA.$$

Now A,T,BA are certainly Γ-Hilbert-Schmidt so that AT is of Γ-trace class. Also, since Q is of order $-k$ and $k > \frac{n}{2}$ the kernel of Q is locally in L^2. Since this kernel has compact support on $\frac{\tilde{X} \times \tilde{X}}{\Gamma}$ it is in L^2 of this space, so that Q is also Γ-Hilbert-Schmidt. Hence $Q(BA)$ is of Γ-trace class and hence A is of Γ-trace class. $\operatorname{Trace}_\Gamma A$ is then computed as in case (i).

So far in this section we have only considered the space $L^2(\tilde{X})$ of scalar-valued functions on $\tilde{X}$. For our applications this needs to be generalized to $L^2(\tilde{X},\tilde{E})$ the L^2-sections of a vector bundle $\tilde{E}$ over $\tilde{X}$, induced from a bundle E over $X = \tilde{X}/\Gamma$. Again Γ acts on $L^2(\tilde{X},\tilde{E})$ and the commuting algebra $\mathcal{A}(E)$ is a von Neumann algebra. In fact, from the measure theory point of view every bundle on X is trivial so that $L^2(\tilde{X},\tilde{E}) \cong L^2(\tilde{X}) \otimes \mathrm{End}\, C^N$, the action of Γ being trivial on $\mathrm{End}\, C^N$. Hence $\mathcal{A}(\tilde{E}) \cong \mathcal{A} \otimes \mathrm{End}\, C^N$ and all the results of this section extend immediately to the algebra $\mathcal{A}(\tilde{E})$ with only minor modifications. Thus in (4.16) the function $A(x)$ must now be replaced by the function on X induced by the Γ-invariant function $\mathrm{tr}\, A(\tilde{x},\tilde{x})$ on $\tilde{X}$, where tr is the usual matrix trace taken in $\mathrm{End}\, E_{\tilde{x}}$.

§ 5 - Proof of the index theorem.

With the technical apparatus of §4 we are now in a position to prove our main result, Theorem (3.8), on the lines indicated in §1.

We recall that we have to deal with an elliptic differential operator

$$\tilde{D} : C^\infty(\tilde{X},\tilde{E}) \to C^\infty(\tilde{X},\tilde{F})$$

which is Γ-invariant. In §2 we saw that the projection operator H_0 onto the space $\mathcal{H}(\tilde{D})$ of L^2-solutions of $\tilde{D}u = 0$ had a C^∞-kernel. Since H_0 is clearly Γ-invariant and positive we can apply lemma (4.16)(i) to deduce that it is of Γ-trace class, and that its Γ-trace is given by the appropriate integral formula over X. This identifies $\mathrm{trace}_\Gamma\, H_0$ with $\dim_\Gamma \mathcal{H}(D)$ as originally defined in §3. Similar remarks apply to the adjoint operator $\tilde{D}^*$. Thus Theorem (3.8) asserts

$$\mathrm{trace}_\Gamma\, H_0 - \mathrm{trace}_\Gamma\, H_1 = \mathrm{index}\, D$$

where H_1 is projection onto $\mathcal{H}(\tilde{D}^*)$, and D is the operator on $X = \tilde{X}/\Gamma$.

We now introduce an almost local parametrix $\tilde{Q}$ for $\tilde{D}$, by lifting up a corresponding parametrix Q for D. Thus we have the equations

$$(5.1)\qquad \begin{aligned} QD &= 1 - S_0\,, & DQ &= 1 - S_1 \\ \tilde{Q}\tilde{D} &= 1 - \tilde{S}_0\,, & \tilde{D}\tilde{Q} &= 1 - \tilde{S}_1 \end{aligned}$$

where the S_i are almost local and with C^∞ kernels, and the $\tilde{S}_i$ their lift to $\tilde{X}$. As explained in §1 we then have the index formula on X :

$$\text{index } D = \text{trace } S_0 - \text{trace } S_1$$

Now the operators $\tilde{S}_i$ have C^∞ kernels which are compactly supported on $\frac{\tilde{X} \times \tilde{X}}{\Gamma}$ (because they are almost local) and so, by Lemma (4.16) (ii), they are of Γ-trace class and

$$\text{trace}_\Gamma\, \tilde{S}_i = \text{trace } S_i$$

(both being given by the same integral over X). Hence to prove Theorem (3.8) it will suffice to show that

$$(5.2)\qquad \text{trace}_\Gamma\, H_0 - \text{trace}_\Gamma\, H_1 = \text{trace}_\Gamma\, \tilde{S}_0 - \text{trace}_\Gamma\, \tilde{S}_1\,.$$

From (5.1) we deduce $H_0 = \tilde{S}_0 H_0$ and $H_1 = H_1 \tilde{S}_1$. Putting

$$(5.3)\qquad T_i = (1 - H_i)\,\tilde{S}_i(1 - H_i)$$

and using (4.13), together with $H_i^2 = H_i$, we see that (5.2) can be rewritten as

$$(5.4)\qquad \text{trace}_\Gamma\, T_0 = \text{trace}_\Gamma\, T_1\,.$$

On the other hand composing (5.1) with $\tilde{D}$ gives

$$\tilde{D}\tilde{S}_0 = \tilde{S}_1\tilde{D}$$

which, using (5.1) again, implies

(5.5) $$\tilde{D}T_0 = T_1\tilde{D} .$$

If $\tilde{D}$ were a bounded invertible operator we could apply $\tilde{D}^{-1}$ to (5.5) and then the basic property (4.13) of trace_Γ would yield (5.4). Since neither $\tilde{D}$ nor its inverse (on $\mathcal{H}(\tilde{D}^*)^\perp$) is bounded we proceed as follows.

The self-adjoint operator $\tilde{D}^*\tilde{D}$ has a unique positive square root A and we can then decompose $\tilde{D}$ (as an L^2-operator with domain as in §3) in the form $\tilde{D} = UA$, where U is a partial isometry with

(5.6) $$U^*U = 1 - H_0 \qquad UU^* = 1 - H_1 .$$

Because this decomposition of $\tilde{D}$ is unique the operators U and A must commute with Γ, and so therefore will the spectral projections of A.
Putting

(5.7) $$T_2 = U^*T_1U = U^*(1-H_1)\tilde{S}_1(1-H_1)U = (1-H_1)U^*\tilde{S}_1U(1-H_1)$$

and using (4.13) we get

(5.8) $$\text{trace}_\Gamma\, T_2 = \text{trace}_\Gamma\, T_1,$$

while (5.5) gives

(5.9) $$AT_0 = T_2A .$$

Now let P_n be the spectral projection of A corresponding to the closed interval $[\frac{1}{n},n]$ and put

$$T_{0n} = P_nT_0P_n , \qquad T_{2n} = P_nT_2P_n ,$$

$$A_n = P_n A P_n + (1 - P_n) .$$

Composing (5.9) on both sides with P_n, and recalling that P_n commutes with A, we get

$$(5.10) \qquad A_n T_{0n} = T_{2n} A_n .$$

Now A_n is, by construction, bounded and invertible, hence (5.10) can be written

$$(5.11) \qquad A_n T_{0n} A_n^{-1} = T_{2n} .$$

Since all operators in this equation are Γ-invariant we can take trace_Γ and use (4.13) to deduce

$$(5.12) \qquad \text{trace}_\Gamma\, T_{0n} = \text{trace}_\Gamma\, T_{2n} .$$

Since (for $i = 0,2$)

$$\text{trace}_\Gamma\, T_{in} = \text{trace}_\Gamma\, P_n(T_i P_n) = \text{trace}_\Gamma T_i P_n^{\,2} = \text{trace}_\Gamma\, T_i P_n$$

and since P_n converges strongly to $(1 - H_0)$ as $n \to \infty$, we can apply the continuity property (4.15) of trace_Γ to obtain

$$(5.13) \qquad \lim_{n\to\infty} \text{trace}_\Gamma\, T_{in} = \text{trace}_\Gamma\, T_i(1 - H_0).$$

Because of the formulae (5.3) and (5.7) for T_0, T_2 we have $T_i(1 - H_0) = T_i$ (for $i = 0,2$). Hence (5.3) together with (5.12) gives

$$\text{trace}_\Gamma\, T_0 = \text{trace}_\Gamma\, T_2$$

which, combined with (5.8), leads to the desired equality

$$\text{trace}_\Gamma\, T_0 = \text{trace}_\Gamma\, T_1$$

and completes the proof of Theorem (3.8).

§ 6 - Further remarks.

We shall now discuss a number of generalizations, applications and open questions.

(6.1) von Neumann bundles.

In our covering situation $\tilde{X} \xrightarrow{\pi} X$ we can view $L^2(\tilde{X})$ as $L^2(X,V)$, where V is the vector bundle over X whose fibre at x is the Hilbert space $L^2(\pi^{-1}(x))$. Since $\pi^{-1}(x)$ is a copy of Γ and is acted on by Γ, we see that V is a bundle of $\mathcal{B}$-modules where $\mathcal{B}$ is the von Neumann algebra generated in $L^2(\Gamma)$ by the action of Γ. Similar remarks hold for $L^2(\tilde{X},\tilde{E})$, V now being $L^2(\pi^{-1}(x),E_x)$. Moreover our elliptic operator $\tilde{D}$ can be viewed as acting on the sections of V (with values in sections of a similar bundle W). In fact V,W are flat bundles and $\tilde{D}$ is the natural extension of D.

We see therefore that the situation we have been studying is a special case of an elliptic operator acting on the sections of a bundle of $\mathcal{B}$-modules, where $\mathcal{B}$ is a von Neumann algebra with a finite trace. One can formulate a general index theorem in this context using the K-theory of von Neumann algebras developed by Breuer [4]. Moreover it seems clear that a K-theory proof can be given by reducing to the usual index theorem. This much has been known to the author and I.M. Singer for some time, but the absence of any natural examples deterred us from working out a detailed proof. The case of infinite coverings now provides a very interesting class of examples and so a presentation of the general case might justify the effort. However, before embarking on it in full generality, it seemed worthwhile to give a self-contained account for the case of coverings. The present treatment should be viewed therefore in this larger context.

(6.2) Heat equation methods.

With a little more work it is possible to apply the heat equation methods of [3] to the infinite covering situation. The main point is to show that the kernel $e(t,\tilde{x},\tilde{y})$ of the fundamental solution of $\frac{\partial^2}{\partial t^2} + \tilde{D}^*\tilde{D}$ on $\tilde{X}$ decays sufficiently fast as the distance $\rho(\tilde{x},\tilde{y}) \to \infty$. One can then construct the corresponding kernel $e(t,x,y)$ for $\frac{\partial^2}{\partial t^2} + D^*D$ on X by summing over Γ. For the asymptotic expansion as $t \to 0$ all terms in this sum arising from elements $\gamma \neq 1$ are exponentially small. Thus e and $\tilde{e}$ have the same asymptotic expansions and this leads to Theorem (3.8). This approach is very close to the Selberg trace formula when $\tilde{X}$ is a homogeneous space. Finally one can hope to apply the heat equation method to the more general von Neumann bundle situation described in (6.1).

(6.3) Non-Galois coverings.

If $\tilde{X} \to X$ is a non-Galois covering, i.e. corresponding to a non-normal subgroup of $\pi_1(X)$, Theorem (3.8) no longer applies. However we may ask whether the weaker assertion

$$\text{index } D > 0 \Rightarrow \mathcal{H}(\tilde{D}) \neq 0$$

still holds. Probably this is false in general but counterexamples are not easy to construct.

(6.4) Betti numbers of coverings.

On a compact Riemannian manifold X the Euler characteristic $E(X)$ is equal to the index of the operator $D = d + d^* : \Omega^{ev} \to \Omega^{odd}$, where d is the exterior derivative on forms, d^* its adjoint and $\Omega^{ev} = \oplus\, \Omega^{2q}$, $\Omega^{odd} = \oplus\, \Omega^{2q+1}$ are the spaces of even and odd degree forms. Applying Theorem (3.8) tells us that

$$\text{index}_\Gamma\, \tilde{D} = \text{index } D = E(X).$$

Now, using (3.1), the null spaces $\mathcal{H}(\tilde{D})$ and $\mathcal{H}(\tilde{D}^*)$ can also be identified with the spaces of L^2-harmonic forms of even and odd degree respectively. Moreover, using (3.1) again one can show that the space $H^q(\tilde{X})$ of L^2-harmonic q-forms on $\tilde{X}$ is naturally isomorphic to the L^2-cohomology group $\mathcal{H}^q(\tilde{X})$, where this is defined as $Z^q/\overline{\mathcal{B}^q}$

$$Z^q = \{L^2 \text{ q-forms } u \text{ with } du = 0\}$$

$$\mathcal{B}^q = \{L^2 \text{ q-forms } u \text{ such that } u = dv, \text{ for some } L^2 \text{ (q-1)-form } v\} .$$

Although the L^2 norms depend on the choice of metric the topology of the Hilbert spaces does not, and so $\mathcal{H}^q(\tilde{X})$ is essentially independent of the metric. Hence

$$\dim_\Gamma H^q(\tilde{X}) = \dim_\Gamma \mathcal{H}^q(\tilde{X}) = B^q_\Gamma(\tilde{X})$$

is independent of the choice of metric. It is a real-valued Betti number of $\tilde{X}$ (relative to Γ) and

$$\sum_q (-1)^q B^q_\Gamma(\tilde{X}) = E(X)$$

is an integer.

Note that Poincaré duality holds (for oriented $\tilde{X}$) i.e. $B^q_\Gamma(\tilde{X}) = B^{n-q}_\Gamma(\tilde{X})$ where $n = \dim \tilde{X}$ and, if $n \equiv 0 \bmod 4$, we also have the signature formula

$$\mathrm{Sign}_\Gamma(\tilde{X}) = \mathrm{Sign}(X)$$

where $\mathrm{Sign}_\Gamma(\tilde{X}) = \beta^{2n}_+(\tilde{X}) - \beta^{2n}_-(\tilde{X})$, and $\beta_\pm$ denote the Γ-dimensions of the eigenspaces of $*$. If $\tilde{X}$ has no compact component then $B^0_\Gamma(\tilde{X}) = B^n_\Gamma(\tilde{X}) = 0$ since a constant function cannot be in L^2 unless it is zero. For example if X is a compact Riemann surface of genus $g \geqslant 2$ and $\Gamma = \pi_1(X)$, $\tilde{X}$ the upper half-plane, then only $B^1_\Gamma(\tilde{X}) \neq 0$ and so we must have

$$B^1_\Gamma(\tilde{X}) = 2g - 2 .$$

These real Betti numbers appear to deserve further study. Some natural questions are :

(i) Triangulate X and compute the simplicial L^2 cohomology of $\tilde{X}$ for the lifted triangulation (using cocycles/closure of coboundaries). Are these groups Γ-isomorphic to our $\mathcal{H}^q(\tilde{X})$? †

(ii) If the answer to (i) is yes, are the $B^q_\Gamma(\tilde{X})$ homotopy invariants of X ?

(iii) A priori the numbers $B^q_\Gamma(\tilde{X})$ are real. Give examples where they are not integral and even perhaps irrational.

REFERENCES

1. M.F. ATIYAH and I.M. SINGER, The index of elliptic operators III, Ann. of Math. 87 (1968), 546-604.

2. M.F. ATIYAH and R. BOTT, A Lefschetz fixed-point formula for elliptic complexes I, Ann. of Math. 86 (1967), 374-407.

3. M.F. ATIYAH, R. BOTT and V.K. PATODI, On the heat equation and the index theorem, Inventiones Math. 19 (1973), 279-330.

4. M. BREUER, Fredholm theories in von Neumann algebras I, Math. Ann. 178 (1968), 243-245.

5. J. DIXMIER, Les algèbres d'opérateurs dans l'espace Hilbertien, Gauthier-Villars Paris, 1969.

6. L. HÖRMANDER, Linear partial differential operators, Springer 1963.

7. L. SCHWARTZ, Théorie des distributions I, Hermann Paris 1950.

8. L. SCHWARTZ, Théorie des distributions à valeurs vectorielles I, Ann. Inst. Fourier, Grenoble, 7 (1957) 1-141.

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† (Added in proof) J.Dodzink has shown that the answer is yes.